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Symmetry of Sampling Problem Based on Epistemic Uncertainty and Ellsberg Urn

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Abstract: A general sampling problem can be described by an Ellsberg urn, which is a mathematical model that assumes that balls are randomly drawn from an urn with an uncertain numbers of colored balls. This means that the Ellsberg urn is essentially an intricate model with simultaneous randomness and epistemic uncertainty, and this is the core problem discussed in this paper. Since practical sampling is usually processed in an intricate environment, the solution for an equivalent mathematical problem is necessary. Suppose an Ellsberg urn contains three unknown numbers of colored balls (i.e., a two-degrees-of-freedom Ellsberg urn), and three balls are randomly drawn from the urn. Compared to the published papers, this paper first constructs a chance space with two-dimensional uncertainty space and three-dimensional probability space to rigorously calculate the color distributions for those drawn balls by uncertainty theory, probability theory, and chance theory. Moreover, it is interesting to find that all cases of the drawn balls are symmetric in such a specific situation of a sample problem with epistemic uncertainty.

Keywords: sampling problem; Ellsberg urn; uncertainty theory; probability theory; chance theory

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1. Introduction

In 1923, Eggenberger and Pólya [1] proposed a famous urn problem, which is currently called the Pólya urn problem. Essentially, a Pólya urn is a kind of known urn problem because each number of colored balls is given. However, for the urn presented by Ellsberg [2], it is assumed that it has unknown numbers of colored balls. A scientific problem is how to calculate the likelihood of all cases existing in the Ellsberg urn by a mathematical formula, rather than through psychological experiments. In fact, getting a colored ball from an Ellsberg urn is equivalent to obtaining a sample from a population. This means that the formula for an Ellsberg urn can also be applied to a sampling problem, and this is an innovative method for sampling problems with epistemic uncertainty. Some new aspects for handling epistemic uncertainty can be found in [3,4].

As an axiomatical mathematical tool to deal with epistemic uncertainty, uncertainty theory was proposed by Liu [5] in 2007 and perfected by Liu [6] in 2009. Recently, many scholars have shown that uncertainty theory is more suitable to model indeterminacy than probability theory in some situations, including COVID-19 spread (Chen et al. [7], Jia and Chen [8], Lio and Liu [9], Liu [10], Ye and Yang [11]), circuit systems (Liu [12]), chemical reactions (Tang and Yang [13]), exchange rates (Ye and Liu [14]), stock prices (Liu and Liu [15]), and birth rates (Ye and Zheng [16]).

Roughly speaking, probability theory is suitable for randomness and frequencies, and uncertainty theory is suitable for (epistemic) uncertainty and degrees of human belief (Liu [17]). Moreover, an intricate system usually contains randomness and (epistemic) uncertainty. To handle such a system, Liu [18] founded chance theory and proposed the concepts of the chance measure, uncertain random variable, and chance distribution. Uncertain random programming is one of the important applications of chance theory, and its developments can be found in the works by Zhou et al. [19], Qin [20], and Ke et al. [21].

Liu [22] demonstrated that the unknown numbers of the colored balls in an Ellsberg urn are uncertain variables, and it is clear that the randomly drawn colored balls are random variables. This means that the Ellsberg urn problem is an uncertain random system and should be handled by chance theory. To discriminate between the Ellsberg urn based on the axiomatical mathematical method and the traditional psychological methods, the urn problem discussed in this paper is also called an uncertain urn problem. The first work about the uncertain urn problem was completed by Liu [22]. After that, Lio and Cheng [23] developed the uncertain urn problem with the assumption that three numbers of colored balls are unknown in the urn, suggesting an uncertain urn problem with two degrees of freedom. As the most general case of Ellsberg urn, Ye and Jia [24] evaluated the color distribution for the uncertain urn problem with k degrees of freedom.

For an Ellsberg urn with multiple drawn balls, Lio and Cheng [25] first used a three-dimensional probability space to characterize the case when three colored balls are randomly drawn, but their method can only solve the one-degree-of-freedom uncertain urn problem. This means that none of the existing works can solve an uncertain urn problem, which is one that has multiple drawn balls and multiple degrees of freedom simultaneously. Therefore, this paper is the first work to provide formulas for the color distributions of three drawn balls from the uncertain urn problem with two degrees of freedom, and it is interesting to find that such an urn problem is equivalent to a symmetric sampling problem. The rest of the paper is organized as follows: Section 2 introduces preliminaries for chance theory to solve the uncertain urn problem. Section 3 constructs a chance space with two-dimensional uncertainty space and three-dimensional probability space to model the uncertain urn problem. From Sections 4–6, results for ten cases of drawn colored balls are obtained based on their chance measures, and the symmetry of the sampling problem is shown. Finally, a brief conclusion is provided in Section 7.

2. Preliminaries

To model an intricate system that contains both uncertainty and randomness, Liu [18] founded chance theory in 2013 and Liu [26] further verified the operational law for the uncertain random system. Some basic definitions and theorems of chance theory are introduced in this section.

Let Γ and Ω be nonempty sets. Let $\mathcal{L}$ and $\mathcal{F}$ be σ-algebras over Γ and Ω, respectively. Let $\mathcal{M}$ be an uncertain measure, and Pr be a probability measure. Then, $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space, $(\Omega, \mathcal{F}, \Pr)$ is a probability space, and

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \Pr) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times \Pr)$$

is a chance space. The concept of the chance measure (Liu [18]) is defined by

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} \mathrm{d}x \tag{1}$$

where Θ is any event in the product σ-algebra $\mathcal{L} \times \mathcal{F}$. The chance measure is proven to satisfy

$$\mathrm{Ch}\{\Gamma \times \Omega\} = 1. \tag{2}$$

In addition, for any event Θ, it is a monotone increasing function with respect to Θ, and self-dual holds, that is,

$$\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1. \tag{3}$$

After that, Hou [27] proved that chance measure satisfies the property of subadditivity, that is, for every countable sequence of events $\Theta_1, \Theta_2, \cdots$, the formula

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\} \tag{4}$$

holds. Based on the chance measure, the concept of uncertain random variable (Liu [18]) can be defined. Essentially, the uncertain random variable is a measurable function defined on chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \Pr)$ and takes a value in the set of real numbers. To describe an uncertain random variable ξ, the concept of chance distribution (Liu [18]) is defined as $\Phi(x) = \mathrm{Ch}\{\xi \le x\}$, $\forall x \in \Re$, and its inverse function $\Phi^{-1}(\alpha)$, $\forall \alpha \in (0,1)$ is called an inverse chance distribution.

For the operational law (Liu [26]) of chance theory, suppose $\eta_1, \eta_2, \cdots, \eta_m$ are independent random variables that have probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and $\tau_1, \tau_2, \cdots, \tau_n$ are independent uncertain variables that have regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots \Upsilon_n$, respectively. For a strictly monotone function f that is increasing with respect to $\tau_1, \cdots, \tau_k$ and decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, the chance distribution of uncertain random variable $\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ can be calculated by

$$\Phi(x) = \int_{\Re^m} g(x; y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{5}$$

where $g(x; y_1, \cdots, y_m)$ represents the root α of

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = x.$$

For the expected value (Liu [18]) of an uncertain random variable ξ, it is determined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \ge x\} \mathrm{d}x - \int_{-\infty}^{0} \mathrm{Ch}\{\xi \le x\} \mathrm{d}x \tag{6}$$

if at least one of the above integrals is finite. Moreover, for the uncertain random variable ξ, which has inverse uncertainty distribution $\Phi^{-1}(\alpha)$, the expected value can be calculated by

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) \mathrm{d}\alpha. \tag{7}$$

3. Problem Description and Formulation

For an uncertain urn problem that includes 100 balls that are either red, black, or yellow, the colored balls are marked from 1 to 100, respectively. Without loss of generality, the order of the balls in the above Ellsberg urn is assumed to be first black, then yellow, and finally red. Then, we draw three balls from the urn, and the core problem is the color distribution of the drawn balls.

To solve this problem, an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is assumed to be

$$\Gamma = \{(\gamma_1, \gamma_2) \mid \gamma_1, \gamma_2 \in \{1, 2, \cdots, 100\}, \gamma_1 + \gamma_2 \le 100\}$$

with power set, and the uncertain measure satisfies

$$\mathcal{M}\{\Lambda\} = \frac{|\Lambda|}{C_{102}^2}$$

where Λ is an event, and $|\Lambda|$ represents its cardinality. Therefore, the number of black balls is an uncertain variable

$$\tau_1(\gamma_1, \gamma_2) = \gamma_1 \tag{8}$$

and the number of yellow balls is an uncertain variable

$$\tau_2(\gamma_1, \gamma_2) = \gamma_2. \tag{9}$$

Since the remaining balls should be red, the number of red balls is an uncertain variable

$$\tau_3(\gamma_1, \gamma_2) = 100 - \gamma_1 - \gamma_2. \tag{10}$$

According to the above assumption for the order of balls, the black balls should be marked from 1 to γ_1, the yellow balls should be marked from γ_1+1 to $\gamma_1+\gamma_2$, and the red balls should be marked from $\gamma_1+\gamma_2+1$ to 100.

Since the balls are drawn from an uncertain urn, this is a random method to obtain three drawn balls. To describe the randomly drawn balls, a probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ is assumed to be

$$\Omega = \{(\omega_1, \omega_2, \omega_3) \mid \omega_1, \omega_2, \omega_3 \in \{1, 2, \cdots, 100\}, \omega_1 \neq \omega_2, \omega_2 \neq \omega_3, \omega_1 \neq \omega_3\}$$

with power set, and the probability measure satisfies

$$\mathrm{Pr}\{\Lambda\} = \frac{|\Lambda|}{100 \times 99 \times 98}$$

where Λ is an event and $|\Lambda|$ represents its cardinality. This means that the uncertain urn problem discussed in this paper should be handled by probability theory (randomly drawn balls) and uncertainty theory (unknown numbers of colored balls), and the events for three drawn balls come from the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \mathrm{Pr})$.

4. Three Cases with the Same Color

This section calculates the color distributions of the cases in which the drawn balls are all the same color, that is, drawing three black balls, three yellow balls, and three red balls.

4.1. Three Drawn Black Balls

This subsection considers the case that there are three black balls drawn from the uncertain urn. The result will show the symmetry of the urn problem discussed in this paper.

Let us represent the event that three black balls are drawn from the uncertain urn by

$$\text{“3b”} = \{(\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \Gamma \times \Omega \mid \omega_1, \omega_2, \omega_3 \leq \gamma_1\}. \tag{11}$$

Then, we calculate the chance measure of event “3b”,

$$\begin{aligned}
&\mathrm{Ch}\{\text{“3b”}\} \\
&= \int_0^1 \mathrm{Pr}\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1, \gamma_2) \in \Gamma \mid (\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \text{“3b”}\} \geq x\}\mathrm{d}x \\
&= 6\int_0^1 \mathrm{Pr}\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1, \gamma_2) \in \Gamma \mid \omega_1 < \omega_2 < \omega_3 \leq \gamma_1\} \geq x\}\mathrm{d}x \\
&= 6\int_0^1 \mathrm{Pr}\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \frac{(102-\omega_3)(101-\omega_3)}{2\mathrm{C}_{102}^2} \geq x,\ \omega_1 < \omega_2 < \omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97} \int_{\frac{(j+1)j}{2\mathrm{C}_{102}^2}}^{\frac{(j+1)(j+2)}{2\mathrm{C}_{102}^2}} \mathrm{Pr}\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \omega_3 \leq 100 - j\}\mathrm{d}x \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \mathrm{Pr}\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \omega_1 = k+1, \omega_3 \leq 100 - j\} \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \frac{(99-j-k)(98-j-k)}{2} \frac{1}{100 \times 99 \times 98} \\
&= \frac{1}{10}.
\end{aligned}$$

4.2. Three Drawn Yellow Balls

This subsection considers the case that there are three yellow balls drawn from the uncertain urn. The result still shows the symmetry of the discussed urn problem.

Let us represent the event that three yellow balls are drawn from the uncertain urn by

$$\text{“3y”} = \{(\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \Gamma \times \Omega \mid \gamma_1 < \omega_1, \omega_2, \omega_3 \le \gamma_1 + \gamma_2\}. \tag{12}$$

Then, we calculate the chance measure of event “3y”,

$$\begin{aligned}
&\mathrm{Ch}\{\text{“3y”}\} \\
&= \int_0^1 \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1, \gamma_2) \in \Gamma \mid (\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \text{“3y”}\} \ge x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \\
&\qquad\qquad \mathcal{M}\{(\gamma_1, \gamma_2) \in \Gamma \mid \gamma_1 < \omega_1 < \omega_2 < \omega_3 \le \gamma_1 + \gamma_2\} \ge x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \frac{\omega_1(101 - \omega_3)}{\mathrm{C}_{102}^2} \ge x,\ \omega_1 < \omega_2 < \omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97}\sum_{k=0}^{97-j}\int_{\frac{(j+1)k}{\mathrm{C}_{102}^2}}^{\frac{(j+1)(k+1)}{\mathrm{C}_{102}^2}} \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \omega_1 = j+1, \omega_3 \le 100 - k\}\mathrm{d}x \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \omega_1 = j+1, \omega_3 \le 100 - k\} \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \frac{(99 - j - k)(98 - j - k)}{2}\frac{1}{100 \times 99 \times 98} \\
&= \frac{1}{10}.
\end{aligned}$$

This means that the chance measure of the event that three yellow balls are drawn is equal to the chance measure of the event that three black balls are drawn.

4.3. Three Drawn Red Balls

This subsection considers the case that there are three red balls drawn from the uncertain urn. The symmetry of the urn problem holds in this situation.

Let us represent the event that three red balls are drawn from the uncertain urn by

$$\text{“3r”} = \{(\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \Gamma \times \Omega \mid \omega_1, \omega_2, \omega_3 > \gamma_1 + \gamma_2\}. \tag{13}$$

Then, we calculate the chance measure of event "3r",

$$
\begin{aligned}
&\mathrm{Ch}\{\text{“3r”}\} \\
&= \int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2) \in \Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3) \in \text{“3r”}\} \geq x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2) \in \Gamma \mid \gamma_1+\gamma_2 < \omega_1 < \omega_2 < \omega_3\} \geq x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \frac{\omega_1(\omega_1+1)}{2\mathrm{C}_{102}^2} \geq x,\ \omega_1 < \omega_2 < \omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97} \int_{\frac{(j+1)j}{2\mathrm{C}_{102}^2}}^{\frac{(j+1)(j+2)}{2\mathrm{C}_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \omega_1 \geq j+1\}\mathrm{d}x \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \omega_1 \geq j+1, \omega_3 = 100-k\} \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \frac{(99-j-k)(98-j-k)}{2}\frac{1}{100\times 99\times 98} \\
&= \frac{1}{10}.
\end{aligned}
$$

It can be concluded that the calculations for drawing three black balls, yellow balls, and red balls are symmetric, and the result is consistent with our intuition.

5. Six Cases with Two Different Colors

This section calculates the color distributions of the cases where the drawn balls are of two different colors, that is, drawing one black ball and two yellow balls, two black balls and one yellow ball, one black ball and two red balls, two black balls and one red ball, one yellow ball and two red balls, and two yellow balls and one red ball.

5.1. One Drawn Black Ball and Two Drawn Yellow Balls

This subsection considers the case where one black ball and two yellow balls are drawn from the uncertain urn. It is interesting that the symmetry of the urn problem still holds, even in this case.

Let us represent the event that one black ball and two yellow balls are drawn from the uncertain urn by

$$
\text{“b2y”} = \bigcup_{i\neq j\neq k,\ i,j,k\in\{1,2,3\}} \{(\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3) \in \Gamma\times\Omega \mid \omega_i \leq \gamma_1 < \omega_j < \omega_k \leq \gamma_1+\gamma_2\}. \tag{14}
$$

Then, we calculate the chance measure of event "b2y",

$$
\begin{aligned}
&\mathrm{Ch}\{\text{"b2y"}\}\\
&= \int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2) \in \Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3) \in \text{"b2y"}\} \geq x\}\mathrm{d}x\\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \\
&\qquad\qquad \mathcal{M}\{(\gamma_1,\gamma_2) \in \Gamma \mid \omega_1 \leq \gamma_1 < \omega_2 < \omega_3 \leq \gamma_1+\gamma_2\} \geq x\}\mathrm{d}x\\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \frac{(101-\omega_3)(\omega_2-\omega_1)}{\mathrm{C}_{102}^2} \geq x,\ \omega_1 < \omega_2 < \omega_3\}\mathrm{d}x\\
&= 6\sum_{j=0}^{97}\sum_{k=0}^{97-j}\int_{\frac{(j+1)k}{\mathrm{C}_{102}^2}}^{\frac{(j+1)(k+1)}{\mathrm{C}_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \omega_2 = \omega_1+j+1, 101-\omega_3 \geq k+1\}\mathrm{d}x\\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \Pr\{(\omega_1,\omega_2,\omega_3) \in \Omega \mid \omega_2 = \omega_1+j+1, 101-\omega_3 \geq k+1\}\\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j} \frac{(99-j-k)(98-j-k)}{2}\frac{1}{100\times 99\times 98}\\
&= \frac{1}{10}.
\end{aligned}
$$

This means that the chance measure of the event that one black ball and two yellow balls are drawn is equal to the chance measure of the event that three black balls, three yellow balls, or three red balls are drawn. This may be a result that should be further illustrated.

5.2. Two Drawn Black Balls and One Drawn Yellow Ball

This subsection considers the case in which two black balls and one yellow ball are drawn from the uncertain urn. This result and the chance measure of the event that one black ball and two yellow balls are drawn should be symmetric.

Let us represent the event that two black balls and one yellow ball are drawn from the uncertain urn by

$$
\text{"2by"} = \bigcup_{i\neq j\neq k,\ i,j,k\in\{1,2,3\}} \{(\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3) \in \Gamma\times\Omega \mid \omega_i < \omega_j \leq \gamma_1 < \omega_k \leq \gamma_1+\gamma_2\}. \tag{15}
$$

Then, we calculate the chance measure of event "2by",

$$
\begin{aligned}
&\mathrm{Ch}\{\text{“2by”}\} \\
&= \int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in \text{“2by”}\}\ge x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \\
&\qquad \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid \omega_1<\omega_2\le\gamma_1<\omega_3\le\gamma_1+\gamma_2\}\ge x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \frac{(101-\omega_3)(\omega_3-\omega_2)}{C_{102}^2}\ge x,\ \omega_1<\omega_2<\omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97}\sum_{k=0}^{97-j}\int_{\frac{(j+1)k}{C_{102}^2}}^{\frac{(j+1)(k+1)}{C_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_3=\omega_2+j+1, 101-\omega_3\ge k+1\}\mathrm{d}x \\
&= \frac{6}{C_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_3=\omega_2+j+1, 101-\omega_3\ge k+1\} \\
&= \frac{6}{C_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\frac{(99-j-k)(98-j-k)}{2}\frac{1}{100\times 99\times 98} \\
&= \frac{1}{10}.
\end{aligned}
$$

5.3. One Drawn Black Ball and Two Drawn Red Balls

This subsection considers the case in which one black ball and two red balls are drawn from the uncertain urn. This result and the chance measure of the event that one black ball and two yellow balls are drawn should also be symmetric.

Let us represent the event that one black ball and two red balls are drawn from the uncertain urn by

$$
\text{“b2r”} = \bigcup_{i\ne j\ne k,\ i,j,k\in\{1,2,3\}} \{(\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in\Gamma\times\Omega \mid \omega_i\le\gamma_1\le\gamma_1+\gamma_2<\omega_j<\omega_k\}. \quad (16)
$$

Then, we calculate the chance measure of event "b2r",

$$
\begin{aligned}
&\mathrm{Ch}\{\text{“b2r”}\} \\
&= \int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in \text{“b2r”}\}\ge x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \\
&\qquad \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid \omega_1\le\gamma_1\le\gamma_1+\gamma_2<\omega_2<\omega_3\}\ge x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \frac{(\omega_2-\omega_1+1)(\omega_2-\omega_1)}{2C_{102}^2}\ge x,\ \omega_1<\omega_2<\omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97}\int_0^{\frac{(j+1)(j+2)}{2C_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_2=\omega_1+j+1\}\mathrm{d}x \\
&= \frac{3}{C_{102}^2}\sum_{j=0}^{97}(j+1)(j+2)\Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_2=\omega_1+j+1\} \\
&= \frac{3}{C_{102}^2}\sum_{j=0}^{97}(j+1)(j+2)\frac{(99-j)(98-j)}{2}\frac{1}{100\times 99\times 98} \\
&= \frac{1}{10}.
\end{aligned}
$$

5.4. Two Drawn Black Balls and One Drawn Red Ball

This subsection considers the case in which two black balls and one red ball are drawn from the uncertain urn. This result and the chance measure of the event that one black ball and two red balls are drawn should be symmetric.

Let us represent the event that two black balls and one red ball are drawn from the uncertain urn by

$$\text{“2br”} = \bigcup_{i \neq j \neq k,\ i,j,k \in \{1,2,3\}} \{(\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \Gamma \times \Omega \mid \omega_i < \omega_j \leq \gamma_1 \leq \gamma_1 + \gamma_2 < \omega_k\}. \quad (17)$$

Then, we calculate the chance measure of event “2br”,

$$\begin{aligned}
&\mathrm{Ch}\{\text{“2br”}\} \\
&= \int_0^1 \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \mathcal{M}\{(\gamma_1, \gamma_2) \in \Gamma \mid (\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \text{“2br”}\} \geq x\}\mathrm{d}x \\
&= 6 \int_0^1 \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \\
&\qquad\qquad \mathcal{M}\{(\gamma_1, \gamma_2) \in \Gamma \mid \omega_1 < \omega_2 \leq \gamma_1 \leq \gamma_1 + \gamma_2 < \omega_3\} \geq x\}\mathrm{d}x \\
&= 6 \int_0^1 \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \frac{(\omega_3 - \omega_2 + 1)(\omega_3 - \omega_2)}{2\mathrm{C}_{102}^2} \geq x,\ \omega_1 < \omega_2 < \omega_3\}\mathrm{d}x \\
&= 6 \sum_{j=0}^{97} \int_0^{\frac{(j+1)(j+2)}{2\mathrm{C}_{102}^2}} \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \omega_3 = \omega_2 + j + 1\}\mathrm{d}x \\
&= \frac{3}{\mathrm{C}_{102}^2} \sum_{j=0}^{97} (j+1)(j+2) \Pr\{(\omega_1, \omega_2, \omega_3) \in \Omega \mid \omega_3 = \omega_2 + j + 1\} \\
&= \frac{3}{\mathrm{C}_{102}^2} \sum_{j=0}^{97} (j+1)(j+2) \frac{(99-j)(98-j)}{2} \frac{1}{100 \times 99 \times 98} \\
&= \frac{1}{10}.
\end{aligned}$$

5.5. One Drawn Yellow Ball and Two Drawn Red Balls

This subsection considers the case in which one yellow ball and two red balls are drawn from the uncertain urn. Similarly, it should be another symmetric case for the uncertain urn problem.

Let us represent the event that one yellow ball and two red balls are drawn from the uncertain urn by

$$\text{“y2r”} = \bigcup_{i \neq j \neq k,\ i,j,k \in \{1,2,3\}} \{(\gamma_1, \gamma_2, \omega_1, \omega_2, \omega_3) \in \Gamma \times \Omega \mid \gamma_1 < \omega_i \leq \gamma_1 + \gamma_2 < \omega_j < \omega_k\}. \quad (18)$$

Then, we calculate the chance measure of event "y2r",

$$\begin{aligned}
&\mathrm{Ch}\{\text{"y2r"}\} \\
&= \int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in \text{"y2r"}\}\geq x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \\
&\qquad \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid \gamma_1<\omega_1\leq\gamma_1+\gamma_2<\omega_2<\omega_3\}\geq x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \frac{\omega_1(\omega_2-\omega_1)}{\mathrm{C}_{102}^2}\geq x,\ \omega_1<\omega_2<\omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97}\sum_{k=0}^{97-j}\int_{\frac{(j+1)k}{\mathrm{C}_{102}^2}}^{\frac{(j+1)(k+1)}{\mathrm{C}_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_2=\omega_1+j+1,\omega_1\geq k+1\}\mathrm{d}x \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_2=\omega_1+j+1,\omega_1\geq k+1\} \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\frac{(99-j-k)(98-j-k)}{2}\frac{1}{100\times 99\times 98} \\
&= \frac{1}{10}.
\end{aligned}$$

5.6. Two Drawn Yellow Balls and One Drawn Red Ball

This subsection considers the case in which two yellow balls and one red ball are drawn from the uncertain urn. This result and the chance measure of the event that one yellow ball and two red balls are drawn should be symmetric.

Let us represent the event that two yellow balls and one red ball are drawn from the uncertain urn by

$$\text{"2yr"} = \bigcup_{i\neq j\neq k,\ i,j,k\in\{1,2,3\}} \{(\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in\Gamma\times\Omega \mid \gamma_1<\omega_i<\omega_j\leq\gamma_1+\gamma_2<\omega_k\}. \tag{19}$$

Then, we calculate the chance measure of event "2yr",

$$\begin{aligned}
&\mathrm{Ch}\{\text{"2yr"}\} \\
&= \int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in \text{"2yr"}\}\geq x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \\
&\qquad \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid \gamma_1<\omega_1<\omega_2\leq\gamma_1+\gamma_2<\omega_3\}\geq x\}\mathrm{d}x \\
&= 6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \frac{\omega_1(\omega_3-\omega_2)}{\mathrm{C}_{102}^2}\geq x,\ \omega_1<\omega_2<\omega_3\}\mathrm{d}x \\
&= 6\sum_{j=0}^{97}\sum_{k=0}^{97-j}\int_{\frac{(j+1)k}{\mathrm{C}_{102}^2}}^{\frac{(j+1)(k+1)}{\mathrm{C}_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_3=\omega_2+j+1,\omega_1\geq k+1\}\mathrm{d}x \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_3=\omega_2+j+1,\omega_1\geq k+1\} \\
&= \frac{6}{\mathrm{C}_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\frac{(99-j-k)(98-j-k)}{2}\frac{1}{100\times 99\times 98} \\
&= \frac{1}{10}.
\end{aligned}$$

6. The Case with One Drawn Black Ball, Yellow Ball, and Red Ball

This section calculates the color distribution of the remaining case, that is, drawing one black ball, one yellow ball, and one red ball. This should be a special situation, but by the duality, it is interesting that the chance measure of the event that one black ball, one yellow ball, and one red ball are drawn and all the other cases are symmetric.

Let us represent the event that one black ball, one yellow ball, and one red ball are drawn from the uncertain urn by

$$\text{“byr”} = \bigcup_{i\neq j\neq k,\ i,j,k\in\{1,2,3\}} \{(\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in\Gamma\times\Omega \mid \omega_i\le\gamma_1<\omega_j\le\gamma_1+\gamma_2<\omega_k\}. \tag{20}$$

Then, we calculate the chance measure of event “byr”,

$$\begin{aligned}
&\mathrm{Ch}\{\text{“byr”}\}\\
&=\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid (\gamma_1,\gamma_2,\omega_1,\omega_2,\omega_3)\in\text{“byr”}\}\ge x\}\mathrm{d}x\\
&=6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid\\
&\qquad\qquad \mathcal{M}\{(\gamma_1,\gamma_2)\in\Gamma \mid \omega_1\le\gamma_1<\omega_2\le\gamma_1+\gamma_2<\omega_3\}\ge x\}\mathrm{d}x\\
&=6\int_0^1 \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \frac{(\omega_3-\omega_2)(\omega_2-\omega_1)}{C_{102}^2}\ge x,\ \omega_1<\omega_2<\omega_3\}\mathrm{d}x\\
&=6\sum_{j=0}^{97}\sum_{k=0}^{97-j}\int_{\frac{(j+1)k}{C_{102}^2}}^{\frac{(j+1)(k+1)}{C_{102}^2}} \Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_2=\omega_1+j+1,\omega_3-\omega_2\ge k+1\}\mathrm{d}x\\
&=\frac{6}{C_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\Pr\{(\omega_1,\omega_2,\omega_3)\in\Omega \mid \omega_2=\omega_1+j+1,\omega_3-\omega_2\ge k+1\}\\
&=\frac{6}{C_{102}^2}\sum_{j=0}^{97}(j+1)\sum_{k=0}^{97-j}\frac{(99-j-k)(98-j-k)}{2}\frac{1}{100\times 99\times 98}\\
&=\frac{1}{10}.
\end{aligned}$$

This means that all chance measures of the ten cases are equal, and the results are symmetric. These counter-intuitive results are caused by the consideration that the numbers of black balls, yellow balls, and red balls are approximately equal. However, in the practical case, since the colored balls in an Ellsberg urn are completely unknown and depend on the human beings who placed the balls inside, it is possible that the urn contains 100 black balls. In this situation, drawing three black balls must be much more possible than any other cases. The results reveal that the measures of the ten cases are symmetric in the sense of the “average”.

Actually, the numbers of black balls, yellow balls, and red balls in the uncertain urn are usually not equal (the above distribution is used because we do not have any other information). It was shown by Liu [22] that probability theory leads to a disastrous decision for the unknown numbers of colored balls in an Ellsberg urn if the practical color distribution is far away from the estimated color distribution. The result obtained by Liu [22] is a special case of the phenomenon described in this paper when two numbers of the colored balls (say black balls and yellow balls) are unknown and one ball is drawn from the uncertain urn.

7. Conclusions

This paper constructed a chance space with two-dimensional uncertainty space and three-dimensional probability space to characterize a two-degrees-of-freedom Ellsberg urn with three randomly drawn balls. Furthermore, by separating the drawn balls into ten cases, a rigorous proof was provided to demonstrate that all the cases have the same chance measure. This result represents a symmetric situation of a sample problem with epistemic uncertainty.

There was a lack of axiomatical mathematical methods in the traditional works on the Ellsberg urn. Recently, uncertainty theory, probability theory, and chance theory have been proposed to solve the Ellsberg urn problem mathematically. However, all the existing works fail to calculate the color distribution when there are multiple drawn balls and unknown numbers of the colored balls. The most important contribution of this paper is the symmetric result for the case with three drawn balls and two degrees of freedom. Unfortunately, we still cannot assure that the result can be extended to high-dimensional cases, such as an urn problem with four drawn balls or three degrees of freedom. This disadvantage leads to a limitation for the obtained color distribution, and future research for the general case (i.e., j drawn balls and k degrees of freedom) should be carried out based on the results provided in this paper. Furthermore, useful future research directions include practical applications for the calculated color distributions of the drawn balls according to data analysis methods such as uncertain regression, uncertain time series, and uncertain differential equation.

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References

1. Eggenberger, F.; Pólya, G. Über die statistik verketteter vorgäge. *Z. Angew. Math. Mech.* **1923**, *3*, 279–289. [CrossRef]
2. Ellsberg, D. Risk, ambiguity, and the savage axioms. *Q. J. Econ.* **1961**, *75*, 643–669. [CrossRef]
3. Bataineh, M.; Alaroud, M.; Al-Omari, S.; Agarwal, P. Series representations for uncertain fractional IVPs in the fuzzy conformable fractional sense. *Entropy* **2021**, *23*, 1646. [CrossRef] [PubMed]
4. Fox, C.; Goedde-Menke, M.; Tannenbaum, D. Ambiguity Aversion and Epistemic Uncertainty. 2021. Available online: https://ssrn.com/abstract=3922716 (accessed on 8 September 2021).
5. Liu, B. *Uncertainty Theory*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2007.
6. Liu, B. Some research problems in uncertainty theory. *J. Uncertain Syst.* **2009**, *3*, 3–10.
7. Chen, X.; Li, J.; Xiao, C.; Yang, P. Numerical solution and parameter estimation for uncertain SIR model with application to COVID-19 pandemic. *Fuzzy Optim. Decis. Mak.* **2021**, *20*, 189–208. [CrossRef]
8. Jia, L.; Chen, W. Uncertain SEIAR model for COVID-19 cases in China. *Fuzzy Optim. Decis. Mak.* **2021**, *20*, 243–259. [CrossRef]
9. Lio, W.; Liu, B. Initial value estimation of uncertain differential equations and zero-day of COVID-19 spread in China. *Fuzzy Optim. Decis. Mak.* **2021**, *20*, 177–188. [CrossRef]
10. Liu, Z. Uncertain growth model for the cumulative number of COVID-19 infections in China. *Fuzzy Optim. Decis. Mak.* **2021**, *20*, 229–242. [CrossRef]
11. Ye, T.; Yang, X. Analysis and prediction of confirmed cases of COVID-19 in China by uncertain time series. *Fuzzy Optim. Decis. Mak.* **2021**, *20*, 209–228. [CrossRef]
12. Liu, Y. Uncertain circuit equation. *J. Uncertain Syst.* **2021**, *14*, 2150018. [CrossRef]
13. Tang, H.; Yang, X. Uncertain chemical reaction equation. *Appl. Math. Comput.* **2021**, *411*, 126479. [CrossRef]
14. Ye, T.; Liu, B. Uncertain hypothesis test for uncertain differential equations. *Fuzzy Optim. Decis. Mak.* **2022**. [CrossRef]

15. Liu, Y.; Liu, B. Residual analysis and parameter estimation of uncertain differential equations. *Fuzzy Optim. Decis. Mak.* **2022**. [CrossRef]
16. Ye, T.; Zheng, H. Analysis of birth rates in China with uncertain time series model. *Tech. Rep.* **2022**.
17. Liu, B. Why is there a need for uncertainty theory? *J. Uncertain Syst.* **2012**, *6*, 3–10.
18. Liu, Y. Uncertain random variables: A mixture of uncertainty and randomness. *Soft Comput.* **2013**, *17*, 625–634. [CrossRef]
19. Zhou, J.; Yang, F.; Wang, K. Multi-objective optimization in uncertain random environments. *Fuzzy Optim. Decis. Mak.* **2014**, *13*, 397–413. [CrossRef]
20. Qin, Z. Uncertain random goal programming. *Fuzzy Optim. Decis. Mak.* **2018**, *17*, 375–386. [CrossRef]
21. Ke, H.; Su, T.; Ni, Y. Uncertain random multilevel programming with application to product control problem. *Soft Comput.* **2015**, *19*, 1739–1746. [CrossRef]
22. Liu, B. Uncertain urn problems and Ellsberg experiment. *Soft Comput.* **2019**, *23*, 6579–6584. [CrossRef]
23. Lio, W.; Cheng, G. T wo-degree-of-freedom Ellsberg urn problem. *Soft Comput.* **2020**, *24*, 6903–6908. [CrossRef]
24. Ye, T.; Jia, L. Ellsberg urn problem with multiple degrees of freedom. *J. Intell. Fuzzy Syst.* **2019**, *37*, 8267–8273. [CrossRef]
25. Lio, W.; Cheng, G. Color distribution of three drawn balls from Ellsberg urn. *J. Ambient. Intell. Humaniz. Comput.* **2021**, *12*, 3169–3176. [CrossRef]
26. Liu, Y. Uncertain random programming with applications. *Fuzzy Optim. Decis. Mak.* **2013**, *12*, 153–169. [CrossRef]
27. Hou, Y. Subadditivity of chance measure. *J. Uncertain. Anal. Appl.* **2014**, *2*, 14. [CrossRef]