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Optimal Test Plan of Step-Stress Model of Alpha Power Weibull Lifetimes under Progressively Type-II Censored Samples

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Abstract: In this study, the estimation of the unknown parameters of an alpha power Weibull (APW) distribution using the concept of an optimal strategy for the step-stress accelerated life testing (SSALT) is investigated from both classical and Bayesian viewpoints. We used progressive type-II censoring and accelerated life testing to reduce testing time and costs, and we used a cumulative exposure model to examine the impact of various stress levels. A log-linear relation between the scale parameter of the APW distribution and the stress model has been proposed. Maximum likelihood estimators for model parameters, as well as approximation and bootstrap confidence intervals (CIs), were calculated. Bayesian estimation of the parameter model was obtained under symmetric and asymmetric loss functions. An optimal test plan was created under typical operating conditions by minimizing the asymptotic variance (AV) of the percentile life. The simulation study is discussed to demonstrate the model's optimality. In addition, real-world data are evaluated to demonstrate the model's versatility.

Keywords: accelerated life testing; step-stress loading; progressively type-II censored samples; alpha power Weibull distribution; interval estimation; optimal design

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1. Introduction

Recently, most produced products are highly reliable with long lifespans, resulting in expensive costs and long experimental durations when tested under usual settings. When standard life-testing is no longer useful, the reliability experimenter may employ accelerated life-testing, in which the experimental units are subjected to higher stress levels than under normal operating settings. Accelerated life tests (ALTs) are used to obtain information on the lifetime distribution of items rapidly by testing them at higher than nominal levels of stress to trigger early failures; for more information on this topic, see [1,2]. Moreover, ALTs enable researchers to investigate the impact of stress factors such as pressure or temperature on the lives of experimental units. The ALT data must be fitted to a model that connects the lifetime to stress and estimates the parameters of the lifetime distribution under normal conditions. This necessitates the development of a model that connects the degrees of stress to the characteristics of the lifetime distribution. The cumulative exposure model, presented in [3] and described further in [4,5], is one such model. ALTs can be performed at increasing or continuous high stress levels. In practice, continual stress ALT results in very few failures during the experimental duration, limiting the efficacy of accelerated testing. The step-stress paradigm, which allows for a change in stress in steps at various intermediate stages of the experiment, is an example of accelerated testing. The step-stress paradigm has been extensively addressed in the literature; for example, [6–10] all considered inferences for the step-stress model assuming exponential lifetimes based on different removing strategies. In the instance of a simple step-stress model $(m = 1)$, [11,12] discussed determining the ideal time τ_1 to change the stress level from x_1 to x_2. The authors of [13] provided both inference and an optimal progressive strategy for progressive type-II censoring. While all of these discussions focused on exponential step-stress models, inferential approaches and optimal progressive plans with Weibull distributed life were researched in [14–16] and used to develop simple and multiple step-stress models, respectively, with lognormally distributed lifetimes and type-I censoring. For a comprehensive evaluation of step-stress models, see [17,18], which also include a comprehensive review of previous work on exact inferential methods for exponential step-stress models, as well as optimal step-stress test design, and [19], which provides an outline of several achievements in progressive censorship.

Step-stress life testing (SSLT) is a type of ALT. The use of an SSLT allows the experimenter to control the level of stress during the experiment. Assume n items are assessed at $s1$ initial stress level. In ALTs, tests are frequently halted before all units fail, resulting in the use of filtered data to decrease test time and expenditure. Type-I and type-II censoring are the two most prevalent censoring systems in life testing or reliability trials. The progressive type-II censoring technique has recently gained popularity. It is frequently used for assessing very reliable data. The key advantage of progressive censoring is that it allows for more information on the lifetimes of the units without exposing all units to high levels of stress, leading to reduced associated costs. This type of censoring technique can be described as follows: In a progressively type-II censored sample, in a life testing experiment, assume n identical objects are tested; $m < n$ is a predetermined number of failures; and $R_1, R_2, \ldots, R_m$ are m prefixed integers satisfying $R_1 + R_2 + \ldots + R_m + m = n$. R_1 of the surviving units is randomly withdrawn at the time of the first failure, $t_{1:m:n}$. Similarly, when the second failure $t_{2:m:n}$ occurs, R_2 of the surviving units is randomly withdrawn, and so on. At the mth failure $t_{m:m:n}$, the experiment is terminated, and all surviving $R_m = n - m - (R_1 + R_2 + \ldots + R_{m-1})$ units are withdrawn. See [20,21] for more information on progressive type-II censoring.

The following is how the document is written: Section 2 presents a description of the lifetime model and test assumptions. The MLEs of model parameters under the simple step-stress ALT are derived in Section 3. Section 4 presents the Bayes estimates (BEs) of model parameters produced using the Markov chain Monte Carlo (MCMC) approach. Section 5 examines a real data set to demonstrate the recommended approaches in Sections 3 and 4. Section 6 constructs the asymptotic, bootstrap, and reasonable confidence intervals for the model parameters. The simulation studies are found in Section 7. In Section 8, the actual real data set is applied to illustrate the flexibility of the model based on progressive type-II censoring. Section 9 has the conclusion.

2. Model Description and Assumptions of Test

The alpha power Weibull (APW) distribution is important because it can simulate monotone and non-monotone failure rate functions, which are prominent in reliability research, and it extends the Weibull distribution. In fact, the APW distribution was inspired by the widespread usage of the Weibull distribution in reliability theory, as well as the fact that the generalization allows for flexibility in analyzing lifetime data. Furthermore, based on the proposed Weibull model, a new generalization of the Weibull distribution, called the APW model, was proposed in [22]. The cumulative distribution (CDF) and probability density function (PDF) are given as follows:

$$F(t) = \frac{1}{1-\alpha}\left(1 - \alpha^{1-e^{-\beta t^{\theta}}}\right), \ \alpha, \beta, \theta > 0, t \geq 0, \tag{1}$$

and

$$f(t) = \frac{log\alpha}{\alpha - 1}\theta\beta t^{\theta-1} e^{-\beta t^{\theta}} \alpha^{1-e^{-\beta t^{\theta}}}, \ \alpha, \beta, \theta > 0, t \geq 0, \tag{2}$$

and the associated hazard rate function (hrf) is given by:

$$h(t) = \frac{log\alpha}{\left(\alpha^{-e^{-\beta t^{\theta}}} - 1\right)}\theta\beta t^{\theta-1}e^{-\beta t^{\theta}},\ \alpha, \beta, \theta > 0, t \geq 0, \tag{3}$$

Assumptions of Test. *Let $S_0 < S_1 < S_2$ represent the stress levels in the test, and let S_0 represent the use-stress or design stress. Assume n identical units are tested under stress level S_1, and the surviving units are tested under stress level S_2 at a predetermined time. The progressive type-II censoring is used as follows: R_1 units are randomly withdrawn from the $n-1$ surviving units at the time of the first failure $t_{1:m:n}$. At the time of the second failure, $t_{2:m:n}$, R_2 units are picked at random from the remaining $n-2-R_1$ units. When the m^{th} failure occurs $t_{m:m:n}$, the test is ended, and all remaining $R_m = n - m - \sum_{j=1}^{m-1} R_j$ units are withdrawn. The complete samples and type-II censored samples are clearly exceptional examples of this technique. Let n_1 be the number of failures prior to time at stress level S_1. The observed progressive censored data are $t_{1:m:n} < t_{2:m:n} < \ldots < t_{n_1:m:n} < \tau < t_{n_1+1:m:n} < \ldots < t_{m:m:n}$ with these notations.*

In the framework of simple step-stress ALT, the following assumptions are employed throughout the article:

1. For stress levels $S_i, i = 0,\ 1,\ 2,$ the failure time T is given by $APW(\alpha, \theta, \beta_i)$.
2. The association between the life feature β and the stress loading S can take one of two forms:
    - The Arrhenius model is as follows: $ln(\beta) = a + b/-S,\ b > 0$, where S is the absolute temperature.
    - The inverse power model is defined as $ln(\beta) = a + b[ln(S)],\ b > 0$, where S is the voltage.
    - Exponential model: $ln(\beta) = a + bS$, where $b > 0$, and S is a weathering variable.

More information on accelerated models can be found in [23]. Thus, for the above three models, $ln(\beta)$ is a linear function of the transformed stress $\omega(S) = 1/-S, ln(S)$, or S.

Furthermore, we assume that the relationship between the parameter β_i and the stress level S_i is linear.

$$ln(\beta_i) = a + b\omega_i, i = 0, 1, 2, \tag{4}$$

where a and b (> 0) are unidentified parameters and $\omega_i = \omega(S_i)$ is an increasing function of S.

3. The cumulative exposure model is still valid, as demonstrated in [23].

The CDF of a test unit under the simple step-stress ALT is calculated using the cumulative exposure model and the CDF presented in Equation (1).

$$G(t) = \begin{cases} F_1(t) & 0 < t < \tau \\ F_2(t - \tau + \tau^*) & t \geq \tau \end{cases}, \tag{5}$$

where $F_j(t) = \frac{1}{1-\alpha}\left(1 - \alpha^{1-e^{-\beta_j t^{\theta}}}\right), j = 1, 2,$ and τ^* is the answer to $F_1(\tau) = F_2(\tau^*)$.

The corresponding PDF is given by

$$g(t) = \begin{cases} f_1(t) & 0 < t < \tau \\ f_2(t - \tau + \tau^*) & t \geq \tau \end{cases}, \tag{6}$$

where $f_j(t) = \frac{log\alpha}{\alpha-1}\theta\beta_j t^{\theta-1}e^{-\beta_j t^{\theta}}\alpha^{1-e^{-\beta_j t^{\theta}}}, j = 1, 2$

3. Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) of the model parameters are obtained in this section. Let $t_{i:m:n} = t_i$ denote the observed lifetime T values acquired from a progressive type-II censoring. The probability function of the four parameters $\alpha,\ \theta,\ \beta_1$, and β_2, based on the progressive type-II censoring sample, is calculated from the CDF in Equation (5) and the associated PDF in Equation (6) as:

$$L(\alpha,\ \theta,\ \beta_1,\ \beta_2) = C\prod_{i=1}^{n_1}\Big\{f_1(t_i)[1-F_1(t_i)]^{R_i}\Big\}\prod_{i=n_1+1}^{m}\Big\{f_2(t-\tau+\tau^*)[1-F_2(t-\tau+\tau^*)]^{R_i}\Big\}, \tag{7}$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\ldots\left(n-m+1-\sum_{j=1}^{m-1}R_j\right)$.

The MLEs of $\alpha,\ \theta,\ \beta_1$, and β_2 exist only if at least one failure occurs before τ and after τ, in which case the logarithm of the likelihood function ℓ of $\alpha,\ \theta,\ \beta_1$, and β_2 is provided by:

$$\begin{aligned}\ell = {} & log(C) + n_1 log(log\alpha) - n_1 log(\alpha-1) + n_1 log\theta + n_1 log(\beta_1) + (\theta-1)\sum_{i=1}^{n_1} log(t_i) - \beta_1\sum_{i=1}^{n_1} {t_i}^{\theta} + \\ & log(\alpha)\sum_{i=1}^{n_1}\left(1-e^{-\beta_1 {t_i}^{\theta}}\right) + R_i log(\alpha) - R_i log(\alpha-1) + R_i log\left(1-\alpha^{-e-\beta_1 {t_i}^{\theta}}\right) + mlog(\theta) \\ & + mlog(\alpha)\ mlog(\beta_2) - mlog(\alpha-1) + (\theta-1)\sum_{i=n_1+1}^{m} log\left(t_i - \frac{\beta_1}{\beta_2}\tau+\tau^*\right) \\ & -\sum_{i=n_1+1}^{m}\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta} + R_i log(\alpha) - R_i log(\alpha-1) + R_i log\left(1-\alpha^{-e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\ \tau+\tau^*\right)^{\theta}}}\right),\end{aligned} \tag{8}$$

The derivatives of ℓ are given as follows:

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} = {} & \frac{n_1}{\theta} + \sum_{i=1}^{n_1}\log(t_i) - \beta_1\sum_{i=1}^{n_1} {t_i}^{\theta}\log(t_i) + \log(\alpha)\sum_{i=1}^{n_1} e^{-\beta_1 {t_i}^{\theta}}\beta_1 {t_i}^{\theta}\log(t_i) - \frac{\alpha^{e^{-\beta_1 {t_i}^{\theta}}} R_i e^{-\beta_1 {t_i}^{\theta}}\beta_1 {t_i}^{\theta}\log(t_i)}{1-\alpha^{e^{-\beta_1 {t_i}^{\theta}}}} + \frac{m}{\theta} \\ & + \left(t_i - \frac{\beta_1}{\beta_2}\tau+\tau^*\right) - \sum_{i=n_1+1}^{m}\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right) \\ & - \frac{R_i\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}} e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)}{\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)}\end{aligned} \tag{9}$$

We now have a system of nonlinear equations with the unknown parameters $\alpha,\ \theta,\ \beta_1$, and β_2. It is evident that obtaining a closed-form solution is quite challenging. As a result, an iterative approach such as the Newton–Raphson method can be employed to provide a numerical solution to the nonlinear system described above.

4. Fisher Information Matrix

The Fisher information matrix is a fundamental statistical construct that describes how much information data provides about an unknown quantity. It can be used to compute an estimator's variance as well as the asymptotic behavior of maximum likelihood estimations. The Fisher information matrix's inverse is an estimator of the asymptotic covariance matrix. The Fisher information matrix is calculated by taking the anticipated values of the loglikelihood function's negative second partial and mixed partial derivatives with respect to $\alpha,\ \theta,\ \beta_1$, and β_2. It is explained further below.

$$\boldsymbol{I_{4\times 4}} = -\boldsymbol{E}\begin{bmatrix}\boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} & \boldsymbol{a_{14}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} & \boldsymbol{a_{24}} \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}} & \boldsymbol{a_{34}} \\ \boldsymbol{a_{41}} & \boldsymbol{a_{42}} & \boldsymbol{a_{43}} & \boldsymbol{a_{44}}\end{bmatrix}, \tag{10}$$

where $\boldsymbol{a_{11}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\alpha^2}\right)$, $\boldsymbol{a_{12}} = \boldsymbol{a_{21}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\alpha\partial\theta}\right)$, $\boldsymbol{a_{13}} = \boldsymbol{a_{31}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\theta\partial\beta_1}\right)$, $\boldsymbol{a_{14}} = \boldsymbol{a_{41}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\theta\partial\ \beta_2}\right)$ $\boldsymbol{a_{22}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\theta^2}\right)$, $\boldsymbol{a_{33}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial{\beta_1}^2}\right)$, $\boldsymbol{a_{32}} = \boldsymbol{a_{23}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\theta\partial\ \beta_1}\right)$, $\boldsymbol{a_{44}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial{\beta_2}^2}\right)$, $\boldsymbol{a_{42}} = \boldsymbol{a_{24}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\theta\partial\ \beta_2}\right)$, and $\boldsymbol{a_{43}} = \boldsymbol{a_{34}} = \boldsymbol{E}\left(\frac{\partial^2\ell}{\partial\beta_1\partial\ \beta_2}\right)$.

The elements of the relevant matrices are computed. As a result, the MLEs' related variance–covariance matrix can be computed as follows:

$$
\begin{aligned}
\frac{\partial^2\ell}{\partial\alpha^2} = &-\frac{n_1(\log\alpha+1)}{(\alpha\log\alpha)^2} + \frac{n_1}{(\alpha-1)^2} - \frac{1}{\alpha^2}\sum_{i=1}^{n_1}\left(1-e^{-\beta_1 t_i\theta}\right) - \frac{R_i}{\alpha^2} + \frac{R_i}{(\alpha-1)^2} \\
&+\frac{R_i e^{-\beta_1 t_i{}^{\theta}}\left[\left(-1+e^{-\beta_1 t_i{}^{\theta}}\right)\alpha^{-2+e^{-\beta_1 t_i}}\left(1-\alpha^{e^{-\beta_1 t_i}}\right)+\alpha^{-1+e^{-\beta_1 t_i}}\right]}{\left(1-\alpha^{e-\beta_1 t_i{}^{\theta}}\right)^2} - \frac{m}{\alpha^2} + \frac{m}{(\alpha-1)^2} - \frac{R_i}{\alpha^2} \\
&+\frac{R_i}{(\alpha-1)^2} \\
&+\frac{R_i e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\left[\left(-1+e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\right)\alpha^{-2+e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}}\left(1-\alpha^{e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}}\right)+\alpha^{-1+e-\beta_1 t_i{}^{\theta}}\right]}{\left(1-\alpha^{e}-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\right)^2}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\frac{\partial^2\ell}{\partial\alpha\partial\beta_2} = &\frac{R_i}{\left(1-\alpha^{e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}}\right)^2}\left\{e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\left[\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\right.\right. \\
&\left.-\theta\beta_1\tau\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}\right]\left(\alpha^{-1+e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}}\right) - \tau\beta_1\theta\left(t_i-\frac{\beta_1}{\beta_2}\tau\right. \\
&\left.\left.+\tau^*\right)^{\theta-1}e^{-2\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\left(\alpha^{-1+e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}}\right)\right\},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&e^{-\beta_1 t_i^{\theta}}t_i{}^{\theta}(\log(t_i))^2\left[1+t_i^{\theta}\right] \\
&\frac{\partial^2\ell}{\partial\theta^2} = -\frac{n_1}{\theta^2} - \beta_1\textstyle\sum_{i=1}^{n_1} t_i{}^{\theta}(\log(t_i))^2 + \beta_1\log\alpha\textstyle\sum_{i=1}^{n_1} + \frac{\alpha^{e^{-\beta_1 t_i\theta}}R_i e^{-\beta_1 t_i\theta}\beta_1 t_i\theta(\log(t_i))^2}{\left(1-\alpha^{e^{-\beta_1 t_i\theta}}\right)^2}\left\{\left(1-\alpha^{e^{-\beta_1 t_i}}\right)\right[1+ \\
&e^{-\beta_1 t_i{}^{\theta}}t_i{}^{\theta} + t_i{}^{\theta} + t_i{}^{\theta}e^{-\beta_1 t_i{}^{\theta}}\alpha^{e^{-\beta_1 t_i\theta}}\left]\right\} - \frac{m}{\theta^2} - \beta_2\textstyle\sum_{i=n_1+1}^{m}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\left[\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)\right]^2 + \\
&\frac{R_i\alpha^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\left[\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)\right]^2}{\left(1-\alpha^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*}\right)^{\theta}\Big)^2}\left\{\left(1-\alpha^{e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*}\Big)^{\theta}}\right)\right[1+ \\
&e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta} + \left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta} + \left(t_i-\frac{\beta_1}{\beta_2}\tau+\right. \\
&\left.\tau^*)^{\theta}e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}\alpha^{e^{-\beta_2(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}}}\right]\Bigg\}.
\end{aligned} \tag{13}
$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\theta\partial\beta_1} ={}& -\sum_{i=1}^{n_1} t_i^{\theta}\log\theta+\log\alpha\sum_{i=1}^{n_1}\left(t_i^{\theta}\log\theta e^{-\beta_1 t_i^{\theta}}-\beta_1 t_i^{\theta}\log\theta e^{-\beta_1 t_i^{\theta}}\right)+\sum_{i=1}^{n_1}\frac{R_i t_i\theta\log(t_i)}{\left(1-\alpha^{e^{-\beta_1 t_i\theta}}\right)^2}\{(1-\\&\alpha^{e^{-\beta_1 t_i^{\theta}}})\left[\beta_1\alpha^{e^{-\beta_1(t_i)^{\theta}}-1}e^{-\beta_1 t_i^{\theta}}+\alpha^{e^{-\beta_1(t_i)^{\theta}}}e^{-\beta_1 t_i^{\theta}}-\alpha^{e^{-\beta_1(t_i)^{\theta}}}e^{-\beta_1 t_i^{\theta}}t_i^{\theta}\right]+\\&\sum_{i=1}^{n_1}\left[\alpha^{-1+e^{-\beta_1 t_i\theta}}\left(\beta_1\alpha^{-\beta_1(t_i)^{\theta}}e^{-\beta_1 t_i\theta}\right)\right]-\frac{\tau}{\beta_2}+\sum_{i=n_1+1}^{m}\theta\tau\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)+\\&\sum_{i=n_1+1}^{m}\theta\tau\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}\}+\sum_{i=n_1+1}^{m}\frac{R_i}{\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)^2}\{\alpha^{-1+e-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)}e^{\theta}e^{-2\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\beta_2(t_i-\\&\frac{\beta_1}{\beta_2}\tau+\tau^*)^{\theta}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)+\tau\theta\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}(t_i-\frac{\beta_1}{\beta_2}\tau+\\&\tau^*)^{2\theta-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)-\frac{\tau\theta}{\beta_2}(t_i-\frac{\beta_1}{\beta_2}\tau+\\&\tau^*)^{\theta-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\alpha^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)-\\&\frac{\tau\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}}{\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)}(1-\\&\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}})\}+R_i\frac{\alpha^{-1+e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}{1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}}.\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\theta\partial\beta_2} ={}& \frac{\beta_1}{\beta_2^2}\tau-\sum_{i=n_1+1}^{m}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{2\theta}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)\left[1+\beta_2\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)\right]+\\&\frac{R_i\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}\log\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)}{\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)}.\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta_1^2} ={}& -\frac{n_1}{\beta_1^2}-\log\alpha\sum_{i=1}^{n_1}t_i^{2\theta}e^{-\beta_1 t_i^{\theta}}+\frac{R_i {t_i}^{2\theta}\log(t_i)e^{-\beta_1 t_i^{\theta}}}{\left(1-\alpha^{e^{-\beta_1 t_i\theta}}\right)^2}\{\left[\alpha^{e^{-\beta_1 t_i\theta}}e^{-\beta_1 t_i^{\theta}}+1\right]\left(1-\alpha^{e^{-\beta_1 t_i}}\right)+\\&\left(\alpha^{e-\beta_1 t_i^{\theta}}\right)^2e^{-\beta_1 t_i\theta}\}+\theta(\theta-1)\sum_{i=n_1+1}^{m}\frac{\tau^2}{\beta_2}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-2}+\\&R_i\frac{\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\tau\theta\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}{\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)^2}[\{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}]-\\&\frac{\tau(\theta-1)}{\beta_2}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{-1}+\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}\}\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)+\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}(t_i-\frac{\beta_1}{\beta_2}\tau+\\&\tau^*)^{\theta-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}].\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2} ={}& -\sum_{i=n_1+1}^{m}\theta\tau^2(\theta-1)\frac{\beta_1}{\beta_2^2}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-2}+\\&\frac{\tau^2\theta\beta_1 R_i\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}{\beta_2^2\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)^2}\{\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)+\theta(t_i-\frac{\beta_1}{\beta_2}\tau+\\&\tau^*)^{-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}+\left[\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{-1}+\tau\beta_1\right]\left(1-\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\right)+\theta\alpha^{e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}}\\&\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta-1}e^{-\beta_2\left(t_i-\frac{\beta_1}{\beta_2}\tau+\tau^*\right)^{\theta}}\}.\end{aligned} \tag{17}$$

5. Confidence Intervals

This section computes the confidence intervals for the parameters (CIs). We should base our confidence intervals on our point estimate because it is the most likely value for the parameter. A confidence interval (CI) is a set of values that serve as good approximations of an unknown population parameter. The first study to use confidence intervals in statistics was [24]. Two types of CIs were estimated in the present study.

5.1. Approximate Confidence Intervals

Asymptotic CIs based on normality do not perform well since the APW distribution's pdf is not symmetric. It is assumed that the underlying distribution is APW. As a result, we choose the parametric bootstrap percentile interval over the nonparametric one. Furthermore, the nonparametric bootstrap percentile interval is generally known to perform badly in general; further material is provided in Section 5.3.1 of [25]. The parametric bootstrap interval can be used with normal approximation or Studentization. However, because this CI is symmetric, it might not be appropriate for our asymmetric case. The MLE results, according to large sample theory, are consistent and regularly distributed, subject to certain regularity constraints. Correct CIs cannot be obtained since parameter MLE values are not in closed form; instead, asymptotic CIs based on the asymptotic normal distribution of MLE values are estimated. Assume that $\varepsilon = (\alpha,\ \theta,\ \beta_1,\ \beta_2)$ is correct. $\left[(\hat{\alpha}-\alpha),\ (\hat{\theta}-\theta),\ (\hat{\beta}_1-\beta_1), (\hat{\beta}_2-\beta_2)\right] \sim N(0,\sigma)$ is known to generate the asymptotic distribution of MLE values of $\alpha,\ \theta,\ \beta_1$, and β_2, where $\sigma = \sigma_{ij}, i, j = 1,2,3,4$, is the variance–covariance matrix of the unknown parameters.

As previously established, the inverse of the Fisher information matrix is an estimator of the AV–covariance matrix. The approximate $100(1-\omega)\%$ two-sided CIs for ε are supplied by

$$(\hat{\varepsilon}_{iL},\ \hat{\varepsilon}_{iU}):\ \hat{\varepsilon}_i \mp z_{1-\frac{\omega}{2}}\sqrt{\hat{\sigma}_{ij}}, i = 1,2,3,4. \tag{18}$$

where $z_{1-\frac{\omega}{2}}$ is the standard normal distribution's th-percentile.

5.2. Bootstrap Confidence Intervals

This subsection focuses on the generation of CIs based on parametric bootstrap sampling with percentile intervals. The Algorithm 1 below is used to compute the bootstrap CIs; for more information, see [26].

Algorithm 1. Bootstrap Algorithm

1: Step 0, basic setup:
2: Set $b = 1$
3: Determine the MLE values of $\varepsilon = (\alpha,\ \theta,\ \beta_1,\ \beta_2)$, as indicated by $\hat{\varepsilon} = (\hat{\alpha},\ \hat{\theta},\ \hat{\beta}_1, \hat{\beta}_2)$.
4: Step 1, sampling:
5: Obtain the b^{th} bootstrap resample t_p^* from $F(.|\hat{\varepsilon})$, where ε is the MLE from Step 0.
6: Step 2, bootstrap estimates:
7: Determine the b^{th} bootstrap estimations.
8: $\hat{\varepsilon}^{*b} = \left(\hat{\alpha}^{*b}, \hat{\beta}^{*b},\ \hat{\beta}_1{}^{*b},\ \hat{\beta}_2{}^{*b}\right)$,
9: Use the t_p^* resample acquired in Step 1.
10: Step 3, repetition:
11: Set $b \leftarrow b+1$,
12: and then repeat Steps 1–3 until b = B.
13: Step 4, begin in ascending order:
14: Arrange the estimates in ascending order so that we have
15: $\left\{\hat{\alpha}^{*[1]},\ \hat{\alpha}^{*[2]}, \ldots,\ \hat{\alpha}^{*[B]}\right\}, \left\{\hat{\theta}^{*[1]},\ \hat{\theta}^{*[2]}, \ldots, \hat{\theta}^{*[B]}\right\}, \left\{\hat{\beta}_1{}^{*[1]},\ \hat{\beta}_1{}^{*[2]}, \ldots,\ \hat{\beta}_1{}^{*[B]}\right\}$ and $\left\{\hat{\beta}_2{}^{*[1]},\ \hat{\beta}_2{}^{*[2]}, \ldots,\ \hat{\beta}_2{}^{*[B]}\right\}$

The $100(1-\omega)\%$ percentile bootstrap CIs for ε are then computed as follows:

$$(\hat{\varepsilon}_{iL},\ \hat{\varepsilon}_{iU}) = \left(\hat{\varepsilon}_i{}^{*[\frac{\omega}{2}B]},\ \hat{\varepsilon}_i{}^{*[(1-\frac{\omega}{2})B]}\right), i = 1,2,3,4, \tag{19}$$

where $\hat{\varepsilon}_1{}^* = \alpha^*,\ \hat{\varepsilon}_2{}^* = \theta^*, \hat{\varepsilon}_3{}^* = \beta_1{}^*$, and $\hat{\varepsilon}_4{}^* = \beta_2{}^*$.

6. Bayesian Estimation

In this section, we look at Bayesian estimators for unknown parameters, which can be thought of as alternatives to the aforementioned MLEs. The prior specifications for the unknown parameters are the starting point for Bayesian analysis. In this study, the parameters $\alpha,\ \theta,\ \beta_1,$ and β_2 are considered to be statistically independent and to be different gamma distributions, indicated by $gamma(a_j, b_j); j = 1, \ldots, 4$, respectively. The joint prior distribution for the APW distribution parameters is provided by

$$\varphi(\alpha,\ \theta,\ \beta_1,\ \beta_2) \propto \alpha^{a_1-1}e^{-b_1\alpha}\ \theta^{a_2-1}e^{-b_2\theta}\ \beta_1^{a_3-1}e^{-b_3\beta_1}\ \beta_2^{a_4-1}e^{-b_4\beta_2}, \tag{20}$$

where $a_j \geq 0$ and $b_j \geq 0; j = 1, \ldots, 4$ represent the previously established hyperparameters that reflect prior knowledge of the unknown parameters. It is worth noting that using independent inverse gamma priors for unknown parameters is not illogical and may result in more expressive posterior density estimates due to the flexible forms of the gamma distribution. The joint posterior distribution of the unknown parameters that results is given by

$$L(\alpha,\ \theta,\ \beta_1,\ \beta_2|\underline{t}) \propto \varphi(\alpha,\ \theta,\ \beta_1,\ \beta_2)\prod_{i=1}^{4}\prod_{j=1}^{n_i} f(t_{ij})(1-F(t_{ij}))^{d_i}, \tag{21}$$

This is unrecognizable, prompting us to use the Metropolis–Hastings (MH) technique to create posterior samples for their conditional posterior distributions. Later, the obtained samples will be used to approximate Bayes estimates and create the parameters' highest posterior density (HPD) credible intervals; for additional information on this approach, see [27,28]. We discuss the symmetric (SLF) and general entropy (GE) loss functions for Bayesian analysis in this study, which are indicated as

$$\ell(\alpha,\widetilde{\alpha}) = (\widetilde{\alpha}-\alpha)^2, \ell\left(\theta,\widetilde{\theta}\right) = \left(\widetilde{\theta}-\theta\right)^2, \ell\left(\beta_1,\widetilde{\beta_1}\right) = \left(\widetilde{\beta_1}-\beta_1\right)^2, \ell\left(\beta_2,\widetilde{\beta_2}\right) = \left(\widetilde{\beta_2}-\beta_2\right)^2 \tag{22}$$

where $\widetilde{\alpha},\ \widetilde{\theta},\ \widetilde{\beta_1}$, and $\widetilde{\beta_2}$ are the estimated posterior means of $\alpha,\ \theta,\ \beta_1,$ and β_2, respectively.

The GE is an asymmetric loss function. This loss is a simple generalization of the entropy loss, which has been utilized by various writers, where the shape parameter q is set to 1, defined by

$$\ell(\varepsilon,\widetilde{\varepsilon}) \propto \left(\frac{\widetilde{\varepsilon}}{\varepsilon}\right)^q - q\ln\left(\frac{\widetilde{\varepsilon}}{\varepsilon}\right) - 1,\ q \neq 1, \tag{23}$$

where $\widetilde{\varepsilon}$ is an estimate of ε. $\widetilde{\varepsilon}$ denotes the Bayes estimator relative to the GE loss function and is given as:

$$\widetilde{\varepsilon}_{GE} = \left[E_\varepsilon\left(\varepsilon^{-q}\right)\right]^{\frac{-1}{q}}, \tag{24}$$

assuming ε^{-q} exists and is finite, where E_ε signifies the expected value. The author of [29] employed the loss function from Equation (24). It should be noted that any other loss function can be easily substituted in a similar manner.

7. Optimization Criterion

In recent years, there has been much interest in finding the optimal censoring scheme in the statistical literature; for example, see [30–33]. For fixed n and m, possible censoring schemes are all $R_1,\ \cdots,\ R_m$ combinations such that $R_1,\ \cdots,\ R_m$, and choosing the best sample technique entails finding the progressive censoring scheme that provides the most information about the unknown parameters among all conceivable progressive censoring schemes. The first difficulty is, of course, determining how to generate unknown parameter information measures based on specific progressive censoring data, and the second is determining how to compare two distinct information measures based on two different progressive censoring techniques; see [34] for more information. The next part of this study covers some of the optimality criteria that were employed in this situation. In practice, we want to select the censoring scheme that delivers the most information about the unknown parameters; see [35] for further information. In our example, Table 1 presents a number of regularly used measures to help us choose the appropriate progressive censoring approach, C_i.

Table 1. Some practical censoring plan optimum criteria.

Criterion	Method
C_1	Maximize trace $[\boldsymbol{I}_{\mathbf{3}\times\mathbf{3}}(.)]$
C_2	Minimize trace $[\boldsymbol{I}_{\mathbf{3}\times\mathbf{3}}(.)]^{-1}$
C_3	Minimize det $[\boldsymbol{I}_{\mathbf{3}\times\mathbf{3}}(.)]^{-1}$
C_4	Minimize Var $\boldsymbol{Var}\left[\boldsymbol{log}\left(\hat{t}_p\right)\right]$, $0 < p < 1$

Regarding C_1, we wish to maximize the observed Fisher $\boldsymbol{I}_{\mathbf{4}\times\mathbf{4}}(.)$ information values. Furthermore, our goal for criteria C_2 and C_3 is to minimize the determinant and trace of $[\boldsymbol{I}_{\mathbf{4}\times\mathbf{4}}(.)]^{-1}$. Comparing multiple criteria is simple when dealing with single-parameter distributions; however, when dealing with unknown multiparameter distributions, comparing the two Fisher information matrices becomes more difficult because criteria C_2 and C_3 are not scale-invariant; however, the optimal censoring scheme of multiparameter distributions can be chosen using the scale-invariant criterion C_4. The criterion C_4, which is dependent on p, clearly tends to minimize the variance of logarithmic MLE of the pth quantile, $log\left(\hat{t}_p\right)$. As a result, the logarithm for $\hat{t}_p$ of the APW distribution is supplied by

$$log\left(\hat{t}_p\right) = log\left\{\frac{-1}{\beta}log\left[1 - \frac{log(1 + p(\alpha - 1))}{log\alpha}\right]\right\}^{\frac{1}{\theta}}, \ 0 < p < 1,$$

From (3), the delta method is used to obtain the approximated variance for $log\left(\hat{t}_p\right)$ of the APW distribution as

$$Var\left(log\left(\hat{t}_p\right)\right) = \left[\nabla log\left(\hat{t}_p\right)\right]^T I^{-1}_{4\times4}\left(\hat{\alpha},\ \hat{\theta},\ \hat{\beta}_1, \hat{\beta}_2\ \right)\left[\nabla log\left(\hat{t}_p\right)\right],$$

where

$$\left[\nabla log\left(\hat{t}_p\right)\right]^T = \left[\frac{\partial}{\partial\alpha}log\left(\hat{t}_p\right), \frac{\partial}{\partial\theta}log\left(\hat{t}_p\right),\ \frac{\partial}{\partial\beta_1}log\left(\hat{t}_p\right), \frac{\partial}{\partial\beta_2}log\left(\hat{t}_p\right)\right]_{(\alpha=\hat{\alpha},\theta=\hat{\theta},\beta_1=\hat{\beta}_1,\beta_2=\hat{\beta}_2)}.$$

The optimum progressive censoring, on the other hand, corresponds to the maximum value of the criterion C_1 and the lowest value of the criteria $C_i i, = 1,\ 2,\ 3,\ 4.$

8. Simulation

For various selections of n, m, and values, simulation tests were carried out to examine the performances of the likelihood and Bayesian estimators under SLF and GE loss function in terms of their values of bias (VB) and values of mean square error (VMSE). Based on the asymptotic distribution of the MLEs, the 95% asymptotic confidence intervals are calculated. Additionally, two MLE bootstrap confidence interval approaches are obtained. The 95% credible confidence intervals are calculated by the highest posterior density (HPD). Since the unclosed forms of the MLE and posterior equations are unknown, we must use appropriate numerical methods, such as the Metropolis–Hastings algorithm for Bayesian analysis and the iterative Newton–Raphson method for MLEs, to solve these nonlinear equations. Considered are two progressive censorship schemes:

Scheme I. $= 0$, and $R_m = n - m$.
Scheme II. $= 0$, and $R_1 = n - m$.

We used different optimization criteria to select the best scheme as determinant and trace of the AV matrices, maximize the principal diagonal elements of the Fisher information matrices, minimize the determinant and trace of the AV matrix, and minimize the variance of the logarithmic MLE of the pth quantile.

The following algorithm is used to carry out the estimation procedure:

1. Give the numbers n, m, and τ. N = 50 and 100; m = 40 and 45 when n = 50, and m = 80 and 90 when n = 100.
2. Give the parameters $\alpha = 1.2$, $\beta_1 = 1.6$, $\beta_2 = 1.6$, and $\theta = 1.5$ and $\alpha = 3$, $\beta_1 = 2$, $\beta_2 = 1.8$, and $\theta = 0.8$ their values.
3. From the random variable t provided by (1), create a sample of the randomness of size n and sort it. The APW distribution random variable is simple to produce. For instance, if the uniform random variable U comes from the range [0, 1], then

$$t = \begin{cases} \left\{ \frac{-1}{\beta_1} \ln\left[1 - \frac{\ln(1-(1-\alpha)u)}{\ln(\alpha)}\right] \right\}^{\frac{1}{\theta}} & t < \tau \\ \tau - \tau^* + \left\{ \frac{-1}{\beta_2} \ln\left[1 - \frac{\ln(1-(1-\alpha)u)}{\ln(\alpha)}\right] \right\}^{\frac{1}{\theta}} & t > \tau \end{cases}$$

4. To generate progressively censored data for given n and m, use the model provided by Equation (6). The set of data can be thought of as:

$$t_{1:m:n} < t_{2:m:n} < \ldots < t_{n_1:m:n} < \tau < t_{n_1+1:m:n} < \ldots < t_{m:m:n}$$

5. To obtain the MLEs of the parameters, the nonlinear system is solved using the Newton–Raphson method.
6. To obtain the Bayesian estimations of the parameters, the posterior distribution is solved using the MCMC method by Metropolis–Hastings algorithm.
7. Repeat steps 3 through 6 for 1000 iterations.
8. Calculate the MILEs and Bayesian parameter-related average values of VB and VMSE.
9. Calculate the various model parameter estimators' confidence intervals.
10. Calculate the various optimization criteria.

The following observations can be drawn from the simulation results shown in Tables 2–7:

- The VB, VMSE, and LCI of the estimates for the two alternative censoring schemes decrease for fixed values of the sample size n by increasing the censored sample size m.
- By increasing the sample size n for fixed values of m, the VB, VMSE, and LCI for various censoring schemes drop.
- Under the cases taken into consideration, the symmetric and asymmetric Bayesian estimations are superior to the MLE in terms of VB and VMSE reduction.
- The LCI dramatically decreases, and the HPD's symmetric and asymmetric Bayesian estimations outperform the MLE's ACI.
- We note that the bootstrap CIs have the smallest CI lengths.

Table 2. Bias, MSE, and length of CI with Scheme I when $\alpha = 1.2$, $\beta_1 = 1.6$, $\beta_2 = 1.6$, and $\theta = 1.5$.

				MLE					**SELF**			**ELF c = −1.25**			**ELF c = 1.25**		
τ	n	m		**Bias**	**MSE**	**LACI**	**LBPCI**	**LBTCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**
0.25	50	40	α	−0.1538	2.0779	5.6213	0.1785	0.1775	0.0221	0.0290	0.6351	0.0236	0.0290	0.6334	0.0077	0.0292	0.6548
			β_1	−0.7315	0.7799	1.9407	0.0641	0.0638	−0.0822	0.0319	0.6075	−0.0808	0.0315	0.6010	−0.0946	0.0361	0.6338
			β_2	−0.0841	0.1824	1.6421	0.0508	0.0502	0.1483	0.0448	0.5792	0.1499	0.0455	0.5823	0.1334	0.0388	0.5583
			θ	−0.0171	0.0467	0.8453	0.0263	0.0263	−0.1120	0.0378	0.6158	−0.1104	0.0372	0.6115	−0.1264	0.0436	0.6466
		45	α	−0.1188	1.4752	4.7407	0.1555	0.1581	0.0193	0.0188	0.5270	0.0203	0.0188	0.5260	0.0102	0.0187	0.5279
			β_1	−0.6927	0.6740	1.7284	0.0548	0.0539	−0.0607	0.0203	0.4822	−0.0599	0.0201	0.4809	−0.0683	0.0220	0.4905
			β_2	−0.0338	0.1480	1.5030	0.0479	0.0473	0.0973	0.0259	0.4811	0.0982	0.0262	0.4832	0.0884	0.0233	0.4682
			θ	−0.0148	0.0390	0.7726	0.0248	0.0248	−0.0790	0.0223	0.4829	−0.0780	0.0220	0.4806	−0.0878	0.0249	0.5031
	100	80	α	−0.3864	1.6118	4.7481	0.2603	0.2511	0.0388	0.0275	0.6051	0.0404	0.0276	0.6049	0.0248	0.0267	0.5888
			β_1	−0.8076	0.8049	1.5345	0.0907	0.0886	−0.1397	0.0375	0.5152	−0.1383	0.0369	0.5117	−0.1524	0.0427	0.5385
			β_2	−0.1827	0.1417	1.2920	0.0708	0.0703	0.2524	0.0831	0.4882	0.2546	0.0845	0.4910	0.2322	0.0707	0.4525
			θ	−0.0276	0.0275	0.6422	0.0363	0.0358	−0.1763	0.0471	0.4742	−0.1744	0.0461	0.4688	−0.1951	0.0612	0.5209
		90	α	−0.3586	1.0389	3.7420	0.1220	0.1221	0.0432	0.0259	0.6013	0.0447	0.0261	0.6021	0.0299	0.0248	0.5990
			β_1	−0.7703	0.7199	1.3950	0.0435	0.0437	−0.1360	0.0369	0.5011	−0.1345	0.0364	0.4992	−0.1487	0.0420	0.5242
			β_2	−0.1065	0.1187	1.2853	0.0399	0.0402	0.2359	0.0728	0.4964	0.2379	0.0740	0.4974	0.2178	0.0626	0.4729
			θ	−0.0214	0.0218	0.5732	0.0187	0.0189	−0.1735	0.0470	0.5183	−0.1717	0.0462	0.5152	−0.1901	0.0554	0.5508
0.7	50	40	α	−0.9983	2.3028	4.8642	0.6904	0.4265	0.0125	0.0117	0.4110	0.0133	0.0117	0.4114	−0.0059	0.0121	0.4106
			β_1	−0.8436	0.9488	1.5382	0.1544	0.1431	−0.0834	0.0243	0.5315	−0.0821	0.0239	0.5284	−0.0953	0.0280	0.5635
			β_2	1.1066	2.3695	4.2305	0.5592	0.5554	−0.0222	0.0321	0.7313	−0.0209	0.0319	0.7277	−0.0344	0.0342	0.7598
			θ	0.6335	0.6156	1.8301	0.2394	0.2323	−0.0925	0.0321	0.7313	−0.0914	0.0222	0.5276	−0.1028	0.0248	0.4666
		45	α	−0.8776	2.0948	4.1432	0.3510	0.3346	0.0031	0.0075	0.3691	0.0041	0.0075	0.3692	0.0055	0.0073	0.3771
			β_1	−0.8919	0.7867	1.0832	0.1020	0.1043	−0.0766	0.0238	0.4806	−0.0754	0.0235	0.4781	−0.0874	0.0266	0.5014
			β_2	0.3129	0.8594	3.4288	0.2355	0.2324	−0.0028	0.0279	0.6203	−0.0013	0.0279	0.6182	−0.0167	0.0289	0.6378
			θ	0.3541	0.2783	1.5363	0.1038	0.1035	−0.0559	0.0224	0.5294	−0.0548	0.0217	0.4586	−0.0656	0.0244	0.5351
	100	80	α	−0.9330	2.3560	4.7935	0.4292	0.3807	0.0194	0.0067	0.3332	0.0201	0.0067	0.3345	0.0134	0.0064	0.3090
			β_1	−0.8437	0.8261	1.3300	0.1106	0.1079	−0.1162	0.0294	0.4980	−0.1151	0.0290	0.4976	−0.1208	0.0330	0.5102
			β_2	0.8341	1.7017	3.9445	0.3058	0.3041	0.0267	0.0251	0.5738	0.0279	0.0252	0.5727	0.0160	0.0249	0.5709
			θ	0.5448	0.5079	1.8070	0.1521	0.1521	−0.1025	0.0241	0.4254	−0.1013	0.0238	0.4224	−0.1127	0.0272	0.4331
		90	α	−1.0459	1.4880	2.4707	0.2820	0.1998	0.0074	0.0063	0.3027	0.0081	0.0063	0.3030	0.0010	0.0063	0.3000
			β_1	−0.9750	0.8200	0.8656	0.0863	0.0772	−0.1101	0.0291	0.4763	−0.1089	0.0287	0.4747	−0.1268	0.0331	0.4947
			β_2	0.0922	0.3430	2.2762	0.2108	0.2058	0.0013	0.0199	0.5037	0.0025	0.0199	0.5035	−0.0096	0.0204	0.5050
			θ	0.2825	0.1573	1.0956	0.0973	0.0992	−0.0959	0.0195	0.3699	−0.0949	0.0193	0.3693	−0.1043	0.0216	0.3794

Table 3. Bias, MSE, and length of CI with Scheme II when $\alpha = 1.2,\ \beta_1 = 1.6,\ \beta_2 = 1.6,$ and $\theta = 1.5$.

				MLE					**SELF**			**ELF c = −1.25**			**ELF c = 1.25**		
τ	n	m		Bias	MSE	LACI	LBPCI	LBTCI	Bias	MSE	LCCI	Bias	MSE	LCCI	Bias	MSE	LCCI
0.25	50	40	α	−0.0976	1.2539	4.3772	0.2034	0.2066	0.0180	0.0283	0.6616	0.0195	0.0284	0.6624	0.0043	0.0284	0.6579
			β_1	−0.6768	0.6263	1.6095	0.0715	0.0700	−0.0757	0.0311	0.6249	−0.0743	0.0307	0.6207	−0.0876	0.0349	0.6573
			β_2	−0.0122	0.1362	1.4473	0.0657	0.0649	0.1158	0.0338	0.5291	0.1172	0.0342	0.5319	0.1031	0.0297	0.5218
			θ	−0.0069	0.0393	0.7770	0.0327	0.0327	−0.0864	0.0316	0.5932	−0.0849	0.0311	0.5922	−0.0997	0.0360	0.6172
		45	α	−0.1614	1.2366	4.3665	0.1423	0.1383	0.0178	0.0265	0.6402	0.0191	0.0265	0.6385	0.0041	0.0269	0.6557
			β_1	−0.6884	0.6163	1.5704	0.0520	0.0523	−0.0748	0.0291	0.5757	−0.0728	0.0287	0.5720	−0.0848	0.0330	0.5975
			β_2	−0.0254	0.1343	1.3480	0.0479	0.0477	0.1130	0.0318	0.5167	0.1132	0.0338	0.5269	0.1018	0.0233	0.5205
			θ	0.0020	0.0347	0.7306	0.0232	0.0229	−0.0849	0.0318	0.5776	−0.0819	0.0314	0.5771	−0.0911	0.0365	0.5997
	100	80	α	−0.2849	0.8869	3.5257	0.2321	0.2180	0.0382	0.0313	0.6588	0.0397	0.0313	0.6590	0.0238	0.0312	0.6753
			β_1	−0.6954	0.6081	1.3860	0.0897	0.0906	−0.1225	0.0291	0.4522	−0.1212	0.0286	0.4475	−0.1342	0.0336	0.4854
			β_2	−0.0266	0.1059	1.2740	0.0834	0.0845	0.1873	0.0497	0.4562	0.1889	0.0504	0.4572	0.1731	0.0433	0.4421
			θ	−0.0159	0.0233	0.5962	0.0371	0.0377	−0.1419	0.0232	0.4902	−0.1403	0.0367	0.4865	−0.1560	0.0432	0.5110
		90	α	−0.2517	0.8791	3.5061	0.1605	0.1627	0.0303	0.0242	0.5772	0.0316	0.0243	0.5783	0.0176	0.0238	0.5817
			β_1	−0.6998	0.6021	1.3421	0.0661	0.0664	−0.1240	0.0236	0.4485	−0.1138	0.0235	0.4384	−0.1252	0.0411	0.5118
			β_2	−0.0391	0.1023	1.2455	0.0568	0.0573	0.1821	0.0460	0.4386	0.1722	0.0503	0.4397	0.1699	0.0415	0.4212
			θ	−0.0227	0.0222	0.5775	0.0256	0.0248	−0.1406	0.0203	0.4720	−0.1358	0.0341	0.4708	−0.1475	0.0419	0.4399
0.7	50	40	α	−0.9948	1.8253	3.5850	0.1160	0.1141	0.0036	0.0108	0.4609	0.0044	0.0108	0.4615	−0.0044	0.0112	0.4652
			β_1	−0.9683	1.0302	1.1935	0.0381	0.0382	−0.0593	0.0248	0.5776	−0.0582	0.0246	0.5768	−0.0691	0.0268	0.5881
			β_2	0.0496	0.3567	2.3343	0.0723	0.0718	0.0195	0.0235	0.5905	0.0208	0.0236	0.5905	0.0080	0.0232	0.5941
			θ	0.2860	0.1713	1.1733	0.0396	0.0390	−0.0512	0.0217	0.5395	−0.0502	0.0216	0.5390	−0.0611	0.0236	0.5549
		45	α	−0.9899	1.6548	3.2220	0.1053	0.1040	0.0035	0.0079	0.3767	0.0070	0.0079	0.3759	0.0008	0.0078	0.3759
			β_1	−0.9749	1.0424	1.1893	0.0381	0.0385	−0.0443	0.0171	0.4654	−0.0435	0.0169	0.4644	−0.0510	0.0181	0.4706
			β_2	0.0386	0.3486	2.3108	0.0740	0.0714	0.0067	0.0161	0.4896	0.0075	0.0162	0.4898	−0.0012	0.0162	0.4928
			θ	0.2735	0.1591	1.1385	0.0367	0.0364	−0.0399	0.0160	0.4526	−0.0391	0.0159	0.4508	−0.0462	0.0170	0.4608
	100	80	α	−1.1083	1.3528	1.3837	0.0583	0.0443	0.0160	0.0066	0.3336	0.0166	0.0066	0.3310	0.0101	0.0064	0.3344
			β_1	−1.0327	1.0892	0.5928	0.0243	0.0238	−0.0956	0.0221	0.4665	−0.0946	0.0219	0.4651	−0.1047	0.0244	0.4786
			β_2	−0.0784	0.1390	1.4294	0.0596	0.0595	0.0323	0.0200	0.5418	0.0333	0.0201	0.5411	0.0226	0.0192	0.5423
			θ	0.2486	0.0987	0.7534	0.0391	0.0385	−0.0733	0.0164	0.3997	−0.0724	0.0162	0.3990	−0.0813	0.0181	0.4088
		90	α	−1.1255	1.2693	0.2016	0.0121	0.0118	0.0105	0.0057	0.3721	0.0112	0.0057	0.3271	0.0047	0.0056	0.3267
			β_1	−1.0583	0.5131	0.4124	0.0212	0.0202	−0.0826	0.0217	0.4421	−0.0816	0.0215	0.4419	−0.0913	0.0239	0.4401
			β_2	−0.1452	0.0925	1.0495	0.0502	0.0496	0.0259	0.0182	0.5014	0.0269	0.0183	0.5011	0.0166	0.0179	0.5054
			θ	0.2417	0.0898	0.6956	0.0244	0.0243	−0.0727	0.0163	0.3917	−0.0727	0.0162	0.3900	−0.0850	0.0179	0.4040

Table 4. Bias, MSE, and length of CI with Scheme I when $\alpha = 3$, $\beta_1 = 2$, $\beta_2 = 1.8$, and $\theta = 0.8$.

				MLE					SELF			ELF c = −1.25			ELF c = 1.25		
τ	n	m		Bias	MSE	LACI	LBPCI	LBTCI	Bias	MSE	LCCI	Bias	MSE	LCCI	Bias	MSE	LCCI
0.25	50	40	α	0.9414	2.8386	3.8474	0.3130	0.3126	−0.0061	0.0309	0.6913	−0.0055	0.0308	0.6907	−0.0120	0.0315	0.7017
			β_1	1.1697	1.9148	2.8996	0.0924	0.0919	−0.0655	0.0328	0.6988	−0.0646	0.0326	0.6989	−0.0728	0.0348	0.7070
			β_2	−0.1820	0.1032	1.0381	0.0335	0.0334	−0.0805	0.0307	0.6690	−0.0796	0.0305	0.6690	−0.0886	0.0331	0.6685
			θ	−0.0560	0.0192	0.4977	0.0159	0.0156	−0.1080	0.0190	0.3310	−0.1062	0.0190	0.3281	−0.1245	0.0258	0.3594
		45	α	0.9517	1.3019	2.6460	0.2598	0.2600	−0.0055	0.0224	0.5740	−0.0050	0.0223	0.5705	−0.0096	0.0226	0.5712
			β_1	1.1301	1.7660	2.7420	0.0848	0.0850	−0.0389	0.0265	0.5946	−0.0378	0.0263	0.5940	−0.0484	0.0286	0.6043
			β_2	−0.1256	0.0700	0.9132	0.0297	0.0292	−0.0066	0.0259	0.5285	−0.0054	0.0256	0.5274	−0.0166	0.0292	0.5484
			θ	−0.0400	0.0191	0.3878	0.0126	0.0126	−0.0941	0.0166	0.3202	−0.0926	0.0161	0.3130	−0.1082	0.0222	0.3598
	100	80	α	0.9154	1.7481	2.9187	0.2476	0.2372	−0.0130	0.0300	0.6706	−0.0057	0.0300	0.6692	−0.0122	0.0305	0.6714
			β_1	1.1945	1.6777	1.9643	0.0636	0.0630	−0.1207	0.0339	0.6496	−0.1197	0.0336	0.6480	−0.1292	0.0376	0.6633
			β_2	−0.1972	0.0705	0.6977	0.0217	0.0217	0.0055	0.0328	0.5840	−0.1167	0.0324	0.5850	−0.1257	0.0359	0.5842
			θ	−0.0760	0.0114	0.2954	0.0097	0.0097	−0.1283	0.0111	0.2034	−0.1271	0.0187	0.2023	−0.1392	0.0227	0.2110
		90	α	0.6449	0.9177	2.1011	0.2116	0.2117	−0.0063	0.0201	0.5409	−0.0051	0.0201	0.5421	−0.0117	0.0204	0.5429
			β_1	1.0994	1.4600	1.9663	0.0624	0.0625	−0.0701	0.0336	0.5493	−0.0690	0.0333	0.5476	−0.0798	0.0365	0.5687
			β_2	−0.1346	0.0417	0.6027	0.0202	0.0202	−0.1176	0.0248	0.5158	0.0065	0.0246	0.5169	−0.0034	0.0262	0.5288
			θ	−0.0508	0.0073	0.2701	0.0086	0.0086	−0.1196	0.0071	0.1912	−0.1184	0.0173	0.1922	−0.1299	0.0214	0.2285
0.5	50	40	α	4.6022	29.0365	10.9926	0.3568	0.3532	0.0017	0.0352	0.6966	0.0021	0.0348	0.6975	−0.0061	0.0419	0.7010
			β_1	1.0747	1.4879	2.2629	0.0722	0.0726	−0.0901	0.0362	0.7075	−0.0893	0.0358	0.7048	−0.0794	0.0394	0.7239
			β_2	−0.5572	0.4593	1.5130	0.0473	0.0458	−0.0855	0.0288	0.6470	−0.0846	0.0288	0.6466	−0.0063	0.0291	0.6544
			θ	−0.0970	0.0272	0.5231	0.0175	0.0172	−0.0593	0.0089	0.2874	−0.0581	0.0088	0.2867	−0.0707	0.0111	0.2787
		45	α	3.8521	20.9405	9.6882	0.2892	0.2879	0.0012	0.0221	0.5780	0.0020	0.0221	0.5781	−0.0024	0.0222	0.5782
			β_1	0.9349	1.1954	2.2238	0.0687	0.0691	−0.0693	0.0284	0.5442	−0.0682	0.0282	0.5415	−0.0976	0.0309	0.5603
			β_2	−0.3477	0.1744	0.9070	0.0284	0.0284	0.0029	0.0282	0.5625	0.0039	0.0279	0.5620	−0.0938	0.0310	0.5853
			θ	−0.0640	0.0146	0.4027	0.0128	0.0128	−0.0550	0.0085	0.2665	−0.0539	0.0083	0.2652	−0.0645	0.0105	0.2985
	100	80	α	4.6348	26.0317	8.3661	0.2871	0.2854	0.0055	0.0288	0.6459	0.0061	0.0288	0.6462	−0.0044	0.0289	0.6470
			β_1	1.0837	1.3192	1.4922	0.0468	0.0468	−0.1154	0.0401	0.6384	−0.1383	0.0397	0.6375	−0.1482	0.0444	0.6543
			β_2	−0.5558	0.3416	0.7091	0.0234	0.0235	−0.1271	0.0360	0.5828	−0.1261	0.0239	0.5825	−0.1365	0.0241	0.5902
			θ	−0.1201	0.0185	0.2514	0.0076	0.0079	−0.0737	0.0077	0.1847	−0.0730	0.0076	0.1848	−0.0803	0.0088	0.1878
		90	α	3.9811	19.9447	7.9374	0.2347	0.2330	−0.0008	0.0191	0.5341	−0.0004	0.0191	0.5347	0.0003	0.0192	0.5358
			β_1	0.9922	1.1096	1.3869	0.0440	0.0440	−0.1039	0.0379	0.5314	−0.1142	0.0374	0.5277	−0.1259	0.0420	0.5520
			β_2	−0.3428	0.1402	0.5913	0.0195	0.0197	−0.0015	0.0239	0.5311	−0.0006	0.0235	0.5307	−0.0098	0.0214	0.5483
			θ	−0.0854	0.0106	0.2243	0.0073	0.0073	−0.0721	0.0069	0.1499	−0.0714	0.0068	0.1500	−0.0788	0.0080	0.1523

Table 5. Bias, MSE, and length of CI with Scheme II when $\alpha = 3,\ \beta_1 = 2,\ \beta_2 = 1.8,$ and $\theta = 0.8$.

				MLE					**SELF**			**ELF c = −1.25**			**ELF c = 1.25**		
τ	n	m		**Bias**	**MSE**	**LACI**	**LBPCI**	**LBTCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**
0.25	50	40	α	2.0064	3.0044	3.6840	1.2511	1.2028	−0.0104	0.0304	0.6579	−0.0098	0.0303	0.6573	−0.0160	0.0313	0.6632
			β_1	0.9564	1.6736	3.4310	0.3479	0.3530	−0.1153	0.0421	0.6372	−0.1142	0.0415	0.6352	−0.1253	0.0474	0.6561
			β_2	−0.1017	0.0939	1.1388	0.1398	0.1398	−0.2211	0.0835	0.6738	−0.2192	0.0822	0.6683	−0.2372	0.0948	0.7169
			θ	0.0168	0.0166	0.5037	0.0476	0.0448	−0.1152	0.0236	0.3450	−0.1120	0.0216	0.3421	−0.1415	0.0433	0.3710
		45	α	1.8139	2.5383	3.5678	0.8749	0.8467	−0.0094	0.0301	0.6254	−0.0088	0.0301	0.6245	−0.0141	0.0302	0.6299
			β_1	1.1395	1.6175	2.6660	0.3382	0.3369	−0.1056	0.0367	0.6071	−0.1046	0.0363	0.6060	−0.1150	0.0401	0.6160
			β_2	−0.1157	0.0744	0.9772	0.1086	0.1087	−0.1875	0.0682	0.6357	−0.1856	0.0670	0.6327	−0.2047	0.0793	0.6619
			θ	−0.0191	0.0065	0.3103	0.0417	0.0424	−0.1051	0.0210	0.3384	−0.1033	0.0201	0.3373	−0.1201	0.0271	0.3814
	100	80	α	1.1551	2.4586	3.0041	0.9165	0.9095	−0.0130	0.0284	0.6773	−0.0123	0.0284	0.6753	−0.0194	0.0288	0.6881
			β_1	1.0838	1.7566	1.9916	0.2634	0.2609	−0.1689	0.0520	0.6223	−0.1676	0.0514	0.6236	−0.1813	0.0586	0.6199
			β_2	−0.1120	1.4052	0.6363	0.0884	0.0886	−0.3136	0.1232	0.5349	−0.3117	0.1216	0.5342	−0.3297	0.1367	0.5515
			θ	−0.0330	0.0061	0.2810	0.0410	0.0411	−0.1046	0.0146	0.2099	−0.1034	0.0143	0.2106	−0.1148	0.0173	0.2047
		90	α	1.0137	1.3916	2.9271	0.2069	0.2072	−0.0023	0.0322	0.6213	−0.0017	0.0216	0.6224	−0.0083	0.0233	0.6133
			β_1	1.0712	0.0352	1.8461	0.0619	0.0628	0.0386	0.0264	0.4687	0.0395	0.0265	0.4655	0.0309	0.0257	0.4962
			β_2	−0.1095	0.0383	0.5960	0.0216	0.0215	−0.0397	0.0198	0.5221	−0.0388	0.0197	0.5185	−0.0476	0.0210	0.5417
			θ	−0.0306	0.0055	0.2648	0.0082	0.0082	0.0017	0.0042	0.1781	0.0025	0.0043	0.1776	−0.0047	0.0037	0.2043
0.5	50	40	α	1.1493	1.9397	2.9250	0.3391	0.3256	0.0030	0.0328	0.6871	0.0036	0.0328	0.6873	−0.0031	0.0330	0.6869
			β_1	0.8450	1.1056	2.4541	0.0741	0.0738	0.0154	0.0300	0.6736	0.0160	0.0300	0.6741	0.0032	0.0301	0.6814
			β_2	−0.2558	0.1263	0.9678	0.0323	0.0326	−0.0268	0.0258	0.6327	−0.0261	0.0257	0.6301	−0.0351	0.0272	0.6393
			θ	−0.0293	0.0131	0.4338	0.0133	0.0135	−0.0068	0.0080	0.3114	−0.0059	0.0081	0.3119	−0.0078	0.0074	0.3117
		45	α	0.7233	1.1610	2.3694	0.2909	0.2914	0.0029	0.0222	0.5789	0.0035	0.0222	0.5790	−0.0011	0.0224	0.5823
			β_1	0.8528	1.0648	2.2785	0.0693	0.0691	0.0118	0.0205	0.5564	0.0127	0.0205	0.5558	0.0030	0.0203	0.5589
			β_2	−0.2507	0.1249	0.9772	0.0315	0.0323	−0.0256	0.0181	0.5202	−0.0245	0.0180	0.5223	−0.0328	0.0188	0.5263
			θ	−0.0308	0.0126	0.4236	0.0133	0.0134	0.0020	0.0057	0.2794	0.0031	0.0056	0.2794	−0.0071	0.0063	0.2788
	100	80	α	0.6045	1.1682	2.0854	0.1939	0.1876	0.0048	0.0309	0.6904	0.0055	0.0308	0.6906	−0.0019	0.0310	0.6967
			β_1	0.8505	0.8734	1.5194	0.0479	0.0482	0.0410	0.0284	0.6407	0.0419	0.0285	0.6418	0.0328	0.0275	0.6312
			β_2	−0.2560	0.0942	0.6641	0.0203	0.0204	−0.0538	0.0232	0.5558	−0.0529	0.0231	0.5532	−0.0623	0.0248	0.5665
			θ	−0.0440	0.0063	0.2604	0.0080	0.0080	−0.0033	0.0032	0.2068	−0.0026	0.0032	0.2068	−0.0096	0.0032	0.2044
		90	α	0.5197	0.8831	1.9005	0.1894	0.1794	−0.0072	0.0206	0.5717	−0.0068	0.0205	0.5704	−0.0011	0.0208	0.5749
			β_1	0.7606	0.7900	1.4300	0.0449	0.0447	0.0283	0.0178	0.4949	0.0289	0.0178	0.4940	0.0230	0.0175	0.4965
			β_2	−0.2537	0.0890	0.6151	0.0201	0.0203	−0.0424	0.0196	0.4903	−0.0417	0.0196	0.4897	−0.0487	0.0202	0.5005
			θ	−0.0486	0.0062	0.2429	0.0078	0.0078	−0.0031	0.0028	0.2069	−0.0022	0.0028	0.2070	−0.0010	0.0029	0.2037

Table 6. Optimization criteria when $\alpha = 1.2$, $\beta_1 = 1.6$, $\beta_2 = 1.6$, and $\theta = 1.5$.

Scheme	τ	n	m	$C1$	$C2$	$C3$	$C4$
I	0.25	50	40	16.3206	0.0061	180.2384	1.7014
			45	6.0414	0.0020	184.6583	1.4621
		100	80	2.2000	0.0004	354.1789	1.8860
			90	1.1367	0.0002	356.2299	1.6011
	0.7	50	40	4.5528	0.0001	758.9591	0.8902
			45	2.3602	0.0001	1041.8315	0.8309
		100	80	1.7095	0.0000	1012.1129	0.8804
			90	0.8120	0.0000	1490.5832	0.8054
II	0.25	50	40	11.6987	0.0048	391.3110	1.3976
			45	2.0260	0.0006	313.1369	1.4117
		100	80	5.3355	0.0003	418.3329	1.3703
			90	0.4786	0.0001	457.4760	1.3855
	0.7	50	40	8.7456	0.0015	1780.0456	1.5195
			45	2.2929	0.0003	1781.0465	1.5845
		100	80	0.3569	0.0000	2361.9466	1.6027
			90	0.2268	0.0000	2927.2421	1.7028

Table 7. Optimization criteria when $\alpha = 3$, $\beta_1 = 2$, $\beta_2 = 1.8$, and $\theta = 0.8$.

Scheme	τ	n	m	$C1$	$C2$	$C3$	$C4$
I	0.25	50	40	27.7201	0.0098	139.3604	0.4791
			45	25.8804	0.0063	146.8403	0.4327
		100	80	26.0761	0.0010	269.3626	0.4494
			90	1.0596	0.0002	284.6315	0.4192
	0.5	50	40	21.4636	0.0047	182.4893	0.9768
			45	19.6311	0.0040	197.9801	0.5763
		100	80	16.5668	0.0003	339.1911	0.7525
			90	13.4178	0.0002	343.7374	0.5483
II	0.25	50	40	30.9083	0.0173	130.2885	0.4187
			45	27.2336	0.0389	148.9823	0.4235
		100	80	27.4830	0.0012	266.5808	0.4100
			90	1.9883	0.0004	295.9238	0.4106
	0.5	50	40	54.6397	0.0178	150.5007	0.5241
			45	19.8762	0.0178	164.4730	0.5210
		100	80	27.8974	0.0007	287.5853	0.4947
			90	21.2904	0.0004	321.4360	0.4917

9. Application of Real Data

In this section, we apply real data to demonstrate the value and adaptability of the APW distribution using various schemes and sampling strategies. The data set, which was used in [22], corresponds to the days between 109 consecutive coal-mining incidents in Great Britain. Measures (AIC, CAIC, BIC, and HQIC), CVM, AD, Kolmogorov–Smirnov (KS), and p-values for the data set are included in Table 8 along with the MLEs and accompanying standard errors (SEs) of each model parameter. Figure 1 shows the relative histogram with the fitted APW distribution, the fitted APW CDF plots with empirical CDF, and the QQ and p-p plots of the data set. The outcomes in Table 8 are also supported by these graphical goodness-of-fit techniques in Figure 1.

Table 8. MLE, SE, and different measures.

	Estimate	SE	AIC	CAIC	BIC	HQIC	CVM	AD	KS	p-Value
par1	209.5080	537.4003	1408.6145	1408.8431	1416.6886	1411.8889	0.0867	0.5489	0.0617	0.8003
par2	0.2054	0.1619								
par3	0.4679	0.1105								

Using two distinct sampling schemes with τ = 200 of step-stress ALT, we create some samples of step-stress ALTs based on progressive type-II censoring with m = 80 and 95 using the real data set. Table 9 reports the generated data and matching censoring scheme. The unknown parameters of the APW under step-stress ALT based on the progressive type-II censoring model's MLEs and Bayesian estimates with various loss functions are calculated using the data sets in Table 9 and are given in Table 10. Bayesian estimation based on different loss functions is presented in Table 10. We create 10,000 MCMC samples using the MCMC algorithm that is discussed. For the sake of using the MCMC sampler algorithm, the initial values of the unknown parameters were assumed to be their MLE estimators.

Table 9. Data based on the model when $\tau = 200$.

m	Scheme	Before τ	After τ
80	I	1 4 4 7 11 13 15 15 17 18 19 19 20 20 22 23 28 29 31 32 36 37 47 48 49 50 54 54 55 59 59 61 61 66 72 72 75 78 78 81 93 96 99 108 113 114 120 120 120 123 124 129 131 137 145 151 156 171 176, 182 188 189 195	203 208 215 217 217 217 224 228 233 255 271 275 275 275 286 291 312
	II	1 4 4 7 11 13 18 19 19 20 20 22 29 31 36 37 47 48 49 50 54 59 61 72 75 78 93 96 99 108 120 120 120 124 129 131 137 145 151 156 171 176 189 195	203 217 217 217 224 228 255 271 275 275 275 286 312 312 315 326 326 330 336 338 345 348 354 361 364 369 378 390 457 498 517 745 871 1312 1357 1613
95	I	1 4 4 7 11 13 15 15 17 18 19 19 20 20 22 23 28 29 31 32 36 37 47 48 49 50 54 54 55 59 59 61 61 66 72 72 75 78 78 81 93 96 99 108 113 114 120 120 120 123 124 129 131 137 145 151 156 171 176 182 188 189 195	203 208 215 217 217 217 224 228 233 255 271 275 275 275 286 291 312 312 312 315 326 326 329 330 336 338 345 348 354 361 364 369
	II	1 4 4 7 11 13 15 17 18 19 19 20 20 22 23 29 31 36 37 47 48 49 50 54 55 59 59 61 61 72 75 78 78 93 96 99 108 113 114 120 120 120 123 124 129 131 137 145 151 156 171 176 188 189 195	203 217 217 217 224 228 233 255 271 275 275 275 286 312 312 315 326 326 329 330 336 338 345 348 354 361 364 369 378 390 457 467 498 517 745 871 1312 1357 1613 1630

Table 10. Parameter estimation of the model when $\tau = 200$.

Scheme		I				II			
m		MLE	SELF	ELF c = −1.25	ELF c = 1.25	MLE	SELF	ELF c = −1.25	ELF c = 1.25
80	α	229.7518	256.6721	259.2273	233.5119	230.6366	283.9013	289.5624	229.1130
	β_1	0.2361	0.2450	0.2459	0.2370	0.1810	0.1936	0.1948	0.1817
	β_2	0.0634	0.0686	0.0693	0.0620	0.0942	0.1032	0.1042	0.0934
	θ	0.5101	0.5049	0.5051	0.5026	0.5369	0.5297	0.5300	0.5263
95	α	229.9271	273.4300	276.5470	243.8001	230.5368	260.5656	264.4054	224.5377
	β_1	0.1928	0.2027	0.2036	0.1945	0.1850	0.1861	0.1871	0.1774
	β_2	0.0882	0.0953	0.0960	0.0889	0.0884	0.0897	0.0906	0.0820
	θ	0.5347	0.5288	0.5290	0.5265	0.5399	0.5424	0.5427	0.5395

Figure 2 shows trace plots of the posterior distributions of parameters tracking the convergence of MCMC outputs. It shows how well the MCMC process converges. Figure 3 also shows the histograms for the marginal posterior density estimates of the parameters based on 10,000 chain values and the Gaussian kernel. The estimations clearly show that all of the generated posteriors are symmetric with respect to the theoretical posterior density functions.

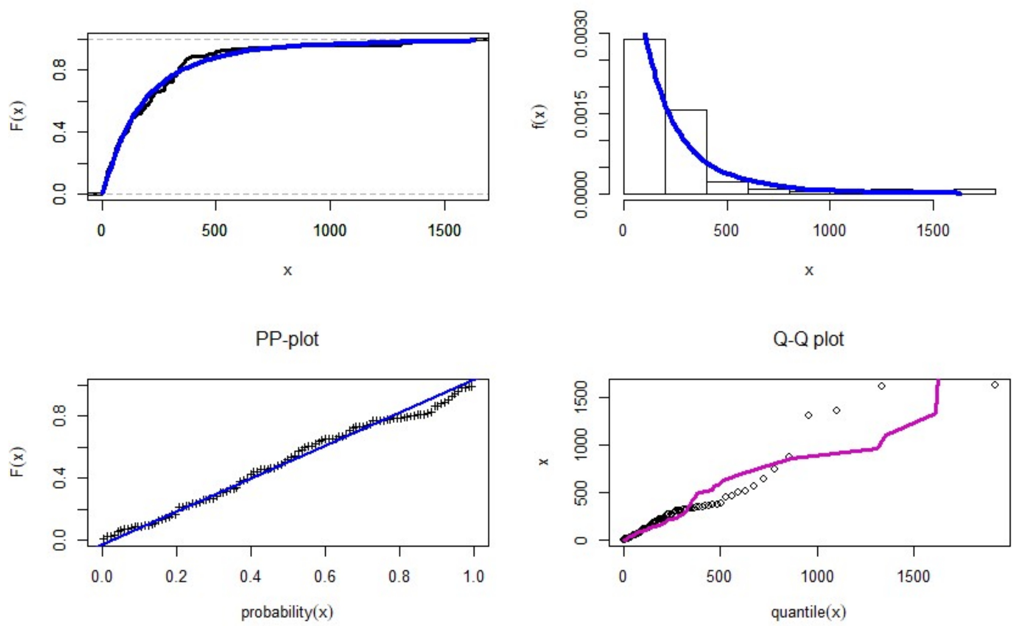

Figure 1. APW plots with different shapes.

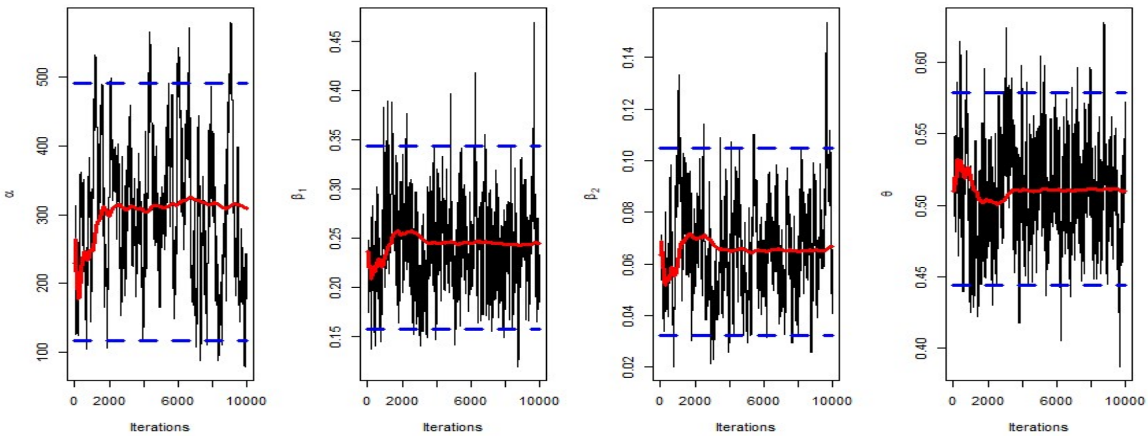

Figure 2. MCMC trace plot.

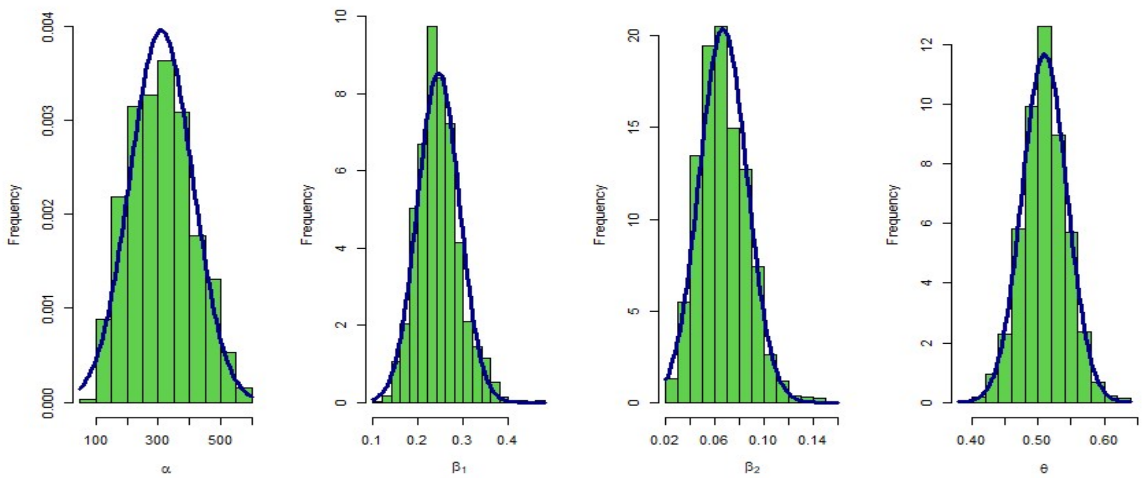

Figure 3. Histogram of MCMC results and kernel density estimates of parameters.

We estimated tau where $F_1(\tau) = F_2(\tau^*)$ and the survival characteristics S1 and S2 for each distribution at distinct mission time τ^*, which are computed and listed in Table 11. The τ^* is near τ, and survival values of the estimates increase with the increase in the censored sample size m. Scheme II is better than Scheme I, according to τ^* and survival values.

Table 11. Estimated τ and survival values of the model.

Scheme		**I**				**II**			
m		**MLE**	**SELF**	**ELF c = −1.25**	**ELF c = 1.25**	**MLE**	**SELF**	**ELF c = −1.25**	**ELF c = 1.25**
80	τ^*	2634.8282	2490.4402	2452.2419	2880.1106	674.7867	656.0487	651.4561	707.7382
	S1	0.000011	0.000017	0.000017	0.000013	0.0137	0.0138	0.0134	0.0173
	S2	0.1490	0.1472	0.1450	0.1675	0.2160	0.2058	0.2014	0.2477
95	τ^*	863.4307	832.6288	827.7715	885.8488	784.8394	767.4241	761.1656	836.1470
	S1	0.0042	0.0047	0.0046	0.0054	0.0063	0.0060	0.0059	0.0067
	S2	0.1864	0.1813	0.1785	0.2077	0.1942	0.1871	0.1838	0.2186

Additionally, two-sided 95% HPD credible intervals, approximative confidence intervals with standard error (SE) of parameters, and several MCMC characteristics such as SE are computed and listed in Table 12. We will demonstrate the idea of optimal censoring under the four criteria listed in Table 1 using the data sets in Table 9. The determinant and trace of the observed V-C matrix of the MLEs (24) can be used to determine the first three requirements with ease in Table 13.

Table 12. SE and confidence interval values of model for MLE and Bayesian estimation methods.

Scheme		**I**						**II**					
		MLE			**Bayesian**			**MLE**			**Bayesian**		
m		**SE**	**Lower**	**Upper**	**SE**	**Lower**	**Upper**	**SE**	**Lower**	**Upper**	**SE**	**Lower**	**Upper**
80	α	118.0578	96.1569	363.3467	73.0245	125.7197	398.2883	186.5222	67.1426	394.1305	113.8806	99.1169	420.4155
	β_1	0.0563	0.1258	0.3464	0.0420	0.1639	0.3223	0.0621	0.0593	0.3026	0.0443	0.1077	0.2820
	β_2	0.0239	0.0166	0.1102	0.0205	0.0332	0.1094	0.0376	0.0206	0.1678	0.0289	0.0461	0.1555
	θ	0.0440	0.4238	0.5964	0.0321	0.4442	0.5647	0.0560	0.4271	0.6467	0.0402	0.4568	0.6118
95	α	137.4981	109.4047	350.4495	82.5557	118.9227	428.0149	157.0162	92.9060	368.1675	89.8255	113.3098	446.9009
	β_1	0.0522	0.0905	0.2951	0.0379	0.1354	0.2819	0.0538	0.0795	0.2904	0.0377	0.1233	0.2716
	β_2	0.0291	0.0311	0.1453	0.0227	0.0541	0.1417	0.0318	0.0260	0.1508	0.0243	0.0437	0.1365
	θ	0.0463	0.4439	0.6254	0.0327	0.4673	0.5891	0.0493	0.4434	0.6365	0.0372	0.4787	0.6225

Table 13. Optimization criteria for data modeling.

m	**Scheme**	***C*1**	***C*2**	***C*3**
80	I	13,937.641	0.0000007	23,735.354
	II	34,790.557	0.0000023	22,146.579
95	I	18,905.736	0.0000007	25,484.337
	II	24,654.097	0.0000009	26,871.390

10. Conclusions

In this study, an optimization method for a step-stress accelerated life test with test unit lifetime is expected to follow an alpha power Weibull distribution. To reduce testing time and expenses, we used progressive type-II censoring and accelerated life testing, and we examined the impact of various stress levels using a cumulative exposure model. A log-linear relationship between the APW distribution's scale parameter and stress has been postulated. Maximum likelihood estimators for model parameters were computed, as were approximation and bootstrap confidence intervals (CIs). Under symmetric and asymmetric loss functions, a Bayesian estimate of the parameter model is obtained. Under normal operating conditions, an optimal test plan was developed by minimizing the asymptotic variance (AV) of the percentile. The simulation research is discussed in order to show that the model is optimal. It is shown that raising the censored sample size m decreases the VB, VMSE, and LCI of the estimates for the two alternative censored methods for fixed values of the sample size n. The VB, VMSE, and LCI for various censoring schemes decrease as sample size n for fixed values of m increases, and these results are consistent with the theoretical results. In terms of VB and VMSE reduction, the symmetric and asymmetric Bayesian estimations outperform the MLE in the instances studied. The LCI drops considerably, and the HPD's symmetric and asymmetric Bayesian estimations exceed the ACI of the MLE. The bootstrap CIs have the shortest CI lengths. Real-world data were also assessed to demonstrate the model's adaptability.

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