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Application of a Multiplier Transformation to Libera Integral Operator Associated with Generalized Distribution

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Abstract: The research presented in this paper deals with analytic p-valent functions related to the generalized probability distribution in the open unit disk U. Using the Hadamard product or convolution, function $f_s(z)$ is defined as involving an analytic p-valent function and generalized distribution expressed in terms of analytic p-valent functions. Neighborhood properties for functions $f_s(z)$ are established. Further, by applying a previously introduced linear transformation to $f_s(z)$ and using an extended Libera integral operator, a new generalized Libera-type operator is defined. Moreover, using the same linear transformation, subclasses of starlike, convex, close-to-convex and spiralike functions are defined and investigated in order to obtain geometrical properties that characterize the new generalized Libera-type operator. Symmetry properties are due to the involvement of the Libera integral operator and convolution transform.

Keywords: p-valent function; starlike function; convex function; close-to-convex function; spiralike function; generalized distribution; neighborhood; Libera operator

MSC: 30C45

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1. Introduction

Let A denote the class of all functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$.

For brevity, let A_p denote the class of all analytic p-valent functions having the form:

$$f_p(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad p \in \mathbb{N}. \tag{2}$$

A function having the form given by (2) is said to be p-valent in the open unit disk U if it is analytic and assumes no value more than p times for $|z| < 1$. The class A_p, which is invariant (or symmetric) under rotations, is subject to investigations at the moment for many researchers, with interesting results related to certain subclasses of $p-$valent functions being obtained in correlation to operators. For instance, in [1], applications of a Salagean operator can be seen in [2], a hypergeometric function is associated with the study, a generalized differential operator is applied in [3,4] and a Dziok–Srivastava operator is used in [5].

In 2018, Porwal [6] introduced and studied a power series whose coefficients are probabilities of the generalized distribution such that

$$g_s(z) = z + \sum_{k=2}^{\infty} \frac{b_{k-1}}{S} z^k \quad p \in \mathbb{N}, \tag{3}$$

where S denotes the sum of the convergent series of the form:

$$S = \sum_{k=0}^{\infty} b_k$$

and $b_k \geq 0, \ \ k \in \mathbb{N}$ (see also [7]).

Here, for convenience, (3) is expressed in terms of analytic p-valent functions, such that

$$g_{p,s}(z) = z^p + \sum_{k=1}^{\infty} \frac{b_{k+p-1}}{S} z^{k+p} \quad p \in \mathbb{N}. \tag{4}$$

By convolution or Hadamard product of two analytic functions f and h, we mean that

$$f(z) * h(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \tag{5}$$

where $h(z) = z + \sum_{k=2}^{\infty} b_k z^k$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

Using the concept defined above using (4) and (5), an analytic function $f_s(z)$ is introduced such that

$$f_s(z) = f_p(z) * g_{p,s}(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} \frac{b_{k+p-1}}{S} z^{k+p} = g_{p,s}(z) * f_p(z) \quad p \in \mathbb{N}. \tag{6}$$

Suppose that $f \in A$ of the form (1) is given. Then f is called starlike, respectively, convex of order σ denoted by $f \in S^*(\sigma)$ and $f \in K(\sigma)$, if the following geometric conditions are satisfied

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma, \quad 0 \leq \sigma < 1, \ \ |z| < 1, \tag{7}$$

and

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \sigma, \quad 0 \leq \sigma < 1, \ \ |z| < 1. \tag{8}$$

Furthermore, let f and g be starlike of order σ, meaning that $f, g \in S^*(\sigma)$. Then f is said to belong to the class of close-to-convex functions of order ρ type σ, denoted by $f \in K(\sigma, \rho)$, if the following geometric condition is satisfied:

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > \rho, \quad 0 \leq \rho < 1, \ \ |z| < 1. \tag{9}$$

Similarly, f is said to belong to the class of spiralike function $f \in S_p(\sigma)$ if the following condition is satisfied

$$\Re\left\{e^{i\theta}\frac{zf'(z)}{f(z)}\right\} > \sigma, \quad 0 \leq \sigma < 1, \quad |\theta| < \frac{\pi}{2}, \quad |z| < 1. \tag{10}$$

The aforementioned geometric conditions (7)–(10) have the following equivalents, respectively:

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \sigma, \tag{11}$$

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| < p - \sigma, \tag{12}$$

$$\left|\frac{f'(z)}{g'(z)} - p\right| < p - \rho, \tag{13}$$

and

$$\left|e^{i\theta}\frac{zf'(z)}{f(z)} - p\right| < p - \sigma. \tag{14}$$

The subclasses that follow have been studied repeatedly by various authors (see [8–14] among others) from different perspectives, and several interesting results were obtained.

$$S^*(\mu;\phi) = \left\{f \in \Gamma : \frac{1}{1-\mu}\left(\frac{zf'(z)}{f(z)} - \mu\right) \prec \phi(z), z \in U\right\},$$

$$C(\mu;\phi) = \left\{f \in \Gamma : \frac{1}{1-\mu}\left(1 + \frac{zf''(z)}{f'(z)} - \mu\right) \prec \phi(z), z \in U\right\},$$

$$K(\mu,\rho;\phi,\varphi) = \left\{f \in \Gamma : \frac{1}{1-\rho}\left(\frac{zf'(z)}{g(z)} - \rho\right) \prec \varphi(z), z \in U,\ g \in S^*(\mu;\phi)\right\}.$$

Let h be univalent in U and f analytic in U, then f is said to be subordinate to h, written as $f \prec h$, if there exists a Schwartz function ω, which is analytic in U, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$ such that $f(z) = h(\omega(z))$. Further, let h be univalent in U, then the following equivalent holds true

$$f \prec h \Leftrightarrow f(0) = h(0) \text{ and } f(U) \subset h(U).$$

Interesting results involving subordination theory can be seen in [13,15–18], among others.

Now, let Ω denote the class of all analytic and univalent functions γ in U for which $\gamma(U)$ is convex with $\gamma(0) = 1$ and $\Re\{\gamma(z)\} > 0, z \in U$. For function $f(z)$ of the form (1), Makinde [19] defined a linear transformation $T^n_\beta f(z), \beta \geq 1, n \geq 0$, such that

$$\begin{aligned} T^0_\beta f(z) &= f(z) \\ T^1_\beta f(z) &= (1-\mu)f(z) + z\mu f'(z) \\ &\vdots \\ T^n_\beta f(z) &= T_\beta(T^{n-1}_\beta f(z)), \end{aligned}$$

where

$$T^n_\beta f(z) = z + \sum_{k=2}^{\infty} \beta\left(\frac{1+\mu(k+\beta-2)}{1+\mu(\beta-1)}\right)^n a_k z^k, \tag{15}$$

and

$$T^{n+1}_\beta f(z) = (1-\mu)T^n_\beta f(z) + z\mu(T^n_\beta f(z))'$$

or

$$z\mu(T^n_\beta f(z))' = T^{n+1}_\beta f(z) - (1-\mu)T^n_\beta f(z), \qquad \mu \in [0,1]. \tag{16}$$

Further, for $\gamma, \phi \in \Omega$, ref. [19] introduced and studied the subclasses of starlike, convex and close-to-convex functions $S^*(\sigma,\gamma), C(\sigma,\gamma)$ and $K(\sigma,\rho;\phi,\gamma)$, respectively, as

$$\begin{aligned} S^n_\beta(\sigma,\gamma) &= \left\{f \in \Gamma : T^n_\beta f(z) \in S^*(\sigma,\gamma)\right\}, \\ C^n_\beta(\sigma,\gamma) &= \left\{f \in \Gamma : T^n_\beta f(z) \in C(\sigma,\gamma)\right\}, \\ K^n_\beta(\sigma,\rho;\psi,\gamma) &= \left\{f \in \Gamma : T^n_\beta f(z) \in K(\sigma,\rho;\psi,\gamma)\right\}. \end{aligned}$$

In addition to these, we define the spiralike class of analytic function $S_p(\sigma,\theta,\gamma)$ such that

$$S^n_{p,\beta}(\sigma,\theta,\gamma) = \left\{f \in \Gamma : T^n_\beta f(z) \in S_p(\sigma,\theta,\gamma)\right\}.$$

Furthermore, Alexander in [20] introduced and studied an integral operator $I(z)$ such that

$$I(z) = \int_0^z \frac{f(t)}{t}dt, \tag{17}$$

for details, see [8,21,22], among others.

Further, Libera [15] defined an integral operator $L(z)$ such that

$$L(z) = \frac{2}{z}\int_0^z f(t)dt. \tag{18}$$

This operator is the solution of the first-order linear differential equation:

$$2f'(z) + f(z) = 2k(z).$$

Obviously, the Libera integral operator is the convolution of the function $f(z)$ given by (1), and the functions $y(z) = z + \sum_{k=2}^{\infty} \frac{2}{k+1} z^k$. That is,

$$L(z) = f(z) * y(z) = y(z) * f(z).$$

Libera integral operator given by (18) maps each of the subclasses of the starlike, convex and close-to-convex functions into itself, which makes the Libera integral operator symmetric in nature. Therefore, if $f(z)$ is close-to-convex with respect to the starlike function $g_*(z)$, $L(z) = \frac{2}{z}\int_0^z f(t)dt$ and $G(z) = \frac{2}{z}\int_0^z g_*(t)dt$, then L is close-to-convex with respect to G (see [11]). Libera integral operator preserves the starlike functions of order $-\frac{1}{2}$, $\frac{1}{2}$ and convex functions of order $-\frac{1}{2}$. It has been established that Libera integral operator converges uniformly, which makes it asymptotic in nature, and coupled with the fact that it is a bounded operator, it is fractional in nature.

Furthermore, certain aspects regarding the convexity of the Libera integral operator were proven in [23], and new operators were defined using it in [24,25].

In particular, the operator $L_a(z)$ $(a \geq 1)$ is defined as follows:

$$L_a(z) = \frac{1+a}{z^a}\int_0^z zf(t)t^{a-1}dt. \tag{19}$$

It is worth noting that the operator $L_a(z)$, given by (19), generalized the previously defined Libera operator (see [11,13,16,21,22], among others).

Here, for $f_s(z)$ of the form (6), the function $T^n_{\beta,p}f_s(z)$ is introduced as follows:

$$T^n_{\beta,p}f_s(z) = z^p + \sum_{k=1}^{\infty} \beta\left(\frac{1+\mu(k+\beta-2)}{1+\mu(\beta-1)}\right)^n a_{k+p}\frac{b_{k+p-1}}{S}z^{k+p}. \tag{20}$$

In Section 3 of the present work, using Equation (20), having considered the extended Libera operator $L_a\left(T^n_{\beta,p}f_s(z)\right)$, where

$$L_a\left(T^n_{\beta,p}f_s(z)\right) = \frac{p+a}{z^a}\int_0^z zT^n_{\beta,p}f_s(t)t^{a-1}dt, \tag{21}$$

we define and study, in terms of the generalized distribution function, the relationship between the properties of the subclasses of starlike functions $S^n_\beta(\sigma,\phi;s)$, convex functions $C^n_\beta(\sigma,\phi;s)$, close-to-convex functions $K^n_\beta(\sigma,\rho,\phi,\psi;s)$ and spiralike functions $S^n_p(\sigma,\beta,\phi,\theta;s)$ such that

$$S^n_{\beta,p}(\sigma,\phi;s) = \left\{f \in \Omega : T^n_{\beta,p}f_s(z) \in S^*(\sigma,\phi;s)\right\},$$

$$C^n_{\beta,p}(\sigma,\phi;s) = \Big\{f \in \Omega : T^n_{\beta,p}f_s(z) \in C(\sigma,\phi;s)\Big\},$$

$$K^n_{\beta,p}(\sigma,\rho,\phi,\psi;s) = \Big\{f \in \Omega : T^n_{\beta,p}f_s(z) \in K(\sigma,\rho;\phi,\psi,s)\Big\}$$

and

$$S^n_p(\sigma,\beta,\theta,\phi;s) = \Big\{f \in \Omega : T^n_{\beta,p}f_s(z) \in S_p(\sigma,\theta,\phi;s)\Big\}.$$

At this juncture, the following Lemmas shall be necessary (see [13–15] to mention but a few).

Lemma 1 ([13])**.** *Let δ_1 be convex and univalent in U with $\delta_1(0) = 1$ and $Re\{t\delta_1(z) + b\} > 0$, $t, b \in \mathbb{C}$. If r is analytic in U with $r(0) = 1$, then*

$$r(z) + \frac{zr'(z)}{tr(z) + b} \prec \delta_1(z), \quad (z \in U),$$

which implies that

$$r(z) \prec \delta_1(U).$$

Lemma 2 ([13])**.** *Let δ_1 be convex and univalent in U with $Re\{\omega(z)\} \geq 0$. If r is analytic in U with $r(0) = \delta_1(0) = 1$, then*

$$r(z) + \omega(z)r'(z) \prec \delta_1, \quad (z \in U),$$

which implies that

$$r(z) \prec \delta_1(U).$$

In Section 2 of the paper, neighborhood properties will be discussed involving the function defined in relation (6). The additional already known results used for the proofs are given at the beginning of Section 2. Section 3 presents some results involving the concept of subordination and the extended Libera operator given in relation (21). The theorems stated there prove the starlikeness, convexity and close-to-convexity characteristics of this operator.

2. Neighborhood of Analytic P-Valent Function Associated with the Generalized Distribution

Next, some results on the neighborhood of the analytic p-valent function associated with the famous generalized probability distribution are presented.

Before proceeding to the main results, the following definitions shall be considered.

Let $f_s(z), G_s(z) \in A_p$, then we say that $f_s(z)$ is $(\alpha,\delta,p,s)-$neighborhood for $G_s(z)$ if it satisfies the condition that

$$\Big|f_s'(z) - e^{i\alpha}G_s'(z)\Big| < \delta \tag{22}$$

for $|\alpha| \leq \pi$, $\delta > p\sqrt{2(1-cos\alpha)}$ and $z \in U$. It implies that $f_s(z) \in (\alpha,\delta,p,s) - N(G_s(z))$.

Similarly, we say that $f_s(z) \in (\alpha,\delta,p,s) - M(G_s(z))$ if it satisfies the condition that

$$\left|\frac{f_s(z)}{z^p} - e^{i\alpha}\frac{G_s(z)}{z^p}\right| < \delta. \tag{23}$$

For recent work in this direction, refer to [17,26,27], among others.

Theorem 1. *Let $f_s(z) \in A_p$ satisfy the inequality*

$$\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\Big|a_{k+p} - e^{i\alpha}c_{k+p}\Big| \leq \delta - p\sqrt{2(1-cos\alpha)}, \tag{24}$$

for $|\alpha| \leq \pi$, $p \in \mathbb{N}$ and $\delta > p\sqrt{2(1-cos\alpha)}$, then $f_s(z) \in (\alpha,\delta,p,s) - N(G_s(z))$.

Proof. From (22), it is observed that

$$\left|f_s'(z)-e^{i\alpha}G_s'(z)\right|=\left|p\left(1-e^{i\alpha}\right)z^{p-1}+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left(a_{k+p}-e^{i\alpha}c_{k+p}\right)z^{k+p-1}\right|$$

$$\leq |p(1-e^{i\alpha})||z|^{p-1}+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}|a_{k+p}-e^{i\alpha}c_{k+p}||z|^{k+p-1}$$

$$< p\sqrt{2(1-cos\alpha)}+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right||z|^{k+p-1}.$$

Now, suppose that

$$\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|\leq \delta-p\sqrt{2(1-cos\alpha)},$$

then we conclude that

$$\left|f_s'(z)-e^{i\alpha}G_s'(z)\right|<\delta, \quad z\in U.$$

Therefore, $f_s(z)\in(\alpha,\delta,p,s)-N(G_s(z))$. □

Consider the following example:

Example 1. *Given that*

$$f_s(z)=z^p+\sum_{k=1}^{\infty}\frac{b_{k+p-1}}{S}a_{k+p}z^{k+p}$$

and

$$G_s(z)=z^p+\sum_{k=1}^{\infty}\frac{b_{k+p-1}}{S}c_{k+p}z^{k+p}$$

with

$$a_{k+p}=\frac{pe^{i\tau}\left[\delta-p\sqrt{2(1-cos\alpha)}\right]}{(k+p-1)(k+p)^2\frac{b_{k+p-1}}{S}}+e^{i\alpha}c_{k+p}, \quad \left(|\alpha|\leq\pi, |\tau|\leq\pi, p\in\mathbb{N}, k\geq 1 \text{ and } z\in U\right).$$

Then

$$\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|$$

$$=\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|\frac{pe^{i\tau}\left[\delta-p\sqrt{2(1-cos\alpha)}\right]}{(k+p-1)(k+p)^2\frac{b_{k+p-1}}{S}}+e^{i\alpha}c_{k+p}-e^{i\alpha}c_{k+p}\right|$$

$$=\sum_{k=1}^{\infty}\frac{p\left|e^{i\tau}\right|\left[\delta-p\sqrt{2(1-cos\alpha)}\right]}{(k+p-1)(k+p)}=p\left[\delta-p\sqrt{2(1-cos\alpha)}\right].\sum_{k=1}^{\infty}\left[\frac{1}{k+p-1}-\frac{1}{k+p}\right]$$

$$=\delta-p\sqrt{2(1-cos\alpha)}, \quad \left(since \ \left[\frac{1}{(k+p-1)(k+p)}\right]_{k=1}^{\infty}\to\frac{1}{p}\right).$$

Therefore, $f_s(z)\in(\alpha,\delta,p,s)-N(G_s(z))$.

Corollary 1. *Let* $f_s(z)\in A_1=A$ *satisfy the inequality*

$$\sum_{k=1}^{\infty}(k+1)\frac{b_k}{S}\left|a_{k+1}-e^{i\alpha}c_{k+1}\right|\leq\delta-\sqrt{2(1-cos\alpha)},$$

for $|\alpha|\leq\pi$, *and* $\delta>\sqrt{2(1-cos\alpha)}$, *then* $f_s(z)\in(\alpha,\delta,1,s)-N(G_s(z))$.

Corollary 2. *Let $f_s(z) \in A_1 = A$ satisfy the inequality*

$$\sum_{k=1}^{\infty} (k+1) \frac{b_k}{S} \left| a_{k+1} - c_{k+1} \right| \leq \delta,$$

for $\delta > 0$, then $f_s(z) \in (0, \delta, 1, s) - N(G_s(z))$.

Corollary 3. *Let $f_s(z) \in A_p$ satisfy the inequality*

$$\sum_{k=1}^{\infty} (k+p) \frac{b_{k+p-1}}{S} \left| |a_{k+p}| - |c_{k+p}| \right| \leq \delta - p\sqrt{2(1 - cos\alpha)},$$

for $|\alpha| \leq \pi$, and $\delta > p\sqrt{2(1 - cos\alpha)}$ and $arg\ a_{k+p} - arg c_{k+p} = u (k \geq 1)$, then $f_s(z) \in (\alpha, \delta, p, s) - N(G_s(z))$.

Proof. From Theorem 1, we have that $f_s(z) \in (\alpha, \delta, p, s) - N(G_s(z))$ if

$$\sum_{k=1}^{\infty} (k+p) \frac{b_{k+p-1}}{S} \left| a_{k+p} - e^{i\alpha} c_{k+p} \right| \leq \delta - p\sqrt{2(1 - cos\alpha)}.$$

Since $\arg a_{p+k} - \arg c_{p+k} = \alpha$, if $\arg a_{p+k} = \theta_n$, then $\arg c_{p+k} = \theta_n - \alpha$. Then

$$\begin{aligned} a_{k+p} - e^{i\alpha} c_{k+p} &= |a_{k+p}| e^{i\theta_n} - |c_{k+p}| e^{i(\theta_n - \alpha)} . e^{i\alpha} \\ &= \Big(|a_{k+p}| - |c_{k+p}| \Big) e^{i\theta}. \end{aligned}$$

Therefore,

$$\left| a_{k+p} - e^{i\alpha} c_{k+p} \right| = \left| |a_{k+p}| - |c_{k+p}| \right| |e^{i\theta}|,$$

and this obviously ends the proof. □

Corollary 4. *Let $f_s(z) \in A_1 = A$ satisfy the inequality*

$$\sum_{k=1}^{\infty} (k+1) \frac{b_k}{S} \left| |a_{k+1}| - |c_{k+1}| \right| \leq \delta - \sqrt{2(1 - cos\alpha)},$$

for $|\alpha| \leq \pi$, and $\delta > \sqrt{2(1 - cos\alpha)}$ and $arg\ a_{k+1} - arg c_{k+1} = u (k \geq 1)$, then $f_s(z) \in (\alpha, \delta, 1, s) - N(G_s(z))$.

Theorem 2. *Let $f_s(z) \in A_p$ satisfy the inequality*

$$\sum_{k=1}^{\infty} \frac{b_{k+p-1}}{S} \left| a_{k+p} - e^{i\alpha} c_{k+p} \right| \leq \delta - p\sqrt{2(1 - cos\alpha)} \tag{25}$$

for $|\alpha| \leq \pi$, $p \in \mathbb{N}$ and $\delta > p\sqrt{2(1 - cos\alpha)}$, then $f_s(z) \in (\alpha, \delta, p, s) - M(G_s(z))$.

Proof. It is easily seen from (23) that

$$\left| \frac{f_s(z)}{z^p} - e^{i\alpha} \frac{G_s(z)}{z^p} \right| = \left| (1 - e^{i\alpha}) + \sum_{k=1}^{\infty} \frac{b_{k+p-1}}{S} \Big(a_{k+p} - e^{i\alpha} c_{k+p} \Big) z^k \right|$$

$$\leq \left| 1 - e^{i\alpha} \right| + \sum_{k=1}^{\infty} \left| \frac{b_{k+p-1}}{S} \right| \left| a_{k+p} - e^{i\alpha} c_{k+p} \right| |z|^k < \sqrt{2(1 - cos\alpha)} + \sum_{k=1}^{\infty} \left| \frac{b_{k+p-1}}{S} \right| \left| a_{k+p} - e^{i\alpha} c_{k+p} \right|.$$

Since
$$\sum_{k=1}^{\infty}\left|\frac{b_{k+p-1}}{S}\right|\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|\leq\delta-\sqrt{2(1-cos\alpha)},$$
then, we conclude that
$$\left|\frac{f_s(z)}{z^p}-e^{i\alpha}\frac{G_s(z)}{z^p}\right|<\delta,\quad z\in U.$$

Therefore, $f_s(z)\in(\alpha,\delta,p,s)-M(G_s(z))$. $\square$

Corollary 5. *Let $f_s(z)\in A_1=A$ satisfy the inequality*
$$\sum_{k=1}^{\infty}\frac{b_k}{S}\left|a_{k+1}-e^{i\alpha}c_{k+1}\right|\leq\delta-\sqrt{2(1-cos\alpha)},$$
for $|\alpha|\leq\pi$ and $\delta>\sqrt{2(1-cos\alpha)}$, then $f_s(z)\in(\alpha,\delta,1,s)-M(G_s(z))$.

Corollary 6. *Let $f_s(z)\in A_1=A$ satisfy the inequality*
$$\sum_{k=1}^{\infty}\frac{b_{k+p-1}}{S}\left|a_{k+1}-c_{k+1}\right|\leq\delta,$$
for $\delta>0$, then $f_s(z)\in(0,\delta,1,s)-M(G_s(z))$.

Theorem 3. *Let $f_s(z)\in(\alpha,\delta,p,s)-N(G_s(z))$ and $arg\left(a_{k+p}-e^{i\alpha}c_{k+p}\right)=(k+p-1)\lambda$, $(k\geq1,\ p\in\mathbb{N})$. Then*
$$\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|\leq\lambda+pcos\alpha-p. \tag{26}$$

Proof. Let $f_s(z)\in(\alpha,\delta,p,s)-N(G_s(z))$, then
$$\left|f_s'(z)-e^{i\alpha}G_s'(z)\right|=\left|p\left(1-e^{i\alpha}\right)z^{p-1}+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|e^{i(k+p-1)\lambda}z^{k+p-1}\right|<\lambda,$$
for all $z\in U$. Further, suppose that we consider z such that
$$argz=-\lambda.$$

Then
$$z^{k+p-1}=\left|z^{k+p-1}\right|.e^{i(k+p-1)\lambda}.$$

We observe that, for this kind of point $z\in U$
$$\left|f_s'(z)-e^{i\alpha}G_s'(z)\right|=\left|p\left(1-e^{i\alpha}\right)z^{p-1}+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|\left|z^{k+p-1}\right|\right|$$
$$=\left|p-pcos\alpha-ipsin\alpha+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right|z^{k+p-1}\right|.$$

It implies that
$$\left|f_s'(z)-e^{i\alpha}G_s'(z)\right|=\left(\left[p+\sum_{k=1}^{\infty}(k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p}-e^{i\alpha}c_{k+p}\right||z|^{k+p-1}-pcos\alpha\right]^2+p^2sin^2\alpha\right)^{\frac{1}{2}}<\lambda,$$

for $z \in U$. That is,

$$p(1 - cos\alpha) + \sum_{k=1}^{\infty} (k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p} - e^{i\alpha}c_{k+p}\right||z|^{k+p-1} < \lambda.$$

Letting $|z|^{k+p-1} \to 1^{-}$, then

$$\sum_{k=1}^{\infty} (k+p)\frac{b_{k+p-1}}{S}\left|a_{k+p} - e^{i\alpha}c_{k+p}\right| \leq \lambda + pcos\alpha - p,$$

and this completes the proof. □

3. Some Results on the Application of a Multiplier Transformation to Libera Integral Operator

Theorem 4. *Suppose that $a > -p$ and let $\gamma \in S$ with $\Re\{(p-\sigma)\gamma(z) + \sigma + \frac{1-\mu}{\mu}\} > 0$. Further, let $T^n_{\beta,p}f_s(z) \in S^n_{\beta,p}(\sigma,\phi;s)$, then $L_a\left(T^n_{\beta,p}f_s(z)\right) \in S^n_{\beta,p}(\sigma,\phi;s)$.*

Proof. If $T^n_{\beta,p}f_s(z) \in S^n_{\beta,p}(\sigma,\phi;s)$, then we have:

$$\left(\frac{1}{p-\sigma}\right)\left(\frac{z(T^n_{\beta,p}f_s(z))'}{T^n_{\beta,p}f_s(z)} - \sigma\right) \prec \phi(z). \tag{27}$$

Now, let

$$r(z) = \left(\frac{1}{p-\sigma}\right)\left(\frac{z(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}f_s(z))} - \sigma\right), \tag{28}$$

where r is analytic in U with $r(0) = 1$.

From (16) and (21), it is observed that

$$z(L_a(T^n_{\beta,p}f_s(z)))' = \frac{(a+p)T^n_{\beta,p}f_s(z) - (1-\mu)L_a(T^n_{\beta,p}f_s(z))}{\mu}. \tag{29}$$

With the aid of (28) and (29), we obtain

$$\frac{(a+p)T^n_{\beta,p}f_s(z)}{\mu L_a(T^n_{\beta,p}f_s(z))} = (p-\sigma)r(z) + \sigma + \frac{1-\mu}{\mu}. \tag{30}$$

Differentiating (30) logarithmically with respect to z and using (28), we have

$$\frac{(T^n_{\beta,p}f_s(z))'}{T^n_{\beta,p}f_s(z)} = \frac{(p-\sigma)r(z)+\sigma}{z} + \frac{(p-\sigma)r'(z)}{(p-\sigma)r(z)+\sigma+\frac{1-\mu}{\mu}}. \tag{31}$$

Simple computations of (31) yields

$$\left(\frac{1}{p-\sigma}\right)\left(\frac{z(T^n_{\beta,p}f_s(z)')}{T^n_{\beta,p}f_s(z)} - \sigma\right) = r(z) + \frac{zr'(z)}{(p-\sigma)r(z)+\sigma+\left(\frac{1-\mu}{\mu}\right)}. \tag{32}$$

We obtain the desired result by applying Lemma 1 to (32) while taking $t = p - \sigma$ and $b = \frac{1-\mu}{\mu}$. □

Theorem 5. *Suppose that $a > -p$ and let $\gamma \in S$ with $\Re\{(p-\sigma)\gamma(z) + \sigma + \frac{1-\mu}{\mu}\} > 0$. Further, let $T^n_{\beta,p}f_s(z) \in C^n_{\beta,p}(\sigma,\phi;s)$, then $L_a\left(T^n_{\beta,p}f_s(z)\right) \in C^n_{\beta,p}(\sigma,\phi;s)$.*

Proof. Since $T^n_{\beta,p}f_s(z) \in C^n_{\beta,p}(\sigma,\phi;s)$ if and only if $z(T^n_{\beta,p}f_s(z))' \in S^n_{\beta,p}(\sigma,\phi;s)$. Now, let $T^n_{\beta,p}f_s(z) \in C^n_{\beta,p}(\sigma,\phi;s)$, then we obtain:

$$\left(\frac{1}{p-\sigma}\right)\left(\frac{z(z(T^n_{\beta,p}f_s(z)'))'}{T^n_{\beta,p}f_s(z)} - \sigma\right) \prec \phi(z), z \in U. \tag{33}$$

Suppose that we set

$$r(z) = \left(\frac{1}{p-\sigma}\right)\left(\frac{z(z(L_a T^n_\beta f_s(z)'))'}{L_a T^n_\beta f_s(z)} - \sigma\right), \tag{34}$$

with r being analytic in U while $r(0) = 1$, then relating (16) and (21) with (34), we obtain

$$\frac{(a+p)(T^n_{\beta,p}f_s(z))'}{\sigma(L_a(T^n_{\beta,p}f_s(z)))'} = (p-\sigma)r(z) + \sigma + \frac{1-\mu}{\mu}. \tag{35}$$

Differentiating (35) logarithmically with respect to z yields

$$\frac{((T^n_{\beta,p}f_s(z))')'}{(T^n_{\beta,p}f_s(z))'} - \frac{((L_a(T^n_{\beta,p}f_s(z)))')'}{(L_a(T^n_{\beta,p}f_s(z)))'} = \frac{(1-\sigma)r'(z)}{(p-\sigma)r(z)+\sigma+\frac{1-\mu}{\mu}}. \tag{36}$$

Using (34) and (35) we obtain

$$\frac{((T^n_{\beta,p}f_s(z))')'}{(T^n_{\beta,p}f_s(z))'} = \frac{(p-\sigma)r(z)+\sigma}{z} + \frac{(p-\sigma)r'(z)}{(p-\sigma)r(z)+\sigma+\frac{1-\mu}{\mu}}. \tag{37}$$

Simple computation of (37) easily yields

$$\left(\frac{1}{p-\sigma}\right)\left(\frac{z((T^n_{\beta,p}f_s(z))')'}{(T^n_{\beta,p}f_s(z))'} - \sigma\right) = r(z) + \frac{zr'(z)}{(p-\sigma)r(z)+\sigma+\left(\frac{1-\mu}{\mu}\right)}. \tag{38}$$

Using (24), (36), Lemma 1, and taking $t = 1-\sigma$ and $b = \frac{1-\mu}{\mu}$, we have shown that $L_a\left(T^n_{\beta,p}f_s(z)\right) \in C^n_{\beta,p}(\sigma,\phi;s)$ and that completes the proof. □

Theorem 6. *Let $c > -p$ and let $\gamma, \psi \in S$ with $Re\{(p-\sigma)\gamma(z) + \sigma + \frac{1-\mu}{\mu}\} > 0$. If $T^n_{\beta,p}f_s(z) \in K^n_{\beta,p}(\sigma,\rho,\phi,\psi;s)$, then, $L_a\left(T^n_{\beta,p}f_s(z)\right) \in K^n_{\beta,p}(\sigma,\rho,\phi,\psi;s)$.*

Proof. Let $T^n_{\beta,p}f_s(z) \in K^n_{\beta,p}(\sigma,\rho,\phi,\psi;s)$, then there exist a function $T^n_{\beta,p}f_s(z) \in S^n_{\beta,p}(\sigma,\phi;s)$ such that

$$\left(\frac{1}{p-\rho}\right)\left(\frac{z(T^n_{\beta,p}f_s(z)')}{T^n_{\beta,p}g_s(z)} - \rho\right) \prec \psi(z). \tag{39}$$

Setting

$$r(z) = \left(\frac{1}{p-\rho}\right)\left(\frac{z(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}g_s(z))} - \rho\right), \tag{40}$$

while r is analytic in U with $r(0) = 1$, then using (16) and (21) with (40), we have

$$\frac{(a+p)T^n_{\beta,p}f_s(z) - (1-\mu)L_a\left(T^n_{\beta,p}f_s(z)\right)}{\mu L_a\left(T^n_{\beta,p}g_s(z)\right)} = (p-\zeta)r(z) + \rho,$$

which yields

$$\frac{(a+p)T^n_{\beta,p}f_s(z)}{\mu L_a T^n_{\beta,p}g_s(z)} = (p-\rho)r(z) + \rho + \frac{(1-\mu)L_a T^n_{\beta,p}f_s(z)}{\mu L_a T^n_{\beta,p}g_s(z)}. \tag{41}$$

(41) can be expressed as

$$\frac{(a+p)T^n_{\beta,p}f_s(z)}{\mu} = \Big[(p-\rho)r(z)+\rho\Big]L_a T^n_{\beta,p}g_s(z) + \frac{1-\mu}{\mu}L_a T^n_{\beta,p}f_s(z).$$

It implies that

$$\frac{(a+p)z(T^n_{\beta,p}f_s(z))'}{\mu} =$$

$$[(p-\rho)zr'(z)]L_a T^n_{\beta,p}g_s(z) + [(p-\rho)r(z)+\rho]z(L_a T^n_{\beta,n}g_s(z))' + \frac{1-\mu}{\mu}z(L_a T^n_{\beta,p}f_s(z))'. \tag{42}$$

Since $T^n_{\beta,p}g_s(z) \in S^n_{\beta,p}(\sigma,\phi;s)$ implies that $L_a\big(T^n_{\beta,p}g_s(z)\big) \in S^n_{\beta,p}(\sigma,\phi;s)$. Then from Theorem 4 and (16) we can write that

$$r(z) = \left(\frac{1}{p-\sigma}\right)\left(\frac{z(L_a(T^n_{\beta,p}g_s(z)))'}{L_a(T^n_{\beta,p}g_s(z))} - \sigma\right) \tag{43}$$

and

$$z(L_a(T^n_{\beta,p}g_s(z)))' = \frac{(a+p)T^n_{\beta,p}g_s(z) - (1-\mu)L_a(T^n_{\beta,p}g_s(z))}{\mu}, \tag{44}$$

respectively.

Now, using (43) in (44) we obtain

$$\frac{(a+p)T^n_{\beta,p}g_s(z)}{\mu L_a\Big(T^n_{\beta,p}g_s(z)\Big)} = (p-\sigma)r(z) + \sigma + \frac{1-\mu}{\mu}, \tag{45}$$

while simple computations from (42) and (45) yields

$$\frac{z(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}g_s(z))} = \frac{(p-\rho)zr'(z)}{(p-\sigma)r(z)+\sigma+\frac{1-\mu}{\mu}} + (p-\rho)r(z) + \rho.$$

This implies that

$$\frac{1}{p-\rho}\left(\frac{z(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}g_s(z))} - \rho\right) = r(z) + \frac{(p-\rho)zr'(z)}{(p-\sigma)r(z)+\sigma+\frac{1-\mu}{\mu}}. \tag{46}$$

Finally, by taking $\omega(z) = \frac{p-\rho}{(p-\sigma)r(z)+\sigma+\frac{1-\mu}{\mu}}$ while relating (43) and (46) and applying Lemma 2, we obtain the desired result. □

Theorem 7. *Let $a > -p$ and $\gamma \in S$ with $\Re\{(p-\sigma)\gamma(z) + \sigma + e^{i\theta}\frac{1-\mu}{\mu}\} > 0$. Further, let $T^n_{\beta,p}f_s(z) \in S^n_p(\sigma,\beta,\theta,\phi;s)$, then $L_a\big(T^n_{\beta,p}f_s(z)\big) \in S^n_p(\sigma,\beta,\theta,\phi;s)$.*

Proof. If $T^n_{\beta,p}f_s(z) \in S^n_p(\sigma,\beta,\theta,\phi;s)$, then we have

$$\left(\frac{1}{p-\sigma}\right)\left(e^{i\theta}\frac{z(T^n_{\beta,p}f_s(z))'}{T^n_{\beta,p}f_s(z)} - \sigma\right) \prec \phi(z). \tag{47}$$

Let
$$r(z) = \left(\frac{1}{p-\sigma}\right)\left(e^{i\theta}\frac{z(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}f_s(z))} - \sigma\right), \tag{48}$$
where r is analytic in U with $r(0) = 1$.

Recall that from (16) and (21), we can write that
$$z(L_a(T^n_{\beta,p}f_s(z)))' = \frac{(a+p)T^n_{\beta,p}f_s(z) - (1-\mu)L_a(T^n_{\beta,p}f_s(z))}{\mu}. \tag{49}$$

Now, appealing to (48) and (49), we obtain
$$e^{i\theta}\frac{(a+p)T^n_{\beta,p}f_s(z)}{\mu L_a(T^n_{\beta,p}f_s(z))} = (p-\sigma)r(z) + \sigma + e^{i\theta}\frac{1-\mu}{\mu}. \tag{50}$$

Differentiating (50) logarithmically with respect to z yields
$$\frac{(T^n_{\beta,p}f_s(z))'}{T^n_{\beta,p}f_s(z)} - \frac{(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}f_s(z))} = \frac{(p-\sigma)r'(z)}{(p-\sigma)r(z) + \sigma + e^{i\theta}\frac{(1-\mu)}{\mu}}. \tag{51}$$

Multiplying through (51) by $e^{i\theta}$, we have
$$e^{i\theta}\frac{(T^n_{\beta,p}f_s(z))'}{T^n_{\beta,p}f_s(z)} = e^{i\theta}\frac{(L_a(T^n_{\beta,p}f_s(z)))'}{L_a(T^n_{\beta,p}f_s(z))} + e^{i\theta}\frac{(p-\sigma)r'(z)}{(p-\sigma)r(z) + \sigma + e^{i\theta}\frac{(1-\mu)}{\mu}}. \tag{52}$$

Applying (48) in (52), we have
$$\frac{(T^n_{\beta,p}f_s(z))'}{T^n_{\beta,p}f_s(z)} = \frac{(p-\sigma)r(z)+\sigma}{z} + \frac{(p-\sigma)r'(z)}{(p-\sigma)r(z) + \sigma + e^{i\theta}\frac{(1-\mu)}{\mu}}. \tag{53}$$

It is easily verified from (53) that
$$\left(\frac{1}{p-\sigma}\right)\left(e^{i\theta}\frac{z(T^n_{\beta,p}f_s(z)')}{T^n_{\beta,p}f_s(z)} - \sigma\right) = r(z) + \frac{e^{i\theta}zr'(z)}{(p-\sigma)r(z) + \sigma + \left(e^{i\theta}\frac{(1-\mu)}{\mu}\right)}. \tag{54}$$

Now, taking $t = p - \sigma$ and $b = e^{i\theta}\frac{(1-\mu)}{\mu}$ while relating (54) and (48) with Lemma 1, the desired result follows. □

4. Conclusions

The study performed in the present paper is related to the intensely investigated class of p-valent functions. The tools involved in the study are convolution, generalized distribution, Libera integral operator and extended forms of this operator, special classes of univalent functions and the theory of differential subordination. Applying the concept of Hadamard product or convolution, in relation (6), a new function $f_s(z)$ is defined using the generalized distribution. Using a linear transformation $T^n_\beta f(z)$ given by (15) and (16), introduced in [19], the spiralike class of analytic function $s_p(\sigma, \theta, \varphi)$ is introduced following the pattern set in [19] where the classes of starlike, convex and close-to-convex functions were previously defined. Furthermore, using the same linear transformation $T^n_\beta f(z)$ and the previously defined generalized Libera operator given in (19), a new generalized Libera-type operator is introduced in (21) involving function $f_s(z)$ given by (6). Investigations on neighborhood properties of function $f_s(z)$ are conducted in Section 2 of the paper. The theorems proven have illustrations through corollaries, and an example is also presented. In Section 3, the new generalized Libera-type operator introduced in (21) is investigated, and the theorems prove that under certain conditions, it has starlikeness, convexity, close-to-convexity and spiralike properties.

In future directions of study, the function defined by (6) could be used related to other operators, such as the linear transformation $T_{\beta}^{n} f(z)$ and obtain potentially interesting operators, which could be further used in different studies for obtaining geometrical properties or for introducing subclasses of univalent functions. Further, the operator given by (21) can be used for investigations, which could lead to introducing new subclasses of univalent functions considering the starlikeness, convexity, close-to-convexity and spiralike properties proven in Section 3.

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