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Some Fractional Integral Inequalities by Way of Raina Fractional Integrals

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Abstract: In this research, some novel Hermite–Hadamard–Fejér-type inequalities using Raina fractional integrals for the class of ϑ-convex functions are obtained. These inequalities are more comprehensive and inclusive than the corresponding ones present in the literature.

Keywords: Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality; ϑ-convex function; Raina fractional integrals; Hölder's integral inequality; power mean inequality

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1. Introduction

Convexity has been known for a long time and has been intensively studied. Convex functions play a significant role in pure and applied mathematics, see [1–3]. To cope with the needs of modern mathematics, various generalizations of convex functions have been presented in the literature, such as the $(\alpha, \beta, \gamma, \delta)$-convex function [4], coordinated convex function [5], harmonically convex function [6], h_1, h_2-convex function [7], GA-convex function [8], biconvex function [9], refined convex function of Raina type [10], s-HH convex function [11], 4-convex function [12], ϑ-convex function [13] and so on. This work utilizes the ϑ-convex function, which is defined as follows.

Definition 1. *A function $p : H \subseteq R \longrightarrow R$ is said to be ϑ-convex on H if there is a function $\vartheta : R \longrightarrow R$ such that H is a ϑ-convex set and the inequality*

$$p(\upsilon\vartheta(\iota) + (1-\upsilon)\vartheta(\kappa)) \leq \upsilon p(\vartheta(\iota)) + (1-\upsilon)p(\vartheta(\kappa)), \quad (0 \leq \upsilon \leq 1), \tag{1}$$

is valid for each $\iota, \kappa \in H$.

If the inequality sign in (1) is reversed, then p is called ϑ-concave on the set H. Every convex function p on a convex set H is a ϑ-convex function provided that $\vartheta(\upsilon) = \upsilon$. The Hermite–Hadamard-type inequality for ϑ-convex functions is given in the following theorem.

Theorem 1. *Suppose that $\vartheta : J \subset R \longrightarrow R$ is a continuous increasing function and $\iota, \kappa \in J$ with $\iota < \kappa$. Moreover, let $p : I \subseteq R \longrightarrow R$ be a ϑ-convex function on $I = [\iota, \kappa]$, then the inequality*

$$p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right) \leq \frac{1}{\vartheta(\kappa)-\vartheta(\iota)}\int_{\vartheta(\iota)}^{\vartheta(\kappa)} p(\vartheta(\upsilon))\, d\vartheta(\upsilon) \leq \frac{p(\vartheta(\iota))+p(\vartheta(\kappa))}{2}, \tag{2}$$

is valid, see [14].

If $\vartheta(v) = v$ in Theorem 1, then Inequality (2) reduces to the classical Hermite–Hadamard inequality

$$p\left(\frac{\iota+\kappa}{2}\right) \leq \int_{\iota}^{\kappa} p(v)dv \leq \frac{p(\iota)+p(\kappa)}{2}, \tag{3}$$

where p is a convex function on $[i,\kappa]$. For a thorough review of recent work related to Hermite–Hadamard-type inequalities, see [15] and the references therein. An extension of Inequality (3) is the classical Hermite–Hadamard–Fejér inequality

$$p\left(\frac{\iota+\kappa}{2}\right)\int_{\iota}^{\kappa} q(v)dv \leq \int_{\iota}^{\kappa} p(v)q(v)dv \leq \frac{p(\iota)+p(\kappa)}{2}\int_{\iota}^{\kappa} q(v)dv, \tag{4}$$

where the function $q : [\iota,\kappa] \longrightarrow R$ is integrable and symmetric with respect to $\frac{\iota+\kappa}{2}$, see [16]. For detailed a investigation of Inequality (4), see [8,17–19]. In the present work, we will extend Inequality (4) for ϑ-convex functions.

It is riveting to study generalized convex functions in the scenario of fractional integral operators, see [20–22] and the references therein. There are various fractional operators inspired by applied problems or analytical approaches, for example, the Caputo–Fabrizio fractional integral [23], generalized fractional operators [24], fractional conformable operators [25], weighted fractional integrals [26], variable order and distributed order fractional operators [27], tempered fractional calculus [28,29] and the Raina fractional integral operator [30,31]. In the present paper, ϑ-convexity is utilized together with Raina fractional integrals. These integrals are defined in the following.

Definition 2. *Suppose that $p \in L(\iota,\kappa)$, then for $\sigma,\rho \in R^+, \delta \in R$, the Raina fractional integrals of p are given as follows*

$$\mathcal{I}^{\varpi}_{\sigma,\rho,\iota+;\delta}p(s) = \int_{\iota}^{s} (s-v)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(s-v)^{\sigma}]p(v)dv, \ \ (s>\iota)$$

and

$$\mathcal{I}^{\varpi}_{\sigma,\rho,\kappa-;\delta}p(s) = \int_{s}^{\kappa} (v-s)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(v-s)^{\sigma}]p(v)dv, \ \ (s<\kappa).$$

Here, $\mathcal{R}^{\varpi}_{\sigma,\rho}(s)$ is the Raina function given as follows

$$\mathcal{R}^{\varpi}_{\sigma,\rho}(s) = \mathcal{R}^{\varpi(0),\varpi(1),\varpi(2),\cdots}_{\sigma,\rho}(s) = \sum_{t=0}^{\infty} \frac{\varpi(t)}{\Gamma(\sigma t+\rho)}s^n, \ \ (\sigma,\rho \in R^+, s \in R),$$

where $\{\varpi(t)\}$ is a bounded sequence of positive real numbers, see [31].

The Raina fractional integrals are highly significant because of their generality. For instance, if we set $t=0$, $\varpi(0)=1$ and $\delta=0$ in Definition 2, then the classical Riemann–Liouville fractional integrals are obtained

$$J^{\rho}_{\iota+}p(s) = \frac{1}{\Gamma(\rho)}\int_{\iota}^{s} (s-v)^{\rho-1}f(v)dv, \ \ (s>\iota)$$

and

$$J^{\rho}_{\kappa-}p(s) = \frac{1}{\Gamma(\rho)}\int_{s}^{\kappa} (v-s)^{\rho-1}f(v)dv, \ \ (s<\kappa).$$

Similarly, various fractional integrals can be obtained by specifying the coefficients $\varpi(t)$. For recent work on inequalities based on Raina fractional integrals, see [20,32] and the references therein. In this article, we obtain Hermite–Hadamard–Fejér-type inequalities for ϑ-convex functions in the context of Raina fractional integrals; therefore, the results will be novel and considerably general.

2. Results

In this section, we establish the Hermite–Hadamard–Fejér inequalities in the setting of ϑ-convex functions and Raina fractional integrals. Firstly, we generalize Inequality (4) for ϑ-convex functions. Then, a Hermite–Hadamard–Fejér-type inequality for ϑ-convex functions involving the Raina fractional integral is obtained. Moreover, the estimates related to the left-hand side of this generalized fractional inequality are provided. The correlation of these results with the contemporary results present in the literature is also determined. We begin with the following result.

Lemma 1. *Suppose that the function $q : [\vartheta(\iota), \vartheta(\kappa)] \longrightarrow R$ is integrable and symmetric with respect to $\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}$, then the following equality*

$$\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) = \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota)) = \frac{1}{2}\Big[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota))\Big],$$

is valid for all $\sigma, \rho \in R^+$ and $\delta \in R$.

Proof. As q is symmetric with respect to $\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}$, we have $q(\vartheta(\iota) + \vartheta(\kappa) - h) = q(h)$ for all $h \in [\vartheta(\iota), \vartheta(\kappa)]$. Consider the left Raina fractional integral

$$\begin{aligned}
&\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) \\
&= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} (\vartheta(\kappa) - v)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\kappa) - v)^{\sigma}]q(v)dv \\
&= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} (h - \vartheta(\iota))^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(h - \vartheta(\iota))^{\sigma}]q(\vartheta(\iota) + \vartheta(\kappa) - h)dh \\
&= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} (h - \vartheta(\iota))^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(h - \vartheta(\iota))^{\sigma}]q(h)dh \\
&= \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota)),
\end{aligned}$$

which is the right Raina integral. □

In the sequel, J represents an interval of non-negative real numbers and $\iota, \kappa \in J$ with $\iota < \kappa$ and I represents the interval of real numbers. The function $q : [\vartheta(\iota), \vartheta(\kappa)] \longrightarrow R$ is an integrable and symmetric function with respect to $\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}$ with $\|q\|_{\infty} = sup_{v\in[\iota,\kappa]}|q(v)|$. Furthermore, the following notation is used to reduce complexity.

$$\Delta := \kappa - \iota, \quad \vartheta(\Delta) := \vartheta(\kappa) - \vartheta(\iota),$$

$$\mathcal{M}(J) := \frac{p(\iota) + p(\kappa)}{2}\Big[J^{\rho}_{\iota+}p(\kappa) + J^{\rho}_{\kappa-}p(\iota)\Big] - \Big[J^{\rho}_{\iota+}pq(\kappa) + J^{\rho}_{\kappa-}pq(\iota)\Big],$$

$$\mathcal{M}(\mathcal{I}) := \frac{p(\iota) + p(\kappa)}{2}\Big[\mathcal{I}^{\varpi}_{\sigma,\rho,\iota+;\delta}p(\kappa) + \mathcal{I}^{\varpi}_{\sigma,\rho,\kappa-;\delta}p(\iota)\Big] - \Big[\mathcal{I}^{\varpi}_{\sigma,\rho,\iota+;\delta}pq(\kappa) + \mathcal{I}^{\varpi}_{\sigma,\rho,\kappa-;\delta}pq(\iota)\Big]$$

and

$$\begin{aligned}
\mathcal{M}(E(\mathcal{J})) = {} & \frac{p(\vartheta(\iota)) + p(\vartheta(\kappa))}{2}\Big[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, p(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, p(\vartheta(\iota))\Big] \\
& - \Big[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, pq(\vartheta(\iota))\Big].
\end{aligned}$$

Theorem 2. *Suppose that $\vartheta : J \longrightarrow R$ is a continuous increasing function and the function $p : I \longrightarrow R$ is such that $p \in L([\vartheta(\iota), \vartheta(\kappa)])$ for $\vartheta(\iota), \vartheta(\kappa) \in I$. If p is a ϑ-convex function on $[\iota, \kappa]$, then the inequality*

$$p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\int_{\vartheta(\iota)}^{\vartheta(\kappa)} q(v)dv \leq \int_{\vartheta(\iota)}^{\vartheta(\kappa)} p(v)q(v)dv \leq \frac{p(\vartheta(\iota))+p(\vartheta(\kappa))}{2}\int_{\vartheta(\iota)}^{\vartheta(\kappa)} q(v)dv,$$

is valid for all $\sigma, \rho \in R^+$ and $\delta \in R$.

Proof. Considering the following integral, using the change in variable and the fact that q is symmetric with respect to $\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}$, we have

$$\begin{aligned}\int_{\vartheta(\iota)}^{\vartheta(\kappa)} p(v)q(v)dv &= \int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}} p(v)q(v)dv + \int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)} p(v)q(v)dv \\ &= \int_0^{\frac{\vartheta(\Delta)}{2}} p(\vartheta(\iota)+h)q(\vartheta(\iota)+h)dh + \int_0^{\frac{\vartheta(\Delta)}{2}} p(\vartheta(\kappa)-h)q(\vartheta(\kappa)-h)dh \\ &= \int_0^{\frac{\vartheta(\Delta)}{2}} [p(\vartheta(\iota)+h)+p(\vartheta(\kappa)-h)]q(\vartheta(\iota)+h)dh. \end{aligned} \tag{5}$$

Since p is convex, for $0 \leq a \leq b$, we have

$$p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}+a\right)+p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}-a\right) \leq p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}+b\right)+p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}-b\right), \tag{6}$$

see ([33] pp. 164–165). On substituting $a = 0$ and $b = \frac{\vartheta(\Delta)}{2} - h \geq 0$ into (6), we have

$$2p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right) \leq p(\vartheta(\kappa)-h)+p(\vartheta(\iota)+h). \tag{7}$$

By using Inequality (7) in Inequality (5), we have

$$\begin{aligned}\int_{\vartheta(\iota)}^{\vartheta(\kappa)} p(v)q(v)dv &\geq 2p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\int_0^{\frac{\vartheta(\Delta)}{2}} q(\vartheta(\iota)+h)dh \\ &= p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\int_{\vartheta(\iota)}^{\vartheta(\kappa)} q(v)dv.\end{aligned} \tag{8}$$

Now, putting $a = \frac{\vartheta(\Delta)}{2} - h \geq 0$ and $b = \frac{\vartheta(\Delta)}{2}$ in (6), we have

$$p(\vartheta(\iota)+h)+p(\vartheta(\kappa)-h) \leq p(\vartheta(\kappa))+p(\vartheta(\iota)). \tag{9}$$

By using Inequality (9) in Inequality (5), we have

$$\begin{aligned}\int_{\vartheta(\iota)}^{\vartheta(\kappa)} p(v)q(v)dv &\leq p(\vartheta(\iota))+p(\vartheta(\kappa))\int_0^{\frac{\vartheta(\Delta)}{2}} q(\vartheta(\iota)+h)dh \\ &= \frac{p(\vartheta(\iota))+p(\vartheta(\kappa))}{2}\int_{\vartheta(\iota)}^{\vartheta(\kappa)} q(v)dv.\end{aligned} \tag{10}$$

By combining Inequality (8) and Inequality (10), we get the required result. □

Remark 1. *If ϑ is taken as an identity function in Theorem 2, then Inequality (4) is retrieved.*

Theorem 3. *Suppose that $\vartheta : J \longrightarrow R$ is a continuous increasing function and the function $p : I \longrightarrow R$ is such that $p \in L([\vartheta(\iota), \vartheta(\kappa)])$ for $\vartheta(\iota), \vartheta(\kappa) \in I$. If p is ϑ-convex on $[\iota, \kappa]$, then for Raina fractional integrals, the inequality*

$$\begin{aligned} &p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota))\right] \\ &\leq \left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, pq(\vartheta(\iota))\right] \\ &\leq \frac{p(\vartheta(\iota))+p(\vartheta(\kappa))}{2}\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota))\right], \end{aligned} \tag{11}$$

is valid for all $\sigma, \rho \in R^+$ and $\delta \in R$.

Proof. Since p is a ϑ-convex function on $[\iota, \kappa]$, for $\vartheta(a), \vartheta(b) \in I$, we have

$$p\left(\frac{\vartheta(a)+\vartheta(b)}{2}\right) \leq \frac{p(\vartheta(a))+p(\vartheta(b))}{2}.$$

By substituting $\vartheta(a) = v\vartheta(\iota) + (1-v)\vartheta(\kappa)$ and $\vartheta(b) = (1-v)\vartheta(\iota) + v\vartheta(\kappa)$, we have

$$2\,p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right) \leq p(v\vartheta(\iota)+(1-v)\vartheta(\kappa)) + p((1-v)\vartheta(\iota)+v\vartheta(\kappa)). \tag{12}$$

On multiplying both sides of (12) by $v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))$ and then integrating the resultant inequality with respect to v over $[0,1]$, we have

$$\begin{aligned} &2p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\int_0^1 v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))dv \\ &\leq \int_0^1 v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]p(v\vartheta(\iota)+(1-v)\vartheta(\kappa))q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))dv \\ &\quad+\int_0^1 v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]p((1-v)\vartheta(\iota)+v\vartheta(\kappa))q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))dv. \end{aligned}$$

By substituting $u = v\vartheta(\iota) + (1-v)\vartheta(\kappa)$, we have

$$\begin{aligned} &p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\int_{\vartheta(\iota)}^{\vartheta(\kappa)}(\vartheta(\kappa)-u)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\kappa)-u)^{\sigma}]q(u)du \\ &\leq \int_{\vartheta(\iota)}^{\vartheta(\kappa)}(\vartheta(\kappa)-u)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\kappa)-u)^{\sigma}]p(u)q(u)du \\ &\quad+\int_{\vartheta(\iota)}^{\vartheta(\kappa)}(\vartheta(\kappa)-u)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\kappa)-u)^{\sigma}]p(\vartheta(\iota)+\vartheta(\kappa)-u)q(u)du. \end{aligned}$$

By substituting $\vartheta(\iota) + \vartheta(\kappa) - u = v$ and then utilizing the symmetry of q, we have

$$\begin{aligned} &p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\int_{\vartheta(\iota)}^{\vartheta(\kappa)}(\vartheta(\kappa)-u)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\kappa)-u)^{\sigma}]q(u)du \\ &\leq \int_{\vartheta(\iota)}^{\vartheta(\kappa)}(\vartheta(\kappa)-u)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\kappa)-u)^{\sigma}]p(u)q(u)du \\ &\quad+\int_{\vartheta(\iota)}^{\vartheta(\kappa)}(v-\vartheta(\iota))^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(v-\vartheta(\iota))^{\sigma}]p(v)q(v)dv. \end{aligned}$$

Now, using Definition 2 and Lemma 1, we have

$$p\left(\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\right)\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa))+\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota))\right]$$
$$\leq \left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa))+\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, pq(\vartheta(\iota))\right]. \quad (13)$$

Considering again the ϑ-convexity of p over the interval $[\iota,\kappa]$, we have

$$p(v\vartheta(\iota)+(1-v)\vartheta(\kappa)) \leq tp(\vartheta(\iota))+(1-v)p(\vartheta(\kappa)) \quad (14)$$

and

$$p((1-v)\vartheta(\iota)+v\vartheta(\kappa)) \leq (1-v)p(\vartheta(\iota))+tp(\vartheta(\kappa)). \quad (15)$$

Adding (14) and (15), we have

$$p(v\vartheta(\iota)+(1-v)\vartheta(\kappa))+p((1-v)\vartheta(\iota)+v\vartheta(\kappa)) \leq p(\vartheta(\iota))+p(\vartheta(\kappa)). \quad (16)$$

On multiplying both sides of (16) by $v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))$ and integrating with respect to v over the interval $[0,1]$, we have

$$\int_0^1 v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]p(v\vartheta(\iota)+(1-v)\vartheta(\kappa))q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))dv$$
$$+\int_0^1 v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]p((1-v)\vartheta(\iota)+v\vartheta(\kappa))q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))dv$$
$$\leq [p(\vartheta(\iota))+p(\vartheta(\kappa))]\int_0^1 v^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}v^{\sigma}]q(v\vartheta(\iota)+(1-v)\vartheta(\kappa))dv.$$

Substituting $u = v\vartheta(\iota)+(1-v)\vartheta(\kappa)$, then utilizing the symmetry of q and finally using Definition 2 and Lemma 1, we have

$$\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa))+\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, pq(\vartheta(\iota))\right]$$
$$\leq \frac{p(\vartheta(\iota))+p(\vartheta(\kappa))}{2}\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa))+\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota))\right]. \quad (17)$$

By combining (13) and (17), we get the required result. □

Remark 2. *If ϑ is taken as an identity function in Theorem 2, then the following inequality is valid for all $\sigma,\rho \in R^{+}$ and $\delta \in R$*

$$p\left(\frac{\iota+\kappa}{2}\right)\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\iota+;\delta}q(\kappa)+\mathcal{I}^{\varpi}_{\sigma,\rho,\kappa-;\delta}q(\iota)\right] \leq \left[\mathcal{I}^{\varpi}_{\sigma,\rho,\iota+;\delta}pq(\kappa)+\mathcal{I}^{\varpi}_{\sigma,\rho,\kappa-;\delta}pq(\iota)\right]$$
$$\leq \frac{p(\iota)+p(\kappa)}{2}\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\iota+;\delta}q(\kappa)+\mathcal{I}^{\varpi}_{\sigma,\rho,\kappa-;\delta}q(\iota)\right].$$

Remark 3. *If ϑ is taken as an identity function and $\varpi(0) = 1$ and $v = 0$ in Theorem 2, then the following inequality is valid*

$$p\left(\frac{\iota+\kappa}{2}\right)\left[J^{\rho}_{\iota+}q(\kappa)+J^{\rho}_{\kappa-}q(\iota)\right] \leq \left[J^{\rho}_{\iota+}pq(\kappa)+J^{\rho}_{\kappa-}pq(\iota)\right]$$
$$\leq \frac{p(\iota)+p(\kappa)}{2}\left[J^{\rho}_{\iota+}q(\kappa)+J^{\rho}_{\kappa-}q(\iota)\right],$$

which is given in [34].

Lemma 2. *Suppose that $\vartheta : J \longrightarrow R$ is a continuous increasing function and $p : I \longrightarrow R$ is a differentiable function on I^o such that $p' \in L([\vartheta(\iota), \vartheta(\kappa)])$ for $\vartheta(\iota), \vartheta(\kappa) \in I$. Then, the equality*

$$\mathcal{M}(\vartheta(\mathcal{R})) = \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left[\int_{\vartheta(\iota)}^{v} (\vartheta(\kappa) - h)^{\rho-1} \mathcal{R}^{\varpi}_{\sigma,\rho+1}[v(\vartheta(\Delta)(\vartheta(\kappa) - h))^{\sigma}] q(h) dh \right.$$
$$\left. - \int_{v}^{\vartheta(\kappa)} (h - \vartheta(\iota))^{\rho-1} \mathcal{R}^{\varpi}_{\sigma,\rho+1}[\delta(\vartheta(\Delta)(h - \vartheta(\iota)))^{\sigma}] q(h) dh \right] p'(v) dv,$$

is valid for all $\sigma, \rho \in R^+$ and $\delta \in R$.

Proof. By applying the integration by parts technique on the following integral, using change in variable and then by applying Definition 2, we have

$$\begin{aligned} I_1 &= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left[\int_{\vartheta(\iota)}^{v} (\vartheta(\kappa) - h)^{\rho-1} \mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa) - h)^{\sigma}] q(h) dh \right] p'(v) dv \\ &= \left(\int_{\vartheta(\iota)}^{\vartheta(\kappa)} (\vartheta(\kappa) - h)^{\rho-1} \mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa) - h)^{\sigma}] q(h) dh \right) p(\vartheta(\kappa)) \\ &\quad - \int_{\vartheta(\iota)}^{\vartheta(\kappa)} (\vartheta(\kappa) - v)^{\rho-1} \mathcal{R}^{\varpi}_{\sigma,\rho+1}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa) - v)^{\sigma}] q(v) p(v) dv \\ &= \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) \quad p(\vartheta(\kappa)) - \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa)) \\ &= \left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota)) \right] \frac{p(\vartheta(\kappa))}{2} - \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa)). \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left[\int_{v}^{\vartheta(\kappa)} (h - \vartheta(\iota))^{\rho} \mathcal{R}^{\varpi}_{\sigma,\rho+1}[\delta(\vartheta(\Delta))^{\sigma}(h - \vartheta(\iota))^{\sigma}] q(h) dh \right] p'(v) dv \\ &= -\left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, q(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, q(\vartheta(\iota)) \right] \frac{p(\vartheta(\iota))}{2} + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, pq(\vartheta(\iota)). \end{aligned}$$

Thus,

$$\begin{aligned} I_1 - I_2 = \frac{p(\vartheta(\iota)) + p(\vartheta(\kappa))}{2} & \left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, p(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, p(\vartheta(\iota)) \right] \\ & - \left[\mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\iota)+;\delta}\, pq(\vartheta(\kappa)) + \mathcal{I}^{\varpi}_{\sigma,\rho,\vartheta(\kappa)-;\delta}\, pq(\vartheta(\iota)) \right] = \mathcal{M}(\vartheta(\mathcal{R})). \end{aligned}$$

□

Theorem 4. *Suppose that $\vartheta : J \longrightarrow R$ is a continuous increasing function and $p : I \longrightarrow R$ is a differentiable function on I^o such that $p' \in L([\vartheta(\iota), \vartheta(\kappa)])$ for $\vartheta(\iota), \vartheta(\kappa) \in I$ and $|p'|$ is ϑ-convex, then the inequality*

$$|\mathcal{M}(\vartheta(\mathcal{R}))| \leq (\vartheta(\Delta))^{\rho+1} \|q\|_{\infty} \mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\vartheta(\Delta))^{2\sigma}] \left(\left| p'(\vartheta(\iota)) \right| + \left| p'(\vartheta(\kappa)) \right| \right),$$

is valid for all $\sigma, \rho \in R^+$ and $\delta \in R$, where $\varpi_1(t) = \varpi(t)\left(1 - \frac{1}{2^{\sigma t+\rho}}\right) for \quad t = 0, 1, 2, \cdots.$

Proof. Since p is ϑ-convex on $[\iota, \kappa]$, then

$$\begin{aligned} |p'(v)| &= \left| p'\left(\frac{\vartheta(\kappa) - v}{\vartheta(\kappa) - \vartheta(\iota)} \vartheta(\iota)) + \frac{v - \vartheta(\iota)}{\vartheta(\kappa) - \vartheta(\iota)} \vartheta(\kappa)) \right) \right| \\ &\leq \frac{\vartheta(\kappa) - v}{\vartheta(\kappa) - \vartheta(\iota)} \left| p'(\vartheta(\iota)) \right| + \frac{v - \vartheta(\iota)}{\vartheta(\kappa) - \vartheta(\iota)} \left| p'(\vartheta(\kappa)) \right|. \end{aligned} \tag{18}$$

As q is symmetric with respect to $\frac{p(\iota)+p(\kappa)}{2}$, then

$$\begin{aligned}&\int_{v}^{\vartheta(\kappa)}(h-\vartheta(\iota))^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(h-\vartheta(\iota)))^{\sigma}]q(h)dh\\&=\int_{\vartheta(\iota)}^{\vartheta(\iota)+\vartheta(\kappa)-v}(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(\vartheta(\kappa)-h))^{\sigma}]q(h)dh.\end{aligned}$$

Consider the integral

$$\begin{aligned}&\bigg|\int_{\vartheta(\iota)}^{v}(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(\vartheta(\kappa)-h))^{\sigma}]q(h)dh\\&\quad-\int_{v}^{\vartheta(\kappa)}(h-\vartheta(\iota))^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(h-\vartheta(\iota))^{\sigma}]q(h)dh\bigg|\\&=\left|\int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v}(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(\vartheta(\kappa)-h))^{\sigma}]q(h)dh\right| \qquad (19)\\&\leq\begin{cases}\int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v}\Big|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(\vartheta(\kappa)-h))^{\sigma}]q(h)\Big|dh & \Big[\vartheta(\iota),\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}\Big]\\\int_{\vartheta(\iota)+\vartheta(\kappa)-v}^{v}\Big|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(\vartheta(\kappa)-h))^{\sigma}]q(h)\Big|dh & \Big[\frac{\vartheta(\iota)+\vartheta(\kappa)}{2},\vartheta(\kappa)\Big]\end{cases}.\end{aligned} \tag{20}$$

Using Lemma 1 and inequalities (18)–(20), we have

$$\begin{aligned}&|\mathcal{M}(\vartheta(\mathcal{R}))|\\&=\int_{\vartheta(\iota)}^{\vartheta(\kappa)}\left|\int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v}(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta)(\vartheta(\kappa)-h))^{\sigma}]q(h)dh\right||p'(v)|dv\\&=\int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}\left(\int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v}\Big|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h)\Big|dh\right)\\&\qquad\left(\frac{\vartheta(\kappa)-v}{\vartheta(\kappa)-\vartheta(\iota)}\Big|p'(\vartheta(\iota))\Big|+\frac{v-\vartheta(\iota)}{\vartheta(\kappa)-\vartheta(\iota)}\Big|p'(\vartheta(\kappa))\Big|\right)dv\\&+\int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)}\left(\int_{\vartheta(\iota)+\vartheta(\kappa)-v}^{v}\Big|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}_{\sigma,\rho}^{\varpi}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h)\Big|dh\right)\\&\qquad\left(\frac{\vartheta(\kappa)-v}{\vartheta(\kappa)-\vartheta(\iota)}\Big|p'(\vartheta(\iota))\Big|+\frac{v-\vartheta(\iota)}{\vartheta(\kappa)-\vartheta(\iota)}\Big|p'(\vartheta(\kappa))\Big|\right)dv\\&\leq\frac{\|q\|_{\infty}}{\vartheta(\Delta)}\sum_{t=0}^{\infty}\frac{\varpi(t)}{\Gamma(\sigma t+\rho+1)}(\delta(\vartheta(\Delta))^{\sigma})^{t}\\&\left[\int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}\{(\vartheta(\kappa)-v)^{\sigma t+\rho}-(v-\vartheta(\iota))^{\sigma t+\rho}\}\Big((\vartheta(\kappa)-v)\Big|p'(\vartheta(\iota))\Big|+(v-\vartheta(\iota))\Big|p'(\vartheta(\kappa))\Big|\Big)dv\right.\\&\left.+\int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)}\{(v-\vartheta(\iota))^{\sigma t+\rho}-(\vartheta(\kappa)-v)^{\sigma t+\rho}\}\Big((\vartheta(\kappa)-v)\Big|p'(\vartheta(\iota))\Big|+(v-\vartheta(\iota))\Big|p'(\vartheta(\kappa))\Big|\Big)dv\right]\end{aligned} \tag{21}$$

and we have

$$\begin{aligned}&\int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}\{(\vartheta(\kappa)-v)^{\sigma t+\rho}-(v-\vartheta(\iota))^{\sigma t+\rho}\}(\vartheta(\kappa)-v)dv\\&=\int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)}\{(v-\vartheta(\iota))^{\sigma t+\rho}-(\vartheta(\kappa)-v)^{\sigma t+\rho}\}(v-\vartheta(\iota))dv\\&=\frac{(\vartheta(\Delta))^{\sigma t+\rho+2}}{(\sigma t+\rho+1)}\left(\frac{\sigma t+\rho+1}{\sigma t+\rho+2}-\frac{1}{2^{\sigma t+\rho+1}}\right).\end{aligned} \tag{22}$$

Also, we have

$$\begin{aligned}
&\int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}} \{(\vartheta(\kappa)-v)^{\sigma t+\rho}-(v-\vartheta(\iota))^{\sigma t+\rho}\}(v-\vartheta(\iota))dv \\
&= \int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)} \{(v-\vartheta(\iota))^{\sigma t+\rho}-(\vartheta(\kappa)-v)^{\sigma t+\rho}\}(\vartheta(\kappa)-v)dv \\
&= \frac{(\vartheta(\Delta))^{\sigma t+\rho+2}}{(\sigma t+\rho+1)}\left(\frac{1}{\sigma t+\rho+2}-\frac{1}{2^{\sigma t+\rho+1}}\right).
\end{aligned} \tag{23}$$

By substituting (22) and (23) into (21), we have

$$\begin{aligned}
&|\mathcal{M}(\vartheta(\mathcal{R}))| \\
&= (\vartheta(\Delta))^{\rho+1}\|q\|_\infty \sum_{t=0}^{\infty} \frac{\varpi(t)}{\Gamma(\sigma t+\rho+2)}\left(\delta(\vartheta(\Delta))^{2\sigma}\right)^t \left(1-\frac{1}{2^{\sigma t+\rho}}\right)\left(\left|p'(\vartheta(\iota))\right|+\left|p'(\vartheta(\kappa))\right|\right) \\
&= (\vartheta(\Delta))^{\rho+1}\|q\|_\infty \sum_{t=0}^{\infty} \frac{\varpi_1(t)}{\Gamma(\sigma t+\rho+2)}\left(\delta(\vartheta(\Delta))^{2\sigma}\right)^t \left(\left|p'(\vartheta(\iota))\right|+\left|p'(\vartheta(\kappa))\right|\right) \\
&= (\vartheta(\Delta))^{\rho+1}\|q\|_\infty \mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\vartheta(\Delta))^{2\sigma}]\left(\left|p'(\vartheta(\iota))\right|+\left|p'(\vartheta(\kappa))\right|\right),
\end{aligned}$$

where

$$\varpi_1(t)=\varpi(t)\left(1-\frac{1}{2^{\sigma t+\rho}}\right) \text{ for } t=0,1,2,\cdots.$$

□

Remark 4. *If ϑ is taken as an identity function in Theorem 4, then the following inequality is valid*

$$|\mathcal{M}(\mathcal{R})| \le (\Delta)^{\rho+1}\|q\|_\infty \mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\Delta)^{2\sigma}]\left(\left|p'(\iota)\right|+\left|p'(\kappa)\right|\right).$$

Remark 5. *If ϑ is taken as an identity function and $\varpi(0)=1$ and $v=0$ in Theorem 4, then the following inequality is valid*

$$|\mathcal{M}(J)| \le \frac{(\Delta)^{\rho+1}\|q\|_\infty}{\Gamma(\rho+2)}\left(1-\frac{1}{2^\rho}\right)\left(\left|p'(\iota)\right|+\left|p'(\kappa)\right|\right),$$

which is given in [34].

Theorem 5. *Suppose that $\vartheta : J \longrightarrow R$ is a continuous increasing function and $p : I \longrightarrow R$ is a differentiable function on I^o such that $p' \in L([\vartheta(\iota),\vartheta(\kappa)])$ for $\vartheta(\iota),\vartheta(\kappa) \in I$. If $\left|p'\right|^n$, $n>1$, is a ϑ-convex function on $[\iota,\kappa]$, then for Raina fractional integrals, the inequality*

$$|\mathcal{M}(\vartheta(\mathcal{R}))| \le 2^{\frac{1}{m}}(\vartheta(\Delta))^{\rho+1}\|q\|_\infty\left(\mathcal{R}^{\varpi_2}_{\sigma,\rho+1}[\delta(\vartheta(\Delta))^{2\sigma}]\right)^{\frac{1}{m}}\left(\frac{\left|p'(\vartheta(\iota))\right|^n+\left|p'(\vartheta(\kappa))\right|^n}{2}\right)^{\frac{1}{n}},$$

is valid for all $\sigma,\rho \in R^+$, $\delta \in R$ and $\frac{1}{m}+\frac{1}{n}=1$. Here, $\varpi_2(t)=\left(\frac{\varpi(t)}{(\sigma t+\rho)m+1}\left(1-\frac{1}{2^{(\sigma t+\rho)m}}\right)\right)^{\frac{1}{m}}$ for $t=0,1,2,\cdots$.

Proof. Using Lemma 2, Inequality (21), the properties of the modulus and the well-known Hölder's inequality,

$$\begin{aligned}
&|\mathcal{M}(\vartheta(\mathcal{R}))| \\
&\leq \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left| \int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v} (\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho+1}\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h)dh \right| p'(v)dv \\
&= \left(\int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left| \int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v} (\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h)dh \right|^{m} dv \right)^{\frac{1}{m}} \\
&\quad \left(\int_{\vartheta(\iota)}^{\vartheta(\kappa)} |p'(v)|^{n} dv \right)^{\frac{1}{n}} \\
&= (I_3)^{\frac{1}{m}}(I_4)^{\frac{1}{n}}.
\end{aligned} \tag{24}$$

Considering the following integral, using Inequality (20) and solving the resultant integrals, we have

$$\begin{aligned}
I_3 &= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left| \int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v} (\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h)dh \right|^{m} dv \\
&\leq \int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}} \left(\int_{v}^{\vartheta(\iota)+\vartheta(\kappa)-v} \left| (\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h) \right|^{m} dh \right) \\
&\quad + \int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)} \left(\int_{\vartheta(\iota)+\vartheta(\kappa)-v}^{v} \left| (\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}]q(h) \right|^{m} dh \right) \\
&\leq \left(\|q\|_{\infty} \sum_{t=0}^{\infty} \frac{\varpi(t)}{\Gamma(\sigma t+\rho+1)} (\delta(\vartheta(\Delta))^{\sigma})^{t} \right)^{m} \left[\int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}} |(\vartheta(\kappa)-v)^{\sigma t+\rho} - (v-\vartheta(\iota))^{\sigma t+\rho}|^{m} dv \right. \\
&\quad \left. + \int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)} |(v-\vartheta(\iota))^{\sigma t+\rho} - (\vartheta(\kappa)-v)^{\sigma t+\rho}|^{m} dv \right] \\
&\leq \left(\|q\|_{\infty} \sum_{t=0}^{\infty} \frac{\varpi(t)}{\Gamma(\sigma t+\rho+1)} (\delta(\vartheta(\Delta))^{\sigma})^{t} \right)^{m} \left[\int_{\vartheta(\iota)}^{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}} \left\{ (\vartheta(\kappa)-v)^{(\sigma t+\rho)m} \right. \right. \\
&\quad \left. \left. -(v-\vartheta(\iota))^{(\sigma t+\rho)m} \right\} dv + \int_{\frac{\vartheta(\iota)+\vartheta(\kappa)}{2}}^{\vartheta(\kappa)} \left\{ (v-\vartheta(\iota))^{(\sigma t+\rho)m} - (\vartheta(\kappa)-v)^{(\sigma t+\rho)m} \right\} dv \right] \\
&\leq \left(\|q\|_{\infty} \sum_{t=0}^{\infty} \frac{\varpi(t)}{\Gamma(\sigma t+\rho+1)} (\delta(\vartheta(\Delta))^{2\sigma})^{t} \right)^{m} \left(\frac{2(\vartheta(\Delta))^{m\rho+1}}{(\sigma t+\rho)m+1} \left(1 - \frac{1}{2^{(\sigma t+\rho)m}} \right) \right) \\
&= 2(\vartheta(\Delta))^{m\rho+1}\|q\|_{\infty}^{m} \left(\mathcal{R}^{\varpi_2}_{\sigma,\rho+1}[\delta(\vartheta(\Delta))^{2\sigma}] \right)^{m},
\end{aligned} \tag{25}$$

where $\varpi_2(t) = \varpi(t)\left(\frac{1}{(\sigma t+\rho)m+1}\left(1 - \frac{1}{2^{(\sigma t+\rho)m}}\right) \right)^{\frac{1}{m}}$. Furthermore, we have used here the fact that $(a-b)^m \leq a^m - b^m$ for $0 \leq b \leq a$ and $m \geq 1$.

Now, considering the following integral and using the fact that $\left|p'\right|^n$ is ϑ-convex on $[\iota, \kappa]$

$$\begin{aligned}
I_4 &= \int_{\vartheta(\iota)}^{\vartheta(\kappa)} |p'(v)|^{n} dv \\
&\leq \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left| p'\left(\frac{\vartheta(\kappa)-v}{\vartheta(\kappa)-\vartheta(\iota)}\vartheta(\iota)) + \frac{v-\vartheta(\iota)}{\vartheta(\kappa)-\vartheta(\iota)}\vartheta(\kappa)) \right) \right|^{n} dv \\
&\leq \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left(\frac{\vartheta(\kappa)-v}{\vartheta(\kappa)-\vartheta(\iota)} \left| p'(\vartheta(\iota)) \right|^{n} + \frac{v-\vartheta(\iota)}{\vartheta(\kappa)-\vartheta(\iota)} \left| p'(\vartheta(\kappa)) \right|^{n} \right) dv
\end{aligned}$$

$$= \vartheta(\Delta)\left(\frac{\left|p'(\vartheta(\iota))\right|^n + \left|p'(\vartheta(\kappa))\right|^n}{2}\right). \tag{26}$$

By substituting the values of integrals I_3 and I_4 into (24), we have the required result. □

Remark 6. *If ϑ is taken as an identity function in Theorem 5, then the following inequality is valid:*

$$|\mathcal{M}(J)| \leq 2^{\frac{1}{m}}(\Delta)^{\rho+1}\|q\|_\infty\left(\mathcal{R}^{\varpi_2}_{\sigma,\rho+1}[\delta(\Delta)^{2\sigma}]\right)^{\frac{1}{m}}\left(\frac{\left|p'(\iota)\right|^n + \left|p'(\kappa)\right|^n}{2}\right)^{\frac{1}{n}}.$$

Remark 7. *If ϑ is taken as an identity function and $\varpi(0) = 1$ and $v = 0$ in Theorem 5, then the following inequality is valid*

$$|\mathcal{M}(\mathcal{R})| \leq \frac{2^{\frac{1}{m}}(\Delta)^{\rho+1}\|q\|_\infty}{\Gamma(\rho+1)}\left(\frac{1}{\rho m+1}\left(1-\frac{1}{2^{\rho m}}\right)\right)^{\frac{1}{m}}\left(\frac{\left|p'(\iota)\right|^n + \left|p'(\kappa)\right|^n}{2}\right)^{\frac{1}{n}},$$

which is given in [34].

Theorem 6. *Suppose that $\vartheta : J \longrightarrow R$ is a continuous increasing function and $p : I \longrightarrow R$ is a differentiable function on I^o such that $p' \in L([\vartheta(\iota), \vartheta(\kappa)])$ for $\vartheta(\iota), \vartheta(\kappa) \in I$. If $\left|p'\right|^n, n > 1$, is a ϑ-convex function on $[\iota, \kappa]$, then the following inequality*

$$|\mathcal{M}(\vartheta(\mathcal{R}))| \leq 2(\vartheta(\Delta))^{\rho+1+\frac{1}{n}}\|q\|_\infty\mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\vartheta(\Delta))^{2\sigma}]\left(\frac{\left|p'(\vartheta(\iota))\right|^n + \left|p'(\vartheta(\kappa))\right|^n}{2}\right)^{\frac{1}{n}},$$

is valid for all $\sigma, \rho \in R^+$ and $\delta \in R$, where ϖ_1 is as defined in Theorem 4.

Proof. Using Lemma 2, Inequality (21), the properties of the modulus and the well-known Hölder's inequality, we have

$$\begin{aligned}
&|\mathcal{M}(\vartheta(\mathcal{R}))| \\
&= \int_{\vartheta(\iota)}^{\vartheta(\kappa)}\left|\int_v^{\vartheta(\iota)+\vartheta(\kappa)-v}(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho+1}\delta(\vartheta(\Delta))^\sigma(\vartheta(\kappa)-h)^\sigma]q(h)dh\right|p'(v)dv \\
&= \left(\int_{\vartheta(\iota)}^{\vartheta(\kappa)}\left|\int_v^{\vartheta(\iota)+\vartheta(\kappa)-v}\left|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^\sigma(\vartheta(\kappa)-h)^\sigma]q(h)\right|dh\right|dv\right)^{1-\frac{1}{n}} \\
&\left(\int_{\vartheta(\iota)}^{\vartheta(\kappa)}\left|\left[\int_v^{\vartheta(\iota)+\vartheta(\kappa)-v}\left|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^\sigma(\vartheta(\kappa)-h)^\sigma]q(h)\right|dh\right]|p'(v)|^n\right|dv\right)^{\frac{1}{n}} \\
&= (I_5)^{1-\frac{1}{n}}(I_6)^{\frac{1}{n}}.
\end{aligned} \tag{27}$$

Considering I_5 and solving it by using (20) and (22)–(23), respectively, we have

$$\begin{aligned}
I_5 &= \int_{\vartheta(\iota)}^{\vartheta(\kappa)}\left|\int_v^{\vartheta(\iota)+\vartheta(\kappa)-v}\left|(\vartheta(\kappa)-h)^{\rho-1}\mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^\sigma(\vartheta(\kappa)-h)^\sigma]q(h)\right|dh\right|dv \\
&= 2(\vartheta(\Delta))^{\rho+1}\|q\|_\infty\mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\vartheta(\Delta))^{2\sigma}].
\end{aligned}$$

Similarly, solving I_6 by using (20), the ϑ-convexity of $|p^{'}|^n$ and (22)–(23), we have

$$I_6 = \int_{\vartheta(\iota)}^{\vartheta(\kappa)} \left| \left[\int_{\upsilon}^{\vartheta(\iota)+\vartheta(\kappa)-\upsilon} \left| (\vartheta(\kappa)-h)^{\rho-1} \mathcal{R}^{\varpi}_{\sigma,\rho}[\delta(\vartheta(\Delta))^{\sigma}(\vartheta(\kappa)-h)^{\sigma}] q(h) \right| dh \right] |p^{'}(\upsilon)|^n \right| d\upsilon$$
$$= (\vartheta(\Delta))^{\rho+2} \|q\|_{\infty} \mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\vartheta(\Delta))^{2\sigma}] \left(\left| p^{'}(\vartheta(\iota)) \right|^n + \left| p^{'}(\vartheta(\kappa)) \right|^n \right). \tag{28}$$

By substituting values of integrals I_5 and I_6 into (27), we get the required result. □

Remark 8. *If ϑ is taken as an identity function in Theorem 6, then the following inequality is valid*

$$|\mathcal{M}(\mathcal{R})| \leq 2(\vartheta(\Delta))^{\rho+1+\frac{1}{n}} \|q\|_{\infty} \mathcal{R}^{\varpi_1}_{\sigma,\rho+2}[\delta(\Delta)^{2\sigma}] \left(\frac{\left| p^{'}(\iota) \right|^n + \left| p^{'}(\kappa) \right|^n}{2} \right)^{\frac{1}{n}}.$$

Remark 9. *If ϑ is taken as an identity function, $\varpi(0) = 1$ and $v = 0$ in Theorem 6, then the following inequality is valid*

$$|\mathcal{M}(\mathcal{R})| \leq \frac{2\ (\Delta)^{\rho+1+\frac{1}{n}} \|q\|_{\infty}}{\Gamma(\rho+2)} \left(1 - \frac{1}{2^{\rho}} \right) \left(\frac{\left| p^{'}(\iota) \right|^n + \left| p^{'}(\kappa) \right|^n}{2} \right)^{\frac{1}{n}},$$

which is given in [34].

3. Conclusions

In the present work, Hermite–Hadamard–Fejér-type inequalities are established by utilizing the Raina fractional integrals for ϑ-convex functions. Primarily, a generalized version of the Hermite–Hadamard–Fejér inequality for ϑ-convex functions is obtained. Moreover, the fractional Hermite–Hadamard–Fejér inequality is also established by using Raina fractional integrals. Furthermore, right-sided estimates are formulated for the said fractional inequality. The backward compatibility of the results obtained in the study shows that these results are a considerable extension of the analogous results present in the literature.

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