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Application of Homotopy Analysis Transform Method for Solving a Fractional Singular One-Dimensional Thermo-Elasticity Coupled System

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Abstract: This article extends the application of fractional-order time derivatives to replace their integer-order counterparts within a system comprising two singular one-dimensional coupled partial differential equations. The resulting model proves invaluable in representing radially symmetric deformation and temperature distribution within a unit disk. The incorporation of fractional-order derivatives in mathematical models is shown to significantly enhance their capacity for characterizing real-life phenomena in comparison to their integer-order counterparts. To address the studied system numerically, we employ the q-homotopy analysis transform method (q-HATM). We evaluate the efficiency of this method in solving the problem through a series of illustrative examples. The convergence of the derived scheme is assessed visually, and we compare the performance of the q-HATM with that of the Laplace decomposition method (LDM). While both methods excel in resolving the majority of the presented examples, a notable divergence arises in the final example: the numerical solutions obtained using q-HATM converge, whereas those derived from LDM exhibit divergence. This discrepancy underscores the remarkable efficiency of the q-HATM in addressing this specific problem.

Keywords: q-homotopy; fractional derivative; coupled system; Laplace transform; decomposition method; auxiliary parameter; symmetric deformation

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1. Introduction

Applied mathematical models play a pivotal role in characterizing and studying a wide array of real-life phenomena across diverse disciplines, including physics, astronomy, chemistry, biology, economics, medicine, disease sciences, and engineering. Historically, ordinary and partial differential equations have been fundamental tools in these domains. However, since many natural phenomena exhibit continuous patterns, mathematical models based on integer-order ordinary and partial differential equations often fall short in providing accurate characterizations. Consequently, the scientific community has increasingly turned its attention to fractional ordinary and partial differential equations, which offer more precise and realistic mathematical models for these phenomena. This shift in focus has garnered significant global interest in recent decades. Within the existing literature, numerous articles are dedicated to presenting fractional mathematical models that describe physical and engineering processes. For a comprehensive overview, please refer to [1–6] and the references therein.

Conversely, obtaining exact analytical solutions for these mathematical models remains a formidable challenge. Consequently, various computational methods and numerical techniques have been developed by numerous researchers to address this issue. These techniques include the Adomian decomposition method (ADM), pioneered by Adomian in 1986 [7–10], the variational iteration method introduced by He [11,12], the finite difference method introduced by Crank-Nicolson in [13], the homotopy perturbation method (HPM) proposed by He in 1999 [14–16], and the Laplace decomposition method (LDM), introduced by Khuri in [17,18]. Additionally, the homotopy analysis method (HAM), developed by Liao in 1992, is presented in [19–23]. A modified version of HAM, known as the q-homotopy analysis method, has been introduced by El-Tavil and Huseen [24,25] to accelerate its convergence, where $q \in [0, \frac{1}{n}]$, and n is a positive integer. Furthermore, a robust analytical numerical technique, the q-homotopy analysis transform method, is presented in [26–28], which combines the q-homotopy analysis method with the Laplace transform method.

The primary objective of this paper is to investigate the application and efficiency of the q-homotopy analysis transform method in solving a fractional linear singular one-dimensional thermo-elasticity coupled system. This mathematical model can be applied to depict radially symmetric deformations and temperature distributions within a unit disk [29]. Specifically, we will employ the q-homotopy analysis transform method to numerically solve the following fractional coupled system:

$$\begin{cases} \dfrac{\partial^{\mu}}{\partial t}\xi(x,t) - \dfrac{d_1}{x}\dfrac{\partial}{\partial x}\Big(x\dfrac{\partial \xi(x,t)}{\partial t}\Big) + cx\dfrac{\partial \vartheta(x,t)}{\partial x} = f_1(x,t), \ 1 < \mu \leq 2, \\[2ex] \dfrac{\partial^{\nu}}{\partial t}\vartheta(x,t) - \dfrac{d_2}{x}\dfrac{\partial}{\partial x}\Big(x\dfrac{\partial \vartheta(x,t)}{\partial t}\Big) + cx\dfrac{\partial^2 \xi(x,t)}{\partial x \partial t} = f_2(x,t), \ 0 < \nu \leq 1, \end{cases} \tag{1}$$

subject to the following initial conditions:

$$\begin{cases} \xi(x,0) = g_1(x), \ \xi_t(x,0) = g_2(x), \ 0 < x < 1 \\[2ex] \vartheta(x,0) = g_3(x), \ 0 < x < 1 \end{cases} \tag{2}$$

that satisfies the boundary conditions:

$$\xi(1,t) = \vartheta(1,t) = 0, \ 0 < t < T, \tag{3}$$

where d_i, $i = 1,2$ and c are positive real numbers, f_i, g_j, $i = 1,2$, $j = 1,2,3$ are known functions, and $\frac{\partial^{\delta}}{\partial t}$ denotes the Caputo fractional derivative of order δ, where δ is a non-integer positive real number. The physical meaning of the functions in Equation (1) can be interpreted as follows: the function ξ represents the displacement, ϑ is the difference in the absolute temperature, f_1 is an external force, f_2 is a heat supply, and d_1, d_2, and c are positive constants.

The Caputo fractional derivative is often used in practical applications, as it enables one to include the traditional initial and boundary conditions in formulating mathematical models. Moreover, as in the integer-order derivative, the Caputo fractional derivative of a constant is zero [30].

Let us mention that integer-order versions of model (1) are considered in [29,31,32].

Definition 1 ([33,34])**.** *The Caputo fractional derivative of a non-integer-order δ of a function $\theta(x,t)$ is defined by:*

$$\frac{\partial^{\delta}}{\partial t^{\delta}}\theta(x,t) = \begin{cases} \frac{1}{\Gamma(n-\delta)} \int_0^t \frac{\partial^n \theta(x,\omega)}{\partial \omega^n}(t-\omega)^{n-\delta-1}\, d\omega, \ n-1 < \delta < n, \\[2ex] \frac{\partial^n}{\partial t^n}\theta(x,t), \qquad \delta = n, \end{cases}$$

where $\Gamma(n-\delta)$ denotes the Gamma function.

Definition 2 ([35,36]). *The Laplace transform $\mathcal{L}$ of the Caputo fractional derivative $\partial_t^\delta \theta(x,t)$, $n-1<\delta<n$, of a function $\theta(x,t)$ is defined as:*

$$\mathcal{L}\Big[\frac{\partial^\delta}{\partial t^\delta}\theta(x,t)\Big] = s^\delta \mathcal{F}(x,s) - \sum_{j=0}^{n-1} s^{\delta-j-1}\theta^{(j)}(x,0^+), \tag{4}$$

where $\mathcal{F}(x,s)$ denotes the Laplace transform of the function $\theta(x,t)$.

The rest of the paper is organized as follows: In Section 2, we present the basic outlines of the q-HATM. In Section 3, we apply the q-HATM to develop a numerical scheme for solving models (1)–(3) as well as utilizing the LDM for the same goal. Section 4 is devoted to the numerical computations, where we give a set of examples to test the efficiency of the resulting iterative scheme. Finally, we discuss our findings and conclusions in Section 5.

2. Basic Outlines of q-HATM

The q-HATM is introduced by El-Tavil and Huseen [24,25], which is a modified version of the HAM. In this section, we present basic concepts of this method. Thus, consider the following general fractional partial differential equation in the Caputo sense:

$$\frac{\partial^\delta}{\partial t^\delta}\vartheta(x,t) + R\vartheta(x,t) + \tilde{N}\vartheta(x,t) = F(x,t),\ n-1<\delta\le n, \tag{5}$$

in which $\vartheta(x,t)$ is a differentiable function, $\frac{\partial^\delta}{\partial t^\delta}$ is the Caputo derivative of order δ, R is a linear differential operator, $\tilde{N}$ denotes a nonlinear differential operator, and $F(x,t)$ is a known function.

Then, applying Laplace transform to both sides of Equation (5) gives:

$$s^\delta \mathcal{L}\{\vartheta(x,t)\} - \sum_{k=0}^{n-1} s^{\delta-k-1}\vartheta^{(k)}(x,0) + \mathcal{L}\{R\vartheta(x,t) + \tilde{N}\vartheta(x,t)\} = \mathcal{L}\{F(x,t)\},$$

or

$$\mathcal{L}\{\vartheta(x,t)\} - \sum_{k=0}^{n-1}\frac{1}{s^{k+1}}\vartheta^{(k)}(x,0) + \frac{1}{s^\delta}\mathcal{L}\{R\vartheta(x,t) + \tilde{N}\vartheta(x,t) - F(x,t)\} = 0.$$

Next, according to the HAM method [19], we define an operator $\mathcal{N}$ as follows:

$$\begin{aligned}\mathcal{N}[\varphi(x,t;q)] \quad = \quad & \mathcal{L}\{\varphi(x,t;q)\} - \textstyle\sum_{k=0}^{n-1}\frac{1}{s^{k+1}}\varphi^{(k)}(x,0;q) + \frac{1}{s^\delta}\mathcal{L}\{R\varphi(x,t;q) \\ & +\tilde{N}\varphi(x,t;q) - F(x,t)\},\end{aligned}$$

where $q\in[0,\frac{1}{n}]$, $n\ge 1$, and φ is a real valued function in x,t and q. Thus, we take the zeroth-order deformation equation to be:

$$(1-nq)\mathcal{L}\Big[\varphi(x,t;q) - \vartheta_0(x,t)\Big] = q\hbar\mathcal{N}[\varphi(x,t;q)], \tag{6}$$

where $\hbar$ is a non-vanishing auxiliary parameter, which is used to control and adjust the convergence region of the desired series solution, $q\in[0,\frac{1}{n}]$ is an embedding parameter, $\mathcal{L}$ denotes the traditional Laplace transform operator, $\vartheta_0(x,t)$ is an initial guess for the exact solution $\vartheta(x,t)$, and $\varphi(x,t;q)$ is an unknown function.

It is clear that at $q=0$ and $q=\frac{1}{n}$, Equation (6) implies:

$$\varphi(x,t;0) = \vartheta_0(x,t) \ \text{ and } \ \varphi(x,t;\frac{1}{n}) = \vartheta(x,t).$$

Thus, as q moves continuously from 0 to $\frac{1}{n}$, the function $\varphi(x,t;q)$ deforms from the initial approximation $\vartheta_0(x,t)$ to the exact solution $\vartheta(x,t)$.

Next, the Taylor series expansion of $\varphi(x,t;q)$ in powers of q implies:

$$\varphi(x,t;q) = \vartheta_0(x,t) + \sum_{m=1}^{\infty} \vartheta_m(x,t) q^m, \tag{7}$$

where

$$\vartheta_m(x,t) = \frac{1}{m!} \frac{\partial^m \varphi(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

As mentioned in [22], if the auxiliary parameter $\hbar$ and the initial guess $\vartheta_0(x,t)$ are properly chosen, then the power series (7) would converge at $q = \frac{1}{n}$ to one of the solutions of the above problem, and it is given as:

$$\vartheta(x,t) = \vartheta_0(x,t) + \sum_{m=1}^{\infty} \left(\frac{1}{n}\right)^m \vartheta_m(x,t). \tag{8}$$

In fact, the existence of the factor $\left(\frac{1}{n}\right)^m$ in the series (8) accelerates the convergence in the q-HATM compared with the HAM.

Next, differentiating the zeroth order deformation Equation (6) m-times with respect to q, dividing by $m!$ and then setting $q = 0$, gives the following mth order deformation equation:

$$\mathcal{L}\{\vartheta_m(x,t) - \chi_m \vartheta_{m-1}(x,t)\} = \hbar \Re(\vec{\vartheta}_{m-1}), \tag{9}$$

where

$$\vec{\vartheta}_k(x,t) = [\vartheta_0(x,t), \vartheta_1(x,t), ..., \vartheta_k(x,t)],$$

and

$$\Re(\vec{\vartheta}_{m-1}) = \frac{1}{(m-1)!} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} \mathcal{N}[\varphi(x,t;q)] \right\}\bigg|_{q=0}.$$

Finally, applying the inverse Laplace transform to both sides of Equation (9) implies that the components $\vartheta_m(x,t)$ can be determined recursively by the iterative scheme:

$$\vartheta_m(x,t) = \chi_m \vartheta_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left[\Re(\vec{\vartheta}_{m-1})\right], \; m = 1, 2, \ldots, \tag{10}$$

where

$$\chi_j = \begin{cases} 0, & j \leq 1, \\ n, & j > 1. \end{cases}$$

3. Application

3.1. Application of the q-HATM

The q-HATM [24,25] is an accurate and efficient computational technique for handling the solution of an integer-order as well as a fractional-order mathematical models. Thus, it is widely used to solve a wide range of mathematical models in different scientific fields: for example, see [26,27,37–40] and the references therein.

To explore the applicability and efficiency of the q-HATM for solving problem (1), we apply Laplace transform to both sides of each equation in (1); then, in view of (2), we obtain:

$$\mathcal{L}[\xi(x,t)] - \frac{1}{s}\xi(x,0) - \frac{1}{s^2}\xi_t(x,0) - \frac{1}{s^\mu}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\xi(x,t)\right) - cx\frac{\partial}{\partial x}\vartheta(x,t) + f_1(x,t)\right] = 0,$$

$$\mathcal{L}[\vartheta(x,t)] - \frac{1}{s}\vartheta(x,0) - \frac{1}{s^\nu}\mathcal{L}\left[\frac{d_2}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\vartheta(x,t)\right) - cx\frac{\partial^2}{\partial x \partial t}\xi(x,t) + f_2(x,t)\right] = 0.$$

Next, we define two operators $\mathcal{N}_1$ and $\mathcal{N}_2$ as follows:

$$
\begin{aligned}
\mathcal{N}_1[\varphi_1(x,t;q),\varphi_2(x,t;q)] &= \mathcal{L}\{\varphi_1(x,t;q)\} - \tfrac{1}{s}\xi(x,0) - \tfrac{1}{s^2}\xi_t(x,0) - \tfrac{1}{s^\mu}\mathcal{L}\{\tfrac{d_1}{x}\tfrac{\partial}{\partial x}\varphi_1(x,t;q) \\
&\quad + d_1\tfrac{\partial^2}{\partial x^2}\varphi_1(x,t;q) - cx\tfrac{\partial}{\partial x}\varphi_2(x,t;q) + f_1(x,t)\}, \\
\mathcal{N}_2[\varphi_1(x,t;q),\varphi_2(x,t;q)] &= \mathcal{L}\{\varphi_2(x,t;q)\} - \tfrac{1}{s}\vartheta(x,0) - \tfrac{1}{s^\nu}\mathcal{L}\{\tfrac{d_2}{x}\tfrac{\partial}{\partial x}\varphi_2(x,t;q) \\
&\quad + d_2\tfrac{\partial^2}{\partial x^2}\varphi_2(x,t;q) - cx\tfrac{\partial}{\partial x\partial t}\varphi_1(x,t;q) + f_2(x,t)\}.
\end{aligned}
$$

Hence, we take the zeroth deformation equations as:

$$
\begin{aligned}
(1-nq)\mathcal{L}\Big[\varphi_1(x,t;q) - \xi_0(x,t)\Big] &= q\hbar_1\mathcal{N}_1[\varphi_1(x,t;q),\varphi_2(x,t;q)], \\
(1-nq)\mathcal{L}\Big[\varphi_2(x,t;q) - \vartheta_0(x,t)\Big] &= q\hbar_2\mathcal{N}_2[\varphi_1(x,t;q),\varphi_2(x,t;q)],
\end{aligned}
$$

which implies that the mth-order deformation equations are given by:

$$
\mathcal{L}[\xi_m(x,t) - \chi_m\xi_{m-1}(x,t)] = \hbar_1\Re_1(\vec{\xi}_{m-1},\vec{\vartheta}_{m-1}),
$$

$$
\mathcal{L}[\vartheta_m(x,t) - \chi_m\vartheta_{m-1}(x,t)] = \hbar_2\Re_2(\vec{\xi}_{m-1},\vec{\vartheta}_{m-1}),
$$

where

$$
\begin{aligned}
\Re_1(\vec{\xi}_{m-1},\vec{\vartheta}_{m-1}) &= \mathcal{L}\{\xi_{m-1}(x,t)\} - (1-\tfrac{\chi_m}{n})\Big(\tfrac{1}{s}\xi(x,0) + \tfrac{1}{s^2}\xi(x,0)\Big) - \tfrac{d_1}{s^\nu}\mathcal{L}\{\tfrac{1}{x}\tfrac{\partial}{\partial x}\xi_{m-1} \\
&\quad + d_1\tfrac{\partial^2}{\partial x^2}\xi_{m-1} - cx\tfrac{\partial}{\partial x}\vartheta_{m-1} + (1-\tfrac{\chi_m}{n})f_1(x,t)\}, \\
\Re_2(\vec{\xi}_{m-1},\vec{\vartheta}_{m-1}) &= \mathcal{L}\{\vartheta_{m-1}(x,t)\} - (1-\tfrac{\chi_m}{n})\tfrac{1}{s}\vartheta(x,0) - \tfrac{d_2}{s^\nu}\mathcal{L}\{\tfrac{1}{x}\tfrac{\partial}{\partial x}\xi_{m-1} \\
&\quad + d_2\tfrac{\partial^2}{\partial x^2}\vartheta_{m-1} - cx\tfrac{\partial}{\partial x\partial t}\xi_{m-1} + (1-\tfrac{\chi_m}{n})f_2(x,t)\}.
\end{aligned}
$$

Therefore, successive terms of the approximate series solution can be computed recursively from the iterative schemes:

$$
\begin{aligned}
\xi_m(x,t) &= \chi_m\xi_{m-1}(x,t) + \hbar_1\mathcal{L}^{-1}\Big[\Re_1(\vec{\xi}_{m-1},\vec{\vartheta}_{m-1})\Big],\ m \geq 1, \\
\vartheta_m(x,t) &= \chi_m\vartheta_{m-1}(x,t) + \hbar_2\mathcal{L}^{-1}\Big[\Re_2(\vec{\xi}_{m-1},\vec{\vartheta}_{m-1})\Big],\ m \geq 1,
\end{aligned}
\tag{11}
$$

and the solution will be given as:

$$
\begin{aligned}
\xi(x,t) &= \xi_0(x,t) + \textstyle\sum_{i=1}^{\infty}(\tfrac{1}{n})^m\xi_i(x,t), \\
\vartheta(x,t) &= \vartheta_0(x,t) + \textstyle\sum_{i=1}^{\infty}(\tfrac{1}{n})^m\vartheta_i(x,t).
\end{aligned}
$$

3.2. Application of the LDM

To use the LDM [17,18] for solving problem (1), again we apply the Laplace transform to both sides of each equation in this problem, taking into account the property (2), to obtain:

$$
\begin{aligned}
&\mathcal{L}[\xi(x,t)] = \tfrac{1}{s}\xi(x,0) + \tfrac{1}{s^2}\xi_t(x,0) + \tfrac{1}{s^\mu}\mathcal{L}\Big[\tfrac{d_1}{x}\tfrac{\partial}{\partial x}\Big(x\tfrac{\partial}{\partial x}\xi(x,t)\Big) - cx\tfrac{\partial}{\partial x}\vartheta(x,t) + f_1(x,t)\Big], \\
&\mathcal{L}[\vartheta(x,t)] = \tfrac{1}{s}\vartheta(x,0) + \tfrac{1}{s^\nu}\mathcal{L}\Big[\tfrac{d_2}{x}\tfrac{\partial}{\partial x}\Big(x\tfrac{\partial}{\partial x}\vartheta(x,t)\Big) - cx\tfrac{\partial^2}{\partial x\partial t}\xi(x,t) + f_2(x,t)\Big].
\end{aligned}
\tag{12}
$$

Then, applying the inverse Laplace transform to each side of the equations in (12), we obtain:

$$\begin{aligned} \xi(x,t) &= \xi(x,0) + \xi_t(x,0)t + \mathcal{L}^{-1}\left[\frac{1}{s^{\mu}}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\xi(x,t)\right) - cx\frac{\partial}{\partial x}\vartheta(x,t) + f_1(x,t)\right]\right], \\ \vartheta(x,t) &= \vartheta(x,0) + \mathcal{L}^{-1}\left[\frac{1}{s^{\nu}}\mathcal{L}\left[\frac{d_2}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\vartheta(x,t)\right) - cx\frac{\partial^2}{\partial x\partial t}\xi(x,t) + f_2(x,t)\right]\right] \end{aligned} \tag{13}$$

Now, the LDM defines the solution of the system (1)–(3) in a series form as:

$$\xi(x,t) = \sum_{i=0}^{\infty}\xi_i(x,t), \text{ and } \vartheta(x,t) = \sum_{i=0}^{\infty}\vartheta_i(x,t). \tag{14}$$

Then, the components of this solution can be determined by substituting the series in (14) into (13) and matching the terms on both sides to obtain the following recursive relations:

$$\begin{aligned} \xi_0(x,t) &= \xi(x,0) + \xi_t(x,0)t + \mathcal{L}^{-1}\left[\frac{1}{s^{\mu}}\mathcal{L}[f_1(x,t)]\right], \\ \xi_i(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^{\mu}}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\xi_{i-1}(x,t)\right) - cx\frac{\partial}{\partial x}\vartheta_{i-1}(x,t)\right]\right], \ i = 1,2,\cdots, \end{aligned} \tag{15}$$

$$\begin{aligned} \vartheta_0(x,t) &= \vartheta(x,0) + \mathcal{L}^{-1}\left[\frac{1}{s^{\nu}}\mathcal{L}[f_2(x,t)]\right], \\ \vartheta_i(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^{\nu}}\mathcal{L}\left[\frac{d_2}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\vartheta_{i-1}(x,t)\right) - cx\frac{\partial^2}{\partial x\partial t}\xi_{i-1}(x,t)\right]\right], \ i = 1,2,\cdots. \end{aligned} \tag{16}$$

4. Numerical Results

In this section, we employ the iterative scheme (11), obtained by applying the q-HATM [26–28], to solve numerically a set of examples to test the efficiency of this scheme in handling the solution of fractional problems of the type (1)–(3), in which the function ξ represents the displacement, and ϑ represents the difference absolute temperature in a unit disk.

Example 1. *Consider Equation (1) with* $d_1 = d_2 = c = 1$*:*

$$\begin{cases} \dfrac{\partial^{\mu}}{\partial t}\xi(x,t) - \dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial\xi(x,t)}{\partial t}\right) + x\dfrac{\partial\vartheta(x,t)}{\partial x} = -3 + 2t - 6t\ln(x), \ 1 < \mu \leq 2, \\ \dfrac{\partial^{\nu}}{\partial t}\vartheta(x,t) - \dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial\vartheta(x,t)}{\partial t}\right) + x\dfrac{\partial^2\xi(x,t)}{\partial x\partial t} = 1 - 3t^2 + 2\ln(x), \ 0 < \nu \leq 1, \end{cases}$$

subject to the following initial conditions:

$$\begin{cases} \xi(x,0) = 0, \ \xi_t(x,0) = \ln(x), \ 0 < x < 1, \\ \vartheta(x,0) = -3\ln(x), \ 0 < x < 1, \end{cases}$$

and satisfies the boundary conditions:

$$\xi(1,t) = \vartheta(1,t) = 0, \ 0 < t < T.$$

Solution.

Let $\xi_0(x,t) = \xi(x,0) = 0$ and $\vartheta_0(x,t) = \vartheta(x,0) = -3\ln(x)$. Then, in view of (11), using $n = 1$ and $\hbar_1 = \hbar_2 = \hbar$, the first few terms of the series solution are given as:

$$\begin{aligned}
\xi_1(x,t) &= \frac{2\hbar t^{1+\mu}}{\Gamma[2+\mu]}(3\ln(x)-1), \\
\vartheta_1(x,t) &= \frac{6\hbar t^{2+\nu}}{\Gamma[3+\nu]} - \frac{\hbar t^{\nu}(1+2\ln(x))}{\Gamma[1+\nu]},
\end{aligned}$$

$$\begin{aligned}
\xi_2(x,t) &= \frac{-2\hbar^2 t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} - \frac{2\hbar^2 t^{1+\mu}(1-3\ln(x))}{\Gamma[2+\mu]} + \frac{2ht^{1+\mu}(3\ln(x)-1)}{\Gamma[2+\mu]}, \\
\vartheta_2(x,t) &= \frac{6ht^{2+\nu}}{\Gamma[3+\nu]} - \frac{ht^{\nu}(1+2\ln(x))}{\Gamma[1+\nu]} - \frac{h^2 t^{\nu}}{\Gamma[1+\nu]} + \frac{6h^2 t^{2+\nu}}{\Gamma[3+\nu]} + \frac{6h^2 t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} \\
&\quad + \frac{6h^2 \mu t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} - \frac{2h^2 t^{\nu}\ln(x)}{\Gamma[1+\nu]}, \\
\xi_3(x,t) &= -\frac{2ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{4h^2 t^{1+\mu}}{\Gamma[2+\mu]} - \frac{2h^3 t^{1+\mu}}{\Gamma[2+\mu]} - \frac{4h^2 t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} - \frac{4h^3 t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} \\
&\quad + \frac{6ht^{1+\mu}\ln(x)}{\Gamma[2+\mu]} + \frac{12h^2 t^{1+\mu}\ln(x)}{\Gamma[2+\mu]} + \frac{6h^3 t^{1+\mu}\ln(x)}{\Gamma[2+\mu]}, \\
\vartheta_3(x,t) &= -\frac{ht^{\nu}}{\Gamma[1+\nu]} - \frac{2h^2 t^{\nu}}{\Gamma[1+\nu]} - \frac{h^3 t^{\nu}}{\Gamma[1+\nu]} + \frac{6ht^{2+\nu}}{\Gamma[3+\nu]} + \frac{12h^2 t^{2+\nu}}{\Gamma[3+\nu]} + \frac{6h^3 t^{2+\nu}}{\Gamma[3+\nu]} \\
&\quad + \frac{12h^2 t^{\nu+\mu}\Gamma[1+r]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} + \frac{12h^3 t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} + \frac{12h^2 \mu t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} \\
&\quad + \frac{12h^3 \mu t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} - \frac{2ht^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{4h^2 t^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{2h^3 t^{\nu}\ln(x)}{\Gamma[1+\nu]}, \\
&\cdots.
\end{aligned}$$

Hence, the solution of the coupled system is given by:

$$\begin{aligned}
\xi(x,t) &= \xi_0(x,t) + \xi_1(x,t) + \xi_2(x,t) + \cdots \\
&= \frac{2\hbar t^{1+\mu}}{\Gamma[2+\mu]}(3\ln(x)-1) \\
&\quad - \frac{2\hbar^2 t^{\mu}(t^{\nu}\Gamma[2+\mu] + t\Gamma[1+\nu+\mu](1-3\ln(x)))}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} + \frac{2\hbar t^{1+\mu}(3\ln(x)-1)}{\Gamma[2+\mu]} \\
&\quad + \cdots,
\end{aligned} \tag{17}$$

$$
\begin{aligned}
\vartheta(x,t) &= \vartheta_0(x,t)+\vartheta_1(x,t)+\vartheta_2(x,t)+\cdots \\
&= -3\ln(x)+\frac{6\hbar t^{2+\nu}}{\Gamma[3+\nu]}-\frac{\hbar t^{\nu}(1+2\ln(x))}{\Gamma[1+\nu]}+\frac{6ht^{2+\nu}}{\Gamma[3+\nu]} \\
&\quad -\frac{ht^{\nu}(1+2\ln(x))}{\Gamma[1+\nu]}-\frac{h^2t^{\nu}}{\Gamma[1+\nu]}+\frac{6h^2t^{2+\nu}}{\Gamma[3+\nu]}+\frac{6h^2t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]} \\
&\quad +\frac{6h^2\mu t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]}-\frac{2h^2t^{\nu}\ln(x)}{\Gamma[1+\nu]} \\
&\quad +\cdots.
\end{aligned} \tag{18}
$$

Figure 1 displays the h-curve corresponding to the 15th-order truncated series solution. It follows from this figure that the values of the parameter $\hbar$ required for the convergence of the series solution are lying in the range $-1.8<\hbar<0$.

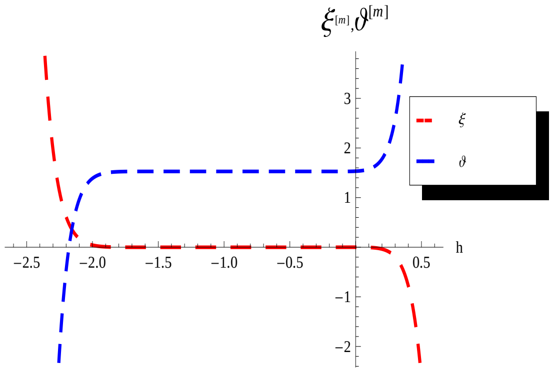

Figure 1. The $\hbar$-curve corresponding to the 15th-order approximate series solution at $x=0.6$, $t=0.01$, $\mu=1.3$, and $\nu=0.2$.

Figure 2 shows the graph of the truncated series solution using a distinct number of terms of the truncated series solution of Example 1 at $x=0.2$, $\hbar=-0.7$, $\mu=1.3$ and $\nu=0.7$. It shows the rapid convergence of these approximate solutions.

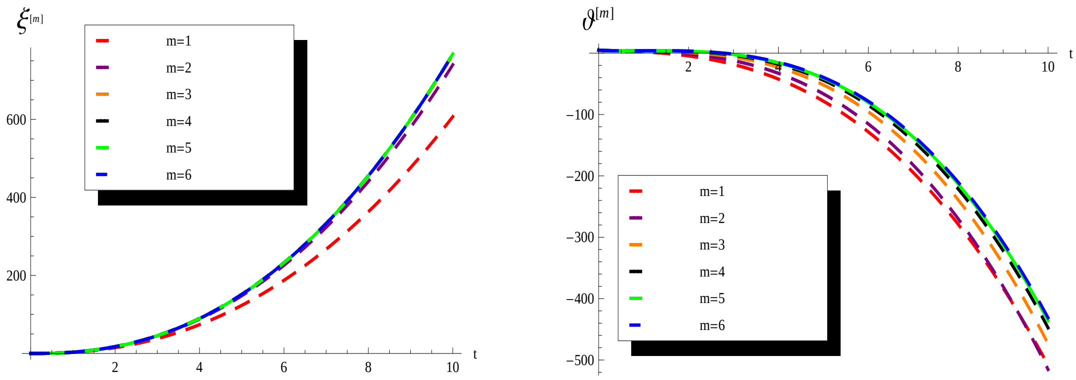

Figure 2. Truncated series solution $\xi^{[m]}(x,t)$ & $\vartheta^{[m]}(x,t)$ of Example 1 using several values of m.

We noticed that for integer values of $\mu = 2$ and $\nu = 1$, and setting $\hbar = -1$, the solution given by the series (17) and (18) reduces to:

$$\begin{aligned}
\xi(x,t) &= \frac{t^3}{3} + t\ln(x) + \frac{2}{3}t^3(-2+3\ln(x)) - t^3(-1+3\ln(x)) \\
&= \left(t - t^3\right)\ln(x), \\
\vartheta(x,t) &= 4t - 3\ln(x) + 6t\ln(x) - 4t(1+\ln(x)) \\
&= (-3+2t)\ln(x),
\end{aligned}$$

which is the exact solution of Example 1 in this case.

In Tables 1 and 2, we present the numerical solutions of Example 1 resulting from the kth-order truncated series solution $\xi^{[k]}\&\vartheta^{[k]}$ generated by the q-HATM and LDM for several values of k, x, and t, rounded to 6 significant digits. These tables illustrate the rapid convergence of these truncated solutions just after few terms. As it appears in these tables, both methods show good performance for this example.

Table 1. Comparative numerical results between q-HATM and LDM at $d_1 = 1$, $d_2 = 1$, $c = 1$, $\mu = 1.3\,\nu = 0.4$, $\hbar = -1$, and different values of k, x, and t.

			t					
			0.1		**1**		**5**	
	k		***HATM***	***LDM***	***HATM***	***LDM***	***HATM***	***LDM***
0.1	1	$\xi^{[k]}$	−0.200720	−0.226554	3.59117	2.2964	227.281	207.308
		$\vartheta^{[k]}$	5.28214	4.91095	0.831902	3.58913	−96.6100	−38.8371
	2	$\xi^{[k]}$	−0.226554	−0.226554	2.29640	2.29640	207.308	207.308
		$\vartheta^{[k]}$	4.91095	4.91095	3.58913	3.58913	−38.8371	−38.8371
	20	$\xi^{[k]}$	−0.226554	−0.226554	2.29640	2.29640	207.308	207.308
		$\vartheta^{[k]}$	4.91095	4.91095	3.58913	3.58913	−38.8371	−38.8371
0.5	1	$\xi^{[k]}$	−0.0578117	−0.0836456	1.60200	0.307238	89.5254	69.5526
		$\vartheta^{[k]}$	1.89810	1.52691	−0.368544	2.38868	−94.5321	−36.7592
	2	$\xi^{[k]}$	−0.0836456	−0.0836456	0.307238	0.307238	69.5526	69.5526
		$\vartheta^{[k]}$	1.52691	1.52691	2.38868	2.38868	−36.7592	−36.7592
	20	$\xi^{[k]}$	−0.0836456	−0.0836456	0.307238	0.307238	69.5526	69.5526
		$\vartheta^{[k]}$	1.52691	1.52691	2.38868	2.38868	−36.7592	−36.7592
0.9	1	$\xi^{[k]}$	−0.00561994	−0.0314538	0.875532	−0.419230	39.2154	19.2426
		$\vartheta^{[k]}$	0.662211	0.291022	−0.806961	1.95026	−93.7732	−36.0004
	2	$\xi^{[k]}$	−0.0314538	−0.0314538	−0.419230	−0.419230	19.2426	19.2426
		$\vartheta^{[k]}$	0.291022	0.291022	1.95026	1.95026	−36.0004	−36.0004
	20	$\xi^{[k]}$	−0.0314538	−0.0314538	−0.419230	−0.419230	19.2426	19.2426
		$\vartheta^{[k]}$	0.291022	0.291022	1.95026	1.95026	−36.0004	−36.0004

Table 2. Comparative numerical results between q-HATM and LDM at $d_1 = 1$, $d_2 = 1$, $c = 1$, $\mu = 1.75$, $\nu = 0.65$, $\hbar = -1$, and different values of k, x, and t.

			t					
			0.1		**0.5**		**5**	
x	***k***		***HATM***	***LDM***	***HATM***	***LDM***	***HATM***	***LDM***
0.1	1	$\xi^{[k]}$	−0.0668385	−0.226571	1.27317	0.602298	287.393	255.466
		$\vartheta^{[k]}$	1.97995	5.76698	1.37805	2.27969	−112.984	−20.3642
	2	$\xi^{[k]}$	−0.0695093	−0.226571	0.602298	0.602298	255.466	255.466
		$\vartheta^{[k]}$	1.73925	5.76698	2.27969	2.27969	−20.3642	−20.3642
	20	$\xi^{[k]}$	−0.0695093	−0.226571	0.602298	0.602298	255.466	255.466
		$\vartheta^{[k]}$	1.73925	5.76698	2.27969	2.27969	−20.3642	−20.3642
0.5	1	$\xi^{[k]}$	−0.0668385	−0.0695093	0.699324	0.0284545	112.934	81.0069
		$\vartheta^{[k]}$	1.97995	1.73925	0.125798	1.02744	−107.633	−15.0128
	2	$\xi^{[k]}$	−0.0695093	−0.0695093	0.0284545	0.0284545	81.0069	81.0069
		$\vartheta^{[k]}$	1.73925	1.73925	1.02744	1.02744	−15.0128	−15.0128
	20	$\xi^{[k]}$	−0.0695093	−0.0695093	0.0284545	0.0284545	81.0069	81.0069
		$\vartheta^{[k]}$	1.73925	1.73925	1.02744	1.02744	−15.0128	−15.0128
0.9	1	$\xi^{[k]}$	−0.00947778	−0.0121486	0.489749	−0.181120	49.2199	17.2923
		$\vartheta^{[k]}$	0.508974	0.268272	−0.331539	0.570103	−105.679	−13.0584
	2	$\xi^{[k]}$	−0.0121486	−0.0121486	−0.181120	−0.181120	17.2923	17.2923
		$\vartheta^{[k]}$	0.268272	0.268272	0.570103	0.570103	−13.0584	−13.0584
	20	$\xi^{[k]}$	−0.0121486	−0.0121486	−0.181120	−0.181120	17.2923	17.2923
		$\vartheta^{[k]}$	0.268272	0.268272	0.570103	0.570103	−13.0584	−13.0584

Example 2. *Consider Equation (1) with $d_1 = 1$, $d_2 = 1$, and $c = 5$:*

$$\begin{cases} \dfrac{\partial^{\mu}}{\partial t}\xi(x,t) - \dfrac{1}{x}\dfrac{\partial}{\partial x}\Big(x\dfrac{\partial \xi(x,t)}{\partial t}\Big) + 5x\dfrac{\partial \vartheta(x,t)}{\partial x} = 5 + 5t + 2\ln(x),\ 1 < \mu \leq 2, \\ \dfrac{\partial^{\nu}}{\partial t}\vartheta(x,t) - \dfrac{1}{x}\dfrac{\partial}{\partial x}\Big(x\dfrac{\partial \vartheta(x,t)}{\partial t}\Big) + 5x\dfrac{\partial^2 \xi(x,t)}{\partial x \partial t} = 10t + \ln(x),\ 0 < \nu \leq 1, \end{cases}$$

subject to the following initial conditions:

$$\begin{cases} \xi(x,0) = \ln(x),\ \xi_t(x,0) = 0,\ 0 < x < 1, \\ \vartheta(x,0) = \ln(x),\ 0 < x < 1, \end{cases}$$

which satisfies the boundary conditions:

$$\xi(1,t) = \vartheta(1,t) = 0,\ 0 < t < T.$$

Solution.

Let $\xi_0(x,t) = \xi(x,0) = \ln(x)$ and $\vartheta_0(x,t) = \vartheta(x,0) = \ln(x)$. Then, in view of (11), using $n = 1$ and $\hbar_1 = \hbar_2 = \hbar$, the first few terms of the series solution are as follows:

$$
\begin{aligned}
\xi_1(x,t) &= -\frac{5ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{2ht^{\mu}\ln(x)}{\Gamma[1+\mu]}, \\
\vartheta_1(x,t) &= -\frac{10ht^{1+\nu}}{\Gamma[2+\nu]} - \frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]},
\end{aligned}
$$

$$
\begin{aligned}
\xi_2(x,t) &= -\frac{5ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{5h^2t^{1+\mu}}{\Gamma[2+\mu]} - \frac{5h^2t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} - \frac{2ht^{\mu}\ln(x)}{\Gamma[1+\mu]} - \frac{2h^2t^{\mu}\ln(x)}{\Gamma[1+\mu]}, \\
\vartheta_2(x,t) &= -\frac{10ht^{1+\nu}}{\Gamma[2+\nu]} - \frac{10h^2t^{1+\nu}}{\Gamma[2+\nu]} - \frac{10h^2\mu t^{-1+\nu+\mu}\Gamma[r]}{\Gamma[1+\mu]\Gamma[\nu+\mu]} - \frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{h^2t^{\nu}\ln(x)}{\Gamma[1+\nu]},
\end{aligned}
$$

$$
\begin{aligned}
\xi_3(x,t) &= -\frac{5ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{10h^2t^{1+\mu}}{\Gamma[2+\mu]} - \frac{5h^3t^{1+\mu}}{\Gamma[2+\mu]} - \frac{10h^2t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} - \frac{10h^3t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} \\
&\quad -\frac{2ht^{\mu}\ln(x)}{\Gamma[1+\mu]} - \frac{4h^2t^{\mu}\ln(x)}{\Gamma[1+\mu]} - \frac{2h^3t^{\mu}\ln(x)}{\Gamma[1+\mu]}, \\
\vartheta_3(x,t) &= -\frac{10ht^{1+\nu}}{\Gamma[2+\nu]} - \frac{20h^2t^{1+\nu}}{\Gamma[2+\nu]} - \frac{10h^3t^{1+\nu}}{\Gamma[2+\nu]} - \frac{20h^2\mu t^{-1+\nu+\mu}\Gamma[r]}{\Gamma[1+\mu]\Gamma[\nu+\mu]} - \frac{20h^3\mu t^{-1+\nu+\mu}\Gamma[\mu]}{\Gamma[1+\mu]\Gamma[\nu+\mu]} \\
&\quad -\frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{2h^2t^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{h^3t^{\nu}\ln(x)}{\Gamma[1+\nu]}, \\
&\;\;\vdots
\end{aligned}
$$

Hence, the solution is given by the following:

$$
\begin{aligned}
\xi(x,t) &= \xi_0(x,t) + \xi_1(x,t) + \xi_2(x,t) + \cdots \\
&= \ln(x) - \frac{5ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{2ht^{\mu}\ln(x)}{\Gamma[1+\mu]} - \frac{5ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{5h^2t^{1+\mu}}{\Gamma[2+\mu]} \\
&\quad -\frac{5h^2t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} - \frac{2ht^{\mu}\ln(x)}{\Gamma[1+\mu]} - \frac{2h^2t^{\mu}\ln(x)}{\Gamma[1+\mu]} + \cdots,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\vartheta(x,t) &= \vartheta_0(x,t) + \vartheta_1(x,t) + \vartheta_2(x,t) + \cdots \\
&= \ln(x) - \frac{10ht^{1+\nu}}{\Gamma[2+\nu]} - \frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{10ht^{1+\nu}}{\Gamma[2+\nu]} - \frac{10h^2t^{1+\nu}}{\Gamma[2+\nu]} \\
&\quad -\frac{10h^2\mu t^{-1+\nu+\mu}\Gamma[r]}{\Gamma[1+\mu]\Gamma[\nu+\mu]} - \frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{h^2t^{\nu}\ln(x)}{\Gamma[1+\nu]} + \cdots .
\end{aligned} \tag{20}
$$

Figure 3 displays the h-curve corresponding to the 16th-order truncated series solution. It follows from this figure that the values of the parameter $\hbar$ required for the convergence of the series solution are in the range $-1.5 < \hbar < -0.4$.

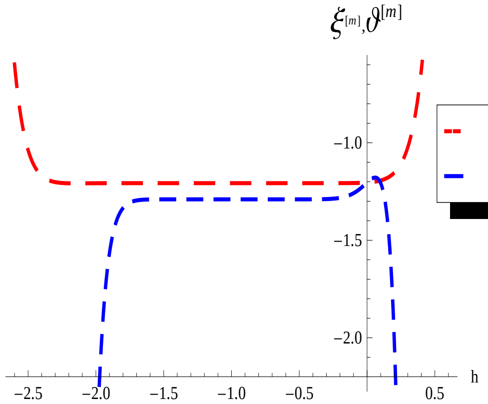

Figure 3. The $\hbar$-curve corresponding to the 16th-order approximate series solution at $x = 0.3$, $t = 0.01$, $\mu = 1.4$, and $\nu = 0.75$.

Figure 4 shows the graph of the truncated series solution using a distinct number of terms of the truncated series solution of Example 2 at $x = 0.5$, $\hbar = -0.8$, $\mu = 1.3$ and $\nu = 0.7$. It shows the rapid convergence of these approximate solutions.

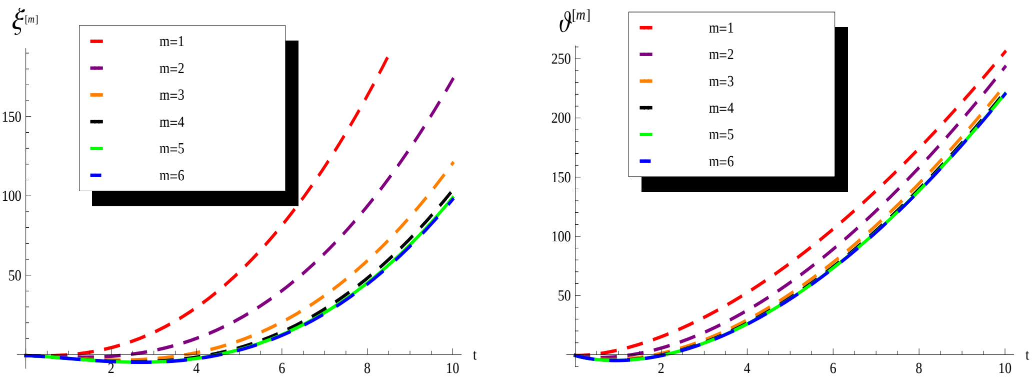

Figure 4. Truncated series solution $\xi^{[k]}(x,t)$ & $\vartheta^{[k]}(x,t)$ of Example 2 using several values of m.

It is noticed that for integer values of $\mu = 2$ and $\nu = 1$, and using $\hbar = -1$, the solution given by the series (19) and (20) reduces to:

$$\begin{aligned}
\xi(x,t) &= -\frac{5h(2+h)t^{1+\mu}}{\Gamma[2+\mu]} - \frac{5h^2 t^{\nu+\mu}}{\Gamma[1+\nu+\mu]} + \frac{(-2h(2+h)t^{\mu} + \Gamma[1+\mu])\ln(x)}{\Gamma[1+\mu]} \\
&= \left(t^2+1\right)\ln(x), \\
\vartheta(x,t) &= -\frac{10h(2+h)t^{1+\nu}}{\Gamma[2+\nu]} - \frac{10h^2\mu t^{-1+\nu+\mu}\Gamma[\mu]}{\Gamma[1+\mu]\Gamma[\nu+\mu]} + \frac{(-h(2+h)t^{\nu} + \Gamma[1+\nu])\ln(x)}{\Gamma[1+\nu]} \\
&= (t+1)\ln(x),
\end{aligned}$$

which is the exact solution of Example 2 in this case.

Tables 3 and 4 present the numerical solutions of Example 2 resulting from the kth-order truncated series solution $\xi^{[k]}\&\vartheta^{[k]}$ generated by the q-HATM and LDM for several values of k, x, and t, rounded to six significant digits. These tables illustrate the rapid convergence of the these truncated solutions just after few terms. Again, as it appears from these tables, both methods perform very well for solving this example.

Table 3. Comparative numerical results between q-HATM and LDM at $d_1 = 1$, $d_2 = 1$, $c = 5$, $\mu = 1.3\, \nu = 0.4\, \hbar = -1$, and different values of k, x, and t.

			t					
			0.1		**1**		**5**	
x	k		$HATM$	LDM	$HATM$	LDM	$HATM$	LDM
0.1	1	$\xi^{[k]}$	−2.49107	−2.55566	−4.38644	−7.62334	41.2062	−8.72583
		$\vartheta^{[k]}$	−3.01524	−5.21112	3.15269	−7.85278	69.3833	35.4296
	2	$\xi^{[k]}$	−2.55566	−2.55566	−7.62334	−7.62334	−8.72583	−8.72583
		$\vartheta^{[k]}$	−5.21112	−5.21112	−7.85278	−7.85278	35.4296	35.4296
	20	$\xi^{[k]}$	−2.55566	−2.55566	−7.62334	−7.62334	−8.72583	−8.72583
		$\vartheta^{[k]}$	−5.21112	−5.21112	−7.85278	−7.85278	35.4296	35.4296
0.5	1	$\xi^{[k]}$	−0.743360	−0.807945	−0.0180716	−3.25498	65.1720	15.2400
		$\vartheta^{[k]}$	−0.683663	−2.87954	6.57607	-4.42941	74.4459	40.4921
	2	$\xi^{[k]}$	−0.807945	−0.807945	−3.25498	−3.25498	15.2400	15.2400
		$\vartheta^{[k]}$	−2.87954	−2.87954	−4.42941	−4.42941	40.4921	40.4921
	20	$\xi^{[k]}$	−0.807945	−0.807945	−3.25498	−3.25498	15.2400	15.2400
		$\vartheta^{[k]}$	−2.87954	−2.87954	−4.42941	−4.42941	40.4921	40.4921
0.9	1	$\xi^{[k]}$	−0.105074	−0.169659	1.57731	−1.65959	73.9246	23.9926
		$\vartheta^{[k]}$	0.167859	−2.02802	7.82632	−3.17915	76.2948	42.3410
	2	$\xi^{[k]}$	−0.169659	−0.169659	−1.65959	−1.65959	23.9926	23.9926
		$\vartheta^{[k]}$	−2.02802	−2.02802	−3.17915	−3.17915	42.3410	42.3410
	20	$\xi^{[k]}$	−0.169659	−0.169659	−1.65959	−1.65959	23.9926	23.9926
		$\vartheta^{[k]}$	−2.02802	−2.02802	−3.17915	−3.17915	42.3410	42.3410

Table 4. Comparative numerical results between q-HATM and LDM at $d_1 = 1$, $d_2 = 1$, $c = 5$, $\mu = 1.75$, $\nu = 0.65$, $\hbar = -1$, and different values of k, x, and t.

			t					
			0.1		**1**		**5**	
x	k		$HATM$	LDM	$HATM$	LDM	$HATM$	LDM
0.1	1	$\xi^{[k]}$	−2.35149	−2.35817	−4.03540	−5.71257	44.3256	−35.4934
		$\vartheta^{[k]}$	−2.72454	−3.04503	1.87245	−6.17798	86.2490	9.62286
	2	$\xi^{[k]}$	−2.35817	−2.35817	−5.71257	−5.71257	−35.4934	−35.4934
		$\vartheta^{[k]}$	−3.04503	−3.04503	1.87245	−6.17798	9.62286	9.62286
	20	$\xi^{[k]}$	−2.35817	−2.35817	−5.71257	−5.71257	−35.4934	−35.4934
		$\vartheta^{[k]}$	−3.04503	−3.04503	1.87245	−6.17798	9.62286	9.62286

Table 4. *Cont.*

			t					
			0.1		**1**		**5**	
x	k		$HATM$	LDM	$HATM$	LDM	$HATM$	LDM
0.5	1	$\xi^{[k]}$	−0.706464	−0.713141	−0.424620	−2.10179	79.3945	−0.424480
		$\vartheta^{[k]}$	−0.714807	−1.03530	5.26992	−2.78051	92.9483	16.3222
	2	$\xi^{[k]}$	−0.713141	−0.713141	−2.10179	−2.10179	−0.424480	−0.424480
		$\vartheta^{[k]}$	−1.03530	−1.03530	−2.78051	−2.78051	16.3222	16.3222
	20	$\xi^{[k]}$	−0.713141	−0.713141	−2.10179	−2.10179	−0.424480	−0.424480
		$\vartheta^{[k]}$	−1.03530	−1.03530	−2.78051	−2.78051	16.3222	16.3222
0.9	1	$\xi^{[k]}$	−0.105680	−0.112357	0.894081	−0.783093	92.2020	12.3831
		$\vartheta^{[k]}$	0.0191709	−0.301323	6.51072	−1.53971	95.3950	
	2	$\xi^{[k]}$	−0.112357	−0.112357	−0.783093	−0.783093	12.3831	12.3831
		$\vartheta^{[k]}$	−0.301323	−0.301323	−1.53971	−1.53971	18.7688	
	20	$\xi^{[k]}$	−0.112357	−0.112357	−0.783093	−0.783093	12.3831	12.3831
		$\vartheta^{[k]}$	−0.301323	−0.301323	−1.53971	−1.53971	18.7688	18.7688

Example 3. *Consider Equation (1) with $d_1 = 2$, $d_2 = 3$, and $b = 5$:*

$$\begin{cases} \dfrac{\partial^\mu}{\partial t}\xi(x,t) - \dfrac{2}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial \xi(x,t)}{\partial t}\right) + 5x\dfrac{\partial \vartheta(x,t)}{\partial x} = 5(1+t+t^2) + 6t\ln(x),\ 1<\mu\leq 2, \\ \dfrac{\partial^\nu}{\partial t}\vartheta(x,t) - \dfrac{3}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial \vartheta(x,t)}{\partial t}\right) + 5x\dfrac{\partial^2 \xi(x,t)}{\partial x \partial t} = 15t^2 - 5 + (1+2t)\ln(x),\ 0<\nu\leq 1, \end{cases}$$

subject to the following initial conditions:

$$\begin{cases} \xi(x,0) = -3\ln(x),\ \xi_t(x,0) = -\ln(x),\ 0<x<1, \\ \vartheta(x,0) = \ln(x),\ 0<x<1, \end{cases}$$

and satisfies the boundary conditions:

$$\xi(1,t) = \vartheta(1,t) = 0,\ 0<t<T.$$

Solution.

Let $\xi_0(x,t) = \xi(x,0) = -3\ln(x)$ and $\vartheta_0(x,t) = \vartheta(x,0) = \ln(x)$. Then, in view of (11), using $n=1$ and $\hbar_1 = \hbar_2 = \hbar$, the first few terms of the series solution are given by:

$$\begin{aligned} \xi_1(x,t) &= -\frac{5ht^{1+\mu}}{\Gamma[2+\mu]} - \frac{10ht^{2+\mu}}{\Gamma[3+\mu]} + ht\ln(x) - \frac{6ht^{1+\mu}\ln(x)}{\Gamma[2+\mu]}, \\ \vartheta_1(x,t) &= \frac{5ht^{\nu}}{\Gamma[1+\nu]} - \frac{30ht^{2+\nu}}{\Gamma[3+\nu]} - \frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]} - \frac{2ht^{1+\nu}\ln(x)}{\Gamma[2+\nu]}, \end{aligned}$$

$$\begin{aligned}
\xi_2(x,t) &= -\frac{5ht^{1+\mu}}{\Gamma[2+\mu]}-\frac{5h^2t^{1+\mu}}{\Gamma[2+\mu]}-\frac{10ht^{2+\mu}}{\Gamma[3+\mu]}-\frac{10h^2t^{2+\mu}}{\Gamma[3+\mu]}-\frac{5h^2t^{\nu+\mu}}{\Gamma[1+\nu+\mu]}-\frac{10h^2t^{1+\nu+\mu}}{\Gamma[2+\nu+\mu]}\\
&\quad +ht\ln(x)+h^2t\ln(x)-\frac{6ht^{1+\mu}\ln(x)}{\Gamma[2+\mu]}-\frac{6h^2t^{1+\mu}\ln(x)}{\Gamma[2+\mu]},\\
\vartheta_2(x,t) &= \frac{5ht^{\nu}}{\Gamma[1+\nu]}+\frac{10h^2t^{\nu}}{\Gamma[1+\nu]}-\frac{30ht^{2+\nu}}{\Gamma[3+\nu]}-\frac{30h^2t^{2+\nu}}{\Gamma[3+\nu]}-\frac{30h^2t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]}\\
&\quad -\frac{30h^2\mu t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]}-\frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]}-\frac{h^2t^{\nu}\ln(x)}{\Gamma[1+\nu]}-\frac{2ht^{1+\nu}\ln(x)}{\Gamma[2+\nu]}-\frac{2h^2t^{1+\nu}\ln(x)}{\Gamma[2+\nu]},\\
&\cdots.
\end{aligned}$$

Thus, the series solution is given by:

$$\begin{aligned}
\xi(x,t) &= \xi_0(x,t)+\xi_1(x,t)+\xi_2(x,t)+\cdots,\\
&= -3\ln(x)-\frac{5ht^{1+\mu}}{\Gamma[2+\mu]}-\frac{10ht^{2+\mu}}{\Gamma[3+\mu]}+ht\ln(x)-\frac{6ht^{1+\mu}\ln(x)}{\Gamma[2+\mu]}\\
&\quad -\frac{5ht^{1+\mu}}{\Gamma[2+\mu]}-\frac{5h^2t^{1+\mu}}{\Gamma[2+\mu]}-\frac{10ht^{2+\mu}}{\Gamma[3+\mu]}-\frac{10h^2t^{2+\mu}}{\Gamma[3+\mu]}-\frac{5h^2t^{\nu+\mu}}{\Gamma[1+\nu+\mu]}-\frac{10h^2t^{1+\nu+\mu}}{\Gamma[2+\nu+\mu]}\\
&\quad +ht\ln(x)+h^2t\ln(x)-\frac{6ht^{1+\mu}\ln(x)}{\Gamma[2+\mu]}-\frac{6h^2t^{1+\mu}\ln(x)}{\Gamma[2+\mu]}\\
&\quad +\cdots,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\vartheta(x,t) &= \vartheta_0(x,t)+\vartheta_1(x,t)+\vartheta_2(x,t)+\cdots\\
&= \ln(x)+\frac{5ht^{\nu}}{\Gamma[1+\nu]}-\frac{30ht^{2+\nu}}{\Gamma[3+\nu]}-\frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]}-\frac{2ht^{1+\nu}\ln(x)}{\Gamma[2+\nu]}\\
&\quad \frac{5ht^{\nu}}{\Gamma[1+\nu]}+\frac{10h^2t^{\nu}}{\Gamma[1+\nu]}-\frac{30ht^{2+\nu}}{\Gamma[3+\nu]}-\frac{30h^2t^{2+\nu}}{\Gamma[3+\nu]}-\frac{30h^2t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]}\\
&\quad -\frac{30h^2\mu t^{\nu+\mu}\Gamma[1+\mu]}{\Gamma[2+\mu]\Gamma[1+\nu+\mu]}-\frac{ht^{\nu}\ln(x)}{\Gamma[1+\nu]}-\frac{h^2t^{\nu}\ln(x)}{\Gamma[1+\nu]}-\frac{2ht^{1+\nu}\ln(x)}{\Gamma[2+\nu]}-\frac{2h^2t^{1+\nu}\ln(x)}{\Gamma[2+\nu]}\\
&\quad +\cdots.
\end{aligned} \tag{22}$$

Figure 5 shows the h-curve corresponding to the 14th-order truncated series solution. It shows that the values of $\hbar$ producing a convergent series solution are located in the range $-1.4 < \hbar < -0.2$.

Figure 6 shows the graph of the truncated series solution using a distinct number of terms of the truncated series solution of Example 3 at $x = 0.4$, $\hbar = -1.1$, $\mu = 1.3$ and $\nu = 0.4$. It shows the rapid convergence of these approximate solutions.

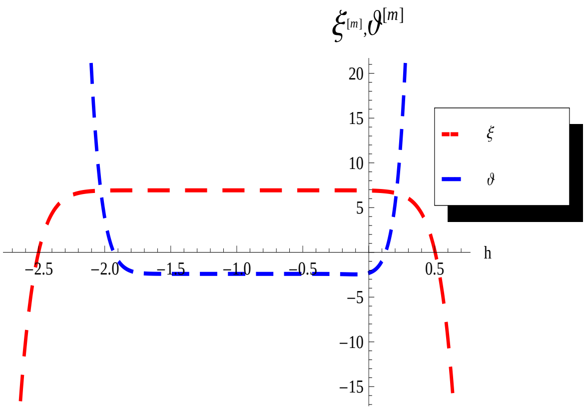

Figure 5. The $\hbar$-curve corresponding to the 14th-order approximate series solution at $x = 0.1, t = 0.01$, $\mu = 1.8$, and $\nu = 0.7$.

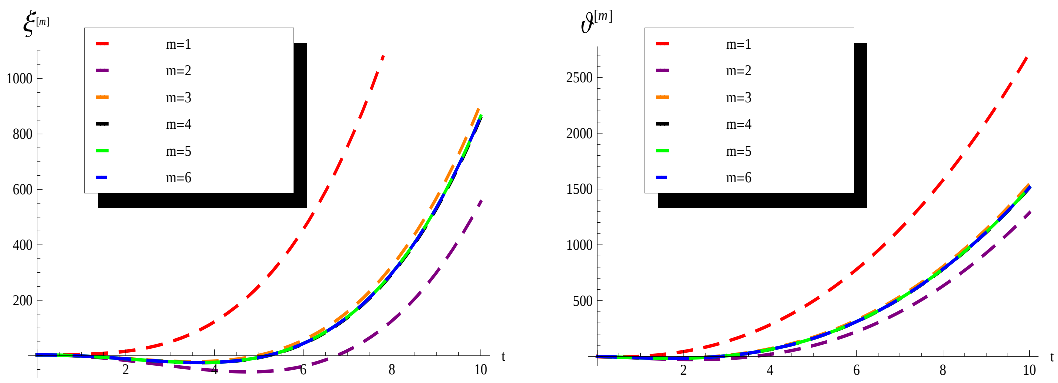

Figure 6. Truncated series solution $\xi^{[k]}(x,t)$ & $\vartheta^{[k]}(x,t)$ of Example 2 using several values of m.

On the other hand, it is found that if $\mu = 2$ and $\nu = 1$ take integer values and $\hbar = -1$, then the solution given by the series (21) and (22) reduces to:

$$\begin{aligned} \xi(x,t) &= -3\ln(x) - t\ln(x) + t^3\ln(x) \\ &= \left(t^3 - t - 3\right)\ln(x), \\ \vartheta(x,t) &= \ln(x) + t\ln(x) + t^2\ln(x) \\ &= \left(t^2 + t + 1\right)\ln(x), \end{aligned}$$

which is the exact solution of Example 3 in this case.

Tables 5 and 6 present the computed approximate numerical solutions of Example 3 generated from the kth-order truncated series solution $\xi^{[k]}\&\vartheta^{[k]}$ obtained by using the q-HATM and LDM for several values of k, x, and t, rounded to six significant digits. It appears from these tables that the approximate solutions generated by the q-HATM converge rapidly to the numerical solution of the fractional system in this example, while those obtained by LDM diverge away from it.

Table 5. Comparative numerical results between q-HATM and LDM at $d_1 = 2$, $d_2 = 3$, $c = 5$, $\mu = 1.3$, $\nu = 0.4$, $\hbar = -1$, and different values of k, x, and t.

			t					
			0.1		**1**		**5**	
x	**k**		***HATM***	***LDM***	***HATM***	***LDM***	***HATM***	***LDM***
0.1	1	$\xi^{[k]}$	7.13659	−5.70225	8.15776	−225.788	95.1809	9327.71
		$\vartheta^{[k]}$	−3.14325	−14.2973	−6.90776	−1000.92	567.526	104,323
	2	$\xi^{[k]}$	7.13571	−48.9027	6.90776	−19793.9	−269.402	2.18094×10^6
		$\vartheta^{[k]}$	−2.64825	−906.041	−6.90776	−57,543.6	−32.4740	-4.48396×10^6
	4	$\xi^{[k]}$	7.13571	−23,084.0	6.90776	-1.49985×10^9	−269.402	-2.03399×10^{12}
		$\vartheta^{[k]}$	−2.64825	-2.83707×10^7	−6.90776	-3.90236×10^{10}	−32.4740	-1.11455×10^{13}
0.5	1	$\xi^{[k]}$	2.14894	1.85859	3.32944	5.00487	283.485	−2324.10
		$\vartheta^{[k]}$	−1.29220	1.57557	−2.07944	−25.9033	590.224	4778.26
	2	$\xi^{[k]}$	2.14806	1.73289	2.07944	−89.7380	−81.0982	42115.8
		$\vartheta^{[k]}$	−0.797202	−25.4133	−2.07944	−192.244	−9.77565	4281.22
	4	$\xi^{[k]}$	2.14806	−8.60542	2.07944	−25,471.4	−81.0982	-2.35197×10^7
		$\vartheta^{[k]}$	−0.797202	−610.033	−2.07944	−9.77565	−175,454	-2.68279×10^7
0.9	1	$\xi^{[k]}$	0.327387	0.401310	1.56608	8.69824	352.256	−2985.78
		$\vartheta^{[k]}$	−0.616177	1.77678	−0.316082	−14.8761	598.514	1485.35
	2	$\xi^{[k]}$	0.326512	0.380032	0.316082	−8.58753	−12.3272	12334.8
		$\vartheta^{[k]}$	−0.121177	−6.34458	−0.316082	−22.0275	−1.48593	4887.13
	4	$\xi^{[k]}$	0.326512	−1.94451	0.316082	−2128.81	−12.3272	319,519
		$\vartheta^{[k]}$	−0.121177	−133.995	−0.316082	−7827.00	−1.48593	−10,299.8

Table 6. Comparative numerical results between q-HATM and LDM at $d_1 = 2$, $d_2 = 3$, $c = 5$, $\mu = 1.75$, $\nu = 0.65$, $\hbar = -1$, and different values of k, x, and t.

			t					
			0.1		**1**		**5**	
x	**k**		***HATM***	***LDM***	***HATM***	***LDM***	***HATM***	***LDM***
0.1	1	$\xi^{[k]}$	7.13458	3.83500	7.82014	−181.595	103.806	9818.09
		$\vartheta^{[k]}$	−4.17120	−1.82618	−5.89382	−641.931	472.924	98,809.2
	2	$\xi^{[k]}$	7.12751	2.12439	5.15639	−6433.88	−210.775	698,279
		$\vartheta^{[k]}$	−2.96769	−116.019	−10.4020	−24,560.5	9.82302	-5.55911×10^6
	4	$\xi^{[k]}$	7.12751	−52.1407	5.15639	-5.85746×10^7	−210.775	-2.95017×10^{12}
		$\vartheta^{[k]}$	−2.96769	−265,147	−10.4020	-6.11186×10^9	9.82302	-1.18729×10^{13}

Table 6. *Cont.*

			t					
			0.1		**1**		**5**	
x	***k***		***HATM***	***LDM***	***HATM***	***LDM***	***HATM***	***LDM***
0.5	1	$\zeta^{[k]}$	2.14920	2.07148	3.56567	1.60583	273.436	−1819.880
		$\vartheta^{[k]}$	−2.11295	1.1827	−0.329041	−13.5076	510.471	4669.25
	2	$\zeta^{[k]}$	2.14213	2.06707	0.901923	−25.8340	−41.1445	28,486.1
		$\vartheta^{[k]}$	−0.909437	−2.73552	−4.83725	−138.091	47.3700	3870.46
	4	$\zeta^{[k]}$	2.14213	1.93882	0.901923	−2014.43	−41.1445	-2.38004×10^7
		$\vartheta^{[k]}$	−0.909437	−8.46116	−4.83725	−27079.7	47.3700	-3.71281×10^7
0.9	1	$\zeta^{[k]}$	0.328481	0.343344	2.01188	4.67504	335.387	−2514.04
		$\vartheta^{[k]}$	−1.36125	1.19248	1.70329	−5.84939	524.184	1366.79
	2	$\zeta^{[k]}$	0.321411	0.342650	−0.651863	−0.372439	20.8067	8949.48
		$\vartheta^{[k]}$	−0.157739	−0.00607621	−2.80492	−28.9225	61.0826	5635.91
	4	$\zeta^{[k]}$	0.321411	0.305315	−0.651863	−202.622	20.8067	−323,568
		$\vartheta^{[k]}$	−0.157739	−1.33735	−2.80492	−1529.67	61.0826	−587,610

5. Conclusions

This article extends integer-order time derivatives in a system of two singular one-dimensional coupled partial differential equations to fractional-order derivatives, utilizing Caputo's sense. The resulting coupled system is numerically solved using the q-HATM method. Three illustrative examples demonstrate the efficiency of this derived numerical scheme. Notably, when the fractional orders are replaced by traditional integer orders, the numerical solutions for these examples converge to their exact solutions. The convergence of these numerical solutions is graphically tested by plotting truncated series solutions with varying numbers of terms, as depicted in Figures 2, 4 and 6. These plots vividly depict the rapid convergence of the resulting numerical solutions after only a few iterations.

Furthermore, we compare the numerical values of the obtained solutions with those obtained by LDM for different values of the fractional orders μ and ν as well as various values of the independent variables x and t. It is evident that when $d_1 = d_2 = 1$, as in Examples 1 and 2, both methods exhibit excellent performance, as evidenced in Tables 1 and 4. However, in Example 3, where these coefficients deviate from unity, the q-HATM method continues to perform admirably, while the numerical values obtained by LDM diverge, as shown in Tables 5 and 6. Consequently, these results underscore the reliability and efficiency of the q-HATM method for solving singular fractional problems of this nature as well as other analogous mathematical problems.

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