symmetry

MDPI

Article

Insight into Spatially Colored Stochastic Heat Equation: Temporal Fractal Nature of the Solution

Wensheng Wang

School of Economics, Hangzhou Dianzi University, Hangzhou 310018, China; wswang@hdu.edu.cn

Abstract: In this paper, the solution to a spatially colored stochastic heat equation (SHE) is studied. This solution is a random function of time and space. For a fixed point in space, the resulting random function of time has exact, dimension-dependent, global continuity moduli, and laws of the iterated logarithm (LILs). It is obtained that the set of fast points at which LILs fail in this process, and occur infinitely often, is a random fractal, the size of which is evaluated by its Hausdorff dimension. These points of this process are everywhere dense with the power of the continuum almost surely, and their hitting probabilities are determined by the packing dimension $\dim_P(E)$ of the target set E.

Keywords: spatially colored stochastic heat equation; space–time white noise; Brownian-time processes; temporal fractal nature; hitting probabilities; moduli of continuity

check for updates

Citation: Wang, W. Insight into Spatially Colored Stochastic Heat Equation: Temporal Fractal Nature of the Solution. *Symmetry* **2023**, *15*, 2181. https://doi.org/10.3390/sym15122181

Academic Editors: Pedro José Fernández de Córdoba Castellá, Juan Carlos Castro-Palacio, Shufei Wu and Miguel Enrique Iglesias Martínez

Received: 26 October 2023
Revised: 4 December 2023
Accepted: 6 December 2023
Published: 10 December 2023

1. Introduction

Mathematical modeling, especially through stochastic partial differential equations (SPDEs), plays a crucial role in understanding systems affected by randomness. These models are fundamental in various disciplines, including physics (see del Castillo-Negrete et al. [1]), engineering (see Kou and Xie [2]), finance (see Bayraktar et al. [3]), and environmental sciences (see Denk et al. [4]). The stochastic heat equation (SHE) is a mathematical model that considers stochastic and deterministic components to explain how a random field evolves. It adds a stochastic element to the classical heat equation to account for random changes in the system regarding heat transport. The spatially colored SHE is an important class of SHEs. This equation is driven by a spatially colored noise, which makes it is possible to solve linear and non-linear equations in the space of real-valued stochastic processes (see Dalang [5]). It has its own importance because it is relevant to the parabolic Anderson localization (see Hu [6], Mueller and Tribe [7]). It is also related to the KPZ equation, which is the field theory of many surface growth models, such as the Eden model, ballistic deposition, and the SOS model (see Bruned et al. [8], Hu [6]).

In this paper, the following d-dimensional SHE is considered:

$$\begin{cases} \dfrac{\partial}{\partial t} u_{\alpha,d}(t,x) = \dfrac{\vartheta}{2}\dfrac{\partial^2}{\partial x^2} u_{\alpha,d}(t,x) + \sigma(u_{\alpha,d}(t,x))\dot{W}_{\alpha,d}, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^d, \\ u_{\alpha,d}(0,x) = w(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

with $\vartheta > 0$ and Gaussian space–time colored noise $W_{\alpha,d}$. The noise $W_{\alpha,d}$ is assumed to have a particular covariance structure (see Dalang [5]),

$$\mathbb{E}[W_{\alpha,d}(t,A)W_{\alpha,d}(s,B)] = (t \wedge s)\int_A \int_B f_{\alpha,d}(x-y)dxdy, \quad t,s \in \mathbb{R}_+, A,B \in \mathcal{B}_b(\mathbb{R}^d), \tag{2}$$

where

$$f_{\alpha,d}(x) = c_{\alpha,d}|x|^{-d+\alpha}, \quad 0 < \alpha < d, \tag{3}$$

with $c_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. The initial condition, $w(x)$, is taken to be bounded and ρ-Hölder continuous. σ is assumed to be Lipschitz continuous; there exists $c_0 \geq 0$ such that $|\sigma(x) - \sigma(y)| \leq c_0|x-y|$ and $|\sigma(x)| \leq c_0(1+|x|)$.

It is known (see Dalang [5], Dalang et al. [9], Khoshnevisan [10], Raluca and Tudor [11], Rippl and Sturm [12], Tudor [13]) that (1) admits a unique mild solution if and only if $d < 2+\alpha$ and this mild solution is interpreted as the solution of the following integral equation:

$$u_{\alpha,d}(t,x) = \int_{\mathbb{R}^d} \mathbb{G}_{t;x,y} u_{\alpha,d}(0,y)dy + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{t-s;x,y}\sigma(u_{\alpha,d}(s,y))W_{\alpha,d}(ds,dy), \tag{4}$$

for $t \in \mathbb{R}_+, x \in \mathbb{R}^d$, where the above integral is a Wiener integral with respect to the noise $W_{\alpha,d}$ (see, e.g., Balan and Tudor [14] for the definition), and $\mathbb{G}$ is the Green kernel of the heat equation given by

$$\mathbb{G}_{t;x,y} = \begin{cases} (2\pi\vartheta t)^{-1/2}e^{-|x-y|^2/(2\vartheta t)} & \text{if } t > 0,\ x,y \in \mathbb{R}^d, \\ 0 & \text{if } t \leq 0,\ x,y \in \mathbb{R}^d. \end{cases} \tag{5}$$

Bezdek [15] investigated the weak convergence of probability measures corresponding to the solution of (1) in $d = 1$. It was shown that probability measures corresponding to $u_{\alpha,1}$ weakly converge to those corresponding to the solution to the SHE with white noise when $\alpha \uparrow 1$, that is, the solution of (1) converges *in the appropriate sense* to the solution of the same equation, but with white noise W instead of colored noise $W_{\alpha,1}$ as $\alpha \uparrow 1$. This means the solution to

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\vartheta}{2}\frac{\partial^2}{\partial x^2}u(t,x) + \sigma(u(t,x))\dot{W}, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}, \\ u(0,x) = w(x), \quad x \in \mathbb{R}, \end{cases} \tag{6}$$

where W denotes white noise. SPDEs such as (6) have been studied in Balan and Tudor [14], Dalang [5], Dalang et al. [9], Pospíšil and Tribe [16], Swanson [17], Tudor [13], and others.

Among others, Tudor and Xiao [18] investigated the exact temporal global continuity modulus and temporal LIL of the process $u_{\alpha,d}$ in time. In fact, they investigated these path properties for a wider class, namely, the solution to the linear SHE driven by a fractional noise in time with a correlated spatial structure. Swanson [17] showed that the solutions of the SHEs in (6) with $\vartheta = \sigma = 1$, in time, had infinite quadratic variation and were not semimartingales, and also investigated central limit theorems for modifications of the quadratic variations of the solutions of the SHEs with white noise. Pospíšil and Tribe [16] showed that the quadratic variations of the solutions of the SHEs in (6) with $\vartheta = \sigma = 1$, in time, had Gaussian asymptotic distributions. Inspired by Swanson [17] and Pospíšil and Tribe [16], Wang [19] showed that the realized power variations of the solutions of the SHEs in (6) with $\vartheta = \sigma = 1$, in time, had Gaussian asymptotic distributions. Wang et al. [20] showed that the realized power variations of the solutions of the SHEs in (1) with spatially colored noise, in time, had infinite quadratic variation and Gaussian asymptotic distributions.

For $\lambda \in (0,1]$ and $[a,b] \subset \mathbb{R}_+$, the set of λ-*fast points* for a process X, is defined to be the set

$$F(\lambda) := \Big\{ t \in [a,b] : \limsup_{h \to 0+} \phi_h |X(t+h) - X(t)| \geq \lambda \Big\}, \tag{7}$$

where ϕ_h is an appropriate regularization constant. The set $F(\lambda)$ is the set of t where the LILs of the process are X. This kind of set is usually called the *fast point set* or the *exceptional time set*. It is interesting to obtain information about the size of $F(\lambda)$. One usually does this by considering their Hausdorff measures. This problem was first studied by Orey and Taylor [21] on the fast set of Brownian motion. In Orey and Taylor [21], it was shown that $F(\lambda)$ is a random fractal with probability 1, $\dim_H(F(\lambda)) = 1 - \lambda^2$. See Mattila [22] for the definition of the Hausdorff dimension. After this famous paper, several papers studied this problem for general Gaussian processes. Among other things, the fractal nature for empirical increments and processes with independent increments was studied in Deheuvels and Mason [23]. The fractal nature for the fast point set of L^p-valued Gaussian processes was studied in Zhang [24]. Khoshnevisan et al. [25] showed that the packing dimension was the right index for deciding which sets intersect $F(\lambda)$. In Khoshnevisan et al. [25], it was shown that for any $\alpha \in (0,1]$ and any analytic set $B \subset \mathbb{R}_+$,

$$P\{F(\alpha) \cap B \neq \emptyset\} = \begin{cases} 1, & \text{if } \dim_p(B) > \alpha^2, \\ 0, & \text{if } \dim_p(B) < \alpha^2. \end{cases} \tag{8}$$

See also Mattila [22] for the definition of packing dimension.

Inspired by the studies of Orey and Taylor [21], Zhang [24], and Khoshnevisan et al. [25], this paper is devoted to establishing a fractal nature for the set of temporal fast points of the spatially colored SHE. In particular, in this paper, Hausdorff dimensions for the sets of temporal fast points of the spatially colored SHE are evaluated, and hitting probabilities of temporal fast points are obtained by using the packing dimension $\dim_p(E)$ of the target set E. On the other hand, the global temporal continuity modulus and temporal LIL for $u_{\alpha,d}(\cdot, x)$ were obtained in Tudor and Xiao [18]. Tudor and Xiao [18] showed the existence of regularization constants for the global temporal continuity modulus and the temporal LIL, but their exact values remain unknown. In this paper, the exact values of these regularization constants are obtained, and the exact, dimension-dependent, global temporal continuity modulus and the temporal LIL for the spatially colored SHE solution $u_{\alpha,d}(t,x)$ are established.

Our proofs are based on the method of Orey and Taylor [21], Zhang [24], and Khoshnevisan et al. [25]. The pinned string process with respect to $u_{\alpha,d}(\cdot, x)$ is used to obtain precise estimations of the mean squares of the process $u_{\alpha,d}$ in time and the exact values of these regularization constants. This work builds upon the recent work on a delicate analysis of the Green kernel of SHEs driven by space–time white noise.

Throughout this paper, an unspecified positive and finite constant will be denoted by c, which may not be the same in each occurrence.

2. Main Results

The gamma function is known as a generalization of the factorial function to non-integer values, and provides a continuous and smooth interpolation between the factorial values of positive integers. The gamma function is crucial in various branches of mathematics, including complex analysis, number theory, and statistics. It has applications in solving definite integrals, evaluating infinite products, and expressing solutions to certain differential equations. For any $\alpha > 0$ and $\alpha < d < \alpha + 2$, let

$$K_{\alpha,d} = \frac{\vartheta^{-\frac{d-\alpha}{2}}}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int_0^\infty \frac{1}{y^{2-\frac{d-\alpha}{2}}}(1 - e^{-y/2})dy, \tag{9}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$, $s > 0$ is the Gamma function. Here, $\alpha < d < \alpha + 2$ ensures that the integral in (9) exists.

The global temporal continuity modulus and temporal LIL for the spatially colored SHE solution are as follows. In fact, Equation (8) below is another form of the global temporal moduli of continuity of the spatially colored SHE, which is slightly different from those obtained by Tudor and Xiao [18].

Theorem 1. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then, with probability 1, for any interval $[a, b] \subset \mathring{\mathbb{R}}_+ = (0, \infty)$,*

$$\lim_{h\to 0+} \sup_{s,t\in[a,b],|t-s|<h} \phi^{-1}_{\alpha,d,h}|u_{\alpha,d}(t, x) - u_{\alpha,d}(s, x)| = 1, \tag{10}$$

where $\phi_{\alpha,d,h} = h^{\frac{1}{2}-\frac{d-\alpha}{4}}\sqrt{2K_{\alpha,d}\log(1/h)}$, and for any fixed $t_0 \in \mathring{\mathbb{R}}_+$,

$$\limsup_{h\to 0+} \sup_{|t_0-s|<h} \hat{\phi}^{-1}_{\alpha,d,h}|u_{\alpha,d}(t_0, x) - u_{\alpha,d}(s, x)| = 1, \tag{11}$$

where $\hat{\phi}_{\alpha,d,h} = h^{\frac{1}{2}-\frac{d-\alpha}{4}}\sqrt{2K_{\alpha,d}\log\log(1/h)}$. Here $K_{\alpha,d}$ is given in (9).

Remark 1. *For the above theorem, it is worth remarking that:*

(1). *Equation (10) is another form of the global temporal modulus of continuity of the spatially colored SHEs, which is slightly different from that obtained by Tudor and Xiao [18]. Equation (10) with $c_3^{(\alpha,d)}|t-s|^{\frac{1}{2}-\frac{d-\alpha}{4}}\sqrt{\log(1/|t-s|)}$ taking the place of $\phi_{\alpha,d,h}$ was established in Proposition 1 of Tudor and Xiao [18], and Equation (11) with $c_4^{(\alpha,d)}h^{\frac{1}{2}-\frac{d-\alpha}{4}}\sqrt{\log\log(1/h)}$ taking the place of $\hat{\phi}_{\alpha,d,h}$ was established in Proposition 2 of Tudor and Xiao [18], where $c_3^{(\alpha,d)} > 0$ and $c_4^{(\alpha,d)} > 0$ are dimension-dependent constants, independent of x, whose exact values remain unknown. Here, in Equations (10) and (11), the exact constants for the global temporal modulus of continuity and temporal LIL of the spatially colored SHEs are obtained. Moreover, by using Lemma 4 below, it is easy to obtain $c_3^{(\alpha,d)} = c_4^{(\alpha,d)} = \sqrt{2K_{\alpha,d}}$ in Tudor and Xiao [18]. In this sense, the results of Theorem 1 generalize those in Tudor and Xiao [18].*

(2). *Equation (10) describes the size of the global maximal temporal oscillation of the spatially colored SHE solution $u_{\alpha,d}(\cdot, x)$ over the interval $[a, b]$ is $\phi_{\alpha,d,h}$. Equation (11) describes the size of the local temporal oscillation of the spatially colored SHE solution $u_{\alpha,d}(\cdot, x)$ at a prescribed time $t_0 \in \mathring{\mathbb{R}}$ is $\hat{\phi}_{\alpha,d,h}$. It is interesting to compare Equations (10) and (11). The latter one states that, at some given point, the LIL of $u_{\alpha,d}(\cdot, x)$ for any fixed x is not more than $\hat{\phi}_{\alpha,d,h}$. On the other hand, the former tells us that the global continuity modulus of $u_{\alpha,d}(\cdot, x)$ can be much larger, namely $\phi_{\alpha,d,h}$.*

(3). *By Equation (11), an application of Fubini's theorem shows that the random time set*

$$F_{\alpha,d,x,+} := \left\{ t \in [a, b] : \limsup_{h\to 0+} \hat{\phi}^{-1}_{\alpha,d,h}|u_{\alpha,d}(t + h, x) - u_{\alpha,d}(t, x)| > 1 \right\}$$

almost surely has Lebesgue measure zero for any $[a,b] \subset \mathring{\mathbb{R}}$. However, $F_{\alpha,d,x,+}$ is not empty: in fact, the set of t satisfying the much stronger growth condition (12) below is almost surely everywhere dense with the power of the continuum.

Fix $x \in \mathbb{R}^d$. For $\lambda \in (0,1]$ and $[a,b] \subset \mathring{\mathbb{R}}$, the set of temporal λ-fast points for the spatially colored SHE, defined by

$$F_{\alpha,d,x}(\lambda; a,b) := \left\{ t \in [a,b] : \limsup_{h\to 0+} \phi^{-1}_{\alpha,d,h} |u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| \geq \lambda \right\}, \quad (12)$$

where $\phi_{\alpha,d,h}$ is given in (10).

The following theorem obtains the Hausdorff dimension of the set of temporal fast points of the spatially colored SHEs.

Theorem 2. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then, for any $\lambda \in [0,1]$ and any $0 < a < b < \infty$, with probability 1,*

$$\dim_H(F_{\alpha,d,x}(\lambda; a,b)) = 1 - \lambda^2. \quad (13)$$

The next theorem shows that the packing dimension is the right index for deciding which sets intersect $F_{\alpha,d,x}(\lambda; a,b)$.

Theorem 3. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1) and $0 < \alpha < d < \alpha + 2$. Then, for any $\lambda \in [0,1]$, $0 < a < b < \infty$ and any analytic set $E \subset \mathbb{R}_{a,+} = [a,\infty)$,*

$$P\{F_{\alpha,d,x}(\lambda; a,b) \cap E \neq \emptyset\} = \begin{cases} 1, & \text{if } \dim_P(E) > \lambda^2, \\ 0, & \text{if } \dim_P(E) < \lambda^2. \end{cases} \quad (14)$$

Remark 2. *Let $a \in \mathring{\mathbb{R}}$. It is easy to see that Equation (14) is equivalent to that, with probability 1, for any analytic set $E \subset \mathbb{R}_{a,+}$,*

$$\sup_{t\in E} \limsup_{h\to 0+} \phi_{\alpha,d,h} |u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| = (\dim_P(E))^{1/2}. \quad (15)$$

Therefore, Equation (15) can be viewed as a probabilistic interpretation of the packing dimension of an analytic set $E \subset \mathbb{R}_{a,+}$ in the sense of spatially colored SHEs.

Remark 3. *Let $a \in \mathring{\mathbb{R}}$. Do as in Khoshnevisan et al. [25]; by reversing the order of* sup *and* lim sup *in Equation (15), the following probabilistic interpretations of the upper and lower Minkowski dimensions of $E \subset \mathbb{R}_{a,+}$, denoted by $\overline{\dim}_M(E)$ and $\underline{\dim}_M(E)$, are obtained, respectively; see Mattila [22] for definitions. For any analytic set $E \subset \mathbb{R}_{a,+}$, with probability 1,*

$$\limsup_{h\to 0+} \sup_{t\in E} \phi_{\alpha,d,h} |u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| = (\overline{\dim}_M(E))^{1/2} \quad (16)$$

and

$$\liminf_{h\to 0+} \sup_{t\in E} \phi_{\alpha,d,h} |u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| = (\underline{\dim}_M(E))^{1/2}. \quad (17)$$

3. Auxiliary Lemmas

To derive some needed estimations on the variance function of increments of some auxiliary Gaussian random fields, the following pinned string process in time $\{U_{\alpha,d}(t), t \geq 0\}$ is introduced:

$$U_{\alpha,d}(t) = \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{t-r;x,y} W_{\alpha,d}(dr,dy) + \int_{-\infty}^0 \int_{\mathbb{R}^d} (\mathbb{G}_{t-r;x,y} - \mathbb{G}_{-r;x,y}) W_{\alpha,d}(dr,dy), \tag{18}$$

where $x \in \mathbb{R}^d$ is fixed. Note that $U_{\alpha,d}(0) = 0$ and $U_{\alpha,d}(t)$ can be expressed as

$$U_{\alpha,d}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} (\mathbb{G}_{(t-r)_+;x,y} - \mathbb{G}_{(-r)_+;x,y}) W_{\alpha,d}(dr,dy). \tag{19}$$

In the above, $z_+ = \max(z,0)$. Now, for all $t \geq 0$, one has the following decomposition:

$$u_{\alpha,d}(t) = U_{\alpha,d}(t) - V_{\alpha,d}(t), \tag{20}$$

where

$$V_{\alpha,d}(t) = \int_{-\infty}^0 \int_{\mathbb{R}^d} (\mathbb{G}_{t-r;x,y} - \mathbb{G}_{-r;x,y}) W_{\alpha,d}(dr,dy). \tag{21}$$

Lemma 1. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then, for all $s,t \in \mathbb{R}_+$,*

$$\mathbb{E}[(U_{\alpha,d}(t,x) - U_{\alpha,d}(s,x))^2] = K_{\alpha,d}|t-s|^{1-\frac{d-\alpha}{2}}, \tag{22}$$

where $K_{\alpha,d}$ is given in (9). Consequently, $U_{\alpha,d}(\cdot,x)$ coincides in distribution with $K_{\alpha,d}^{1/2} B^H(\cdot)$ with $H := \frac{1}{2} - \frac{d-\alpha}{4}$, where B^H is a fractional Brownian motion with Hurst parameter H.

Proof. Denote by $\mu(d\xi) = |\xi|^{-\alpha} d\xi$ the tempered non-negative measure on $\mathbb{R}^d$, and $\hat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(r)\, e^{-\mathbf{i}\langle \xi,r\rangle} dr$ the Fourier transform of the function $r \mapsto \varphi(r)$, and f the Riesz kernel defined in (3). Then, for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ (see, e.g., Tudor [13], Tudor and Xiao [18]),

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(r) f(r-y) \psi(y) dr dy = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} \mu(d\xi). \tag{23}$$

It follows from (23) that, for any $0 \leq s < t$,

$$\begin{aligned}
&\mathbb{E}[|U_{\alpha,d}(t) - U_{\alpha,d}(s)|^2] \\
&= \mathbb{E}\Big[\Big(\int_{\mathbb{R}} \int_{\mathbb{R}^d} (\mathbb{G}_{t-r;x,y}\mathbb{I}_{\{t>r\}} - \mathbb{G}_{s-r;x,y}\mathbb{I}_{\{s>r\}}) W_{\alpha,d}(dr,dy)\Big)^2\Big] \\
&= \int_{\mathbb{R}} dr \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\mathbb{G}_{t-r;x,y}\mathbb{I}_{\{t>r\}} - \mathbb{G}_{s-r;x,y}\mathbb{I}_{\{s>r\}}) f(y-y') \\
&\quad \times (\mathbb{G}_{t-r;x,y'}\mathbb{I}_{\{t>r\}} - \mathbb{G}_{s-r;x,y'}\mathbb{I}_{\{s>r\}}) dy dy' \\
&= (2\pi)^{-d} \int_{\mathbb{R}} dr \int_{\mathbb{R}^d} (\hat{\mathbb{G}}_{t-r;x,\cdot}\mathbb{I}_{\{t>r\}} - \hat{\mathbb{G}}_{s-r;x,\cdot}\mathbb{I}_{\{s>r\}})(\xi) \\
&\quad \times \overline{(\hat{\mathbb{G}}_{t-r;x,\cdot}\mathbb{I}_{\{t>r\}} - \hat{\mathbb{G}}_{s-r;x,\cdot}\mathbb{I}_{\{s>r\}})(\xi)} \mu(d\xi).
\end{aligned} \tag{24}$$

Since the Fourier transform of the Green kernel

$$\hat{\mathbb{G}}_{t;x,\cdot}(\xi) = e^{-\mathbf{i}\vartheta\langle x,\xi\rangle} \exp\Big(-\frac{t\vartheta|\xi|^2}{2}\Big), \tag{25}$$

Equation (24) becomes

$$
\begin{aligned}
&\mathbb{E}[|U_{\alpha,d}(t) - U_{\alpha,d}(s)|^2] \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(d\xi) \int_{\mathbb{R}} (e^{-(t-r)\vartheta|\xi|^2/2}\mathbb{I}_{\{t>r\}} - e^{-(s-r)\vartheta|\xi|^2/2}\mathbb{I}_{\{s>r\}})^2 dr \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^{-\alpha} d\xi \int_{\mathbb{R}} (e^{-(t-r)\vartheta|\xi|^2}\mathbb{I}_{\{t>r\}} + e^{-(s-r)\vartheta|\xi|^2}\mathbb{I}_{\{s>r\}} - 2e^{-(t+s-2r)\vartheta|\xi|^2/2}\mathbb{I}_{\{s>r\}}) dr \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{2}{\vartheta|\xi|^{2+\alpha}} (1 - e^{-(t-s)\vartheta|\xi|^2/2}) d\xi \\
&= 2(2\pi)^{-d} (t-s)^{1-\frac{d-\alpha}{2}} \vartheta^{-\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} \frac{1}{|\zeta|^{2+\alpha}} (1 - e^{-|\zeta|^2/2}) d\zeta,
\end{aligned}
\tag{26}
$$

where the last equality follows from the change of variable $\xi \mapsto \zeta : \zeta = \sqrt{(t-s)\vartheta}\xi$. By the following integral formula (see Corollary on page 23 in Fang et al. [26]):

$$
\int_{\mathbb{R}^d} f\Big(\sum_{i=1}^d x_i^2\Big) dx_1 \cdots dx_d = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty y^{d/2-1} f(y) dy, \tag{27}
$$

Equation (26) becomes

$$
\begin{aligned}
&\mathbb{E}[|U_{\alpha,d}(t) - U_{\alpha,d}(s)|^2] \\
&= (t-s)^{1-\frac{d-\alpha}{2}} \frac{\vartheta^{-\frac{d-\alpha}{2}}}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int_0^\infty \frac{1}{y^{2-\frac{d-\alpha}{2}}} (1 - e^{-y/2}) dy.
\end{aligned}
\tag{28}
$$

This completes the proof. □

Lemma 2. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then, for any $s, t \in \mathbb{R}_{a,+}$ with some $a > 0$, there exists a constant $c = c(\alpha, d, \vartheta, a) > 0$, independent of s, t and x, such that*

$$
\mathbb{E}[(V_{\alpha,d}(t,x) - V_{\alpha,d}(s,x))^2] \le c|t-s|^2. \tag{29}
$$

Proof. It follows from (23) and (25) that, for any $0 \le s < t$,

$$
\begin{aligned}
&\mathbb{E}[|V_{\alpha,d}(t) - V_{\alpha,d}(s)|^2] \\
&= \mathbb{E}\Big[\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^d} (\mathbb{G}_{t-r;x,y}\mathbb{I}_{\{0>r\}} - \mathbb{G}_{s-r;x,y}\mathbb{I}_{\{0>r\}}) W_{\alpha,d}(dr,dy)\Big)^2\Big] \\
&= \int_0^\infty dr \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\mathbb{G}_{t+r;x,y} - \mathbb{G}_{s+r;x,y}\mathbb{I}_{\{0>r\}}) f(y-y') \\
&\quad \times (\mathbb{G}_{t+r;x,y'} - \mathbb{G}_{s+r;x,y'}) dy dy' \\
&= (2\pi)^{-d} \int_0^\infty dr \int_{\mathbb{R}^d} (\hat{\mathbb{G}}_{t+r;x,\cdot} - \hat{\mathbb{G}}_{s+r;x,\cdot})(\xi) \overline{(\hat{\mathbb{G}}_{t+r;x,\cdot} - \hat{\mathbb{G}}_{s+r;x,\cdot})(\xi)} \mu(d\xi) \\
&= \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{-\alpha} d\xi \int_0^\infty (e^{-(t+r)\vartheta|\xi|^2/2} - e^{-(s+r)\vartheta|\xi|^2/2})^2 dr \\
&= \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{-\alpha} d\xi \int_0^\infty e^{-(s+r)\vartheta|\xi|^2} (e^{-(t-s)\vartheta|\xi|^2/2} - 1)^2 dr \\
&= \frac{1}{4\vartheta(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{-2-\alpha} e^{-s\vartheta|\xi|^2} (e^{-(t-s)\vartheta|\xi|^2/2} - 1)^2 d\xi.
\end{aligned}
\tag{30}
$$

Since $|1 - e^{-z}| \leq 2z$ for all $z \geq 0$, by (27), one has for all $a \leq s < t$,

$$\begin{aligned}
&\mathbb{E}[|V_{\alpha,d}(t) - V_{\alpha,d}(s)|^2] \\
&\leq \frac{\vartheta}{4(2\pi)^d}(t-s)^2 \int_{\mathbb{R}^d} |\xi|^{2-\alpha} e^{-a\vartheta|\xi|^2} d\xi \\
&= \frac{\vartheta}{4(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2)} (t-s)^2 \int_0^\infty y^{(d-\alpha)/2} e^{-a\vartheta y} dy \\
&= c(t-s)^2.
\end{aligned} \tag{31}$$

The proof is completed. □

The following exact large deviation estimation for spatially colored SHEs is needed.

Lemma 3. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then, for any $t \in \mathbb{R}_{a,+}$ with some $a > 0$, there exists a constant $h_0 = h_0(\alpha, d, \vartheta, a) > 0$ such that for any $0 < h < h_0$,*

$$\lim_{z \to \infty} z^{-2} \log \mathbb{P}\Big(|u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| \geq z K_{\alpha,d}^{1/2} h^{\frac{1}{2} - \frac{d-\alpha}{4}}\Big) = -\frac{1}{2}. \tag{32}$$

Proof. Since u and V are independent, by Lemmas 1 and 2,

$$\begin{aligned}
&\mathbb{E}[(u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x))^2] \\
&= \mathbb{E}[(U_{\alpha,d}(t+h,x) - U_{\alpha,d}(t,x))^2] - \mathbb{E}[(V_{\alpha,d}(t+h,x) - V_{\alpha,d}(t,x))^2] \\
&= (K_{\alpha,d} + o(1)) h^{1-(d-\alpha)/2}.
\end{aligned}$$

Thus, by the well known estimation (cf., e.g., Csörgő and Révész [27], p.23),

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{y} - \frac{1}{y^3}\Big) e^{-y^2/2} \leq 1 - \Phi(y) \leq \frac{1}{\sqrt{2\pi} y} e^{-y^2/2}, \quad \forall y > 0, \tag{33}$$

(32) is obtained immediately, where $\Phi(y)$ is the standard normal distribution function. The proof is completed. □

The following Fernique-type inequality for spatially colored SHEs is also needed.

Lemma 4. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then for any $\epsilon > 0$ and $a > 0$, there exist constants $h_0 = h_0(\alpha, d, \vartheta, a, \epsilon) > 0$ and $c = c(\alpha, d, \vartheta, a, \epsilon) > 0$, independent of x, such that, for any $T > 0$, $0 < h < h_0$ and $z > 0$,*

$$\mathbb{P}\Big(\sup_{a \leq t \leq a+T} \sup_{0 \leq s \leq h} |u_{\alpha,d}(t+s,x) - u_{\alpha,d}(t,x)| \geq z K_{\alpha,d}^{1/2} h^{\frac{1}{2} - \frac{d-\alpha}{4}}\Big) \leq \frac{cT}{h} e^{-\frac{z^2}{2+\epsilon}}. \tag{34}$$

Proof. By using Lemma 3, following the same lines as the proof of Proposition 3.3 in Meerschaert et al. [28], (34) is obtained. This completes the proof. □

4. Proofs

Proof of Theorem 1. By using (34), following the same lines as the proof of Theorems 1.4 and 1.7 in Tudor and Xiao [18], (10) and (11) are obtained. This completes the proof. □

Proof of Theorem 2. By Remark 2, it suffices to show (15). By using Lemma 4 and following the same lines in the proof of Theorem 2 of Orey and Taylor [21], p. 180, it is easy to show that, with probability 1,

$$\forall \lambda \in [0,1], \quad \dim_H(F_{\alpha,d,x}(\lambda; a, b)) \leq 1 - \lambda^2. \tag{35}$$

That is, the upper bound of Equation (15) is validated.

It now turns to the proof of the opposite inequality. It suffices to show that, with probability 1,

$$\forall \lambda \in [0,1], \quad \dim_H(F_{\alpha,d,x}(\lambda; a, b)) \geq 1 - \lambda^2. \tag{36}$$

The method of proof is similar to those of Theorem 2 of Orey and Taylor [21] and Theorem 1.1 of Zhang [24], but is more complicated in our SHE with the spatially colored noise case.

This time, $0 < \lambda < 1$ is assumed, as otherwise there is nothing to prove. For each fixed $0 < \lambda_0 < \lambda < 1$, it suffices to show that $F_{\alpha,d,x}(\lambda; a, b)$ contains a Cantor-like subset of dimension at least $\eta - 2\epsilon$, where $0 < \epsilon < \eta/2 < 1$ and $\eta = 1 - \lambda_0^2$. The result then follows by taking a sequence of values of λ_0 converging to λ and ϵ converging to 0. The proof is devoted to the construction of this Cantor-like subset and was inspired by, and is an accurate generalized version of, the arguments in the proofs of Zhang [24] and Orey and Taylor [21]. □

The following lemma is required in the proof (see Zhang [24]).

Lemma 5. *Suppose that $g : [a, b] \to [0, \infty)$ is a continuous function. Let $F \subset [a, b]$ be such that $F = \cap_{m=1}^{\infty} F_m$, where $F_1 \supset \cdots \supset F_m \cdots$ for $m = 1, 2, \ldots$, and $F_m = \cup_{k=1}^{N_m} I_{m,i}$ with $\{I_{m,i} : 1 \leq i \leq N_m\}$ being, for each $m \geq 1$, a collection of disjoint closed subintervals of $[a, b]$. Then, if there exist two constants $\delta > 0$ and $C > 0$ such that, for every interval $I \subset [a, b]$ with $|I| \leq \delta$ there is a constant $m(I)$ such that for all $m \geq m(I)$,*

$$N_m(I) =: \#\{I_{m,i} \subset I; 1 \leq i \leq N_m\} \leq Cg(|I|)N_m, \tag{37}$$

it holds that $\mu_g(F) > 0$.

Let $\mathcal{T}$ denote the collection of intervals $[s, t] \subset [a, b]$ such that

$$u_{\alpha,d}(t, x) - u_{\alpha,d}(s, x) \geq \lambda \phi_{|t-s|}.$$

The modulus of continuity (10) tells us that

$$|u_{\alpha,d}(t, x) - u_{\alpha,d}(s, x)| \leq \sqrt{2}\phi_{|t-s|} \tag{38}$$

for all $s, t \in [a, b]$ with $|s - t|$ being sufficiently small. Hence, there exists $0 < K < 1$, depending only on λ and λ_0 such that, for every sufficiently small $I_{\text{time}} = [s, t] \subset [a, b]$,

$$u_{\alpha,d}(t, x) - u_{\alpha,d}(s, x) \geq \lambda_0 \phi_{|t-s|} \tag{39}$$

implies that $[v, t] \in \mathcal{T}$ for all $v \in I_{\text{time}}(K) = [s, s + K(t - s)]$. For convenience, K is assumed to be the reciprocal of an integer.

Suppose that r_n is the reciprocal of an integer, $r_{n+1} < Kr_n$, and Kr_n/r_{n+1} is an integer for $n = 1, 2, \ldots$ Let τ be a positive number such that $\tau < \epsilon/16$. For each $n \geq 1$, define $\nu_n = \lfloor r_n^{-\tau} \rfloor$, $\varrho_n = \lfloor (r_n^{-1} - 1)/\nu_n \rfloor + 1$, $h_n = r_n(b-a)$ and

$$t_{n,i} = t_{n,i}(a,b) = a + i\nu_n h_n, \quad i = 0, 1, \ldots, \varrho_n - 1,$$

$$\mathcal{J}_n = \{[t_{n,i}, t_{n,i} + h_n]; i = 0, 1, \ldots, \varrho_n - 1\}.$$

For each $n \geq 1$ and any $I_{\text{time}} = [t_{n,i}, t_{n,i} + h_n] \in \mathcal{J}_n$, define

$$\Xi_{\alpha,d,x}(n, I_{\text{time}}) = \gamma_{h_n}^{-1}(u_{\alpha,d}(t_{n,i} + h_n, x) - u_{\alpha,d}(t_{n,i}, x)),$$

where $\gamma_h = K_{\alpha,d}^{1/2} h^{\frac{1}{2} - \frac{d-\alpha}{4}}$. Moreover, define

$$\begin{aligned}
&\mathcal{J}_{n,+} = \{I_{\text{time}} \in \mathcal{J}_n; \Xi_{\alpha,d,x}(n, I_{\text{time}}) > \lambda(2\log(1/h_n))^{1/2}\},\\
&\mathcal{J}_{n,+}(K) = \{I_{\text{time}}(K) = [s, s + K(t-s)], I_{\text{time}} = [s,t] \in \mathcal{J}_{n,+}\},\\
&\rho_n(J_{\text{time}}) = \#\{I_{\text{time}} \in \mathcal{J}_{n,+}, I_{\text{time}} \subset J_{\text{time}}\}, \quad \rho_n = \rho_n([a,b]),\\
&\varrho_n(J_{\text{time}}) = \#\{I_{\text{time}} \in \mathcal{J}_n, I_{\text{time}} \subset J_{\text{time}}\}, \quad \varrho_n = \varrho_n([a,b]),\\
&h_n^{1-\eta(n)} = P(N(0,1) > \lambda(2\log(1/h_n))^{1/2}),
\end{aligned}$$

where $0 < \eta(n) \to \eta := 1 - \lambda_0^2$ as $n \to \infty$.

From (10), it follows that, for large enough n , $I_{\text{time}} = [s,t] \in \mathcal{J}_{n,+}$ implies (39), and then $[v,t] \in \mathcal{T}$ for any $s \in I_{\text{time}}(K) \in \mathcal{J}_{n,+}(K)$.

Lemma 6. *Let $\vartheta > 0$ and $x \in \mathbb{R}^d$ be fixed. Assume that $w = 0$ and $\sigma = 1$ in (1), and $0 < \alpha < d < \alpha + 2$. Then, there exists a constant $c = c(\alpha, d, \vartheta, a, b) > 0$, independent of x, such that for any $I_{\text{time}} = [t_{n,i}, t_{n,i} + h_n] \in \mathcal{J}_n$ and $J_{\text{time}} = [t_{n,j}, t_{n,j} + h_n] \in \mathcal{J}_n$ with $I_{\text{time}} \cap J_{\text{time}} = \emptyset$, and any $n \geq n_0$ with some $n_0 > 0$,*

$$\mathbb{E}[\Xi_{\alpha,d,x}(n, I_{\text{time}})\Xi_{\alpha,d,x}(n, J_{\text{time}})] \leq c\nu_n^{-\frac{d-\alpha}{2}}. \tag{40}$$

Proof. Without loss of generality, it is assumed that $j > i > 0$. For brevity, define by $Z_{\xi,x}(\cdot,\cdot)$ the increments of the process $\xi(\cdot,\cdot)$:

$$Z_{\xi,x}(s,t) = \xi(t,x) - \xi(s,x), \quad t,s \in \mathbb{R}_+, x \in \mathbb{R}^d. \tag{41}$$

Then, for any $j > i > 0$,

$$\begin{aligned}
&\mathbb{E}[Z_{\xi,x}(i\nu_n, i\nu_n + 1)Z_{\xi,x}(j\nu_n, j\nu_n + 1)]\\
&= \mathbb{E}[(Z_{\xi,x}(j\nu_n, i\nu_n + 1))^2] - \mathbb{E}[(Z_{\xi,x}(j\nu_n, i\nu_n))^2]\\
&\quad - \mathbb{E}[(Z_{\xi,x}(j\nu_n + 1, i\nu_n + 1))^2] + \mathbb{E}[(Z_{\xi,x}(j\nu_n + 1, i\nu_n))^2].
\end{aligned} \tag{42}$$

It follows from (42) and Lemma 1 that, for $j > i > 0$ and large n,

$$\begin{aligned}
&\mathbb{E}[Z_{U_{\alpha,d},x}(i\nu_n, i\nu_n + 1)Z_{U_{\alpha,d},x}(j\nu_n, j\nu_n + 1)]\\
&= K_{\alpha,d}\nu_n^{1-\frac{d-\alpha}{2}}[(j - i + 1/\nu_n)^{1-\frac{d-\alpha}{2}} - 2(j-i)^{1-\frac{d-\alpha}{2}} + (j - i - 1/\nu_n)^{1-\frac{d-\alpha}{2}}],
\end{aligned} \tag{43}$$

where $U_{\alpha,d}(\cdot,\cdot)$ is given in (18). Let $f(x) = x^{1-\frac{d-\alpha}{2}}$ for $x > 0$. Then, $f''(x) < 0$ for $x > 0$. This, together with (43) and the Lagrange mean value theorem, yields that

$$\mathbb{E}[Z_{U_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{U_{\alpha,d},x}(j\nu_n, j\nu_n+1)] < 0. \tag{44}$$

It follows from (30) and (27) that, for any $a \le s < t$,

$$\begin{aligned}
&\mathbb{E}[(Z_{V_{\alpha,d},x}(s,t))^2] \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{e^{-s\vartheta|\xi|^2}}{\vartheta|\xi|^{\alpha+2}}(1-e^{-(t-s)\vartheta|\xi|^2/2})^2 d\xi \\
&= \frac{1}{2^d\pi^{d/2}\Gamma(d/2)\vartheta}\int_0^\infty \zeta^{\frac{d-\alpha}{2}-2}e^{-s\vartheta\zeta}(1-e^{-(t-s)\vartheta\zeta/2})^2 d\zeta \\
&= \frac{1}{2^d\pi^{d/2}\Gamma(d/2)\vartheta}\int_0^\infty \zeta^{\frac{d-\alpha}{2}-2}(e^{-t\vartheta\zeta}+e^{-s\vartheta\zeta}-2e^{-(t+s)\vartheta\zeta/2})d\zeta.
\end{aligned} \tag{45}$$

This, together with (42), yields that

$$\begin{aligned}
&\mathbb{E}[Z_{V_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{V_{\alpha,d},x}(j\nu_n, j\nu_n+1)] \\
&= \frac{1}{2^d\pi^{d/2}\Gamma(d/2)\vartheta}\int_0^\infty \zeta^{\frac{d-\alpha}{2}-2}\Big\{-2e^{-((j+i)\nu_n+1)\vartheta\zeta/2}+2e^{-((j+i)\nu_n)\vartheta\zeta/2} \\
&\quad +2e^{-((j+i)\nu_n+2)\vartheta\zeta/2}-2e^{-((j+i)\nu_n+1)\vartheta\zeta/2}\Big\}d\zeta.
\end{aligned} \tag{46}$$

By the changes of variables, (46) becomes

$$\begin{aligned}
&\mathbb{E}[Z_{V_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{V_{\alpha,d},x}(j\nu_n, j\nu_n+1)] \\
&= \frac{2}{2^d\pi^{d/2}\Gamma(d/2)\vartheta}((j+i)\nu_n)^{1-\frac{d-\alpha}{2}}\left\{1-2\left(1+\frac{1}{(j+i)\nu_n}\right)^{1-\frac{d-\alpha}{2}}\right. \\
&\quad \left.+\left(1+\frac{2}{(j+i)\nu_n}\right)^{1-\frac{d-\alpha}{2}}\right\}\int_0^\infty \zeta^{\frac{d-\alpha}{2}-2}e^{-\vartheta\zeta/2}d\zeta.
\end{aligned} \tag{47}$$

By Taylor expansion of $(1\pm z)^s$, one has that, for any $s > 0$ and $z \in (0,1/4)$, $1-2(1+z)^s+(1+2z)^s = -s(1+\eta_1 z)^{s-1}z + s(1+z+\eta_2 z)^{s-1}z = s(s-1)(1+\eta_2-\eta_1)(1+\eta_3 z)^{s-2}z^2$, where $\eta_1, \eta_2 \in [0,1]$ and $\eta_3 \in [0,2]$. This, together with (47), taking $s = 1-\frac{d-\alpha}{2}$ and $z = \frac{1}{(j+i)\nu_n}$, yields that

$$|\mathbb{E}[Z_{V_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{V_{\alpha,d},x}(j\nu_n, j\nu_n+1)]| \le c\nu_n^{-\frac{d-\alpha}{2}}. \tag{48}$$

Since the Gaussian process $\{u_{\alpha,d}(t,x), t \ge 0\}$ is self-similar with index $\frac{1}{2}-\frac{d-\alpha}{4}$ (see Tudor and Xiao [18]),

$$\mathbb{E}[\Xi_{\alpha,d,x}(n, I_{\text{time}})\Xi_{\alpha,d,x}(n, J_{\text{time}})] = \mathbb{E}[Z_{u_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{u_{\alpha,d},x}(j\nu_n, j\nu_n+1)]. \tag{49}$$

Since $u_{\alpha,d}(t,x) = U_{\alpha,d}(t,x) - V_{\alpha,d}(t,x)$, and $u_{\alpha,d}(t,x)$ and $V_\alpha(t,x)$ are independent for $(t,x) \in \mathring{\mathbb{R}}_+ \times \mathbb{R}^d$, the Equation (49) becomes

$$\begin{aligned}
&\mathbb{E}[\Xi_{\alpha,d,x}(n, I_{\text{time}})\Xi_{\alpha,d,x}(n, J_{\text{time}})] \\
&= \mathbb{E}[Z_{U_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{U_{\alpha,d},x}(j\nu_n, j\nu_n+1)] - \mathbb{E}[Z_{V_{\alpha,d},x}(i\nu_n, i\nu_n+1)Z_{V_{\alpha,d},x}(j\nu_n, j\nu_n+1)].
\end{aligned} \tag{50}$$

By (44), (48) and (50), (40) is obtained. The proof is completed. □

The following three lemmas are needed.

Lemma 7. *For any $0 < \zeta \le 1/2$, there exists an integer n_0 such that*

$$\mathbb{P}(|\rho_n(J_{\text{time}}) - \mathbb{E}[\rho_n(J_{\text{time}})]| \ge \gamma\mathbb{E}[\rho_n(J_{\text{time}})]) \le 2\exp(-\zeta(1-\zeta)(\gamma - 2\zeta)\mathbb{E}[\rho_n(J_{\text{time}})]) + h_n^3, \quad (51)$$

for all $J_{\text{time}} \subseteq [a,b]$, $n \ge n_0$ and $\gamma > 0$.

Proof. For brevity, denote by $u_{n,i} = \gamma_{h_n}^{-1} Z_{u_{\alpha,d},x}(t_{n,i}, t_{n,i} + h_n)$, $U_{n,i} = \gamma_{h_n}^{-1} Z_{U_{\alpha,d},x}(t_{n,i}, t_{n,i} + h_n)$ and $V_{n,i} = \gamma_{h_n}^{-1} Z_{V_{\alpha,d},x}(t_{n,i}, t_{n,i} + h_n)$, where $Z_{\xi,x}$, $U_{\alpha,d}$ and $V_{\alpha,d}$ are defined in (41), (19) and (21), respectively, and by $\varrho_n = \varrho_n(J_{\text{time}})$, $\ell_n = (2\log(1/h_n))^{1/2}$ and $\delta_n = \nu_n^{-\frac{d-\alpha}{4}}$. Note that

$$\rho_n(J_{\text{time}}) = \sum_{i=1}^{\varrho_n(J_{\text{time}})} \mathbb{I}(u_{n,i} > \lambda\ell_n). \quad (52)$$

Let $\{\varsigma_n, u_{n,i}^*, i = 1, \ldots, \varrho_n\}$ are independent mean zero Gaussian random variables with $\mathbb{E}[\varsigma_n^2] = \delta_n$ and $\mathbb{E}[(u_{n,i}^*)^2] = 1 - \delta_n - \mathbb{E}[V_{n,i}^2]$. It follows from Lemma 1 that $\mathbb{E}[u_{n,i}^2] = \mathbb{E}[U_{n,i}^2] - \mathbb{E}[V_{n,i}^2] = 1 - \mathbb{E}[V_{n,i}^2]$ and $\mathbb{E}[(\varsigma_n + u_{n,i}^*)^2] = \mathbb{E}[\varsigma_n^2] + \mathbb{E}[(u_{n,i}^*)^2] = 1 - \mathbb{E}[V_{n,i}^2] = \mathbb{E}[u_{n,i}^2]$. Moreover, by Lemma 6, one has $\mathbb{E}[u_{n,i}u_{n,j}] \le \mathbb{E}[(\varsigma_n + u_{n,i}^*)(\varsigma_n + u_{n,j}^*)] = \mathbb{E}[\varsigma_n^2] = \delta_n$ $(i \ne j)$.

Let

$$f(z) = \begin{cases} e^z & \text{if } 0 \le z \le q_n \\ e^{q_n}(z - q_n + 1) & \text{if } z \ge q_n \end{cases}$$

with $q_n = \zeta(\gamma+1)\mathbb{E}[\rho_n(J_{\text{time}})]$, and let $g(u_{n,1}, \ldots, u_{n,\varrho_n}) = f(\zeta\rho_n(J_{\text{time}}))$. Then, $g(u_{n,1}, \ldots, u_{n,\varrho_n}) \le e^{\zeta\rho_n(J_{\text{time}})} \vee \varrho_n e^{q_n}$. By the well-known comparison property (cf. Theorem 3.11 of Ledoux and Talagrand [29], p. 74 or Lemma 2.1 of Zhang [24]), one has

$$\mathbb{E}[g(u_{n,1}, \ldots, u_{n,\varrho_n})] \le \mathbb{E}[g(\varsigma_n + u_{n,1}^*, \ldots, \varsigma_n + u_{n,\varrho_n}^*)].$$

Thus, it follows that

$$\begin{aligned}
&\mathbb{P}(\rho_n(J_{\text{time}}) - \mathbb{E}[\rho_n(J_{\text{time}})] \ge \gamma\mathbb{E}[\rho_n(J_{\text{time}})]) \\
&= \mathbb{P}(f(\zeta\rho_n(J_{\text{time}})) \ge f(q_n))) \\
&= \mathbb{P}(g(u_{n,1}, \ldots, u_{n,\varrho_n}) \ge e^{q_n})) \\
&\le e^{-q_n}\mathbb{E}[g(u_{n,1}, \ldots, u_{n,\varrho_n})] \\
&\le e^{-q_n}\mathbb{E}[g(\varsigma_n + u_{n,1}^*, \ldots, \varsigma_n + u_{n,\varrho_n}^*)] \\
&\le e^{-q_n}\Big\{\mathbb{E}[e^{\zeta\sum_{i=1}^{\varrho_n}\mathbb{I}\{\varsigma_n + u_{n,i}^* > \lambda\ell_n\}}\mathbb{I}(\varsigma_n \le 2\delta_n^{1/2}\ell_n)] + \varrho_n e^{q_n}\mathbb{P}(\varsigma_n > 2\delta_n^{1/2}\ell_n)\Big\} \\
&\le e^{-q_n}\mathbb{E}[e^{\zeta\sum_{i=1}^{\varrho_n}\mathbb{I}\{u_{n,i}^* > (\lambda - 2\delta_n^{1/2})\ell_n\}}] + \varrho_n\mathbb{P}(\varsigma_n > 2\delta_n^{1/2}\ell_n)\}.
\end{aligned} \quad (53)$$

Since $1 - e^{-x} \le 2x$ for all $x > 0$, by (45), one has for any $t, s \in \mathbb{R}_{a,+}$,

$$\begin{aligned}
&\mathbb{E}[(Z_{V_{\alpha,d},x}(s,t))^2] \\
&= \frac{1}{2^d\pi^{d/2}\Gamma(d/2)\vartheta}\int_0^\infty \zeta^{\frac{d-\alpha}{2}-2}e^{-s\vartheta\zeta}(1 - e^{-(t-s)\vartheta\zeta/2})^2 d\zeta \\
&\le \frac{\vartheta}{2^d\pi^{d/2}\Gamma(d/2)}(t-s)^2\int_0^\infty \zeta^{\frac{d-\alpha}{2}}e^{-a\vartheta\zeta}d\zeta \\
&= c(t-s)^2.
\end{aligned}$$

This yields that $\mathbb{E}[V_{n,i}^2] \leq c r_n^{1+\frac{d-\alpha}{2}} \leq \delta_n$ for any $1 \leq i \leq \varrho_n$. Thus, $1-2\delta_n \leq \mathbb{E}[(u_{n,i}^*)^2] = 1-\delta_n - \mathbb{E}[V_{n,i}^2] \leq 1-\delta_n$ for any $1 \leq i \leq \varrho_n$. For an n large enough, denote by $p_{n,0} = p_n(\lambda)$, $p_{n,1} = p_n(\frac{\lambda - 2\delta_n^{1/2}}{(1-\delta_n)^{1/2}})$ and $p_{n,2} = p_n(\frac{\lambda}{(1-\delta_n)^{1/2}})$ where the following notation is used:

$$p_n(z) = \mathbb{P}(N(0,1) > z\ell_n), \;\; z > 0.$$

Then, it follows from (52) that

$$\varrho_n p_{n,0} \leq \mathbb{E}[\rho_n(J_{\rm time})] = \sum_{i=1}^{\varrho_n(J_{\rm time})} \mathbb{P}\Big(\frac{u_{n,i}}{\sqrt{1-\mathbb{E}[V_{n,i}^2]}} > \frac{\lambda}{\sqrt{1-\mathbb{E}[V_{n,i}^2]}}\ell_n\Big) \leq \varrho_n p_{n,2}.$$

Thus, $q_n = \zeta(\gamma+1)\mathbb{E}[\rho_n(J_{\rm time})] \geq \zeta(\gamma+1)\varrho_n p_{n,0}$.

By the fact that $\{u_{n,i}^*, i = 1, \ldots, \varrho_n\}$ are independent, one has

$$\begin{aligned}
&\mathbb{E}[e^{\zeta \sum_{i=1}^{\varrho_n} \mathbb{I}\{u_{n,i}^* > (\lambda - 2\delta_n^{1/2})\ell_n\}}] \\
&\leq \mathbb{E}\Big[e^{\zeta \sum_{i=1}^{\varrho_n} \mathbb{I}\{\frac{u_{n,i}^*}{\sqrt{1-\delta_n-\mathbb{E}[V_{n,i}^2]}} > \frac{\lambda-2\delta_n^{1/2}}{\sqrt{1-\delta_n}}\ell_n\}}\Big] \\
&= e^{\zeta \varrho_n p_{n,1}} \Big(\mathbb{E}\Big[e^{\zeta(\mathbb{I}\{\frac{u_{n,1}^*}{\sqrt{1-\delta_n-\mathbb{E}[V_{n,1}^2]}} > \frac{\lambda-2\delta_n^{1/2}}{\sqrt{1-\delta_n}}\ell_n\} - p_{n,1})}\Big]\Big)^{\varrho_n} \\
&\leq e^{\zeta \varrho_n p_{n,1}} (1 + p_{n,1}(1-p_{n,1})\zeta^2)^{\varrho_n} \\
&\leq e^{\zeta \varrho_n p_{n,1} + \zeta^2 \varrho_n p_{n,1}(1-p_{n,1})} \\
&\leq e^{\zeta \varrho_n p_{n,1} + \zeta^2 \varrho_n p_{n,1}}.
\end{aligned}$$

Then, it follows from (53) that

$$\begin{aligned}
&\mathbb{P}(\rho_n(J_{\rm time}) - \mathbb{E}[\rho_n(J_{\rm time})] \geq \gamma \mathbb{E}[\rho_n(J_{\rm time})]) \\
&\leq e^{-\zeta \varrho_n((\gamma+1)p_{n,0} - (1+\zeta)p_{n,1})} + \varrho_n \mathbb{P}(\varsigma_n > 2\delta_n^{1/2}\ell_n).
\end{aligned} \tag{54}$$

It follows from (33) that $p_{n,0} \sim p_{n,1} \sim p_{n,2}$ as $n \to \infty$. This implies that $(1+\zeta)p_{n,1} \leq (1+2\zeta)p_{n,0}$ and $p_{n,0} \geq (1-\zeta)p_{n,2}$. Thus, (54) becomes

$$\begin{aligned}
&\mathbb{P}(\rho_n(J_{\rm time}) - \mathbb{E}[\rho_n(J_{\rm time})] \geq \gamma \mathbb{E}[\rho_n(J_{\rm time})]) \\
&\leq e^{-\zeta \varrho_n(\gamma - 2\zeta)p_{n,0}} + h_n^3 \\
&\leq e^{-\zeta(1-\zeta)(\gamma-2\zeta)\varrho_n p_{n,2}} + h_n^3.
\end{aligned} \tag{55}$$

Similarly to (55), by choosing $q_n = \zeta((\gamma-1)\varrho_n p_{n,0} + \varrho_n)$, one has

$$\begin{aligned}
&\mathbb{P}(\mathbb{E}[\rho_n(J_{\rm time})] - \rho_n(J_{\rm time}) \geq \gamma \mathbb{E}[\rho_n(J_{\rm time})]) \\
&\leq e^{-\zeta(1-\zeta)(\gamma-2\zeta)\varrho_n p_{n,0}} + h_n^3.
\end{aligned} \tag{56}$$

This, together with (55), yields (51). The proof is completed. □

Lemma 8. *Given $\epsilon > 0$, $\delta > 0$, with probability 1 there exists an integer n_0 such that*

$$|\rho_n(J_{\rm time}) - \mathbb{E}[\rho_n(J_{\rm time})]| \leq \epsilon \mathbb{E}[\rho_n(J_{\rm time})] \tag{57}$$

for all $J_{\rm time} \subseteq [a,b]$ such that $|J_{\rm time}| \geq \delta$, and all $n \geq n_0(\epsilon, \delta)$.

Proof. It follows from (33) that $p_{n,0} = h_n^{\lambda^2(1+\tau_n)}$, where $\tau_n \to 0$ as $n \to \infty$. This, together with Lemma 7 and the Borel–Cantelli argument, yields (57). The proof is completed. $\square$

Lemma 9. *Given $\eta' < \eta = 1 - \lambda^2$, there is an absolute constant c such that, with probability 1, there is n_1 such that*

$$\rho_n(J_{\text{time}}) \le c|J_{\text{time}}|^{\eta'}\rho_n([a,b]) \tag{58}$$

for all $J_{\text{time}} \subseteq [a,b]$, $n \ge n_1$.

Proof. By Lemma 8, it is sufficient to show that

$$\rho_n(J_{\text{time}}) \le c|J_{\text{time}}|^{\eta'}\mathbb{E}[\rho_n([a,b])] = c|J_{\text{time}}|^{\eta'}\varrho_n h_n^{1-\eta(n)} \tag{59}$$

for $n \ge n_1$. Note that $|J_{\text{time}}| < h_n$, implies $\rho_n(J_{\text{time}}) = 0$, $h_n \le |J_{\text{time}}| < \nu_n h_n$, implies $\rho_n(J_{\text{time}}) \le 1$ and $|J_{\text{time}}|^{\eta'}\varrho_n h_n^{1-\eta(n)} \ge c r_n^{\delta+\eta'-\eta(n)} \to \infty$, it needs only to consider the case of $|J_{\text{time}}| \ge \nu_n h_n$. It is clearly sufficient to consider only the class $\mathcal{D}_n$ of intervals $[a+ih_n, a+jh_n]$, where i,j are integers and $0 \le i < j \le (\nu_n h_n)^{-1}$. Note that $\varrho_n \sim \nu_n^{-1}h_n^{-1} \sim (b-a)^{-1}r_n^{\tau-1}$ and $\varrho_n(J_{\text{time}}) = |J_{\text{time}}|\varrho_n$. It is deduced from Lemma 7 that, for an n large enough

$$\begin{aligned}
&\mathbb{P}(\rho_n(J_{\text{time}}) > c|J_{\text{time}}|^{\eta'}\varrho_n h_n^{1-\eta(n)}, J \in \mathcal{D}_n) \\
&\le h_n^{-2}\exp(-c|h_n|^{\eta'}\varrho_n h_n^{1-\eta(n)}) + h_n \\
&\le h_n^{-2}\exp(-c r_n^{\tau+\eta'-\eta(n)}) + h_n.
\end{aligned}$$

Since $\tau + \eta' - \eta(n) \to \tau + \eta' - \eta < 0$, it follows that

$$\sum_{n=1}^{\infty}\mathbb{P}(\rho_n(J_{\text{time}}) > c|J_{\text{time}}|^{\eta'}\varrho_n h_n^{1-\eta(n)}, J \in \mathcal{D}_n) < \infty,$$

which implies almost surely there exists $n_1 = n_1(\eta') > 0$ such that (59) holds. This completes the proof of the lemma. $\square$

We are now ready to show that there exists a sequence of sets $F_1 \supset F_2 \supset \cdots$ fulfilling the assumptions of Lemma 5, such that $F = \cap_{n=1}^{\infty}F_n \subset F_{\alpha,d,x}(\lambda; a,b)$. Since only a countable number steps of the construction are needed and each step can be carried out with probability 1, one can assume that all the steps are carried out in the same probability 1 set. Choose $\eta' = \eta - \frac{1}{4}\epsilon$ and define $n_1 =: n_1(\eta')$ such that (58) is valid for $n \ge n_1$. Suppose that (ϵ_k) is a sequence of positive numbers with $\sum\epsilon_k < \infty$. In the first step, by applying Lemma 8, there exists an integer $m_1 \ge n_1$ such that

$$|\rho_n - \mathbb{E}[\rho_n]| < \epsilon_1\mathbb{E}[\rho_n] \ \ (n \ge m_1).$$

Then, one will define an increasing sequence $m_1, m_2, \ldots$ inductively and define for $k \ge 1$.

$$\begin{aligned}
&\{I_{k,i}, 1 \le i \le Q_k\} = \{I_{\text{time}}(K) \in \mathcal{J}_{m_k,+}, I_{\text{time}}(K) \subset F_{k-1}\}, \\
&F_0 = [a,b], \ \ F_k = \cup_{i=1}^{Q_k} I_{k,i}, \\
&Q_k(J_{\text{time}}) = \#\{i, I_{k,i} \subset J_{\text{time}}\} \ \text{ for } J_{\text{time}} \subset [a,b], Q_k = Q_k([a,b]), \\
&\varsigma(k) = \eta(m_k), \ \ \tau(k) = 1 - \varsigma(k), \ \ R_k = |I_{k,i}| = K(b-a)r_{m_k}.
\end{aligned}$$

For $k \geq 2$, suppose that m_{k-1} has been defined; one can define m_k large enough to ensure

$$m_k \geq n_0(\epsilon, R_{k-1}^{2\tau(k-1)/\epsilon}), \ \ m_k = n_0(\epsilon_k, R_{k-1}),$$
$$m_k \geq 2m_{k-1}, \ \ r_{m_k} \leq r_{m_{k-1}}^2,$$

where $n_0(\epsilon, \delta)$ is the integer determined in Lemma 8 to invalidate (57), and

$$R_k^{1/(2\epsilon)} \leq K^{2\eta}(b-a)^{2\eta} \prod_{i=1}^{k-1} R_i^{\tau(i)} K^{\varsigma(i)}(b-a)^{\varsigma(i)}. \tag{60}$$

Then,

$$|\rho_n(J_{\text{time}}) - \mathbb{E}\rho_n(J_{\text{time}})| \leq \epsilon_k \mathbb{E}[\rho_n(J_{\text{time}})] \tag{61}$$

for all $\subseteq [a,b]$ such that $|J_{\text{time}}| \geq R_{k-1}$, and all $n \geq n_k$.

By using (60), (61) and Lemmas 8 and 9, following the same lines as the proof of (2.23) in Zhang [24], one has

$$Q_{k+j}(J_{\text{time}}) \leq c\Big(\prod_{i=1}^{k} \nu_{m_i}\Big) R_k^{\epsilon} |J_{\text{time}}|^{\eta-2\epsilon} Q_{k+j} \tag{62}$$

for all $R_{k+1} < |J_{\text{time}}| \leq R_k$, $k \geq 1$, $j \geq 1$.

Noting that

$$r_{m_k}^2 \leq r_{m_k}^{1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}} \leq r_{m_k} r_{m_{k-1}} \cdots r_{m_1}$$

and

$$\prod_{i=1}^{k} \nu_{m_i} \leq \Big(\prod_{i=1}^{k} r_{m_i}\Big)^{-\tau},$$

by (62), one has

$$Q_{k+j}(J_{\text{time}}) \leq c(K(b-a))^{\epsilon} r_{m_k}^{\epsilon-2\tau} |J_{\text{time}}|^{\eta-2\epsilon} Q_{k+j}$$

for all $R_{k+1} < |J_{\text{time}}| \leq R_k$, $k \geq 1$, $j \geq 1$. Thus, it follows from Lemma 5 and the fact that $r_{m_k}^{\epsilon-2\tau} \to 0$ $(k \to \infty)$ that, with probability 1,

$$\mu_{s^{\eta-2\epsilon}}(F_{\alpha,d,x}(\lambda; a, b)) > 0. \tag{63}$$

Hence, (36) is proved. The proof is completed.

Proof of Theorem 3. By Remark 2, it is sufficient to show Equation (15). By using (10) and Lemma 4, following the same lines as the proof of the upper bound of Theorem 2.1 in Khoshnevisan et al. [25], one has, with probability 1,

$$\sup_{t\in E} \limsup_{h\to 0+} \phi_{\alpha,d,h}|u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| \leq (\dim_p(E))^{1/2}. \tag{64}$$

It now turns to the proof of the opposite inequality. That is, it is sufficient to prove that, with probability 1,

$$\sup_{t\in E} \limsup_{h\to 0+} \phi_{\alpha,d,h}|u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)| \geq (\dim_p(E))^{1/2}. \tag{65}$$

Fix ϖ such that $\dim_p(E) > \varpi$. For any integer $n \geq 1$, let $\mathcal{Q}_n$ denote the set of all intervals of the form $[a + m2^{-n}(b-a), a + (m+1)2^{-n}(b-a)]$, $m \in \mathbb{Z}_+$. In words, $\mathcal{Q}_n$ denotes the totality of all intervals. For all $I_{\text{time}} \in \mathcal{Q}_n$, define $\pi_n(I_{\text{time}}) = a + m2^{-n}(b-a)$ to be the smallest element in I_{time}. For $I_{\text{time}} \in \mathcal{Q}_n$, denote by $\omega_n(I_{\text{time}})$ the indicator function of the event $(\Theta_{\alpha,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1}) > \varpi^{1/2})$, where the following notation is used:

$$\Theta_{\alpha,d,x}(t,h) = \phi_{\alpha,d,h}|u_{\alpha,d}(t+h,x) - u_{\alpha,d}(t,x)|. \tag{66}$$

In words, $\omega_n(I_{\text{time}})$ is a Bernoulli random variable whose values take 1 or 0 according to whether

$$\Theta_{\alpha,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1}) > \varpi^{1/2}.$$

Define by $D := \limsup_n D(n)$ a discrete limsup random fractal, where

$$D(n) = \bigcup_{I_{\text{time}} \in \mathcal{Q}_n : \omega_n(I_{\text{time}})=1} I^0_{\text{time}},$$

where I^0_{time} denotes the interior of I_{time}. It is claimed that, whenever $\dim_p(E) > \varpi$, then

$$\mathbb{P}(D \cap E \neq \emptyset) = 1. \tag{67}$$

The verification of (67) is postponed and (65) is proved first and thereby the proof is completed.

Since $\dim_p(E) > \varpi$, (67) implies that there exists $t \in E$ a.s. such that $\Theta_{\alpha,d,x}(a + 2^{-n}[t2^n](b-a), 2^{-n}(\log)^{-1}) \geq \varpi$ for infinitely many n. In particular,

$$\sup_{t\in E} \limsup_{n\to\infty} \Theta_{\alpha,d,x}(a + 2^{-n}[t2^n](b-a), 2^{-n}(\log)^{-1}) \geq \varpi \quad \text{a.s.}$$

By (13),

$$\lim_{n\to\infty} \sup_{t\in I_{\text{time}}:I_{\text{time}}\in\mathcal{Q}_n} |\Theta_{\alpha,d,x}(t, 2^{-n}(\log)^{-1}) - \Theta_{\alpha,d,x}(a+2^{-n}[t2^n](b-a), 2^{-n}(\log)^{-1})| = 0 \quad \text{a.s.}$$

Thus, if $\dim_{\mathrm{P}}(E) > \varpi$, then (65) holds and thereby (15) holds.

It remains to verify (67). Fix small $\epsilon > 0$ such that $\dim_p(E) > \varpi + \epsilon$. By Joyce and Preiss [30], there is a closed $E_\star \subset E$, such that for all open sets O, whenever $E_\star \cap O \neq \emptyset$, then $\overline{\dim}_M(E_\star \cap O) > \varpi + \epsilon$ (see Mattila [22] for the definition of upper Minkowski dimension). It is enough to show that $D \cap E_\star \neq \emptyset$, a.s. Fix an open set O such that $E_\star \cap O \neq \emptyset$. It is claimed that, with probability 1,

$$D(n) \cap E_\star \cap O \neq \emptyset \text{ for infinitely many } n. \tag{68}$$

Define by $V(n) := \cup_{k=n}^\infty D(k)$, $n \geq 1$, the open sets. Then, this claim implies that, with probability 1, $V(n) \cap E_\star \cap O \neq \emptyset$ for all n; by letting O run over a countable base for the open sets, one has that $V(n) \cap E_\star$ is a.s. dense in (the complete metric space) $E_\star$. By Baire's category theorem (see Munkres [31]), one has that $\cap_{n=1}^\infty V(n) \cap E_\star$ is dense in $E_\star$ and in particular, nonempty. Since $D = \cap_{n=1}^\infty V(n)$, one has that $D \cap E_\star \neq \emptyset$, a.s.; which, in turn, (67) holds and the result follows.

Fix an open set O satisfying $E_\star \cap O \neq \emptyset$. Denote by $\mathcal{N}_n$ the total number of intervals $I_{\text{time}} \in \mathcal{Q}_n$ satisfying $I_{\text{time}} \cap E_\star \cap O \neq \emptyset$. Since $\overline{\dim}_M(E_\star \cap O) > \varpi + \epsilon$, by the definition of the upper Minkowski dimension, there exists $\varpi_1 > \varpi + \epsilon$, such that $\mathcal{N}_n \geq 2^{n\varpi_1}$ for infinitely many integers n. Thus, $\#(\aleph) = \infty$, where

$$\aleph := \left\{ n \geq 1 : \mathcal{N}_n \geq 2^{n\varpi_1} \right\}. \tag{69}$$

Denote by $\Pi_n := \sum \omega_n(I_{\text{time}})$ the total number of intervals $I_{\text{time}} \in \mathcal{Q}_n$ such that $I_{\text{time}} \cap E_\star \cap O \cap D(n) \neq \emptyset$, where the sum is taken over all $I_{\text{time}} \in \mathcal{Q}_n$ such that $I_{\text{time}} \cap E_\star \cap O \neq \emptyset$; that is,

$$\Pi_n = \#\{I_{\text{time}} \in \mathcal{Q}_n : I_{\text{time}} \cap E_\star \cap O \neq \emptyset, \Theta_{\alpha,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1}) > \varpi^{1/2}\}.$$

In order to show (68); with probability 1, $D(n) \cap E_\star \cap O \neq \emptyset$ for infinitely many n, it suffices to show that $\Pi_n > 0$ for infinitely many n, a.s. That is, it is enough to show that

$$P(\Pi_n > 0 \ \text{ i.o.}) = 1, \tag{70}$$

where i.o. means infinitely often.

It follows from (66) and (53) that $\mathbb{E}[\Theta^2_{\alpha,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1})] = \frac{1}{2}(1 + \tau_n)$, where $\tau_n \to 0$ as $n \to \infty$. Thus, $p_n := \mathbb{P}(\omega_n(I_{\text{time}}) = 1) = 2^{-n(\varpi+\theta_n)}$, where $\theta_n \to 0$ as $n \to \infty$. Hence, $\mathbb{E}[\Pi_n] = \mathcal{N}_n p_n \geq 2^{n(\varpi_1 - \varpi - \theta_n)}$. Thus, it follows from Lemma 8 that, with probability 1, $\Pi_n \geq c2^{n(\varpi_1 - \varpi - \theta_n)}$. Since, $\varpi_1 - \varpi - \theta_n \to \varpi_1 - \varpi < 0$, it follows that, with probability 1, $\Pi_n \to \infty$, which implies that $\mathbb{P}(\Pi_n = 0) \to 0$ as $n \to \infty$. By Fatou's lemma, one has

$$\mathbb{P}(\Pi_n > 0 \ \text{ i.o.}) \geq \limsup_{n\to\infty} \mathbb{P}(\Pi_n > 0) = 1.$$

This yields (70). This completes the proof. □

5. Conclusions

In this paper, the temporal fractal nature of the solution to a spatially colored SHE has been investigated. The Hausdorff dimensions and hitting probabilities of the sets of temporal fast points for the spatially colored SHEs in time variable t have been obtained. It has been confirmed that these points of the spatially colored SHEs, in time, are everywhere dense with power of the continuum almost surely, and their hitting probabilities are determined by the packing dimension $\dim_p(E)$ of the target set E. The research findings are as follows:

i. The spatially colored SHEs have the exact, dimension-dependent, temporal moduli of continuity and temporal LIL. The exact values of the regularization constants of these results are the same.
ii. The set of temporal fast points for the spatially colored SHEs in time variable t is a random fractal, the size of which is measured by its Hausdorff dimension.
iii. The packing dimension of the target analytic set determines the probability that the intersection of the set of temporal fast points for the spatially colored SHEs in time variable t and this analytic set is non-empty.

Funding: This work was supported by Humanities and Social Sciences of Ministry of Education Planning Fund of China grant 21YJA910005 and National Natural Science Foundation of China under grant No. 11671115.

Data Availability Statement: Data is contained within the article.

Acknowledgments: The author wishes to express his deep gratitude to a referee for his/her valuable comments on an earlier version which improved the quality of this paper.

Conflicts of Interest: The author declares no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. del Castillo-Negrete, D.; Carreras, B.A.; Lynch, V.E. Front dynamics in reaction-diffusion systems with Levy ights: A fractional diffusion approach. *Phys. Rev. Lett.* **2003**, *91*, 018302. [CrossRef]
2. Kou, S.C.; Xie, X.S. Generalized Langevin equation with fractional Gaussian noise: Subdiffusion within a single protein molecule. *Phys. Rev. Lett.* **2004**, *93*, 180603. [CrossRef]
3. Bayraktar, E.; Poor, V; Sircar, R. Estimating the fractal dimension of the SP 500 index using wavelet analysis. *Int. J. Theor. Appl. Financ.* **2004**, *7*, 615–643. [CrossRef]
4. Denk, G.; Meintrup, D.; Schaffer, S. Modeling, simulation and optimization of integrated circuits. *Int. Ser. Numer. Math.* **2004**, *146*, 251–267.
5. Dalang, R.C. Extending martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **1999**, *4*, 1–29; Erratum in *Electron. J. Probab.* **2001**, *6*, 1–5. [CrossRef]
6. Hu, Y. Some recent progress on stochastic heat equations. *Acta Math. Sci.* **2019**, *39*, 874–914. [CrossRef]
7. Mueller, C.; Tribe, R. A singular parabolic Anderson model. *Electron. J. Probab.* **2004**, *9*, 98–144. [CrossRef]
8. Bruned, Y.; Gabriel, F.; Hairer, M.; Zambotti, L. Geometric stochastic heat equations. *J. Am. Math. Soc.* **2022**, *35*, 1–80. [CrossRef]
9. Dalang, R.C.; Khoshnevisan, D.; Rassoul-Agha, F. *A Minicourse on Stochastic Partial Differential Equations*; Lecture Notes in Mathematics, 1962; Springer: Berlin/Heidelberg, Germany, 2009.
10. Khoshnevisan, D. *Analysis of Stochastic Partial Differential Equations*; CBMS Regional Conference Series in Mathematics, Amer. Math. Soc., 119; CBMS: Providence, DK, USA, 2014.
11. Raluca, M.B.; Tudor, C.A. Stochastic heat equation with multiplicative fractional-colored noise. *J. Theoret. Probab.* **2010**, *23*, 834–870.
12. Rippl, T.; Sturm, A. New results on pathwise uniqueness for the heat equation with colored noise. *Electron. J. Probab.* **2013**, *18*, 1–46. [CrossRef]
13. Tudor, C.A. *Analysis of Variations for Self-Similar Processes—A Stochastic Calculus Approach*; Springer: Cham, Switzerland, 2013.
14. Balan, R.M.; Tudor, C.A. The stochastic wave equation with fractional noise: A random field approach. *Stoch. Process. Appl.* **2010**, *120*, 2468–2494. [CrossRef]
15. Bezdek, P. On weak convergence of stochastic heat equation with colored noise. *Stoch. Process. Appl.* **2016**, *126*, 2860–2875. [CrossRef]
16. Pospíšil, J.; Tribe, R. Parameter estimates and exact variations for stochastic heat equations driven by space-time white noise. *Stoch. Anal. Appl.* **2007**, *25*, 593–611. [CrossRef]
17. Swanson, J. Variations of the solution to a stochastic heat equation. *Ann. Probab.* **2007**, *35*, 2122–2159. [CrossRef]
18. Tudor, C.A.; Xiao, Y. Sample path properties of the solution to the fractional-colored stochastic heat equation. *Stoch. Dyn.* **2017**, *17*, 1750004. [CrossRef]
19. Wang, W. Asymptotic distributions for power variation of the solution to a stochastic heat equation. *Acta Math. Sin. Engl. Ser.* **2021**, *37*, 1367–1383. [CrossRef]
20. Wang, W.; Chang, X.; Wang, L. Asymptotic Distributions for Power Variations of the Solution to the Spatially Colored Stochastic Heat Equation. *Dis. Dyn. Nat. Soc.* **2021**, *2021*, 8208934. [CrossRef]
21. Orey, S.; Taylor, S.T. How often on a Brownian path does the iterated logarithm fail? *Proc. Lond. Math. Sot.* **1974**, *28*, 174–192. [CrossRef]
22. Mattila, P. *Geometry of Sets and Measures in Euclidean Spaces*; Cambridge University Press: Cambridge, UK, 1995.
23. Deheuvels, P.; Mason, P. On the fractal nature of empirical increments. *Ann. Probab.* **1995**, *23*, 355–387. [CrossRef]
24. Zhang, L.X. On the fractal nature of increments of ℓ^p-valued Gaussian processes. *Stoch. Process. Appl.* **1997**, *71*, 91–110. [CrossRef]
25. Khoshnevisan, D.; Peres, Y.; Xiao, Y. Limsup random fractals. *Electron. J. Probab.* **2000**, *5*, 1–24. [CrossRef]
26. Fang, K.T.; Kotz, S.; Ng, K.W. *Symmetric Multivariate and Related Distribution*; Chapman and Hall Ltd.: London, UK, 1990.
27. Csörgő, M.; Révész, P. *Strong Approxiamtions in Probability and Statistics*; Academic Press: New York, NY, USA, 1981.

28. Meerschaert, M.M.; Wang, W.; Xiao, Y. Fernique type inequality and moduli of continuity for anisotropic Gaussian random fields. *Trans. Am. Math. Soc.* **2013**, *365*, 1081–1107. [CrossRef] [PubMed]
29. Ledoux, M.; Talagrand, M. *Probability in Banach Spaces*; Springer: Berlin/Heidelberg, Germany, 1991.
30. Joyce, H; Preiss, D. On the existence of subsets of finite positive packing measure. *Mathematika* **1995**, *42*, 15–24. [CrossRef]
31. Munkres, J.R. *Topology: A First Course*; Prentice-Hall Inc.: Englewood Cliffs, NJ, USA, 1975.