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Empirical Bayes Decision for a Generalized Exponential Distribution with Contaminated Data

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Abstract: The two-sided and one-sided empirical Bayes test (EBT) rules for the parameter of a generalized exponential distribution with contaminated data (errors in variables) are constructed by a deconvolution kernel method, respectively. Under the type of the supersmooth error distributions and the supersmooth errors with the error level can be controlled situations, the asymptotically optimal uniformly over a class of prior distributions and uniform rates of convergence of the corresponding regret for the proposed EBT rules are obtained with suitable conditions. The example study shows that the assumptions and conditions of the main results of this paper are satisfied easily by calculating.

Keywords: generalized exponential distribution; empirical Bayes decision; deconvolution kernel method; contaminated data

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1. Introduction

Much of the more recent literature has looked at the empirical Bayes test (EBT). EBT for the parameter of some common distributions are investigated [1–4]. Under non-identical components case, empirical Bayes testing for a lifetime guarantee is considered for the double parameter exponential distribution [5]. Merging Bayesian and empirical Bayes posterior distributions in total variation is discussed [6]. A double empirical Bayes decision is obtained for multi-experiment studies by means of an empirical Bayes analytical method [7]. In order to study the relationship between empirical Bayes posterior distributions and false discovery rate control, a spike and slab empirical Bayes multiple testing is constructed [8]. An empirical Bayes multiple testing procedure for the sparse sequence model is investigated [9]. In earlier times, on the EBT for the continuous one-parameter exponential family has lots of work and is asymptotically optimal and the optimal convergence rates of EBT are obtained [10–14]. Most of the studies have discussed the empirical Bayes decision problem in the case of non-contaminated data, which is not the case for pure data cases. However, in practical application problems, contaminated data (errors in variables) are involved in many fields, and it has been widely studied [15–19]. In recent literature, the one-sided empirical Bayes decision problem is investigated for the continuous one-parameter exponential family with contaminated data [20].

Suppose that the random variable X has the generalized exponential distribution (GED) with probability density function (PDF) of the following forms [21]

$$f(x|\theta) = (\mu x + \theta)\exp(-\theta x - \mu x^2/2), \tag{1}$$

where θ is a unknown parameter with the natural space $\Theta = \{\theta > 0| \int_\Omega f(x|\theta)\,dx = 1\}$. In this article, we assume $\mu > 0$ is a known constant, and the sample space is $\Omega = \{x|x > 0\}$.

GED is also called linear exponential distribution, it is a combinatorial distribution, and the exponential and Rayleigh distributions are considered as special cases of GED when $\theta = 0$ and $\mu = 0$, respectively. The hazard function of GED is a linear function about time and age in linear exponential models, and it is one of the reasonable models for lifetime distributions of random phenomena. Progressive type-II censored competing risks data when the lifetimes are assumed to be a linear exponential distribution [21]. Recurrence relations for single and product moments of generalized order statistics have been derived with the linear exponential distribution [22]. The linear exponential distribution has been used in the area of reliability and life-testing see, for example, Broadbent [23] and Bain [24].

The two-sided and one-sided EBT rules are constructed for GED with contaminated data in this article. Deconvolution kernel method is employed to develop the two-sided and one-sided EBT rules with contaminated data, respectively. For errors in the variables model, deconvolution kernel method can eliminate the effect of the additive noise kernel density estimation. Furthermore, under the supersmooth error distributions and the supersmooth errors, the error level can be controlled, the asymptotically optimal and uniform rates of convergence are obtained with suitable conditions.

In practical problems, we often encounter measurement errors due to observation conditions, so the analysis of contaminated data is very important. For the pair random variables (X, θ), assume that θ has a prior distribution $G(\theta)$, X is one dimensional real random variable with a marginal density function $f_{X|\theta}(x)$ when θ is given, (X, θ) is not directly observable. We observe only Y, where $Y = X + \varepsilon$, and ε are the random error. Suppose that ε follows a known distribution F_ε on $(-\infty, \infty)$, and independent on (X, θ).

Firstly, we consider the two-sided test problem as follows

$$H_0 : \theta_1 \leqslant \theta \leqslant \theta_2 \leftrightarrow H_1 : \theta < \theta_1 \quad or \quad \theta > \theta_2 \tag{2}$$

where θ_1 and θ_2 are known constants.

Define $\theta_0 = (\theta_1 + \theta_2)/2$, $\gamma_0 = (\theta_2 - \theta_1)/2$, then (2) is equivalent to

$$H_0^* : |\theta - \theta_0| \leqslant \gamma_0 \leftrightarrow H_1^* : |\theta - \theta_0| > \gamma_0 \tag{3}$$

For hypothesis test (3), let $i = 0, 1$. taking 0–1 weighted square loss function in the following

$$L_i(\theta, d_i) = (1 - i)a[(\theta - \theta_0)^2 - \gamma_0^2] I_{[|\theta - \theta_0| > \gamma_0]} + ia[\gamma_0^2 - (\theta - \theta_0)^2] I_{[|\theta - \theta_0| \leqslant \gamma_0]},$$

where $a > 0$ is a constant, and $d = \{d_0, d_1\}$ is the decision space, d_0 indicates accepting H_0^*, d_1 indicates rejecting H_0^*.

When $i = 0$, then we obtain $L_0(\theta, d_0) = a\left[(\theta - \theta_0)^2 - \gamma_0^2\right] I_{[|\theta - \theta_0|\gamma_0]}$; when $i = 1$ then we have $L_1(\theta, d_1) = a\left[\gamma_0^2 - (\theta - \theta_0)^2\right] I_{[|\theta - \theta_0| \leq \gamma_0]}$.

Let the parameter θ be distributed according to an unknown prior $G(\theta)$, and assume that $G(\theta)$ belongs to the following class of distributions

$$\vartheta = \{G : G \text{ is a prior on } \Omega \text{ such that } \sup_x \left| f_X^{(m)}(x) \right| \leq B\}, \tag{4}$$

where $f_x^m(x)$ denotes the m order derivative of $f_X(x) = \int_\Theta f_{X|\theta}(x)\, dG(\theta)$, which is the marginal density of X, and $m \geqslant 2$ is an integer, $B > 0$ is a constant.

We define the randomized decision rule for hypothesis test (3) as follows

$$\delta(y) = P(accepting H_0^* | Y = y). \tag{5}$$

Then, the Bayes risk of $\delta(y)$ is given by

$$\begin{aligned} R(\delta(y), G(\theta)) &= \int_{-\infty}^{\infty} \int_\Theta [L_0(\theta, d_0)\delta(y) + L_1(\theta, d_1)(1 - \delta(y)] f_{Y|\theta}(y)\, dG(\theta)\, dy \\ &= a \int_{-\infty}^{\infty} \beta_G(y)\delta(y)\, dy + C_G, \end{aligned} \tag{6}$$

where

$$C_G = \int_{\Theta} L_1(\theta, d_1)\, dG(\theta), \beta_G(y) = \int_{\Theta} [(\theta-\theta_0)^2 - \gamma_0^2] f_{Y|\theta}(y)\, dG(\theta) \tag{7}$$

with

$$f_Y(y) = \int_{\Theta} f_{Y|\theta}(y)\, dG(\theta). \tag{8}$$

and $f_{Y|\theta}(y)$ denotes the density of Y given θ, i.e., $f_{Y|\theta}(y) = \int f_{X|\theta}(y-x)\, dF_{\varepsilon}(x)$.

Let $P_X(x) = \int_{\Theta} e^{-\theta x - \frac{1}{2}\mu x^2}\, dG(\theta)$, and

$$P_X^{(1)}(x) = -\int_{\Theta} (\mu x + \theta) e^{-\theta x - \frac{1}{2}\mu x^2}\, dG(\theta) = -f_X(x),$$

thus, we have $\int_x^{\infty} f_X(x)\, dx = P_X(x)$.

From (7), we obtain

$$\begin{aligned}\beta_G(y) = &\int_{-\infty}^{\infty} f_X^{(2)}(y-x)\, dF_{\varepsilon}(x) + \int_{-\infty}^{\infty} Q(y-x) f_X^{(1)}(y-x)\, dF_{\varepsilon}(x)\\ &+ \int_{-\infty}^{\infty} \phi(y-x) f_X(y-x)\, dF_{\varepsilon}(x) - \mu \int_{-\infty}^{\infty} Q(y-x) p_X(y-x)\, dF_{\varepsilon}(x),\end{aligned} \tag{9}$$

where $f_X(x) = \int_{\theta} f_{X|\theta}(x) dG(\theta)$ is the marginal PDF of random variable X, $f_X^{(1)}(x)$ and $f_X^{(2)}(x)$ denote the first order and the second order derivative of $f_X(x)$, respectively.

In (9), let

$$\begin{aligned}Q(y-x) &= 2u(y-x) + 2\theta_0 \ \textit{and}\\ \phi(y-x) &= \mu^2(y-x)^2 + 2\mu\theta_0(y-x) + 3\mu + \theta_0^2 - \gamma_0^2.\end{aligned}$$

So from (9), we define the best Bayes decision minimizing $R(\delta(y), G(\theta))$ as follows

$$\delta_G(y) = \begin{cases} 1, \text{ if } \beta_G(y) \le 0, \\ 0, \text{ elsewhere.} \end{cases} \tag{10}$$

A test is called a Bayes test with respect to $G(\theta)$ if

$$R(\delta_G, G) = \inf_{\delta'} R(\delta', G) = a \int_{-\infty}^{\infty} \beta_G(y) \delta_G(y) dy + C_G. \tag{11}$$

Since $G(\theta)$ is unknown in this paper, $\delta_G(y)$ is unavailable to use, so this leads us to use the empirical Bayes approach in the following.

The rest of this article is organized as follows. In Section 2, the two-sided EBT rule for GED with contaminated data is proposed; Section 3 is devoted to obtaining asymptotic properties and the uniform convergence rate of two-sided EBT rule; the main results of two-sided EBT are proved in Section 4; Section 5 investigated one-sided EBT rule for GED with contaminated data; an example study is presented in Section 6.

2. The Proposed Two-Sided EBT Rule of GED with Contaminated Data

It is well-known that we usually make the following assumptions in the empirical Bayes framework, let (Y_1, θ_1), (Y_2, θ_2), $\cdots$, (Y_n, θ_n), and (Y, θ) be independent pair of random variables, the parameters θ_i $(1 \le i \le n)$ and θ have a common prior distribution $G(\theta)$; $Y_i (1 \le i \le n)$ and Y are distributed according to the same marginal distribution F_Y with density function $f_Y(y) = \int_{\Theta} f_{Y|\theta}(y) dG(\theta)$, $Y_1, \cdots, Y_n$ denotes historical samples and Y is called the present sample.

Deconvolution is a very important problem. It is often encountered when modeling unobservable data or to estimate conditional moments useful in likelihood calculations. When dealing with non-parametric estimation of priors or in measurement error models, the sample data are noisy because of the measurement error; deconvolution kernel method is adopted to eliminate the effect of the additive noise kernel density estimation. In order to obtain the empirical Bayes decision, we employ the deconvolution kernel method in the following by Fan [17–19].

Let $\varphi_Y(t)$ and $\varphi_\varepsilon(t)$ be the characteristic function (c.f.) of Y and ε, respectively. Note that

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx)\frac{\varphi_Y(t)}{\varphi_\varepsilon(t)}dt \tag{12}$$

Thus, a deconvoluted kernel density estimation of $f_X^{(r)}(x)(r = 0,1,2)$ is defined by

$$f_n^{(r)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx)(-it)^r\varphi_K(th_n)\frac{\varphi_n(t)}{\varphi_\varepsilon(t)}dt, \tag{13}$$

where $0 < h_n \to 0$ as $n \to \infty$ and $\varphi_n(t) = \frac{1}{n}\sum_{j=1}^{n}\exp(itY_j)$ is called the empirical c.f. of random variable Y. Note that $f_X^{(0)}(x) = f_X(x)$ and $f_n^{(0)}(x) = f_n(x)$.

We can also rewrite (13) as kernel type of estimation as follows

$$f_n^{(r)}(x) = \frac{1}{nh_n^{1+r}}\sum_{j=1}^{n} K_{nr}\left(\frac{x - Y_j}{h_n}\right), \quad r = 0,1,2. \tag{14}$$

where

$$K_{nr}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx)(-it)^r\frac{\varphi_K(t)}{\varphi_\varepsilon(t/h_n)}dt$$

We define an estimator of the $p_X(x)$ of the random variable X by

$$p_n(x) = \int_{-M_n}^{x} f_n(t)dt, \tag{15}$$

where $f_n(x)$ is the kernel density estimator given by (13), and $M_n(\to \infty)$ is a sequence of constants.

Hence, we define an estimator of the $\beta_G(y)$ as follows

$$\begin{aligned}\beta_n(y) = &\int_{-\infty}^{\infty} f_n^{(2)}(y-x)dF_\varepsilon(x) + \int_{-\infty}^{\infty} Q(y-x)f_n^{(1)}(y-x)dF_\varepsilon(x)\\ &+ \int_{-\infty}^{\infty}\phi(y-x)f_n(y-x)dF_\varepsilon(x) - \mu\int_{-\infty}^{\infty} Q(y-x)p_n(y-x)dF_\varepsilon(x).\end{aligned} \tag{16}$$

Furthermore, an empirical Byes test rule is defined as

$$\delta_n(y) = \begin{cases} 1, & \text{if } \beta_n(y) \le 0 \\ 0, & \text{elsewhere.}\end{cases} \tag{17}$$

In the following, let E be the expectation with respect to the joint distribution of $(Y_1, Y_2, \cdots, Y_n)$. Then, the overall Bayes risk of $\delta_n(y)$ would be

$$R(\delta_n, G) = a\int_{-\infty}^{\infty}\beta_G(y)E[\delta_n(y)]dy + C_G. \tag{18}$$

By the definition, for any $G(\theta) \in \vartheta$, if $\sup_{G\in\vartheta}(R(\delta_n, G) - R(\delta_G, G)) = O(n^{-q})$, where $q > 0$, then $\delta_n(y)$ is called asymptotically optimal uniformly with uniform convergence rate $O(n^{-q})$.

3. Asymptotic Properties of Two-Sided EBT Rule

In this section, asymptotic properties of $R(\delta_n, G) - R(\delta_G, G)$ be investigated, some assumptions on the kernel function $K(x)$ and the error variable ε are given in the following.

(A1) The K(x) is a symmetric function about zero on $(-\infty, +\infty)$ and satisfies $\int_{-\infty}^{\infty} K(x)dx = 1$, $\int_{-\infty}^{\infty} x^j K(x)dx = 0$ for $j = 1, \cdots, (r-1)$ and $\int_{-\infty}^{\infty} x^r K(x)dx \neq 0$ for some integer $r > 0$,

(A2) $\varphi_K(t)$ is a symmetric function, having $m+2$ bounded integral derivatives on $(-\infty, +\infty)$,

(A3) $\varphi_K(t) = 1 + O(|t|^m)$ as $t \to 0$,

(A4) The characteristic function of ε satisfies $\varphi_\varepsilon(t) \neq 0$ for any t,

(A5) $\int_\Theta \int_{-\infty}^{\infty} \theta^2 f_{X|\theta}(y-x)dF_\varepsilon(x)dG(\theta) < \infty$ uniformly in y, and

(A6) for some $0 < \lambda < 1$, $\int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} dy < \infty$ and $\int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \left[\int_{-\infty}^{\infty} |Q(y-x)|dF_\varepsilon(x)\right]^\lambda dy < \infty$ and $\int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \left[\int_{-\infty}^{\infty} |\phi(y-x)|dF_\varepsilon(x)\right]^\lambda dy < \infty$, where $\beta_G(y)$ is given by (9).

Next, theorem below about the two-sided EBT establish the rates of convergence of the regret $R(\delta_n, G) - R(\delta_G, G)$, where $R(\delta_G, G)$ and $R(\delta_n, G)$ are given by (11) and (18), respectively.

Theorem 1. *For some integer $m \geq 2$ and constants $0 < \lambda < 1$, ϑ is defined by (4). Suppose that $K(x)$ and $F_\varepsilon(x)$ are such that (A1)–(A6) hold, and the following conditions are satisfied:*

(B1) $\varphi_K(t) = 0$ for $|t| \geq 1$,
(B2) $|\varphi_s(t)||t|^{-\beta_0} \exp(|t|^{-\beta}/\gamma) \geq \gamma_0$ as $|t| \to \infty$ for some positive constants β, γ, γ_0 and a constant β_0.

Then, by the choosing the bandwidth $h_n = (4/\gamma)^{1/\beta}(\log n)^{-1/\beta}$, we obtain

$$\sup_{G \in \vartheta}(R(\delta_n, G) - R(\delta_G, G)) = O\Big((\log n)^{-\lambda(m-2)/\beta}\Big). \tag{19}$$

Remark 1. *If its characteristic function $\varphi_\varepsilon(t)$ satisfies condition (B2) of Theorem 1, then the distribution of a random variable ε is called supersmooth of order β. The common examples of supersmooth distributions are normal, Cauchy, mixture normal, etc. In practice, the conditions of Theorem 1 are easy to verify. It can be seen from the result of the Theorem 1 that the rate of convergence of EBT is very slow for very common error distributions, such as normal. Fan [17,18] pointed out the supersmooth error distribution will result in a worse convergence rate than of the smooth distribution.*

It appears that the optimal rate of convergence for Gaussian deconvolution is extremely slow. Since the normal distribution is frequently used in applications, we need to study how to large a noise level is acceptable. Thus, considering the following model, let us assume that the data $Y_1, \cdots, Y_n$ are independent identical distribution samples from

$$Y = X + \varepsilon, \tag{20}$$

where $\varepsilon = \sigma_0\tilde{\varepsilon}$, σ_0 parameterizes the noise level.

Theorem 2. *For some integer $m \geq 2$ and constants $0 < \lambda < 1$, ϑ is defined by (4). Suppose that $K(x)$ and $F_\varepsilon(x)$ are such that (A1)–(A6) hold with $\varepsilon = \sigma_0 \tilde{\varepsilon}$. Then, let $\sigma_0 = O\left(n^{-1/(2m+1)}\right)$ and by choosing the bandwidth $h_n = O\left(n^{-1/(2m+1)}\right)$, we have*

$$\sup_{G \in \vartheta} (R_n - R_G) = O\left(n^{-\lambda(m-2)/(2m+1)}\right).$$

Remark 2. *Although all the data are contaminated with supersmooth errors, the results of Theorem 2 can also be as good as that of the uncontaminated data case. Suppose that all the data are contaminated with supersmooth errors, while the error level can be controlled, namely, $\varphi_\varepsilon(t) = \varphi_{\tilde{\varepsilon}}(\sigma_0 t)$. Fan [19] had been considered model (20). Theorem 2 indicates that the convergence rate is also very slow. The result of the following Lemma 3 is as good as ordinary smooth errors distribution but the result of the following Lemma 4 cause to the worse convergence rate of empirical Bayes estimator.*

4. Proofs

In this section, first we need some lemmas to prove the main results of this paper. Lemmas 1 and 2 are due to Fan [17,18], Lemmas 3, and 4 are due to Fan [19]. The proof of Lemma 4 can be found in Johns and Van Ryzin [10]. Theorems 1 and 2 shall be proved, since the proofs of Theorems 1 and 2 are similar, only Theorem 1 is proved in detail. In the following, $c, c_1, c_2, \cdots$ always stand for some positive constants and may be different even with the same notations.

Lemma 1. *Let $f_n^{(r)}(x)$ be given by (14), under the assumptions (A1)–(A4) and the conditions (B1)–(B2) of Theorem 1 are satisfied, by the choosing the bandwidth $h_n = (4/\gamma)^{1/\beta}(\log n)^{-1/\beta}$, we have*

$$\sup_x \sup_{G \in \vartheta} E\left(f_n^{(r)}(x) - f_X^{(r)}(x)\right)^2 \leq c(\log n)^{-2(m-r)/\beta}, \tag{21}$$

where $f_X^{(r)}(x)$ denotes the r order derivative of $f_X(x)$ and ϑ is given by (4).

Lemma 2. *Let $p_n(x)$ be given by (15), suppose that $\varphi_K(t)$ is a symmetric function, having $m+3$ bounded integrable derivatives on $(-\infty, \infty)$, and satisfying $\varphi_K(t) = 1 + O(|t|^{m+1})$ as $t \to 0$. Under the assumption (A4) and the conditions (B1)–(B2) of Theorem 1 hold, with the choice $h_n = (4/\gamma)^{-1/\beta}(\log n)^{-1/\beta}$ of the bandwidth and $M_n = n^{1/3}$, we have*

$$\sup_{P_X \in \Omega^*} \sup_{G \in \vartheta} E(p_n(x) - p_X(x))^2 \leq c(\log n)^{-2(m+1)/\beta} \tag{22}$$

where ϑ is given by (4), and $\Omega^* = \left\{p : p'_X(x) \in \vartheta, p(-n) + 1 - p(n) = o\left((\log n)^{-(m+1)/\beta}\right)\right\}$.

Lemma 3. *Let $f_X^{(r)}(x)$ be given by (14). If the assumptions of (A1)–(A4) hold, let $\sigma_0 = O\left(n^{-1/(2m+1)}\right)$, then by choosing the bandwidth $h_n = O\left(n^{-1/(2m+1)}\right)$, we have*

$$\sup_{x \in \Omega} \sup_{G \in \vartheta} E_n\left(f_n^{(r)}(x) - f_X^{(r)}(x)\right)^2 \leq c\left(n^{-2(m-r)/(2m+1)}\right), \tag{23}$$

where $f_X^{(r)}(x)$ denote the r order derivative of $f_X(x)$ and ϑ is given by (4).

Lemma 4. *Let $p_n(x)$ is given by (15), suppose that $\phi''_K(\cdot)$ and $\phi''_\varepsilon(\cdot)$are bounded, respectively. Let $K(x)$ satisfy (A1)–(A4) with $m = 2$, and $\sigma_0 = O\left(n^{-1/5}\right)$, then we have*

$$\sup_{x \in \Omega} \sup_{G \in \vartheta} E_n(p_n(x) - p_X(x))^2 \leq cn^{-1}. \tag{24}$$

Lemma 5. *Let $R(\delta_G, G)$ and $R(\delta_n, G)$ be defined by (11) and (18), respectively, then*

$$0 \leq R(\delta_n, G) - R(\delta_G, G) \leq a \int_{-\infty}^{\infty} |\beta_G(y)| p(|\beta_n(y) - \beta_G(y)| \geq |\beta_G(y)|) dy,$$

where $\beta_G(y)$ and $\beta_n(y)$ are given by (9) and (16), respectively.

Proof of Theorem 1. By Lemma 5 and by the Markov inequality, for any $0 < \lambda < 1$,

$$\begin{aligned} 0 \leq R(\delta_n, G) - R(\delta_G, G) &\leq a \int_{-\infty}^{\infty} |\beta_G(y)| p(|\beta_n(y) - \beta_G(y)| \geq |\beta_G(y)|) dy \\ &\leq a \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} E|\beta_n(y) - \beta_G(y)|^{\lambda} dy. \end{aligned} \tag{25}$$

By applying the C_r-inequality followed by Lyapunov's inequality and using Fubini's Theorem, we obtain

$$\begin{aligned} E|\beta_n(y) - \beta_G(y)|^{\lambda} \leq c_1 \Bigg\{ & E\left| \int_{-\infty}^{\infty} \left(f_n^{(2)}(y-x) - f_G^{(2)}(y-x) \right) dF_\varepsilon(x) \right|^{\lambda} \\ & + E\left| \int_{-\infty}^{\infty} Q(y-x) \left(f_n^{(1)}(y-x) - f_G^{(1)}(y-x) \right) dF_\varepsilon(x) \right|^{\lambda} \\ & + E\left| \int_{-\infty}^{\infty} \phi(y-x)(f_n(y-x) - f_G(y-x)) dF_\varepsilon(x) \right|^{\lambda} \\ & - \mu E\left| \int_{-\infty}^{\infty} Q(y-x)(p_n(y-x) - p_G(y-x)) dF_\varepsilon(x) \right|^{\lambda} \Bigg\} \\ & \leq c_1 \left[\int_{-\infty}^{\infty} E\left| f_n^{(2)}(y-x) - f_G^{(2)}(y-x) \right| dF_\varepsilon(x) \right]^{\lambda} \\ & + c_1 \left[\int_{-\infty}^{\infty} |Q(y-x)| E\left| f_n^{(1)}(y-x) - f_G^{(1)}(y-x) \right| dF_\varepsilon(x) \right]^{\lambda} \\ & + c_1 \left[\int_{-\infty}^{\infty} |\phi(y-x)| E|f_n(y-x) - f_G(y-x)| dF_\varepsilon(x) \right]^{\lambda} \\ & + c_2 \left[\int_{-\infty}^{\infty} |Q(y-x)| E|p_n(y-x) - p_G(y-x)| dF_\varepsilon(x) \right]^{\lambda}. \end{aligned} \tag{26}$$

Furthermore, by (25) and (26), we obtain

$$\begin{aligned} & \sup_{G \in \vartheta} (R(\delta_n, G) - R(\delta_G, G)) \leq \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \sup_{G \in \vartheta} E|\beta_n(y) - \beta_G(y)|^{\lambda} dy \\ & \leq c_1 \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} \sup_{G \in \vartheta} E\left| f_n^{(2)}(y-x) - f_G^{(2)}(y-x) \right| dF_\varepsilon(x) \right]^{\lambda} dy \\ & + c_1 \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} |Q(y-x)| \sup_{G \in \vartheta} E\left| f_n^{(1)}(y-x) - f_G^{(1)}(y-x) \right| dF_\varepsilon(x) \right]^{\lambda} dy \\ & + c_1 \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} |\phi(y-x)| \sup_{G \in \vartheta} E|f_n(y-x) - f_G(y-x)| dF_\varepsilon(x) \right]^{\lambda} dy \\ & + c_2 \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} |Q(y-x)| \sup_{G \in \vartheta} E|p_n(y-x) - p_G(y-x)| dF_\varepsilon(x) \right]^{\lambda} dy \\ & = A_n + B_n + C_n + D_n \end{aligned} \tag{27}$$

From Lemmas 1 and 2, by the assumption conditions of Theorem 1, we have

$$A_n \leq c_3(\log n)^{-\lambda(m-2)/\beta} \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} dy \leq c(\log n)^{-\lambda(m-2)/\beta}, \tag{28}$$

$$B_n \leq c_4(\log n)^{-\lambda(m-1)/\beta} \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \left[\int_{-\infty}^{\infty} |Q(y-x)| dF_\varepsilon(x)\right]^{\lambda} dy$$
$$\leq c(\log n)^{-\lambda(m-1)/\beta}, \tag{29}$$

$$C_n \leq c_5(\log n)^{-\lambda m/\beta} \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \left[\int_{-\infty}^{\infty} |\phi(y-x)| dF_\varepsilon(x)\right]^{\lambda} dy \leq c(\log n)^{-\lambda m/\beta}, \tag{30}$$

$$D_n \leq c_6(\log n)^{-\lambda(m+1)/\beta} \int_{-\infty}^{\infty} |\beta_G(y)|^{1-\lambda} \left[\int_{-\infty}^{\infty} |Q(y-x)| dF_\varepsilon(x)\right]^{\lambda} dy$$
$$\leq c(\log n)^{-\lambda(m+1)/\beta} \tag{31}$$

Substituted (27) by (28) to (31), we obtain

$$\sup_{G \in \vartheta} (R(\delta_n, G) - R(\delta_G, G)) = O\Big((\log n)^{-\lambda(m-2)/\beta}\Big)$$

So the proof of Theorem 1 was completed. □

Proof of Theorem 2. The proof is similar to that of Theorem 1 above, except that we let Lemmas 3 and 4 in the place of Lemmas 1 and 2 in the proof of Theorem 1, respectively. □

5. One-Sided EBT Rule and Its Asymptotic Properties

In this section, we study one-sided EBT for the parameter θ of GED with contaminated data. Considering the problem of testing the hypotheses $H_0' : \theta \leq \theta_0$ versus $H_1' : \theta > \theta_0$, where θ_0 be a known positive constant. Let linear loss function of testing the hypotheses as follows

$$L_0(\theta, d_0') = a(\theta - \theta_0) I(\theta > \theta_0), \quad L_1(\theta, d_1') = a(\theta - \theta_0) I(\theta \leq \theta_0), \tag{32}$$

where a is a positive constant and $d = \{d_0', d_1'\}$ is the action space, d_0' indicates accepting H_0', d_1' indicates rejecting H_0', $I_{[A]}$ is the indicator of the set A.

The same as above, assume that X is not directly observable and because of measurement error or the nature of environment, we can only observe $Y = X + \varepsilon$, where the error variable ε has a known distribution F_ε on $(-\infty, \infty)$. It is assumed that ε and (X, θ) are independent. It is assumed that the parameter θ is a realization of a random variable having an unknown prior distribution $G(\theta)$ over the natural parameter space Θ. Let randomized decision rule for the preceding testing problem is $\delta^*(y) = P\{accepting\, H_0' \mid Y = y\}$. For one-sided test, we assume that $G(\theta)$ belongs to the following class of distributions

$$\vartheta^* = \left\{ G : G \text{ is a prior on } \Omega \text{ such that } \sup_x \left| f_X^{(m)}(x) \right| \leq B \right\}, \tag{33}$$

where $f_X^{(m)}(x)$ denotes the m order derivative of $f_X(x) = \int_\Theta f_{X|\theta}(x) dG(\theta)$, which is the marginal density of X , and $m \geq 1$ is an integer, $B > 0$ is a constant.

Let $R(\delta^*, G)$ denotes the Bayes risk of the test δ^* when G is the prior distribution, it can be expressed as

$$R(\delta^*(y), G(\theta)) = \int_\Theta \int_\Omega [L_0(\theta, d_0)\delta^*(y) + L_1(\theta, d_1)(1 - \delta^*(y))] f_{Y|\theta}(y) dy dG(\theta)$$
$$= a \int_\Omega \beta_G^*(y) \delta^*(y) dy + C_G, \tag{34}$$

where

$$C_G = \int_\Theta L_1(\theta, d_1) dG(\theta), \beta^*_G(y) = \int_\Theta (\theta - \theta_0) f_{Y|\theta}(y) dG(\theta). \tag{35}$$

From (35), we obtain

$$\begin{aligned} \beta^*_G(y) = {} & \mu \int_{-\infty}^{\infty} p_X(y-x) dF_\varepsilon(x) - \int_{-\infty}^{\infty} \tau(y-x) f_X(y-x) dF_\varepsilon(x) \\ & - \int_{-\infty}^{\infty} f_X^{(1)}(y-x) dF_\varepsilon(x), \end{aligned} \tag{36}$$

where $\tau(y-x) = \mu(y-x) + \theta_0$.

Therefore, the Bayes test δ^*_G can be presented as

$$\delta^*_G(y) = \begin{cases} 1, \text{ if } \beta^*_G(y) \le 0 \\ 0, \text{ if } \beta^*_G(y) > 0 \end{cases}. \tag{37}$$

The Bayes risk of $\delta^*_G(y)$ is

$$R(\delta^*_G, G) = \inf_{\delta^*} R(\delta^*, G) = a \int_\Omega \beta^*_G(y) \delta^*_G(y) dy + C_G. \tag{38}$$

Thus, we defined the estimation of $\beta^*_G(y)$ as

$$\begin{aligned} \beta^*_n(y) = {} & \mu \int_{-\infty}^{\infty} p_n(y-x) dF_\varepsilon(x) - \int_{-\infty}^{\infty} \tau(y-x) f_n(y-x) dF_\varepsilon(x) \\ & - \int_{-\infty}^{\infty} f_n^{(1)}(y-x) dF_\varepsilon(x). \end{aligned} \tag{39}$$

Furthermore, one-sided empirical Byes test rule is defined by

$$\delta^*_n(y) = \begin{cases} 1, & \text{if } \beta^*_n(y) \le 0 \\ 0, & \text{elsewhere.} \end{cases} \tag{40}$$

Then, the overall Bayes risks of $\delta^*_n(y)$ would be

$$R(\delta^*_n, G) = a \int_\Omega \beta^*_n(y) E[\delta^*_n(y)] dy + C_G. \tag{41}$$

It is necessary state that Lemmas 1–4 still hold over a class of new prior distributions ϑ^* for one-sided EB decision problem. So by Lemmas 1–5, Theorem below establish the rates of convergence of the regret $R(\delta^*_n, G) - R(\delta^*_G, G)$, where $R\big(\delta^*_G, G\big)$ and $R(\delta^*_n, G)$ are given by (38) and (41), respectively. For one-sided EBT, we assume that the following conditions are satisfied:

(C1) $\int_\Theta \int_{-\infty}^{\infty} |\theta| f_{X|\theta}(y-x) dF_\varepsilon(x) dG(\theta) < \infty$ uniformly in y, and

(C2) for some $0 < \lambda < 1$, $\int_{-\infty}^{\infty} |\beta^*_G(y)|^{1-\lambda} dy < \infty$ and

$\int_{-\infty}^{\infty} |\beta^*_G(y)|^{1-\lambda} \Big[\int_{-\infty}^{\infty} |\tau(y-x)| dF_\varepsilon(x)\Big]^\lambda dy < \infty$, where $\beta^*_G(y)$ is given by (36).

Theorem 3. *For any $0 < \lambda < 1$, let ϑ^* be defined by (33), suppose that $K(x)$, such that (A1)–(A4) and (B1)–(B2) of Theorem 1 hold and satisfying conditions (C1) and (C2). Then, by choosing the bandwidth $h_n = (4/\gamma)^{1/\beta} (\log n)^{-1/\beta}$, we obtain*

$$\sup_{G \in g^*} (R(\delta^*_n, G) - R(\delta^*_G, G)) = O\Big((\log n)^{-\lambda(m-1)/\beta}\Big). \tag{42}$$

Theorem 4. *For any $0 < \lambda < 1$ and some integer $m \geq 1$, let ϑ^* be defined by (33), suppose that $K(x)$ and $F_\varepsilon(x)$ are such that (A1)–(A4) hold with $\varepsilon = \sigma_0\tilde{\varepsilon}$, and satisfying conditions (C1) and (C2). Then, let $\sigma_0 = O\left(n^{-1/(2m+1)}\right)$ and by choosing the bandwidth $h_n = O\left(n^{-1/(2m+1)}\right)$, we have*

$$\sup_{G\in\vartheta^*} (R(\delta_n^*, G) - R(\delta_G^*, G)) = O\left(n^{-\lambda(m-1)/(2m+1)}\right). \tag{43}$$

Remark 3. *For one-sided EBT, Similar to Theorem 1, the supersmooth distribution of a random variable ε is also considered to Theorem 3, its characteristic function $\varphi_\varepsilon(t)$ satisfies condition (B2). Under all the data are contaminated while the error level can be controlled situation, for model (20), by Lemmas 1–5, Theorem 4, Theorem 4 obtained the rate of convergence of one-sided EBT, this result can also be as good as that of the uncontaminated data case.*

Proof of Theorem 3. By Lemma 5 and by the Markov inequality, for $0 < \lambda < 1$,

$$\begin{aligned} 0 \leq R(\delta_n^*, G) - R(\delta_G^*, G) &\leq a\int_{-\infty}^{\infty} |\beta_G^*(y)| p(|\beta_n^*(y) - \beta_G^*(y)| \geq |\beta_G^*(y)|) dy \\ &\leq a\int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} E|\beta_n^*(y) - \beta_G^*(y)|^\lambda dy. \end{aligned} \tag{44}$$

By applying the C_r-inequality followed by Lyapunov's inequality and using Fubini's Theorem, we obtain

$$\begin{aligned} E|\beta_n^*(y) - \beta_G^*(y)|^\lambda \leq & c_1\left\{ E\left|\int_{-\infty}^{\infty}\left(f_n^{(1)}(y-x) - f_G^{(1)}(y-x)\right) dF_\varepsilon(x)\right|^\lambda \right. \\ & + E\left|\int_{-\infty}^{\infty} \tau(y-x)(f_n(y-x) - f_G(y-x)) dF_\varepsilon(x)\right|^\lambda \\ & \left. -\mu E\left|\int_{-\infty}^{\infty} (p_n(y-x) - p_G(y-x)) dF_\varepsilon(x)\right|^\lambda \right\} \\ \leq & c_1\left[\int_{-\infty}^{\infty} E\left|f_n^{(1)}(y-x) - f_G^{(1)}(y-x)\right| dF_\varepsilon(x)\right]^\lambda \\ & + c_1\left[\int_{-\infty}^{\infty} |\tau(y-x)| E|f_n(y-x) - f_G(y-x)| dF_\varepsilon(x)\right]^\lambda \\ & + c_2\left[\int_{-\infty}^{\infty} E|p_n(y-x) - p_G(y-x)| dF_\varepsilon(x)\right]^\lambda. \end{aligned} \tag{45}$$

Furthermore, by (44) and (45), we obtain

$$\begin{aligned} & \sup_{G\in\vartheta^*} (R(\delta_n^*, G) - R(\delta_G^*, G)) \leq \int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} \sup_{G\in\vartheta^*} E|\beta_n^*(y) - \beta_G^*(y)|^\lambda dy \\ \leq & c_1\int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} \sup_{G\in\vartheta^*} E\left|f_n^{(1)}(y-x) - f_G^{(1)}(y-x)\right| dF_\varepsilon(x)\right]^\lambda dy \\ & + c_1\int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} |\tau(y-x)| \sup_{G\in\vartheta^*} E|f_n(y-x) - f_G(y-x)| dF_\varepsilon(x)\right]^\lambda dy \\ & + c_2\int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} \times \left[\int_{-\infty}^{\infty} \sup_{G\in\vartheta^*} E|p_n(y-x) - p_G(y-x)| dF_\varepsilon(x)\right]^\lambda dy \\ & = A_n + B_n + C_n. \end{aligned} \tag{46}$$

From Lemmas 1 and 2, by the assumption conditions of Theorem 3, we have

$$A_n \leq c_3(\log n)^{-\lambda(m-1)/\beta} \int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} dy \leq c(\log n)^{-\lambda(m-1)/\beta}, \quad (47)$$

$$B_n \leq c_4(\log n)^{-\lambda m/\beta} \int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} \left[\int_{-\infty}^{\infty} |\tau(y-x)| dF_\varepsilon(x)\right]^\lambda dy \leq c(\log n)^{-\lambda m/\beta}, \quad (48)$$

$$C_n \leq c_6(\log n)^{-\lambda(m+1)/\beta} \int_{-\infty}^{\infty} |\beta_G^*(y)|^{1-\lambda} dy \leq c(\log n)^{-\lambda(m+1)/\beta}. \quad (49)$$

Substituted (46) by (47) to (49), we obtain

$$\sup_{G \in \vartheta^*} (R(\delta_n^*, G) - R(\delta_G^*, G)) = O\Big((\log n)^{-\lambda(m-1)/\beta}\Big).$$

So the proof of Theorem 3 was completed. □

Proof of Theorem 4. The proof is similar to that of Theorem 3 above, except that we let Lemmas 3 and 4 in the place of Lemmas 1 and 2 in the proof of Theorem 3, respectively. □

6. An Example Study

In this section, an example study is presented to verify the GED and the prior distribution which satisfies theorems in this paper exist. Suppose that the probability density function of random variable X as follows

$$f(x \mid \theta) = (\theta + 2x)e^{-\left(\theta x + x^2\right)}, \quad (50)$$

where θ is a given parameter, and the sample space is $\Omega = \{x \mid x > 0\}$, the parameter space is $\Theta = \{\theta \mid \theta > 0\}$. Let the prior distribution of parameter θ is

$$g(\theta) = \frac{1}{\Gamma(r)} \theta^{-(r+1)} e^{-1/\theta}, \quad (51)$$

where r is a positive known parameter and θ is a positive unknown parameter. By calculating we obtain

$$f_X(x) = \int_0^\infty f(x \mid \theta) g(\theta) d\theta = -e^{-x^2}\left[\frac{r}{(x+1)^{r+1}} + \frac{2x}{(x+1)^r}\right] = -e^{-x^2} q(x),$$

where $q(x) = \frac{r}{(x+1)^{r+1}} + \frac{2x}{(x+1)^r}$.

Obviously, $f_X^{(m)}(x)$ is existence, and $f_X^{(m)}(x) = \frac{-e^{-x^2} p(x)}{(x+1)^{2^m(r+1)}}$, where $p(x)$ is polynomial with respect to x and $\partial(p(x)) \leq 2^{m-1}(r+1) - 1$. Since $\lim_{x \to \infty} f_X^{(m)}(x) = 0$, $\left|f_X^{(m)}(x)\right|$ is bounded on $x \in \Omega$, where $m \geq 1$ is an integer. Thus, $G(\theta) \in \vartheta$ and $G(\theta) \in \vartheta^*$ are satisfied.

Let the supersmooth error distribution F_ε be N(0,1), it is easy to check that $\varphi_\varepsilon(t)$ satisfies the condition (B2) of Theorem 1. Moreover, we can take $b_n = \sqrt{2}(\log n)^{-1/2}$.

For the two-sided EBT case, we used the following kernel function

$$K(x) = \frac{6144 \sin x}{\pi x^5} + \frac{18320 \cos x}{\pi x^6} - \frac{3225600 \sin x}{\pi x^7} + \cdots - \frac{3360 \times 13!}{\pi x^{16}} \cos x, \quad (52)$$

where $-\infty < x < \infty$, and we choose the Fourier transform of the above kernel is

$$\varphi_K(t) = \left(1 - t^2\right)^4 I_{[|t| \leq 1]}.$$

Then, the deconvolution kernel density estimators (14) are the following kernels, for the type of supersmooth error distribution case,

$$K_{nl}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cos tx - i \sin tx)(it)^l \left(1 - t^4\right)^4 \exp\left(\frac{t^2}{2h_n^2}\right) dt \tag{53}$$

Similar to [20], it is easily shown that assumptions and conditions of Theorems 1 and 2 are satisfied with the above specifications.

For one-sided EBT case, we choose $\varphi_K(t) = \left(1 - t^2\right)^3 I_{[|t| \le 1]}$, then the Fourier transform of $\varphi_K(t)$ is a second order kernel as follows

$$K(x) = \frac{48 \cos x}{\pi x^4}\left(1 - \frac{15}{x^2}\right) - \frac{144 \sin x}{\pi x^5}\left(2 - \frac{15}{x^2}\right), -\infty < x < \infty. \tag{54}$$

The corresponding deconvolution kernel density estimators (14) are kernel in the following

$$K_{nl}(x) = \frac{(-1)^l}{\pi} \int_0^1 t^l (\cos tx)^{1-l} (\sin tx)^l \left(1 - t^2\right)^3 \exp\left(\frac{t^2}{2h_n^2}\right) dt, l = 0, 1. \tag{55}$$

Then, similar to literature [20], it is easily shown that assumptions and conditions of Theorem 3 and 4 are satisfied with the above specifications.

Actually, we can take $\varphi_K(t) = \left(1 - t^2\right)^k I_{[|t| \leqslant 1]}$, when $k \ge 4$, at the same time it may suit for one-sided and two-sided EBT. However, if $k = 3$, the second order kernel (53) only satisfies kernel conditions of Theorems 3 and 4.

7. Conclusions

In this paper, we had studied the empirical Bayes decision for the parameter of a generalized exponential distribution with contaminated data, two-sided and one-sided empirical Bayes test rules were constructed by a deconvolution kernel method, respectively. For the type of the supersmooth error distributions the asymptotically optimal uniformly over a class of prior distributions and uniform rates of convergence of the corresponding regret for the proposed EBT rules are obtained under the conditions of Theorems 1 and 3. Furthermore, we also investigated the supersmooth errors with the error level can be controlled case, $Y = X + \varepsilon$, where $\varepsilon = \sigma_0 \tilde{\varepsilon}$, σ_0 parameterizes the noise level, that is, $\varphi_\varepsilon(t) = \varphi_{\tilde{\varepsilon}}(\sigma_0 t)$, and obtained Theorems 2 and 4. As an example, let the supersmooth error distribution F_ε be N(0,1) , we proved the assumptions and conditions of the main results of this paper are satisfied easily by calculating.

In many practical problems, not all the observations are contaminated, but there may be a partially contaminated case. Suppose that only $100p\%$ $(0 < p < 1)$ of the data are measured with error and the remaining data are error free. We consider the mode $Y = X + \varepsilon$, taking $P(\varepsilon = 0) = 1 - p$ and $P(\varepsilon = \varepsilon^*) = p$, where ε^* is an error variable with distribution F_{ε^*} and the characteristic function φ_{ε^*}. Thus, the characteristic function of ε is denoted by $\varphi_\varepsilon(t) = (1 - p) + p\varphi_{\varepsilon^*}(t)$. In this regard, we can consider extending the current research work to this situation, which is believed to be a very interesting topic.

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