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A Novel Fuzzy Covering Rough Set Model Based on Generalized Overlap Functions and Its Application in MCDM

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Abstract: As nonassociative fuzzy logic connectives, it is important to study fuzzy rough set models using overlap functions that replace the role of t-norms. Overlap functions and t-norms are logical operators with symmetry. Recently, intuitionistic fuzzy rough set and multi-granulation fuzzy rough set models have been proposed based on overlap functions. However, some results (that contain five propositions, two definitions, six examples and a proof) must be improved. In this work, we improved the existing results. Moreover, to extend the existing fuzzy rough sets, a new fuzzy covering rough set model was constructed by using the generalized overlap function, and it was applied to the diagnosis of medical diseases. First, we improve some existing results. Then, in order to overcome the limitations of the fuzzy covering rough set model based on overlap functions, a fuzzy β-covering rough set model based on generalized overlap functions was established. Third, some properties of the fuzzy β-covering rough set model based on generalized overlap functions are discussed. Finally, a multi-criteria decision-making (MCDM) method of the fuzzy β-covering rough set based on generalized overlap functions was proposed. Taking medical disease diagnosis as an example, the comparison with other methods shows that the proposed method is feasible and effective.

Keywords: rough set; overlap function; fuzzy set

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1. Introduction

In order to solve the limitation problem of classical rough sets in processing truth-type data, D. Dubois and H. Prade [1] proposed a fuzzy rough set model in 1992 by using the pair of fuzzy operators of the "minimum" and "maximum". Subsequently, in order to expand the application ability of the "minimum" and "maximum" operators in fuzzy rough sets, N.N. Morsi and M.M. Yakout [2] constructed new fuzzy rough sets using continuous triangular modules and their induced residual implication. This attracted the attention of many scholars. Therefore, all kinds of existing generalized fuzzy rough set models and corresponding theories were also developed around continuous triangular modules [3,4]. In practical applications, fuzzy rough sets have made remarkable achievements in knowledge reduction, fault diagnosis, management decision, etc. For example, J.Q. Wang et al. [5] used three-way fuzzy rough sets in MCDM. Y.J. Lin et al. [6] applied fuzzy rough sets to multi-label learning. The existing fuzzy rough set models are mainly based on fuzzy relations and fuzzy coverings. Since any fuzzy covering can induce the corresponding fuzzy relation, and the fuzzy neighborhood induced by fuzzy covering has good noise reduction function, the study of rough sets based on fuzzy covering has been extensive.

Recently, fuzzy covering rough set theory [3,7,8] was generalized to fuzzy β-covering rough set theory by L.M. Ma [9] by replacing 1 with a parameter β ($\beta \in (0,1]$). Based on Ma's work, more and more researchers were attracted for fuzzy β-covering rough set theory. For example, several types of fuzzy covering-based rough set models were constructed [10,11]: attribute reduction (i.e., feature selection) and decision making were studied under fuzzy β-covering rough sets [4,12], and others [13]. However, the existing research focuses on associative fuzzy operators (triangular modules), which have certain limitations when processing unassociative data. It is necessary to further establish a fuzzy rough set model based on unassociative logical operators.

The overlap function was proposed by H. Bustince et al. [14] in 2009, mainly arising from practical problems, such as image processing and classification. In theory, B. Bedregal et al. [15] studied some important properties of overlap functions, such as migration, idempotence and homogeneity. G.P. Dimuro and B. Bedregal [16] studied the Archimedean property, elimination law and limiting properties of overlap functions. As a nonassociative binary function, an overlap function can overcome the limitation of associativity in continuous triangular modules in practical problems. At present, some scholars have begun to study the fuzzy rough set model based on overlap functions. In particular, X.F. Wen and X.H. Zhang [17] presented four types of fuzzy β-covering rough sets under overlap functions, which extended the existing models. In [18], the authors extended overlap functions and fuzzy β-covering rough sets to the intuitionistic fuzzy (IF) environment [19]. The research ideas of Refs. [17,18] are very important, but we found that several results (including five propositions, two definitions, six examples and a proof) were incorrect after checking the paper carefully. These results were the basis of a rough set model, and the inaccuracy of the results led to the reader's incorrect understanding and application of the model.Moreover, the existing work used overlap functions to established fuzzy rough sets. As a generalization of overlap functions, generalized overlap functions have a stronger application ability. Therefore, the generalization of overlap functions is used to extend fuzzy rough set theory in this paper, which is the main motivation of this paper.

In this paper, some results, including four propositions, two definitions, six examples and a proof for [17,18], were improved. Moreover, the generalized overlap function has a stronger application ability, and if it is combined with a fuzzy β-covering rough set to build a more generalized fuzzy β-covering rough set model, the practical application range of the fuzzy rough set will be expanded. Therefore, on the basis of previous studies, the work in this paper expands the existing model from the perspective of generalized overlap functions and fuzzy β-covering, and it illustrates the feasibility and advantages of the new model through its application in multi-attribute decision making.

The rest of this paper is organized as follows. Section 2 reviews some fundamental definitions about overlap functions, fuzzy sets and fuzzy covering-based rough sets. In Section 3, we improve some results from [17,18]. In Section 4, a fuzzy β-covering rough set model based on generalized overlap functions is established, and its corresponding properties are proposed. Section 5, a decision-making method for the fuzzy β-covering rough set based on generalized overlap functions is proposed. Section 6 summarizes the full text and proposes follow-up research ideas.

2. Basic Definitions

This section recalls some fundamental definitions related to overlap functions, fuzzy sets and fuzzy covering-based rough sets. In the following we suppose that U is a nonempty and finite set called the universe.

2.1. Overlap Functions and Fuzzy Sets

Definition 1 ([20])**.** *A bivariate function $O : [0,1]^2 \to [0,1]$ is called an overlap function if for every $a, b, c \in [0,1]$, the following conditions holds:*

(O1) $O(a,b) = O(b,a)$ (symmetry);

(O2) $O(a,b) = 0 \Leftrightarrow ab = 0$;

(O3) $O(a,b) = 1 \Leftrightarrow ab = 1$;

(O4) $O(a,b) \leq O(a,c)$ if $b \leq c$;

(O5) O is continuous.

Definition 2 ([21]). *Let $O : [0,1]^2 \to [0,1]$ be an overlap function, then, for every $a, b \in [0,1]$, the bivariate function $R_O : [0,1]^2 \to [0,1]$ is defined by*

$$R_O(a,b) = max\{c \in [0,1] \mid O(a,c) \le b\}$$

where R_O is the residual implication induced from the overlap function O.

In [22], a mapping $A : U \to [0,1]$ is defined as a fuzzy set, where $A(x)$ is the degree of membership of $x \in U$ to A. Moreover, the fuzzy power set of U is denoted by $F(U)$.

Some basic operations on $F(U)$ are shown as follows [23]: If $A, B \in F(U)$, then

(1) $A \subseteq B$ iff $A(x) \le B(x)$ for all $x \in U$;
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(3) $A \cup B = \{\langle x, A(x) \vee B(x)\rangle : x \in U\}$;
(4) $A \cap B = \{\langle x, A(x) \wedge B(x)\rangle : x \in U\}$;
(5) $A' = \{\langle x, 1 - A(x) : x \in U\}$.

2.2. Fuzzy Rough Sets Based on Overlap Functions

Ma [9] presented the notion of fuzzy β-covering approximation space.

Definition 3 ([9]). *Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. For each $\beta \in (0,1]$, we call $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$, with $C_i \in F(U)$ $(i = 1, 2, \ldots, m)$, a fuzzy β-covering of U if $(\bigcup_{i=1}^{m} C_i)(x) \ge \beta$ for each $x \in U$. We also call $(U, \widehat{C})$ a fuzzy β-covering approximation space (β-FCAS for short).*

Under a fuzzy β-covering approximation space, Wen and Zhang [17] proposed (multi-granulation) fuzzy β-covering rough sets as follows:

Definition 4 ([17]). *Let $(U, \widetilde{C})$ be a β-FCAS, O be an overlap function and R_O be the residual implication induced from the overlap function O. For any $A \in F(U)$, the lower approximation $\widetilde{C}^{-}_{\widetilde{N}_x^\beta}(A)$ and upper approximation $\widetilde{C}^{+}_{\widetilde{N}_x^\beta}(A)$ of A related to O under $\widetilde{C}$ are denoted as follows: for any $x \in U$,*

$$\widetilde{C}^{-}_{\widetilde{N}_x^\beta}(A)(x) = \inf_{y \in U} R_O(\widetilde{N}_x^\beta(y), A(y)),$$

$$\widetilde{C}^{+}_{\widetilde{N}_x^\beta}(A)(x) = \sup_{y \in U} O(\widetilde{N}_x^\beta(y), A(y)),$$

where $\widetilde{N}_x^\beta = \bigcap\{C \in \widetilde{C} : C(x) \ge \beta\}$.

Definition 5 ([17]). *Let $(U, \widetilde{C}_i)$ $(i = 1, 2, 3, \cdots, m)$ be a β-FCAS, O be an overlap function and R_O be the residual implication induced from the overlap function O. For any $A \in F(U)$, the multi-granulation optimistic fuzzy lower approximation $\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$ and upper approximation $\mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$ of A related to O under $\widetilde{C}$ are denoted as follows: for any $x \in U$,*

$$\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)(x) = \bigvee_{i=1}^{m} \inf_{y \in U} R_O((\widetilde{N}_x^\beta)_i(y), A(y)),$$

$$\mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)(x) = \bigwedge_{i=1}^{m} \sup_{y \in U} O((\widetilde{N}_x^\beta)_i(y), A(y)),$$

where $(\widetilde{N}_x^\beta)_i = \{C \in \widetilde{C}_i : C(x) \ge \beta\}$.

Definition 6 ([17]). *Let $(U, \widetilde{C}_i)$ $(i = 1, 2, 3, \cdots, m)$ be a β-FCAS, O be an overlap function and R_O be the residual implication induced from the overlap function O. For any $A \in F(U)$, the* *multi-granulation pessimistic fuzzy lower approximation* $\mathfrak{C}^{-p}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$ *and upper approximation* $\mathfrak{C}^{+p}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$ *of A related to O under* $\widetilde{C}$ *are denoted as follows: for any* $x \in U$,

$$\mathfrak{C}^{-p}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)(x) = \bigwedge_{i=1}^{m} \inf_{y \in U} R_O((\widetilde{N}_x^\beta)_i(y), A(y)),$$

$$\mathfrak{C}^{+p}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)(x) = \bigvee_{i=1}^{m} \sup_{y \in U} O((\widetilde{N}_x^\beta)_i(y), A(y)),$$

where $(\widetilde{N}_x^\beta)_i = \{C \in \widetilde{C}_i : C(x) \geq \beta\}$.

In [18], Wen et al. extended overlap functions and fuzzy rough sets to the intuitionistic fuzzy statement.

3. A Further Study on Fuzzy Rough Sets Based on Overlap Functions in [17,18]

Several fuzzy rough sets based on overlap functions have been established in [17,18]. However, in [17], we found that Propositions 4(i), 4(vii), 5(i), 5(vi) and 5(vii); Definition 11; and Example 3 contained mistakes after checking the paper carefully. We give some corrections of them in this section. In the following, O is the overlap function and R_O is the residual implication of O. Firstly, we show that Proposition 4(i) in [17] is incorrect.

(Proposition 4(i) in [17]). *Let* $(U, \widetilde{C})$ *be an FCAS. For each* $A \in F(U)$, $\widetilde{C}^{-}_{\widetilde{N}_x^\beta}(A) \subseteq A \subseteq \widetilde{C}^{+}_{\widetilde{N}_x^\beta}(A)$.

Example 1. *Let* $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\widetilde{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ *and* $\beta = 0.5$, *where*

$$C_1 = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.8}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.8}{x_6},$$
$$C_2 = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6},$$
$$C_3 = \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4} + \frac{0.3}{x_5} + \frac{0.7}{x_6},$$
$$C_4 = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6},$$
$$C_5 = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6},$$
$$C_6 = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.7}{x_6}.$$

Hence, $\widetilde{C}$ *is a fuzzy* 0.5*-covering. Therefore, we can calculate all* $\widetilde{N}_{x_i}^{0.5}$ $(i = 1, 2, \cdots, 6)$ *as follows:*

$$\widetilde{N}_{x_1}^{0.5} = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.6}{x_6},$$
$$\widetilde{N}_{x_2}^{0.5} = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6},$$
$$\widetilde{N}_{x_3}^{0.5} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.6}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.6}{x_6},$$
$$\widetilde{N}_{x_4}^{0.5} = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6},$$
$$\widetilde{N}_{x_5}^{0.5} = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.6}{x_6},$$
$$\widetilde{N}_{x_6}^{0.5} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.6}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.6}{x_6}.$$

For any $a, b \in [0,1]$, *suppose an overlap function* $O = a^2b^2$ *and its residual implication* $R_O = \begin{cases} 1, & a^2 \leq b; \\ \sqrt{\frac{b}{a^2}}, & a^2 > b. \end{cases}$ *Hence, for* $A = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.3}{x_6}$, *we have*

$$\widetilde{C}^{-}_{\widetilde{N}^{0.5}_x}(A) = \frac{0.8944}{x_1} + \frac{1.0000}{x_2} + \frac{0.9129}{x_3} + \frac{0.7825}{x_4} + \frac{0.8944}{x_5} + \frac{0.9129}{x_6},$$

$$\widetilde{C}^{+}_{\widetilde{N}^{0.5}_x}(A) = \frac{0.1764}{x_1} + \frac{0.0441}{x_2} + \frac{0.1764}{x_3} + \frac{0.0900}{x_4} + \frac{0.1764}{x_5} + \frac{0.1764}{x_6}.$$

Therefore, $\widetilde{C}^{-}_{\widetilde{N}^{\beta}_x}(A) \nsubseteq A$, $A \nsubseteq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A)$ *and* $\widetilde{C}^{-}_{\widetilde{N}^{\beta}_x}(A) \nsubseteq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A)$*, which illustrates that Proposition 4(i) in [17] is incorrect.*

Next, we present a condition under which $\widetilde{C}^{-}_{\widetilde{N}^{\beta}_x}(A) \subseteq A \subseteq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A)$ for each $A \in F(U)$.

Proposition 1. *Let $(U, \widetilde{C})$ be an FCAS. If $O(1,a) \geq a$ ($\forall a \in [0,1]$) and $\widetilde{N}^{\beta}_x(x) = 1$ ($\forall x \in U$), then for each $A \in F(U)$, $\widetilde{C}^{-}_{\widetilde{N}^{\beta}_x}(A) \subseteq A \subseteq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A)$.*

Proof. For any $x \in U$, we have $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A)(x) = \sup_{y \in U} O(\widetilde{N}^{\beta}_x(y), A(y)) \geq O(\widetilde{N}^{\beta}_x(x), A(x)) = O(1, A(x)) \geq A(x)$. Hence, $A \subseteq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A)$.

On the other hand, for any $x \in U$, $R_O(1,x) = max\{z : O(1,z) \leq x\}$, denote $R_O(1,x) = z_0$, then $O(1,z_0) \leq x$. If $z_0 > x$, then $O(1,z_0) \geq z_0 x$, which is contrary to $z_0 > x$. Hence, $z_0 \leq x$, i.e., $R_O(1,x) = max\{z : O(1,z) \leq x\} \leq x$. Therefore, $\widetilde{C}^{-}_{\widetilde{N}^{\beta}_x}(A)(x) = \inf_{y \in U} R_O(\widetilde{N}^{\beta}_x(y), A(y)) \leq R_O(\widetilde{N}^{\beta}_x(x), A(x)) = R_O(1, A(x)) \leq A(x)$, i.e., $\widetilde{C}^{-}_{\widetilde{N}^{\beta}_x}(A) \subseteq A$. □

(Proposition 4(vii) in [17]). *Let $(U, \widetilde{C})$ be an FCAS, $A_i \in F(U)$ $(i \in I)$. If O is continuous and monotonic, then $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A_i)$, $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A_i)$.*

Example 2 (Continued from Example 1)**.** *Let $B = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.8}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5} + \frac{0.7}{x_6}$. Then we have*

$$\widetilde{C}^{-}_{\widetilde{N}^{0.5}_x}(B) = \frac{1.0000}{x_1} + \frac{1.0000}{x_2} + \frac{1.0000}{x_3} + \frac{1.0000}{x_4} + \frac{1.0000}{x_5} + \frac{1.0000}{x_6},$$

$$\widetilde{C}^{+}_{\widetilde{N}^{0.5}_x}(B) = \frac{0.2304}{x_1} + \frac{0.0900}{x_2} + \frac{0.2304}{x_3} + \frac{0.1764}{x_4} + \frac{0.2304}{x_5} + \frac{0.2304}{x_6}.$$

Therefore,

$$\widetilde{C}^{-}_{\widetilde{N}^{0.5}_x}(A \bigcup B) = \frac{1.0000}{x_1} + \frac{1.0000}{x_2} + \frac{1.0000}{x_3} + \frac{1.0000}{x_4} + \frac{1.0000}{x_5} + \frac{1.0000}{x_6},$$

$$\widetilde{C}^{+}_{\widetilde{N}^{0.5}_x}(A \bigcup B) = \frac{0.2304}{x_1} + \frac{0.0900}{x_2} + \frac{0.2304}{x_3} + \frac{0.1764}{x_4} + \frac{0.2304}{x_5} + \frac{0.2304}{x_6},$$

$$\widetilde{C}^{-}_{\widetilde{N}^{0.5}_x}(A \bigcap B) = \frac{0.8944}{x_1} + \frac{1.0000}{x_2} + \frac{0.9129}{x_3} + \frac{0.7825}{x_4} + \frac{0.8944}{x_5} + \frac{0.9129}{x_6},$$

$$\widetilde{C}^{+}_{\widetilde{N}^{0.5}_x}(A \bigcap B) = \frac{0.1764}{x_1} + \frac{0.0441}{x_2} + \frac{0.1764}{x_3} + \frac{0.0441}{x_4} + \frac{0.1764}{x_5} + \frac{0.1764}{x_6}.$$

So we have $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A \bigcup B) \neq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A) \bigcap \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(B)$, $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A \bigcap B) \neq \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A) \bigcap \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(B)$.

Proposition 2. *Let $(U, \widetilde{C})$ be an FCAS, $A_i \in F(U)$ $(i \in I)$. If O is continuous and monotonic, then $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A_i)$, $\widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} \widetilde{C}^{+}_{\widetilde{N}^{\beta}_x}(A_i)$.*

Proof. The proof of Proposition 2 is trivial. □

(Proposition 5(i) in [17]). *Let $\widetilde{C}_i$ $(i = 1, 2, \cdots, m)$ be an FCAS with $\mathfrak{C} = \{\widetilde{C}_1, \cdots, \widetilde{C}_m\}$. For each $A \in F(U)$, $\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}^{\beta}_x)_i}(A) \subseteq A \subseteq \mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}^{\beta}_x)_i}(A)$, where $(\widetilde{N}^{\beta}_x)_i$ is the fuzzy β-neighborhood of $\widetilde{C}_i$.*

Example 3. *Let* $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\widetilde{C}_1 = \{C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, C_{16}\}$, $\widetilde{C}_2 = \{C_{21}, C_{22}, C_{23}, C_{24}, C_{25}, C_{26}\}$ *and* $\beta = 0.5$, *where*

$$C_{11} = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.8}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.8}{x_6}, C_{21} = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.6}{x_6};$$
$$C_{12} = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6}, C_{22} = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6};$$
$$C_{13} = \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4} + \frac{0.3}{x_5} + \frac{0.7}{x_6}, C_{23} = \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.2}{x_5} + \frac{0.4}{x_6};$$
$$C_{14} = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6}, C_{24} = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.6}{x_5} + \frac{0.1}{x_6};$$
$$C_{15} = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, C_{25} = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4} + \frac{0.1}{x_5} + \frac{0.5}{x_6};$$
$$C_{16} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.7}{x_6}, C_{26} = \frac{0.8}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.3}{x_5} + \frac{0.7}{x_6}.$$

Hence, $\widetilde{C}_1$ *and* $\widetilde{C}_2$ *are fuzzy* 0.5*-coverings. For any* $a, b \in [0,1]$, *suppose an overlap function* $O = a^2b^2$ *and its residual implication* $R_O = \begin{cases} 1, & a^2 \le b; \\ \sqrt{\frac{b}{a^2}}, & a^2 > b. \end{cases}$ *Hence, for* $A = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.3}{x_6}$, *we have*

$$\widetilde{C_1}^{-}_{\widetilde{N}_x^{0.5}}(A) = \frac{0.8944}{x_1} + \frac{1.0000}{x_2} + \frac{0.9129}{x_3} + \frac{0.7825}{x_4} + \frac{0.8944}{x_5} + \frac{0.9129}{x_6},$$
$$\widetilde{C_1}^{+}_{\widetilde{N}_x^{0.5}}(A) = \frac{0.1764}{x_1} + \frac{0.0441}{x_2} + \frac{0.1764}{x_3} + \frac{0.0900}{x_4} + \frac{0.1764}{x_5} + \frac{0.1764}{x_6},$$
$$\widetilde{C_2}^{-}_{\widetilde{N}_x^{0.5}}(A) = \frac{0.9129}{x_1} + \frac{1.0000}{x_2} + \frac{1.0000}{x_3} + \frac{1.0000}{x_4} + \frac{0.7454}{x_5} + \frac{1,0000}{x_6},$$
$$\widetilde{C_2}^{+}_{\widetilde{N}_x^{0.5}}(A) = \frac{0.0625}{x_1} + \frac{0.0784}{x_2} + \frac{0.1225}{x_3} + \frac{0.0900}{x_4} + \frac{0.3136}{x_5} + \frac{0.0441}{x_6}.$$

Therefore,

$$\mathfrak{C}^{-o}_{\sum_i^2 (\widetilde{N}_x^\beta)_i}(A) = \frac{0.9129}{x_1} + \frac{1.0000}{x_2} + \frac{1.0000}{x_3} + \frac{1.0000}{x_4} + \frac{0.8944}{x_5} + \frac{1.0000}{x_6},$$
$$\mathfrak{C}^{+o}_{\sum_i^2 (\widetilde{N}_x^\beta)_i}(A) = \frac{0.0625}{x_1} + \frac{0.0441}{x_2} + \frac{0.1225}{x_3} + \frac{0.0900}{x_4} + \frac{0.1764}{x_5} + \frac{0.0441}{x_6}.$$

Hence, $\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A) \nsubseteq A$, $A \nsubseteq \mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$ *and* $\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A) \nsubseteq \mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$.

Proposition 3. *Let* $\widetilde{C}_i$ $(i = 1, 2, \cdots, m)$ *be an FCAS with* $\mathfrak{C} = \{\widetilde{C}_1, \cdots, \widetilde{C}_m\}$. *If* $O(1, a) \ge a$ $(\forall a \in [0,1])$ *and* $\widetilde{N}_x^\beta(x) = 1$ $(\forall x \in U)$, *then for each* $A \in F(U)$, $\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A) \subseteq A \subseteq \mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A)$, *where* $(\widetilde{N}_x^\beta)_i$ *is the fuzzy* β*-neighborhood of* $\widetilde{C}_i$.

Proof. By Proposition 1, the proof is immediate. □

In Proposition 5(vi) and 5(vii) in [17], the authors gave some single inclusion relations. Inspired by the related properties in rough sets and fuzzy rough sets, we found that they could be improved.

(Proposition 5(vi) and 5(vii) in [17]). *Let* $(U, \widetilde{C})$ *be an FCAS,* $A, B \in F(U)$. *The following statements hold:*

(vi) If R_O *is continuous and right monotonic, then* $\mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A \bigcap B) \subseteq \mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A) \bigcap \mathfrak{C}^{-o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(B)$.

(vii) If O is continuous and monotonic, then $\mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A \bigcup B) \supseteq \mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(A) \bigcup \mathfrak{C}^{+o}_{\sum_i^m (\widetilde{N}_x^\beta)_i}(B)$.

Proposition 4. *Let $(U, \widetilde{C})$ be an FCAS, $A, B \in F(U)$. The following statements hold:*

(1) If R_O is continuous and right monotonic, then $\mathfrak{C}^{-o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A \bigcap B) = \mathfrak{C}^{-o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A) \bigcap \mathfrak{C}^{-o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(B)$.

(2) If O is continuous and monotonic, then $\mathfrak{C}^{+o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A \bigcup B) = \mathfrak{C}^{+o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A) \bigcup \mathfrak{C}^{+o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(B)$.

Proof. The proof of Proposition 4 is trivial. □

Finally, we improved other results in [17], which are listed as follows:

(1) **In Definition 11 on page 6**, "$\widetilde{P}^-(A)(x) = \bigwedge_{y \in U}(1 - \widetilde{N}_x^\beta(y) \bigvee A(y))$" should be changed to "$\widetilde{P}^-(A)(x) = \bigwedge_{y \in U}((1 - \widetilde{N}_x^\beta(y)) \bigvee A(y))$".

(2) **In Example 3 on page 8**, "$\widetilde{C}^-_{\widetilde{N}_x^\beta}(A)(x) = \frac{0.04}{x_1} + \frac{0.04}{x_2} + \frac{0.04}{x_3} + \frac{0.04}{x_4} + \frac{0.04}{x_5} + \frac{0.04}{x_6}$" should be changed to "$\widetilde{C}^-_{\widetilde{N}_x^\beta}(A) = \frac{0.04}{x_1} + \frac{0.04}{x_2} + \frac{0.04}{x_3} + \frac{0.04}{x_4} + \frac{0.04}{x_5} + \frac{0.04}{x_6}$".

(3) **In Example 3 on page 8**, "$\widetilde{C}^+_{\widetilde{N}_x^\beta}(A)(x) = \frac{0.71}{x_1} + \frac{0.63}{x_2} + \frac{0.71}{x_3} + \frac{0.71}{x_4} + \frac{0.84}{x_5} + \frac{0.71}{x_6}$" should be changed to "$\widetilde{C}^+_{\widetilde{N}_x^\beta}(A) = \frac{0.71}{x_1} + \frac{0.63}{x_2} + \frac{0.71}{x_3} + \frac{0.71}{x_4} + \frac{0.84}{x_5} + \frac{0.71}{x_6}$".

(4) **In Example 4 on page 10**, "$\widetilde{C}^+_{\widetilde{N}_x^\beta}(A)(x)$" and "$\widetilde{C}^+_{\widetilde{N}_x^\beta}(\widetilde{C}^+_{\widetilde{N}_x^\beta}(A)(x))$" should be changed to "$\widetilde{C}^+_{\widetilde{N}_x^\beta}(A)$" and "$\widetilde{C}^+_{\widetilde{N}_x^\beta}(\widetilde{C}^+_{\widetilde{N}_x^\beta}(A))$". The similar problems in Examples 5 and 6 in [17] are as follows.

(5) **In Example 5 on page 11**, the authors used a fuzzy β-covering with different values for β, $\beta = 0.5$ and $\beta = 0.6$, to calculate $\mathfrak{C}^{-o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A)$ and $\mathfrak{C}^{+o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A)$. This is incorrect, and it can be explained as follows.

In Example 5 in [17], the authors used the fuzzy β-covering $\widetilde{C} = \{C_1, C_2, C_3, C_4, C_5\}$ as follows:

$$C_1 = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.6}{x_6},$$
$$C_2 = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6},$$
$$C_3 = \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.2}{x_5} + \frac{0.4}{x_6},$$
$$C_4 = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.6}{x_5} + \frac{0.1}{x_6},$$
$$C_5 = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4} + \frac{0.1}{x_5} + \frac{0.5}{x_6}.$$

Then the authors state that $\widetilde{C}$ is a fuzzy 0.5-covering and also a fuzzy 0.6-covering. By $\widetilde{N}_x^{0.6}$ and $\widetilde{N}_x^{0.5}$, the authors calculated

$$\mathfrak{C}^{-o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A)(x) = \frac{0.04}{x_1} + \frac{0.04}{x_2} + \frac{0.04}{x_3} + \frac{0.04}{x_4} + \frac{0.04}{x_5} + \frac{0.04}{x_6};$$
$$\mathfrak{C}^{+o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A)(x) = \frac{0.71}{x_1} + \frac{0.63}{x_2} + \frac{0.71}{x_3} + \frac{0.71}{x_4} + \frac{0.84}{x_5} + \frac{0.71}{x_6}.$$

In fact, the process of Example 5 in [17] is incorrect. In Definition 18 in [17], the authors gave $\mathfrak{C}^{-o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A) = \bigvee_{i=1}^{m} \inf_{y \in U} R_O((\widetilde{N}_x^\beta)_i(y), A(y))$ and $\mathfrak{C}^{+o}_{\sum_i^m(\widetilde{N}_x^\beta)_i}(A) = \bigwedge_{i=1}^{m} \sup_{y \in U} O((\widetilde{N}_x^\beta)_i(y), A(y))$, which implies different fuzzy β-coverings $\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_m$ with the same β. However, the authors used the same fuzzy coverings with different values for β, which is contradictory with Definition 18 in [17]. Hence, Example 5 in [17] is incorrect.

Several results in [18] must be improved, we list them as follows:

(1) **In Definition 11 on page 5**, "$sup\, x \in UI(R(x,y), A(x))$" should be changed to "$sup\, x \in UT(R(x,y), A(x))$".

(2) **In the proof of Proposition 1 on page 6**, "$y = (y_1, y_2)$" in $(\widetilde{O}5)$ should be changed to "$y_i = (y_{i1}, y_{i2})$".

(3) **In Example 1 on page 6**, "for $x = (x_1, x_2), y = (y_1, y_2)$" should by changed to "for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$".

(4) **In Example 2 on page 6**, "for $x =< x_1, x_2), y = (y_1, y_2 >$" should by changed to "for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$".

(5) **In Example 2 on page 6**,

$$\text{“}R_{\tilde{o}}(x,y) = \begin{cases} \langle 1,0\rangle, & x_1^3 \le y_1 \text{ and } x_2^3 \le y_2; \\ \langle 1-y_2^{1/3}, y_2^{1/3}\rangle, & x_1^3 \le y_1 \text{ and } x_2^3 < y_2; \\ \langle y_1^{1/3}, 0\rangle, & x_1^3 > y_1 \text{ and } x_2^3 \ge y_2; \\ \langle y_1^{1/3}, y_2^{1/3}\rangle, & x_1^3 > y_1 \text{ and } x_2^3 < y_2.\text{"} \end{cases} \quad \text{should be changed to}$$

$$\text{“}R_{\tilde{o}}(x,y) = \begin{cases} \langle 1,0\rangle, & x_1^3 \le y_1 \text{ and } x_2^3 \le y_2; \\ \langle 1-y_2^{1/3}, y_2^{1/3}\rangle, & x_1^3 \le y_1 \text{ and } x_2^3 > y_2; \\ \langle y_1^{1/3}, 0\rangle, & x_1^3 > y_1 \text{ and } x_2^3 \ge y_2; \\ \langle y_1^{1/3}, y_2^{1/3}\rangle, & x_1^3 > y_1 \text{ and } x_2^3 < y_2.\text{"} \end{cases}$$

(6) **In Example 2 on page 6**,

$$\text{“}R_{\tilde{o}}(x,y) = \begin{cases} \langle 1,0\rangle, & x_1 \le y_1^2 \text{ and } x_2 \le y_2^2; \\ \langle 1-y_2^2, y_2^2\rangle, & x_1 \le y_1^2 \text{ and } x_2 < y_2^2; \\ \langle y_1^2, 0\rangle, & x_1 > y_1^2 \text{ and } x_2 \ge y_2^2; \\ \langle y_1^2, y_2^2\rangle, & x_1 > y_1^2 \text{ and } x_2 < y_2^2.\text{"} \end{cases} \quad \text{should be changed to}$$

$$\text{“}R_{\tilde{o}}(x,y) = \begin{cases} \langle 1,0\rangle, & x_1 \le y_1^2 \text{ and } x_2 \le y_2^2; \\ \langle 1-y_2^2, y_2^2\rangle, & x_1 \le y_1^2 \text{ and } x_2 > y_2^2; \\ \langle y_1^2, 0\rangle, & x_1 > y_1^2 \text{ and } x_2 \ge y_2^2; \\ \langle y_1^2, y_2^2\rangle, & x_1 > y_1^2 \text{ and } x_2 < y_2^2.\text{"} \end{cases}$$

4. A Novel Fuzzy Covering Rough Set Model Based on Generalized Overlap Functions

Based on [17,18], we extended the existing fuzzy β-covering rough sets to a novel fuzzy covering rough set model based on generalized overlap functions. This section mainly takes the generalized overlap function as the bridge, establishes the fuzzy β-covering rough set model based on the generalized overlap function and studies the corresponding properties. Firstly, the notion of generalized overlap function is presented as follows.

Definition 7 ([20]). *A bivariate function $O' : [0,1]^2 \to [0,1]$ is called a generalized overlap function if, for every $a, b, c \in [0,1]$, the following conditions hold:*
(O1) $O'(a,b) = O'(b,a)$ (symmetry);

(O2) if $ab = 0$, then $O'(a,b) = 0$;

(O3) if $ab = 1$, then $O'(a,b) = 1$;

(O4) if $b \le c$, then $O'(a,b) \le O'(a,c)$;

(O5) O' is continuous.

Let $O' : [0,1]^2 \to [0,1]$ be a generalized overlap function, then, for every $a, b \in [0,1]$, the bivariate function $R_{O'} : [0,1]^2 \to [0,1]$ is defined by

$$I_{O'}(a,b) = max\{c \in [0,1] \mid O'(a,c) \leq b\},$$

where $I_{O'}$ is the residual implication induced from the generalized overlap function O'. For example, for any $a, b \in [0,1]$, $O'(a,b) = max\{0, x^2 + y^2 - 1\}$ is a generalized overlap function $I_{O'}(a,b) = \begin{cases} 1, & x^2 \leq y; \\ \sqrt{1 + y - x^2}, & \text{otherwise.} \end{cases}$

Then, the fuzzy β-covering rough set model based on the generalized overlap function is established as follows.

Definition 8. *Let $(U, \widetilde{C})$ be a β-FCAS, O' be a generalized overlap function and $I_{O'}$ be the residual implication induced from the generalized overlap function O'. For any $A \in F(U)$, the lower approximation $\underline{C}(A)$ and upper approximation $\overline{C}(A)$ of A related to O' under $\widetilde{C}$ are denoted as follows: for any $x \in U$,*

$$\overline{C}(A)(x) = sup_{y \in U} O'(\widetilde{N}_x^\beta(y), A(y)),$$

$$\underline{C}(A)(x) = inf_{y \in U} I_{O'}(\widetilde{N}_x^\beta(y), A(y)).$$

Example 4. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\widetilde{C} = \{C_1, C_2, C_3, C_4, C_5\}$ and $\beta = 0.5$, where*

$$C_1 = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.6}{x_6}, C_2 = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6},$$
$$C_3 = \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.2}{x_5} + \frac{0.4}{x_6}, C_4 = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.6}{x_5} + \frac{0.1}{x_6},$$
$$C_5 = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4} + \frac{0.1}{x_5} + \frac{0.5}{x_6}.$$

From Definition 8, we have $\widetilde{C}$ is a fuzzy β-covering. Assume $A = \frac{0.27}{x_1} + \frac{0.54}{x_2} + \frac{0.95}{x_3} + \frac{0.96}{x_4} + \frac{0.15}{x_5} + \frac{0.97}{x_6}$, $O'(x,y) = min\{\sqrt{x}, \sqrt{y}\}$ and $I_{O'}(x,y) = \begin{cases} 1, & x^2 \leq y, \\ y^2, & \text{others.} \end{cases}$ Then,

$$\overline{C}(A) = \frac{0.7746}{x_1} + \frac{0.7348}{x_2} + \frac{0.7071}{x_3} + \frac{0.7071}{x_4} + \frac{0.8944}{x_5} + \frac{0.7071}{x_6},$$

$$\underline{C}(A) = \frac{0.0225}{x_1} + \frac{0.0225}{x_2} + \frac{0.0225}{x_3} + \frac{0.0225}{x_4} + \frac{0.0225}{x_5} + \frac{0.0225}{x_6}.$$

Proposition 5. *Let $\widetilde{C}$ be a fuzzy β-covering, $A \in F(U)$ be a generalized overlap function and $I_{O'}$ be the residual implication induced from the generalized overlap function O'. Then we have*
(1) $\overline{C}(\emptyset) = \emptyset$,
(2) $\underline{C}(U) = U$.

Proof. (1) For any $x \in U$, we have $\overline{C}(\emptyset)(x) = sup_{y \in U} O'(\widetilde{N}_x^\beta(y), \emptyset(y)) = sup_{y \in U} O'(\widetilde{N}_x^\beta(y), 0) = 0$. Therefore, $\overline{C}(\emptyset) = \emptyset$.
(2) For any $x \in U$, we have $I_{O'}(x, 1) = 1$. Then, $\underline{C}(U)(x) = inf_{y \in U} I_{O'}(\widetilde{N}_x^\beta(y), U(y)) = inf_{y \in U} I_{O'}(\widetilde{N}_x^\beta(y), 1) = 1$. That is, $\underline{C}(U) = U$. □

Theorem 1. *Let $\widetilde{C}$ be a fuzzy β-covering, $A \in F(U)$, O' be a generalized overlap function and $I_{O'}$ be the residual implication induced from the generalized overlap function O'. If for any $x \in U$, $O'(1, x) \geq x$ and $\overline{N}^\beta_{\overline{C}(x)} = 1$, then $\underline{C}(A) \subseteq A \subseteq \overline{C}(A)$.*

Proof. For any $x \in U$, $\overline{C}(A)(x) = sup_{y\in U}O'(\widetilde{N}_x^{\beta}(x), A(y)) \geq O'(\widetilde{N}_x^{\beta}(x), A(y)) = O'(1, A(x)) \geq A(X)$. Then, $A \subseteq \overline{C}(A)$. Next we only need to prove that $\underline{C}(A) \subseteq A$. For any $x \in U$, denote $I_{O'}(1,x) = max\{z : O'(1,z) \leq x\} = z_0$, then $O'(1,z_0) \leq x$. Assume $z_0 > x$, then $O'(1,z_0) \geq z_0 > x$, which contradicts $O'(1,z_0) \leq x$. Therefore, $z_0 \leq x$. That is, $I_{O'}(1,x) = max\{z : O'(1,z) \leq x\} \leq x$. Then, by Definition 8, we have $\underline{C}(A)(x) = inf_{y\in U}I_{O'}(\widetilde{N}_x^{\beta}(y), A(y)) \leq I_{O'}(\widetilde{N}_x^{\beta}(x), A(x)) = I_{O'}(1, A(x)) \leq A(x)$, that is, $\underline{C}(A) \subseteq A$. □

Proposition 6. *Let $\widetilde{C}$ be a fuzzy β-covering, $A, B \in F(U)$, O' be a generalized overlap function and $I_{O'}$ be the residual implication induced from the generalized overlap function O'. If $A \subseteq B$, then we have:*
(1) $\overline{C}(A) \subseteq \overline{C}(B)$,
(2) $\underline{C}(A) \subseteq \underline{C}(B)$.

Proof. Since $A \subseteq B$, then for any $x \in U$, we have $A(x) \leq B(x)$. It follows that $\overline{C}(A)(x) = sup_{y\in U}O'(\widetilde{N}_x^{\beta}(y), A(y)) \leq sup_{y\in U}O'(\widetilde{N}_x^{\beta}(y), B(y)) = \overline{C}(B)(x)$. So, $\overline{C}(A) \subseteq \overline{C}(B)$. Similarly, we have $\underline{C}(A) \subseteq \underline{C}(B)$. □

Proposition 7. *Let $\widetilde{C}$ be a fuzzy β-covering, $A, B \in F(U)$, O' be a generalized overlap function and $I_{O'}$ be the residual implication induced from the generalized overlap function O'. If $A \subseteq B$, then we have the following:*
(1) $\overline{C}(A \cup B) = \overline{C}(A) \cup \overline{C}(B)$,
(2) $\underline{C}(A \cap B) = \underline{C}(A) \cap \underline{C}(B)$,
(3) $\overline{C}(A \cap B) \subseteq \overline{C}(A) \cap \overline{C}(B)$,
(4) $\underline{C}(A \cup B) \supseteq \underline{C}(A) \cup \underline{C}(B)$.

Proof. From Definition 8, the statements (1) and (2) are immediate.
(3) From Proposition 6, we have $\overline{C}(A \cap B) \subseteq \overline{C}(A)$ and $\overline{C}(A \cap B) \subseteq \overline{C}(B)$, so $\overline{C}(A \cap B) \subseteq \overline{C}(A) \cap \overline{C}(B)$.
(4) From Proposition 6, we have $\underline{C}(A \cup B) \supseteq \underline{C}(A)$ and $\underline{C}(A \cup B) \supseteq \underline{C}(B)$, so $\underline{C}(A \cup B) \supseteq \underline{C}(A) \cup \underline{C}(B)$. □

5. Decision-Making Methods under the Fuzzy Covering Rough Set Model with Generalized Overlap Functions

5.1. The Background Description of Decision Making

Let the universe $U = \{x_i : i = 1, 2, 3, \cdots, m\}$ be the type set of pneumonia, $V = \{y_j : 1 = 1, 2, 3, \cdots, n\}$ be the set of characteristics of the pneumonia disease (such as cough, vomiting, fever, chest pain and fatigue). Suppose doctors diagnose each case x_i.

Suppose doctors assign a characteristic value $C_j(x_i)$ to the symptoms x_i of each type of pneumonia y_j, where $C_j(x_i) \in [0, 1]$ is the degree that doctors think each of the symptoms x_i is caused by the type of pneumonia y_j. Let $\beta \in (0, 1]$. If there is at least one feature $y_j \in V$ that makes the evaluation value $C_j(x_i)$ not less than β for any $x_i \in U$, it is a fuzzy β-covering information table.

For the introduction of a new case B, the doctor considers that the degree of it belonging to x_i is $B(x_i)$. Then, how can one make a decision about the newly introduced case B by using the fuzzy β-covering information table above, that is, which type of pneumonia does the introduced case belong to?

5.2. The Novel Decision-Making Method

In this subsection, a new decision-making method under a fuzzy β-covering rough set model with generalized overlap functions is proposed as follows (Algorithm 1).

Algorithm 1 A decision-making method of fuzzy β-covering rough sets based on generalized overlap functions

Input: Fuzzy β-covering information table $(U, \widetilde{C}, \beta, B)$.
Output: Which type of pneumonia does sample B belong to.
Step 1: For any $x_i \in U(i = 1, 2, 3, \cdots, m)$, calculate the fuzzy β-covering neighborhood $\widetilde{N}^{\beta}_{x_i}$;
Step 2: Calculate the upper approximation $\overline{C}(B)$ and lower approximation $\underline{C}(B)$ of the β-covering;
Step 3: Give the weight $\zeta \in [0, 1]$;
Step 4: Calculate $S = \zeta\underline{C}(B) + (1 - \zeta)\overline{C}(B)$;
Step 5: Determine the type of pneumonia in case B according to the value of $S(x_i)$.

5.3. Application Examples and Comparative Analysis

In this section, the fuzzy β-covering rough set model and corresponding decision-making method based on generalized overlap functions are used to give the relevant numerical calculation methods and comparative analysis through examples. The experiments were carried out on a personal computer with 64-bit Windows 10, a ADM Ryzen 7 3700X 8-Core Processor 3.59 GHz and 16 GB of memory. The programming language was Matlab r2016a.

Example 5. *An fuzzy β-covering information table $(U, \widetilde{C}, \beta, B)$ is given, where $\beta = 0.5$, $B = \frac{0.2785}{x_1} + \frac{0.5469}{x_2} + \frac{0.9575}{x_3} + \frac{0.9649}{x_4} + \frac{0.1576}{x_5} + \frac{0.9705}{x_6}$ and $(U, \widetilde{C})$ is shown in Table 1.*

Table 1. A fuzzy β-covering.

U	C_1	C_2	C_3	C_4	C_5
x_1	0.7	0.5	0.4	0.3	0.2
x_2	0.6	0.3	0.3	0.7	0.3
x_3	0.4	0.3	0.5	0.8	0.5
x_4	0.5	0.7	0.5	0.2	0.6
x_5	0.1	0.4	0.2	0.6	0.1
x_6	0.6	0.8	0.4	0.1	0.5

Step 1: *For any $x_i \in U(i = 1, 2, \cdots, 6)$, calculate the fuzzy β-covering neighborhoods $\widetilde{N}^{\beta}_{x_i}$, as shown in Table 2.*

Table 2. All fuzzy β-covering neighborhoods $\widetilde{N}^{\beta}_{x_i}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\widetilde{N}^{\beta}_{x_1}$	0,5	0.3	0.2	0.2	0.3	0.2
$\widetilde{N}^{\beta}_{x_2}$	0.3	0.6	0.3	0.3	0.7	0.3
$\widetilde{N}^{\beta}_{x_3}$	0.3	0.4	0.5	0.3	0.8	0.3
$\widetilde{N}^{\beta}_{x_4}$	0.5	0.2	0.2	0.5	0.2	0.5
$\widetilde{N}^{\beta}_{x_5}$	0.1	0.1	0.1	0.1	0.6	0.1
$\widetilde{N}^{\beta}_{x_6}$	0.6	0.1	0.1	0.4	0.1	0.5

Step 2: *Suppose $O'(a, b) = max\{0, x^2 + y^2 - 1\}$ and $I_{O'}(a, b) = \begin{cases} 1, & x^2 \leq y; \\ \sqrt{1 + y - x^2}, & otherwise. \end{cases}$ Then*

$$\overline{C}(B) = \frac{0.3019}{x_1} + \frac{0.0768}{x_2} + \frac{0.1688}{x_3} + \frac{0.1810}{x_4} + \frac{0.5568}{x_5} + \frac{0.1919}{x_6},$$
$$\underline{C}(B) = \frac{1.000}{x_1} + \frac{1.000}{x_2} + \frac{1.000}{x_3} + \frac{1.000}{x_4} + \frac{0.8931}{x_5} + \frac{1.000}{x_6}.$$

Step 3: *Give the weight $\zeta = 0.1$.*

Step 4: *Calculate* $S = \zeta\underline{C}(B) + (1-\zeta)\overline{C}(B) = \frac{0.3717}{x_1} + \frac{0.1691}{x_2} + \frac{0.2501}{x_3} + \frac{0.2629}{x_4} + \frac{0.5904}{x_5} + \frac{0.2727}{x_6}$.

Step 5: *Since* $S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$, *the type of pneumonia in case* B *is* x_5.

On the basis of Example 5, we make the following comparative analysis with the existing methods. First, in the case of $\beta = 0.5$ and $\zeta = 0.1$, we combine the existing fuzzy rough set models (references [9,17,24,25]) with the decision method proposed in this paper to illustrate the advantages of the model proposed in this paper, which is shown in Table 3.

Table 3. First comparative analysis.

Different Fuzzy Rough Set Models for Decision Making	Decision Ordering
Fuzzy rough set models for decision making in [9,24].	$S(x_3) \approx S(x_4) \approx S(x_6) \prec S(x_2) \prec S(x_1) \prec S(x_5)$
Fuzzy rough set models for decision making in [17].	$S(x_3) \approx S(x_4) \approx S(x_6) \prec S(x_2) \prec S(x_1) \prec S(x_5)$
Fuzzy rough set models for decision making in [25].	$S(x_3) \approx S(x_4) \approx S(x_6) \prec S(x_2) \prec S(x_1) \prec S(x_5)$
The proposed models for decision making in this paper.	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$

As can be seen in Table 3, the decision result proposed in this paper is "the pneumonia type of case B is x_5", which is consistent with the decision result corresponding to the model proposed in [9,17,24,25]. Therefore, the decision method proposed in this paper based on the generalized overlap function of the fuzzy β-covering rough set is effective. In the decision-making process corresponding to the model proposed in [9,17,24,25], "$S(x_3) \approx S(x_4) \approx S(x_6)$" makes it impossible for decision makers to accurately distinguish x_3, x_4 and x_6, but the decision values under the model proposed in this paper are not equal, which is good for a decision maker so they can find their difference. From this viewpoint, the used methodology is advantageous in comparison to the current state-of-the-art methods [9,17,24,25]. Ref. [24] proposes a multi-granularity fuzzy covering rough set model, which is a generalized form of Ref. [9]. The rough set model based on a fuzzy relation is proposed in [17]. The fuzzy relation used in this example is the fuzzy relation induced by a fuzzy neighborhood, i.e., $R(x,y) = \widetilde{N}_x^\beta(y)$. Since Refs. [17,25] are all models based on overlap functions, the overlap functions selected in this example are $O(x,y) = min\{\sqrt{x}, \sqrt{y}\}$ and its residual implication $R_O = \begin{cases} 1, & x^2 \leq y; \\ y^2, & x^2 > y. \end{cases}$. It can be seen from the experimental results that the fuzzy β-covering rough set model based on generalized overlap functions has a better application effect in decision making, since the generalized overlap functions weaken the boundary condition and improve the application ability of them.

In order to further illustrate the stability of the model built in this paper, different threshold values were selected for comparative experiments, and the results are shown in Table 4.

As can be seen from Table 4, for different $\zeta \in [0,1]$, decision makers still cannot accurately distinguish x_3, x_4 and x_6 in the decision-making process corresponding to the model proposed in [9,24]. This is because their decision values are all equal, while the decision values in the model proposed in this paper are not equal, showing a good degree of differentiation. When $\zeta = 0.1, 0.2, \cdots, 0.7$, under the model proposed in this paper, the decision result is "the type of pneumonia in case B is x_5", while under the model proposed in [9,24], the decision result is "the type of pneumonia in case B is x_3, x_4, x_5 or x_6", which indicates that the decision method established under the model proposed in this paper has good stability and robustness.

Table 4. Second comparative analysis.

Different $\zeta \in [0,1]$	Decision Making in [9,24]	Decision Making in This Paper
$\zeta = 0.1$	$S(x_3) \approx S(x_4) \approx S(x_6) \prec S(x_2) \prec S(x_1) \prec S(x_5)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.2$	$S(x_3) \approx S(x_4) \approx S(x_6) \prec S(x_2) \prec S(x_1) \prec S(x_5)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.3$	$S(x_3) \approx S(x_4) \approx S(x_6) \prec S(x_2) \prec S(x_1) \prec S(x_5)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.4$	$S(x_2) \prec S(x_1) \prec S(x_5) \prec S(x_3) \approx S(x_4) \approx S(x_6)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.5$	$S(x_2) \prec S(x_1) \prec S(x_5) \prec S(x_3) \approx S(x_4) \approx S(x_6)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.6$	$S(x_2) \prec S(x_1) \prec S(x_5) \prec S(x_3) \approx S(x_4) \approx S(x_6)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.7$	$S(x_2) \prec S(x_1) \prec S(x_5) \prec S(x_3) \approx S(x_4) \approx S(x_6)$	$S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1) \prec S(x_5)$
$\zeta = 0.8$	$S(x_2) \prec S(x_1) \prec S(x_5) \prec S(x_3) \approx S(x_4) \approx S(x_6)$	$S(x_2) \prec S(x_5) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1)$
$\zeta = 0.9$	$S(x_2) \prec S(x_1) \prec S(x_5) \prec S(x_3) \approx S(x_4) \approx S(x_6)$	$S(x_5) \prec S(x_2) \prec S(x_3) \prec S(x_4) \prec S(x_6) \prec S(x_1)$

6. Conclusions

In this paper, the basic properties of fuzzy β-covering rough upper and lower approximation operators based on generalized overlap functions are studied, and a new multi-attribute decision-making method is proposed, which solves the problem that attribute importance degree is difficult to obtain in existing decision-making methods. Its advantages are mainly embodied by the following two aspects:

(1) The model has the important properties of the original fuzzy rough set model. The model is an extended form of the existing rough set model based on fuzzy relations;

(2) This model expands the application ability of fuzzy rough sets in MCDM. The feasibility and advantages of the new decision-making method are illustrated through the comparative analysis of concrete examples.

As the subject of subsequent research, the combination of fuzzy rough sets based on generalized overlap functions will be discussed, and relevant theoretical research results will be applied to knowledge discovery and data mining and other fields.

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