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Applying an Extended β-ϕ-Geraghty Contraction for Solving Coupled Ordinary Differential Equations

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Abstract: In this paper, we introduce a new class of mappings called "generalized β-ϕ-Geraghty contraction-type mappings". We use our new class to formulate and prove some coupled fixed points in the setting of partially ordered metric spaces. Our results generalize and unite several findings known in the literature. We also provide some examples to support and illustrate our theoretical results. Furthermore, we apply our results to discuss the existence and uniqueness of a solution to a coupled ordinary differential equation as an application of our finding.

Keywords: extended β-ϕ-Geraghty contraction mapping; generalized triangular β-admissible mapping; coupled fixed point; ordinary differential equation

MSC: 46T99; 47H10; 46J10; 46J15

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1. Introduction

The natural sciences are completely related to each other, and mathematics plays a crucial role in the development of other sciences. Therefore, mathematicians are always working on developing mechanisms and techniques in order to provide suitable ways to improve other sciences. The fixed-point technique is one of the most powerful methods to help mathematicians provide mechanisms for solving some models in ordinary and partial differential equations prepared by engineers, chemists, or physicists. Also, due to the symmetric property of metric spaces, fixed-point theory is still considered an important tool in developing studies in many fields and various disciplines such as topology, game theory, optimal control, artificial intelligence, logic programming, dynamical systems (and chaos), functional analysis, differential equations, and economics.

The fixed-point technique has been used by some mathematicians to find analytical and numerical solutions to Fredholm integral equations; for example, see [1–5].

It is noteworthy that Banach's contraction theorem (BCT) [6] was the first discovery in mathematics to initiate the study of fixed points (FPs) for mapping under a specific type of contraction condition. Due to the importance of fixed points, mathematicians began to extend the Banach contraction theorem in many directions; some of them extended and generalized the Banach contraction condition in many ways, while others extended metric spaces to new spaces and generalized the Banach contraction theory to new forms. Moreover, others introduced more general contraction conditions to provide new fixed-point results; for example, see [7–18].

In 1973, Geraghty [19] presented an interesting contraction condition called the Geraghty contraction and highlighted some FPs under this condition by generalizing BCT in a complete metric space (CMS). Geraghty's results have been given much attention by several authors; for example, Caballero et al. [20] studied best proximity point theorems for Geraghty contractions, Bilgili et al. [21] generalized the best proximity point under the same conditions of [20], Bae et al. [11] introduced interesting results concerned with FP consequences via the concept of α-Geraghty contraction-type maps in metric spaces, and Gordji et al. [22] discussed an extension of the result of Geraghty in a partially ordered metric space.

Samet et al. [23] introduced the concept of α-admissible mapping and adopted its concept to present some new fixed-point results to give a generalization of a BFT. Recently, Karapínar et al. [9] introduced the concepts of triangular α-admissible mappings and α-ψ-Meir–Keeler contractive mappings and presented some new fixed-point results. Some other authors obtained several results in this direction; see [10,23–25].

The concepts of mixed monotone property and coupled fixed point (CFP) were introduced by Bhaskar and Lakshmikantham [26]. Next, they presented some CFPs for the mapping $\mho : \Lambda \times \Lambda \to \Lambda$ under appropriate contraction conditions. They also supported their main results by providing an application for partial differential equations. Subsequently, some authors have adopted these concepts to give some interesting CFP results; for example, see [27–29].

The aim of this paper is to present the concepts of generalized triangular β-admissible mappings and generalized β-admissible mappings. Next, we study some new CFP results. We also support our results by introducing some examples. Next, we present an application of coupled ordinary differential equations (CODEs).

2. Preliminaries

In this section, we consider some basic definitions and previous results that will help us in obtaining our results.

Let Π be a class of all functions $\pi : [0,\infty) \to [0,1)$ such that the condition below holds:

$$\lim_{i\to\infty} \pi(\tau_i) = 1 \text{ implies } \lim_{i\to\infty} \tau_i = 0.$$

Theorem 1 ([19]). *Let (Λ, ϑ) be a CMS and $\mho : \Lambda \to \Lambda$ be a given mapping. Then $\mho$ has a unique FP provided that the following inequality*

$$\vartheta(\mho\varpi, \mho\sigma) \leq \pi(\vartheta(\varpi,\sigma))\vartheta(\varpi,\sigma)),$$

holds for any $\varpi, \sigma \in \Lambda$, where $\pi \in \Pi$.

The notions of α-admissible and $\alpha - \psi$-contractive mappings were introduced by Samet et al. [23] as follows:

Definition 1 ([23]). *For a non-empty set Λ, let $\mho : \Lambda \to \Lambda$ be a given mapping and $\alpha : \Lambda \times \Lambda \to \mathbb{R}$ be a function. The function $\mho$ is called an α-admissible if*

$$\alpha(\varpi,\sigma) \geq 1 \Rightarrow \alpha(\mho\varpi, \mho\sigma)) \geq 1,$$

holds for all $\sigma \in \Lambda$.

Definition 2 ([23]). *Let (Λ, ϑ) be a metric space. A mapping $\mho : \Lambda \to \Lambda$ is said to be an $\alpha - \psi$-contractive mapping, if there exist two functions $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ and $\pi \in \Pi$ such that*

$$\alpha(\varpi,\sigma)\psi(\vartheta(\mho\varpi, \mho\sigma)) \leq \psi(\vartheta(\varpi,\sigma)),$$

holds for all $\varpi, \sigma \in \Lambda$, where $\psi \in \Phi$ (Φ is defined in the next section).

Samet et al. [23] presented and proved the following interesting theorem:

Theorem 2 ([23]). *Let (Λ, ϑ) be a metric space and $\mho : \Lambda \to \Lambda$ be an $\alpha - \psi$-contractive mapping. Assume the following hypotheses:*

(i) $\mho$ *is α-admissible.*
(ii) *There is $\varpi_0 \in \chi$ so that $\alpha(\varpi_0, \mho\varpi_0) \geq 1$.*
(iii) $\mho$ *is continuous.*

Then $\mho$ has a FP.

The concept of triangular α-admissible for an α-admissible mapping $\Lambda \times \Lambda \to [0, +\infty)$ was given by Karapinar et al. [9] as follows:

$$\text{If } \varpi, \sigma, \rho \in \Lambda \text{ such that } \alpha(\varpi, \sigma) \geq 1,\ \alpha(\sigma, \rho) \geq 1,\ \text{then } \alpha(\varpi, \rho) \geq 1.$$

The concepts of the CFP and mixed monotone property are presented in [26] as follows:

Definition 3 ([26]). *Let Λ be a non-empty set. A pair $(\varpi, \sigma) \in \Lambda \times \Lambda$ is called a CFP of the mapping $\mho : \Lambda \times \Lambda \to \Lambda$ if $\varpi = \mho(\varpi, \sigma)$ and $\sigma = \mho(\sigma, \varpi)$.*

Definition 4 ([26]). *Let $(\Lambda, \leq)$ be a partially ordered set (POS) and $\mho : \Lambda \times \Lambda \to \Lambda$ be a given map. Then $\mho$ has a mixed monotone property if for any $\varpi, \sigma \in \Lambda$,*

$$\varpi_1, \varpi_2 \in \Lambda,\ \varpi_1 \leq \varpi_2 \Rightarrow \mho(\varpi_1, \sigma) \leq \mho(\varpi_2, \sigma),$$

and

$$\sigma_1, \sigma_2 \in \Lambda,\ \sigma_1 \leq \sigma_2 \Rightarrow \mho(\varpi, \sigma_1) \geq \mho(\varpi, \sigma_2).$$

3. Main Results

We begin our work by considering the following definition.

Definition 5. *Let Λ be a non-empty set, $\mho : \Lambda^2 \to \Lambda$ and $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$. We say that $\mho$ is a generalized triangular β-admissible mapping (Shortly ${}^{GT}\beta_{AM}$) if for all $\varpi, \sigma, \varrho, \upsilon, s, t \in \Lambda$,*

(G_i) $\beta((\varpi, \sigma), (\varrho, \upsilon)) \geq 1$ implies $\beta(\mho(\varpi, \sigma), \mho(\varrho, \upsilon)) \geq 1$ and
(G_{ii}) $\beta((\varpi, \sigma), (\varrho, \upsilon)) \geq 1$ and $\beta((\varrho, \upsilon)(s, t)) \geq 1$ imply $\beta((\varpi, \sigma), (s, t)) \geq 1$.

To support the above definition, we give the following examples:

Example 1. *Let $\Lambda = \mathbb{R}$. Define $\mho : \Lambda^2 \to \Lambda$ by $\mho(\varpi, \sigma) = \sqrt[3]{\varpi\sigma}$ and $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$ by $\beta((\varpi, \sigma), (\varrho, \upsilon)) = e^{\varpi\sigma - \varrho\upsilon}$. If $\beta((\varpi, \sigma), (\varrho, \upsilon)) \geq 1$, then $\varpi\sigma \geq \varrho\upsilon$, which implies that $\mho(\varpi, \sigma) = \sqrt[3]{\varpi\sigma} \geq \sqrt[3]{\varrho\upsilon} = \mho(\varrho, \upsilon)$; that is,*

$$\beta(\mho(\varpi, \sigma), \mho(\varrho, \upsilon)) = e^{\sqrt[3]{\varpi\sigma} - \sqrt[3]{\varrho\upsilon}} \geq 1.$$

Also, if $\left\{ \begin{array}{l} \beta((\varpi, \sigma), (\varrho, \upsilon)) \geq 1, \\ \beta((\varrho, \upsilon)(s, t)) \geq 1, \end{array} \right.$ *then* $\left\{ \begin{array}{l} \varpi\sigma - \varrho\upsilon \geq 0, \\ \varrho\upsilon - st \geq 0 \end{array} \right.$*. Hence, $\varpi\sigma - st \geq 0$ and so*

$$\beta((\varpi, \sigma)(s, t)) = e^{\varpi\sigma - st} \geq 1.$$

Therefore, $\mho$ is a ${}^{GT}\beta_{AM}$.

Example 2. *Let $\Lambda = \mathbb{R}$. Define $\mho : \Lambda^2 \to \Lambda$ by $\mho(\varpi, \sigma) = e^{(\varpi\sigma)^7}$ and $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$ by $\beta((\varpi, \sigma), (\varrho, \upsilon)) = \sqrt[5]{\varpi\sigma - \varrho\upsilon} + 1$.*
If $\beta((\varpi, \sigma), (\varrho, \upsilon)) \geq 1$, then $\varpi\sigma \geq \varrho\upsilon$, which leads to $\mho(\varpi, \sigma) = e^{(\varpi\sigma)^7} \geq e^{(\varrho\upsilon)^7} = \mho(\varrho, \upsilon)$; that is,

$$\beta(\mho(\varpi, \sigma), \mho(\varrho, \upsilon)) = \sqrt[5]{e^{(\varpi\sigma)^7} - e^{(\varrho\upsilon)^7}} + 1 \geq 1.$$

Moreover, if $\left\{ \begin{array}{l} \beta((\varpi,\sigma),(\varrho,\upsilon)) \geq 1, \\ \beta((\varrho,\upsilon)(s,t)) \geq 1, \end{array} \right.$ *then* $\left\{ \begin{array}{l} \varpi\sigma - \varrho\upsilon \geq 0, \\ \varrho\upsilon - st \geq 0 \end{array} \right.$ *; that is,* $\varpi\sigma - st \geq 0$*, and hence*

$$\beta((\varpi,\sigma)(s,t)) \geq 1.$$

Therefore , $\mho$ *is a* ${}^{TC}\beta_{AM}$.

Example 3. *Let* $\Lambda = \mathbb{R}$. *Define* $\mho : \Lambda^2 \to \Lambda$ *by* $\mho(\varpi,\sigma) = (\varpi\sigma)^4 + \ln\Big(1 + (\varpi\sigma)^2\Big)$ *and* $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$ *by*

$$\beta((\varpi,\sigma),(\varrho,\upsilon)) = \frac{(\varpi\sigma)^3}{1+(\varpi\sigma)^3} - \frac{(\varrho\upsilon)^3}{1+(\varrho\upsilon)^3} + 1.$$

Then $\mho$ *is a* ${}^{GT}\beta_{AM}$. *In fact, if* $\beta((\varpi,\sigma),(\varrho,\upsilon)) \geq 1$*, then* $\varpi\sigma \geq \varrho\upsilon$ *and hence*

$$\begin{aligned} \mho(\varpi,\sigma) &= (\varpi\sigma)^4 + \ln\Big(1+(\varpi\sigma)^2\Big) \\ &\geq (\varrho\upsilon)^4 + \ln\Big(1+(\varrho\upsilon)^2\Big) = \mho(\varrho,\upsilon). \end{aligned}$$

This implies $\beta(\mho(\varpi,\sigma),\mho(\varrho,\upsilon)) \geq 1$. *Also,*

$$\begin{aligned} & \beta((\varpi,\sigma),(\varrho,\upsilon)) + \beta((\varrho,\upsilon),(s,t)) \\ = {} & \frac{(\varpi\sigma)^3}{1+(\varpi\sigma)^3} - \frac{(\varrho\upsilon)^3}{1+(\varrho\upsilon)^3} + 1 + \frac{(\varrho\upsilon)^3}{1+(\varrho\upsilon)^3} - \frac{(st)^3}{1+(st)^3} + 1 \\ = {} & \frac{(\varpi\sigma)^3}{1+(\varpi\sigma)^3} - \frac{(st)^3}{1+(st)^3} + 2 \\ \leq {} & 2\left(\frac{(\varpi\sigma)^3}{1+(\varpi\sigma)^3} - \frac{(st)^3}{1+(st)^3} + 1\right) = 2\beta((\varpi,\sigma),(s,t)). \end{aligned}$$

Thus,

$$\beta((\varpi,\sigma),(\varrho,\upsilon)) + \beta((\varrho,\upsilon),(s,t)) \leq 2\beta((\varpi,\sigma),(s,t)).$$

Now, if $\left\{ \begin{array}{l} \beta((\varpi,\sigma),(\varrho,\upsilon)) \geq 1, \\ \beta((\varrho,\upsilon)(s,t)) \geq 1, \end{array} \right.$ *, then* $\beta((\varpi,\sigma),(s,t)) \geq 1$.

Example 4. *Let* $\Lambda = \mathbb{R}$. *Define* $\mho : \Lambda^2 \to \Lambda$ *by* $\mho(\varpi,\sigma) = (\varpi\sigma)^3 + \sqrt[7]{\varpi\sigma}$ *and* $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$ *by* $\beta((\varpi,\sigma),(\varrho,\upsilon)) = (\varpi\sigma)^5 - (\varrho\upsilon)^5 + 1$. *Then* $\mho$ *is a* ${}^{GT}\beta_{AM}$.

Example 5. *Let* $\Lambda = \mathbb{R}^+$. *Define* $\mho : \Lambda^2 \to \Lambda$ *by* $\mho(\varpi,\sigma) = (\varpi\sigma)^2 + e^{\varpi\sigma}$ *and* $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$ *by*

$$\beta((\varpi,\sigma),(\varrho,\upsilon)) = \left\{ \begin{array}{ll} 1, & \text{if } (a,b),(u,v) \in [0,1]^2 \times [0,1]^2, \\ 0, & \text{otherwise.} \end{array} \right.$$

Then one can easily show that $\mho$ *is a* ${}^{GT}\beta_{AM}$.

Lemma 1. *Let* $\mho$ *be a* ${}^{GT}\beta_{AM}$. *Assume that there exists* $\varpi_0,\sigma_0 \in \Lambda$ *so that*

$$\beta((\varpi_0,\sigma_0),(\mho(\varpi_0,\sigma_0),\mho(\sigma_0,\varpi_0))) \geq 1.$$

Define two sequences $\{\varpi_i\}$ *and* $\{\sigma_i\}$ *in* Λ *by* $\varpi_i = \mho^i(\varpi_0,\sigma_0)$ *and* $\sigma_i = \mho^i(\sigma_0,\varpi_0)$. *Then*

$$\beta\big((\varpi_j,\sigma_j),(\varpi_i,\sigma_i)\big) \geq 1 \text{ and } \beta\big((\sigma_j,\varpi_j),(\sigma_i,\varpi_i)\big) \geq 1,$$

for $i,j \in \mathbb{N}$ *with* $j < i$.

Proof. Let $\varpi_0, \sigma_0 \in \Lambda$. Then

$$\beta((\varpi_0, \sigma_0), (\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0))) \geq 1.$$

Thus, condition (G_i) implies that

$$\beta((\varpi_1, \sigma_1), (\varpi_2, \sigma_2)) = \beta\Big((\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0)), \Big(\mho^2(\varpi_0, \sigma_0), \mho^2(\sigma_0, \varpi_0)\Big)\Big).$$

Continuing with the same scenario, we conclude that

$$\eta((\varpi_i, \sigma_i), (\varpi_{i+1}, \sigma_{i+1})) \geq 1, \text{ for all } i \geq 0.$$

Similarly, one can prove that

$$\eta((\sigma_i, \varpi_i), (\sigma_{i+1}, \varpi_{i+1})) \geq 1, \text{ for all } i \geq 0.$$

Assume that $j < i$. Since $\beta\big((\varpi_j, \sigma_j), (\varpi_{j+1}, \sigma_{j+1})\big) \geq 1$ and $\beta\big((\varpi_{j+1}, \sigma_{j+1}), (\varpi_{j+2}, \sigma_{j+2})\big) \geq 1$, then (G_{ii}) implies $\beta\big((\varpi_j, \sigma_j), (\varpi_{j+2}, \sigma_{j+2})\big) \geq 1$.
Also, since $\beta\big((\varpi_j, \sigma_j), (\varpi_{j+2}, \sigma_{j+2})\big) \geq 1$ and $\beta\big((\varpi_{j+2}, \sigma_{j+2}), (\varpi_{j+3}, \sigma_{j+3})\big) \geq 1$, then we have $\beta\big((\varpi_j, \sigma_j), (\varpi_{j+3}, \sigma_{j+3})\big) \geq 1$.

Continuing with the same approach, we conclude that $\beta\big((\varpi_j, \sigma_j), (\varpi_i, \sigma_i)\big) \geq 1$. Analogously, we can show that $\beta\big((\sigma_j, \varpi_j), (\sigma_i, \varpi_i)\big) \geq 1$. □

Definition 6. *Let $\mho : \Lambda \times \Lambda \to \Lambda$ and $\beta : \Lambda^2 \times \Lambda^2 \to [0, \infty)$ be two mappings. We say that $\mho$ is a generalized β-admissible if for each $\varpi, \sigma, \varrho, v \in \Lambda$,*

$$\beta((\varpi, \sigma), (\varrho, v)) \geq 1 \Rightarrow \beta((\mho(\varpi, \sigma), \mho(\sigma, \varpi)), (\mho(\varrho, v), \mho(v, \varrho))) \geq 1.$$

From now on, Φ denotes the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that ϕ satisfies the following hypotheses:

(i) $\phi(\tau) = 0$ if $\tau = 0$;
(ii) ϕ is continuous;
(iii) $\phi(a) + \phi(b) \geq \phi(a + b)$
(iii) ϕ is nondecreasing.

Now, we introduce our contraction mapping as follows:

Definition 7. *Let (Λ, ξ) be a partially ordered metric space and $\mho : \Lambda \times \Lambda \to \Lambda$ be a mapping. We say that $\mho$ is an extended β-ϕ-Geraghty contraction (shortly ${}^E\beta$-ϕ_{GC}) if there are two functions $\beta : \Lambda^2 \times \Lambda^2 \to [0, \infty)$ and $\phi \in \Phi$ such that*

$$\beta((\varpi, \sigma), (\varrho, v))\phi(\xi(\mho(\varpi, \sigma), \mho(\varrho, v))) \leq \pi(\phi(\Re(\varpi, \sigma, \varrho, v)))\phi(\Re(\varpi, \sigma, \varrho, v)), \quad (1)$$

for any $\varpi, \sigma, \varrho, v \in \Lambda$ with $\varpi \geq \varrho$ and $\sigma \leq v$, where $\pi \in \Pi$ and

$$\Re(\varpi, \sigma, \varrho, v) = \max\left\{ \begin{array}{c} \xi(\varpi, \varrho), \xi(\sigma, v), \xi(\varpi, \mho(\varpi, \sigma)), \xi(\sigma, \mho(\sigma, \varpi)), \\ \xi(\varrho, \mho(\varrho, v)), \xi(v, \mho(v, \varrho)) \end{array} \right\}.$$

Remark 1. *If $\varpi, \sigma, \varrho, v \in \Lambda$ with $\varpi \neq \sigma \neq \varrho \neq v$, then*

$$\beta((\varpi, \sigma), (\varrho, v))\phi(\xi(\mho(\varpi, \sigma), \mho(\varrho, v))) < \phi(\Re(\varpi, \sigma, \varrho, v)).$$

We have furnished the necessary background to present and prove our first main result.

Theorem 3. *Let $(\Lambda, \leq)$ be a POS and (Λ, ξ) be a complete metric space (CMS). Assume the mapping $\mho : \Lambda \times \Lambda \to \Lambda$ satisfies the following hypotheses:*

(i) *$\mho$ is ${}^{E}\beta$-ϕ_{GC}.*
(ii) *$\mho$ has the mixed monotone property.*
(iii) *$\mho$ is ${}^{GT}\beta_{AM}$.*
(iv) *There are $\varpi_0, \sigma_0 \in \Lambda$ such that*

$$\beta((\varpi_0, \sigma_0), (\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0))) \geq 1 \text{ and } \beta((\sigma_0, \varpi_0), (\mho(\sigma_0, \varpi_0), \mho(\varpi_0, \sigma_0))) \geq 1.$$

(v) *$\mho$ is continuous.*

If there are $\varpi_0, \sigma_0 \in \Lambda$ so that $\varpi_0 \leq \mho(\varpi_0, \sigma_0)$ and $\sigma_0 \geq \mho(\sigma_0, \varpi_0)$, then $\mho$ has a CFP.

Proof. Choose $\varpi_0, \sigma_0 \in \Lambda$ such that $\beta((\varpi_0, \sigma_0), (\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0))) \geq 1$, $\beta((\sigma_0, \varpi_0), (\mho(\sigma_0, \varpi_0), \mho(\varpi_0, \sigma_0))) \geq 1$, $\varpi_0 \leq \mho(\varpi_0, \sigma_0) = \varpi_1$ and $\sigma_0 \geq \mho(\sigma_0, \varpi_0) = \sigma_1$. Now, we choose $\varpi_2, \sigma_2 \in \Lambda$ such that $\mho(\varpi_1, \sigma_1) = \varpi_2$ and $\mho(\sigma_1, \varpi_1) = \sigma_2$. Continuing in this way, we can construct two sequences $\{\varpi_i\}$ and $\{\sigma_i\}$ in Λ such that

$$\varpi_{i+1} = \mho(\varpi_i, \sigma_i) \text{ and } \sigma_{i+1} = \mho(\sigma_i, \varpi_i), \text{ for all } i \geq 0.$$

Next, we use the mathematical induction to prove

$$\varpi_i \leq \varpi_{i+1} \text{ and } \sigma_i \geq \sigma_{i+1}, \text{ for all } i \geq 0. \tag{2}$$

(a) Since $\varpi_0 \leq \mho(\varpi_0, \sigma_0)$, $\sigma_0 \geq \mho(\sigma_0, \varpi_0)$, $\mho(\varpi_0, \sigma_0) = \varpi_1$ and $\mho(\sigma_0, \varpi_0) = \sigma_1$, then $\varpi_0 \leq \varpi_1$ and $\sigma_0 \geq \sigma_1$. Thus (2) holds for $i = 0$.
(b) Assume (2) holds for some fixed i with $i \geq 0$.
(c) Now, we will prove (2) holds for any i. The mixed monotone property of $\mho$ and (b) we imply that

$$\varpi_{i+2} = \mho(\varpi_{i+1}, \sigma_{i+1}) \geq \mho(\varpi_i, \sigma_{i+1}) \geq \mho(\varpi_i, \sigma_i) = \varpi_{i+1},$$

and

$$\sigma_{i+2} = \mho(\sigma_{i+1}, \varpi_{i+1}) \leq \mho(\sigma_i, \varpi_{i+1}) \leq \mho(\sigma_i, \varpi_i) = \sigma_{i+1}.$$

This leads to

$$\varpi_{i+2} \geq \varpi_{i+1} \text{ and } \sigma_{i+2} \leq \sigma_{i+1}.$$

Thus, we conclude that (2) holds for all $i \geq 0$.

If $(\varpi_{i+1}, \sigma_{i+1}) = (\varpi_i, \sigma_i)$ for some $i \geq 0$, then $\varpi_i = \mho(\varpi_i, \sigma_i)$ and $\sigma_i = \mho(\sigma_i, \varpi_i)$; that is, $\mho$ has a CFP. Thus, we may assume that $(\varpi_{i+1}, \sigma_{i+1}) \neq (\varpi_i, \sigma_i)$ for all $i \geq 0$. Since $\mho$ is a ${}^{GT}\beta_{AM}$, then Lemma 1 implies

$$\beta((\varpi_i, \sigma_i), (\varpi_{i+1}, \sigma_{i+1})) \geq 1 \text{ and } \beta((\sigma_i, \varpi_i), (\sigma_{i+1}, \varpi_{i+1})) \geq 1 \text{ for all } i \geq 0. \tag{3}$$

Taking (1) and (3) into account, we derive

$$\begin{aligned} \phi(\xi(\varpi_i, \varpi_{i+1})) &= \phi(\xi(\mho(\varpi_{i-1}, \sigma_{i-1}), \mho(\varpi_i, \sigma_i))) \\ &\leq \beta((\varpi_{i-1}, \sigma_{i-1}), (\varpi_i, \sigma_i))\phi(\xi(\mho(\varpi_{i-1}, \sigma_{i-1}), \mho(\varpi_i, \sigma_i))) \\ &\leq \pi(\phi(\Re(\varpi_{i-1}, \sigma_{i-1}, \varpi_i, \sigma_i)))\phi(\Re(\varpi_{i-1}, \sigma_{i-1}, \varpi_i, \sigma_i)), \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Re(\varpi_{i-1}, \sigma_{i-1}, \varpi_i, \sigma_i) &= \max\left\{ \begin{array}{c} \xi(\varpi_{i-1}, \varpi_i), \xi(\sigma_{i-1}, \sigma_i), \xi(\varpi_{i-1}, \mho(\varpi_{i-1}, \sigma_{i-1})), \\ \xi(\sigma_{i-1}, \mho(\sigma_{i-1}, \varpi_{i-1})), \xi(\varpi_i, \mho(\varpi_i, \sigma_i)), \xi(\sigma_i, \mho(\sigma_i, \varpi_i)) \end{array} \right\} \\ &= \max\{\xi(\varpi_{i-1}, \varpi_i), \xi(\sigma_{i-1}, \sigma_i), \xi(\varpi_i, \varpi_{i+1}), \xi(\sigma_i, \sigma_{i+1})\}. \end{aligned}$$

Again, from (1) and (3), we can write

$$\begin{aligned}\phi(\xi(\sigma_i,\sigma_{i+1})) &= \phi(\xi(\mho(\sigma_{i-1},\varpi_{i-1}),\mho(\sigma_i,\varpi_i))) \\ &\leq \beta((\sigma_{i-1},\varpi_{i-1}),(\sigma_i,\varpi_i))\phi(\xi(\mho(\sigma_{i-1},\varpi_{i-1}),\mho(\sigma_i,\varpi_i))) \\ &\leq \pi(\phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i)))\phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i)),\end{aligned} \tag{5}$$

where

$$\begin{aligned}\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i) &= \max\left\{\max\left\{\begin{array}{c}\xi(\sigma_{i-1},\sigma_i),\xi(\varpi_{i-1},\varpi_i),\xi(\sigma_{i-1},\mho(\sigma_{i-1},\varpi_{i-1})),\\ \xi(\varpi_{i-1},\mho(\varpi_{i-1},\sigma_{i-1})),\xi(\sigma_i,\mho(\sigma_i,\varpi_i)),\xi(\varpi_i,\mho(\varpi_i,\sigma_i))\end{array}\right\}\right\} \\ &= \max\{\xi(\varpi_{i-1},\varpi_i),\xi(\sigma_{i-1},\sigma_i),\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\}.\end{aligned}$$

Setting

$$z_i = \max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\}.$$

From (4) and (5), we obtain

$$\begin{aligned}\phi(z_i) &= \phi(\max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\}) \\ &= \max\{\phi(\xi(\varpi_i,\varpi_{i+1})),\phi(\xi(\sigma_i,\sigma_{i+1}))\} \\ &\leq \max\left\{\begin{array}{c}\pi(\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i)))\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i)),\\ \pi(\phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i)))\phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i))\end{array}\right\} \\ &= \max\left\{\begin{array}{c}\pi(\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i)))\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i)),\\ \pi(\phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i)))\phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i))\end{array}\right\}.\end{aligned} \tag{6}$$

It is clear that the case of

$$\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i) = \Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i) = \max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\}$$

is impossible due to the definition of π. Indeed,

$$\begin{aligned}&\max\{\phi(\xi(\varpi_i,\varpi_{i+1})),\phi(\xi(\sigma_i,\sigma_{i+1}))\} \\ \leq\ &\max\{\pi(\phi(\max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\}))\phi(\max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\})\} \\ <\ &\max\{\phi(\max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\})\} \\ =\ &\max\{\phi(\xi(\varpi_i,\varpi_{i+1})),\phi(\xi(\sigma_i,\sigma_{i+1}))\}.\end{aligned}$$

So, (6) reduces to

$$\begin{aligned}\phi(z_i) &= \phi(\max\{\xi(\varpi_i,\varpi_{i+1}),\xi(\sigma_i,\sigma_{i+1})\}) \\ &= \max\{\phi(\xi(\varpi_i,\varpi_{i+1})),\phi(\xi(\sigma_i,\sigma_{i+1}))\} \\ &\leq \max\{\pi(\phi(\max\{\xi(\varpi_{i-1},\varpi_i),\xi(\sigma_{i-1},\sigma_i)\}))\phi(\max\{\xi(\varpi_{i-1},\varpi_i),\xi(\sigma_{i-1},\sigma_i)\})\} \\ &< \phi(\max\{\xi(\varpi_{i-1},\varpi_i),\xi(\sigma_{i-1},\sigma_i)\}) = \phi(z_{i-1}).\end{aligned}$$

It follows from definition ϕ that $z_i < z_{i-1}$, for all $i \in \mathbb{N}$. Hence $\xi(\varpi_i,\varpi_{i+1}) < \xi(\varpi_{i-1},\varpi_i)$ and $\xi(\sigma_i,\sigma_{i+1}) < \xi(\sigma_{i-1},\sigma_i)$. This implies that $\{\xi(\varpi_i,\varpi_{i+1})\}$ and $\{\xi(\sigma_i,\sigma_{i+1})\}$ are nonincreasing sequences. Accordingly, there exist $\ell,\ell^* \geq 0$ such that $\ell = \lim_{i\to\infty}\xi(\varpi_i,\varpi_{i+1})$ and $\ell^* = \lim_{i\to\infty}\xi(\sigma_i,\sigma_{i+1})$. We shall show that $\ell = 0 = \ell^*$. Suppose the opposite is true; that is, $\ell,\ell^* > 0$. From (4), we get

$$\frac{\phi(\xi(\varpi_i,\varpi_{i+1}))}{\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i))} \leq \pi(\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i))) < 1.$$

Consequently, $\lim_{i\to\infty} \pi(\phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i))) = 1$. Due to the definition of π, we conclude that $\lim_{i\to\infty} \phi(\Re(\varpi_{i-1},\sigma_{i-1},\varpi_i,\sigma_i)) = 0$. Similarly, one can show that $\lim_{i\to\infty} \phi(\Re(\sigma_{i-1},\varpi_{i-1},\sigma_i,\varpi_i)) = 0$. Hence,

$$\ell = \lim_{i\to\infty} \xi(\varpi_i,\varpi_{i+1}) = 0 \text{ and } \ell^* = \lim_{i\to\infty} \xi(\sigma_i,\sigma_{i+1}) = 0. \tag{7}$$

For $j < i$, we have

$$\begin{aligned} \Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i) &= \Re(\sigma_j,\varpi_j,\sigma_i,\varpi_i) \\ &= \max\left\{ \begin{array}{c} \xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i),\xi(\varpi_j,\mho(\varpi_j,\sigma_j)),\xi(\sigma_j,\mho(\sigma_j,\varpi_j)), \\ \xi(\varpi_i,\mho(\varpi_i,\sigma_i)),\xi(\sigma_i,\mho(\sigma_i,\varpi_i)) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i),\xi(\varpi_j,\varpi_{j+1}),\xi(\sigma_j,\sigma_{j+1})), \\ \xi(\varpi_i,\varpi_{i-1}),\xi(\sigma_i,\sigma_{i-1}) \end{array} \right\}. \end{aligned}$$

Allowing $i,j \to +\infty$ in above inequality, we get

$$\lim_{i,j\to\infty} \Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i) = \lim_{i,j\to\infty} \Re(\sigma_j,\varpi_j,\sigma_i,\varpi_i) = \lim_{i,j\to\infty} \max\{\xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i)\}. \tag{8}$$

Now we shall show that $\{\varpi_i\}$ and $\{\sigma_i\}$ are Cauchy sequences. Suppose the contrary; that is, there exists $\epsilon > 0$ such that

$$\limsup_{i,j\to\infty} \max\{\xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i)\} = \epsilon. \tag{9}$$

The triangular inequality implies

$$\xi(\varpi_j,\varpi_i) \leq \xi(\varpi_j,\varpi_{j+1}) + \xi(\varpi_{j+1},\varpi_{i+1}) + \xi(\varpi_{i+1},\varpi_i), \tag{10}$$

and

$$\xi(\sigma_j,\sigma_i) \leq \xi(\sigma_j,\sigma_{j+1}) + \xi(\sigma_{j+1},\sigma_{i+1}) + \xi(\sigma_{i+1},\sigma_i), \tag{11}$$

From (1) and (10), and the properties of ϕ, we have

$$\begin{aligned} \phi(\xi(\varpi_j,\varpi_i)) &\leq \phi(\xi(\varpi_j,\varpi_{j+1}) + \xi(\varpi_{j+1},\varpi_{i+1}) + \xi(\varpi_{i+1},\varpi_i)) \\ &\leq \phi(\xi(\varpi_j,\varpi_{j+1})) + \phi(\xi(\mho(\varpi_j,\sigma_j),\mho(\varpi_i,\sigma_i))) + \phi(\xi(\varpi_{i+1},\varpi_i)) \\ &\leq \phi(\xi(\varpi_j,\varpi_{j+1})) + \pi(\phi(\Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i)))\phi(\Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i)) \\ &\quad +\phi(\xi(\varpi_{i+1},\varpi_i)). \end{aligned} \tag{12}$$

Similarly, from (11), we get

$$\begin{aligned} \phi(\xi(\sigma_j,\sigma_i)) &\leq \phi(\xi(\sigma_j,\sigma_{j+1}) + \xi(\sigma_{j+1},\sigma_{i+1}) + \xi(\sigma_{i+1},\sigma_i)) \\ &\leq \phi(\xi(\sigma_j,\sigma_{j+1})) + \phi(\xi(\mho(\sigma_j,\varpi_j),\mho(\sigma_i,\varpi_i))) + \phi(\xi(\sigma_{i+1},\sigma_i)) \\ &\leq \phi(\xi(\sigma_j,\sigma_{j+1})) + \pi(\phi(\Re(\sigma_j,\varpi_j,\sigma_i,\varpi_i)))\phi(\Re(\sigma_j,\varpi_j,\sigma_i,\varpi_i)) \\ &\quad +\phi(\xi(\sigma_{i+1},\sigma_i)). \end{aligned} \tag{13}$$

With the help of (8), (7), (14) and (13), we find that

$$\begin{aligned} &\lim_{i,j\to\infty} \max\{\phi(\xi(\varpi_j,\varpi_i)),\phi(\xi(\sigma_j,\sigma_i))\} \\ \leq &\lim_{i,j\to\infty} \pi(\phi(\max\{\xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i)\}))\phi(\max\{\xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i)\}). \end{aligned}$$

From (9), we get

$$\begin{aligned} 1 &\leq \lim_{i,j\to\infty} \pi\big(\phi\big(\max\{\xi(\varpi_j,\varpi_i),\xi(\sigma_j,\sigma_i)\}\big)\big) \\ &= \lim_{i,j\to\infty} \pi\big(\phi\big(\Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i)\big)\big), \end{aligned}$$

and hence $\lim_{i,j\to\infty} \pi\big(\phi\big(\Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i)\big)\big) = 1$. Thus, $\lim_{i,j\to\infty} \Re(\varpi_j,\sigma_j,\varpi_i,\sigma_i) = 0$ and hence $\lim_{i,j\to\infty} \xi(\varpi_j,\varpi_i) = 0$ and $\lim_{i,j\to+\infty} \xi(\sigma_j,\sigma_i) = 0$, a contradiction. Therefore $\{\varpi_i\}$ and $\{\sigma_i\}$ are Cauchy sequences in (Λ,ξ). The completeness of Λ implies that there exist $\varpi^*,\sigma^* \in \Lambda$ such that

$$\lim_{i\to\infty} \varpi_i = \varpi^* \text{ and } \lim_{i\to\infty} \sigma_i = \sigma^*.$$

Since $\varpi_{i+1} = \mho(\varpi_i,\sigma_i)$ and $\sigma_{i+1} = \mho(\sigma_i,\varpi_i)$, then by allowing $i \to +\infty$ and using the continuity of $\mho$, we have

$$\varpi^* = \lim_{i\to\infty} \varpi_i = \lim_{i\to\infty} \mho(\varpi_{i-1},\sigma_{i-1}) = \mho(\varpi^*,\sigma^*),$$

and

$$\sigma^* = \lim_{i\to\infty} \sigma_i = \lim_{i\to\infty} \mho(\sigma_{i-1},\varpi_{i-1}) = \mho(\sigma^*,\varpi^*).$$

Thus, $\mho$ has a CFP. □

In the following result, we replace the continuity of the mapping $\mho$ in Theorem 3 by a suitable condition. For this purpose, we present the following definition:

Definition 8. *Let (Λ,ξ) be a CMS, $\beta : \Lambda^2 \times \Lambda^2 \to \mathbb{R}$ be a function and $\mho : \Lambda \times \Lambda \to \Lambda$ be a mapping. We say that two sequences $\{\varpi_i\}$ and $\{\sigma_i\}$ in Λ are β-regular if $\beta((\varpi_i,\sigma_i),(\varpi_{i+1},\sigma_{i+1})) \geq 1$ and $\beta((\sigma_i,\varpi_i),(\sigma_{i+1},\varpi_{i+1})) \geq 1$ for all i, $\lim_{i\to\infty} \varpi_i = \varpi$ and $\lim_{i\to\infty} \sigma_i = \sigma$ for all $\varpi,\sigma \in \Lambda$, then there exist a subsequence $\{\varpi_{i(k)}\}$ of $\{\varpi_i\}$ and a subsequence $\{\sigma_{i(k)}\}$ of $\{\sigma_i\}$ such that $\beta\Big((\varpi_{i(k)},\sigma_{i(k)}),(\varpi,\sigma)\Big) \geq 1$ and $\beta\Big((\sigma_{i(k)},\varpi_{i(k)}),(\sigma,\varpi)\Big) \geq 1$ for all k.*

Theorem 4. *Let $(\Lambda,\leq)$ be a POS and (Λ,ξ) be a CMS. Assume the mapping $\mho : \Lambda \times \Lambda \to \Lambda$ satisfies the following hypotheses:*

(i) $\mho$ *is an* ${}^{E}\beta$-ϕ_{GC}.
(ii) $\mho$ *has the mixed monotone property.*
(iii) $\mho$ *is* ${}^{GT}\beta_{AM}$.
(iv) *There exist* $\varpi_0,\sigma_0 \in \Lambda$ *such that*

$$\beta((\varpi_0,\sigma_0),(\mho(\varpi_0,\sigma_0),\mho(\sigma_0,\varpi_0))) \geq 1 \text{ and } \beta((\sigma_0,\varpi_0),(\mho(\sigma_0,\varpi_0),\mho(\varpi_0,\sigma_0))) \geq 1.$$

(v) *The two sequences* $\{\varpi_n\}$ *and* $\{\sigma_n\}$ *are* β*-regular.*
If there exist $\varpi_0,\sigma_0 \in \Lambda$ *such that* $\varpi_0 \leq \mho(\varpi_0,\sigma_0)$ *and* $\sigma_0 \geq \mho(\sigma_0,\varpi_0)$*, then* $\mho$ *has a CFP.*

Proof. Following the same proof for Theorem 3, we construct two sequences $\{\varpi_i\}$ and $\{\sigma_i\}$ defined by $\varpi_{i+1} = \mho(\varpi_i,\sigma_i)$ and $\sigma_{i+1} = \mho(\sigma_i,\varpi_i)$ such that

$$\beta((\varpi_i,\sigma_i),(\varpi_{i+1},\sigma_{i+1})) \geq 1 \text{ and } \beta((\sigma_i,\varpi_i),(\sigma_{i+1},\varpi_{i+1})) \geq 1 \text{ for all } i \geq 0, \tag{14}$$

$\sigma_i \to \varpi^* \in \Lambda$ and $\sigma_i \to \sigma^* \in \lambda$. By using (14) and (v), we choose a subsequence $\{\varpi_{i(k)}\}$ of $\{\varpi_i\}$ and a subsequence $\{\sigma_{i(k)}\}$ of $\{\sigma_i\}$ such that $\lim_{k\to\infty} \beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big) \geq 1$ and $\lim_{k\to\infty} \beta\Big((\sigma_{i(k)}, \varpi_{i(k)}), (\sigma^*, \varpi^*)\Big) \geq 1$. For $k \in \mathbb{N}$, (1) implies that

$$\begin{aligned} & \beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big)\phi\Big(\xi\Big(\varpi_{i(k)+1}, \mho(\varpi^*, \sigma^*)\Big)\Big) \\ = \; & \beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big)\phi\Big(\xi\Big(\mho(\varpi_{i(k)}, \sigma_{i(k)}), \mho(\varpi^*, \sigma^*)\Big)\Big) \\ \leq \; & \pi\Big(\phi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big)\Big)\phi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*) &= \max\left\{ \begin{array}{c} \xi(\varpi_{i(k)}, \varpi^*), \xi(\sigma_{i(k)}, \sigma^*), \xi(\varpi_{i(k)}, \mho(\varpi_{i(k)}, \sigma_{i(k)})), \\ \xi(\sigma_{i(k)}, \mho(\sigma_{i(k)}, \varpi_{i(k)})), \xi(\varpi^*, \mho(\varpi^*, \sigma^*)), \xi(\sigma^*, \mho(\sigma^*, \varpi^*)) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \xi(\varpi_{i(k)}, \varpi^*), \xi(\sigma_{i(k)}, \sigma^*), \xi(\varpi_{i(k)}, \varpi_{i(k)+1}), \xi(\sigma_{i(k)}, \sigma_{i(k)+1})) \\ , \xi(\varpi^*, \mho(\varpi^*, \sigma^*)), \xi(\sigma^*, \mho(\sigma^*, \varpi^*)) \end{array} \right\}. \end{aligned}$$

Hence

$$\lim_{k\to\infty} \phi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big) = \phi(\max\{\xi(\varpi^*, \mho(\varpi^*, \sigma^*)), \xi(\sigma^*, \mho(\sigma^*, \varpi^*)\}). \tag{16}$$

Similarly, one can show that

$$\lim_{k\to\infty} \phi\Big(\Re(\sigma_{i(k)}, \varpi_{i(k)}, \sigma^*, \varpi^*)\Big) = \phi(\max\{\xi(\sigma^*, \mho(\sigma^*, \varpi^*), \xi(\varpi^*, \mho(\varpi^*, \sigma^*))\}). \tag{17}$$

From (15), we have

$$\beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big) \frac{\phi\Big(\xi\Big(\varpi_{i(k)+1}, \mho(\varpi^*, \sigma^*)\Big)\Big)}{\phi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big)} \leq \pi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big) < 1.$$

Allowing $k \to \infty$ in the above inequality, we get $\lim_{k\to\infty} \pi\Big(\phi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big)\Big) = 1$. Therefore $\lim_{k\to\infty} \phi\Big(\Re(\varpi_{i(k)}, \sigma_{i(k)}, \varpi^*, \sigma^*)\Big) = 0$. By (16), we have

$$\begin{aligned} & \phi(\max\{\xi(\sigma^*, \mho(\sigma^*, \varpi^*), \xi(\varpi^*, \mho(\varpi^*, \sigma^*)\}) \\ \leq \; & \max\{\xi(\sigma^*, \mho(\sigma^*, \varpi^*), \xi(\varpi^*, \mho(\varpi^*, \sigma^*)\} = 0. \end{aligned}$$

Hence, $\xi(\varpi^*, \mho(\varpi^*, \sigma^*)) = 0$ and $\xi(\sigma^*, \mho(\sigma^*, \varpi^*)) = 0$ and so $\varpi^* = \mho(\varpi^*, \sigma^*)$ and $\sigma^* = \mho(\sigma^*, \varpi^*)$. Thus (ϖ^*, σ^*) is a CFP of $\mho$. □

To ensure the uniqueness of the CFP in Theorems 3 and 4, we need to add the following condition:

(C) If (ϖ, σ) and (ϖ^*, σ^*) are CFPs of $\mho$, then there is $(\ell_1, \ell_2) \in \Lambda \times \Lambda$ so that

$$\beta((\varpi, \sigma), (\ell_1, \ell_2)) \geq 1 \text{ and } \beta((\varpi^*, \sigma^*), (\ell_1, \ell_2)) \geq 1.$$

Theorem 5. *The CFP (ϖ^*, σ^*) of $\mho$ in Theorems 3 and 4 is unique if condition (C) is added to the hypotheses of Theorems 3 and 4.*

Proof. Based on Theorem 3 (resp. Theorem 4), the mapping $\mho$ has a CFP, say $(\varpi^*, \sigma^*) \in \Lambda \times \Lambda$. Let $(s^*, t^*) \in \Lambda \times \Lambda$ be another CFP of $\mho$. Then by (C), there is $(\ell_1, \ell_2) \in \Lambda \times \Lambda$ such that

$$\beta((\varpi^*, \sigma^*), (\ell_1, \ell_2)) \geq 1 \text{ and } \beta((s^*, t^*), (\ell_1, \ell_2)) \geq 1. \tag{18}$$

Since $\mho$ is β-admissible, we get

$$\beta\Big((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)\Big) \geq 1 \text{ and } \beta\Big((s^*,t^*),\mho^i(\ell_1,\ell_2)\Big) \geq 1, \text{ for all } i.$$

Hence, we obtain

$$\begin{aligned}
\xi\Big((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)\Big) &\leq \beta\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\xi\Big(\mho(\varpi^*,\sigma^*),\mho\mho^i(\ell_1,\ell_2)\Big) \\
&\leq \pi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big)\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big) \\
&< \xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big), \text{ for all } i \in \mathbb{N}. \qquad (19)
\end{aligned}$$

Thus, the sequence $\{\xi((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2))\}$ is nonincearsing. Therefore, there exists $\varrho \geq 0$ such that $\lim_{i\to\infty}\xi((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)) = \varrho$. By (19), we get

$$\frac{\xi(\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2))}{\xi((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2))} \leq \pi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big).$$

By allowing $i \to +\infty$ in above inequality, we reach to

$$\lim_{i\to+\infty}\pi\Big(\phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big)\Big) = 1.$$

Thus, $\lim_{i\to\infty}\xi((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)) = 0$. Thus, $\lim_{i\to\infty}\mho^i(\ell_1,\ell_2) = (\varpi^*,\sigma^*)$. Analogously, one can obtain $\lim_{i\to\infty}\mho^i(\ell_1,\ell_2) = (s^*,t^*)$. Thus, we have $(\varpi^*,\sigma^*) = (s^*,t^*)$; that is, the CFP of $\mho$ is unique. □

4. Some Related Results

We dedicate this section to extracting some new results using our results in the previous section. We begin with the following important definition:

Definition 9. *Let $(\Lambda,\leq)$ be a POS and (Λ,ξ) be a metric space. We say that a mapping $\mho : \Lambda\times\Lambda \to \Lambda$ is a β-ϕ-Geraghty contraction if there exist $\pi \in \Pi$ and $\phi \in \Phi$ such that for all $\varpi,\sigma,\varrho,\upsilon \in \Lambda$ with $\varpi \geq \varrho$ and $\sigma \leq \upsilon$, we have*

$$\beta((\varpi,\sigma),(\varrho,\upsilon))\phi(\xi(\mho(\varpi,\sigma),\mho(\varrho,\upsilon))) \leq \pi\left(\phi\left(\frac{\xi(\varpi,\varrho)+\xi(\sigma,\upsilon)}{2}\right)\right)\phi\left(\frac{\xi(\varpi,\varrho)+\xi(\sigma,\upsilon)}{2}\right). \qquad (20)$$

Theorem 6. *Let $(\Lambda,\leq)$ be a POS and (Λ,ξ) be a CMS. Assume the mapping $\mho : \Lambda\times\Lambda \to \Lambda$ satisfies the following conditions:*

(i) $\mho$ is a β-ϕ-Geraghty contraction mapping.
(ii) $\mho$ has the mixed monotone property.
(iii) $\mho$ is ${}^{GT}\beta_{AM}$.
(iv) There exist $\varpi_0,\sigma_0 \in \Lambda$ such that

$$\beta((\varpi_0,\sigma_0),(\mho(\varpi_0,\sigma_0),\mho(\sigma_0,\varpi_0))) \geq 1 \textit{ and } \beta((\sigma_0,\varpi_0),(\mho(\sigma_0,\varpi_0),\mho(\varpi_0,\sigma_0))) \geq 1.$$

(v) $\mho$ is continuous.

If there exist $\varpi_0,\sigma_0 \in \Lambda$ such that $\varpi_0 \leq \mho(\varpi_0,\sigma_0)$ and $\sigma_0 \geq \mho(\sigma_0,\varpi_0)$, then $\mho$ has a CFP.

Proof. Theorem 3 ensures that the sequence $\{\varpi_i\}$ defined by $\varpi_{i+1} = \mho(\varpi_i,\sigma_i)$ is convergent to some $\varpi^* \in \Lambda$ and the sequence $\{\sigma_i\}$ defined by $\sigma_{i+1} = \mho(\sigma_i,\varpi_i)$ is convergent to some $\sigma^* \in \Lambda$. Also, for each i, we have $\beta((\varpi_i,\sigma_i),(\varpi_{i+1},\sigma_{i+1})) \geq 1$ and $\beta((\sigma_i,\varpi_i),(\sigma_{i+1},\varpi_{i+1})) \geq 1$. Then the continuity of $\mho$ implies that $\mho$ has a CFP in $\Lambda\times\Lambda$. □

Theorem 7. *Let $(\Lambda, \leq)$ be a POS and (Λ, ξ) be a CMS. Assume the mapping $\mho : \Lambda \times \Lambda \to \Lambda$ satisfies the following conditions:*

(i) $\mho$ *is a β-ϕ-Geraghty contraction mapping.*
(ii) $\mho$ *has the mixed monotone property.*
(iii) $\mho$ *is* ${}^{GT}\beta_{AM}$.
(iv) *There exist $\varpi_0, \sigma_0 \in \Lambda$ such that*

$$\beta((\varpi_0, \sigma_0), (\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0))) \geq 1 \text{ and } \beta((\sigma_0, \varpi_0), (\mho(\sigma_0, \varpi_0), \mho(\varpi_0, \sigma_0))) \geq 1.$$

(v) *The sequences $\{\varpi_n\}$ and $\{\sigma_n\}$ are β-regular.*
If there exist $\varpi_0, \sigma_0 \in \Lambda$ such that $\varpi_0 \leq \mho(\varpi_0, \sigma_0)$ and $\sigma_0 \geq \mho(\sigma_0, \varpi_0)$, then $\mho$ has a CFP.

Proof. Choose $\varpi_0, \sigma_0 \in \Lambda$ be such that

$$\beta((\varpi_0, \sigma_0), (\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0))) \geq 1 \text{ and } \beta((\sigma_0, \varpi_0), (\mho(\sigma_0, \varpi_0), \mho(\varpi_0, \sigma_0))) \geq 1.$$

Based on the proof of Theorem 3, we can find a sequence $\{\varpi_i\}$ defined by $\varpi_{i+1} = \mho(\varpi_i, \sigma_i)$ convergent to some $\varpi^* \in \Lambda$ and a sequence $\{\sigma_i\}$ and $\sigma_{i+1} = \mho(\sigma_i, \varpi_i)$ convergent to some $\sigma^* \in \Lambda$. Also, for $i \in \mathbb{N}$, we have $\beta((\varpi_i, \sigma_i), (\varpi_{i+1}, \sigma_{i+1})) \geq 1$ and $\beta((\sigma_i, \varpi_i), (\sigma_{i+1}, \varpi_{i+1})) \geq 1$.

From condition (iii), we get $\lim_{i\to\infty} \sup \beta((\varpi_i, \sigma_i), (\varpi^*, \sigma^*)) > 0$ and $\lim_{i\to\infty} \sup \beta(\sigma(\sigma_i, \varpi_i), (\sigma^*, \varpi^*)) > 0$. Thus, there exist a sub-sequence $\{\varpi_{i(k)}\}$ of $\{\varpi_i\}$ and a sub-sequence $\{\sigma_{i(k)}\}$ of $\{\sigma_i\}$ such that

$$\lim_{k\to\infty} \beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big) = p > 0$$

and

$$\lim_{k\to\infty} \beta\Big((\sigma_{i(k)}, \varpi_{i(k)}), (\sigma^*, \varpi^*)\Big) = q > 0.$$

Then we get

$$\begin{aligned}
& \phi\Big(\xi\Big(\Big(\varpi_{i(k)+1}, \sigma_{i(k)+1}\Big), \mho(\varpi^*, \sigma^*)\Big)\Big) \\
= \;& \phi\Big(\xi\Big(\mho\Big(\varpi_{i(k)}, \sigma_{i(k)}\Big), \mho(\varpi^*, \sigma^*)\Big)\Big) \\
\leq \;& \frac{1}{\beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big)} \times \\
& \pi\left(\phi\left(\frac{\xi(\varpi_{i(k)}, \varpi^*) + \xi(\sigma_{i(k)}, \sigma^*)}{2}\right)\right)\phi\left(\frac{\xi(\varpi_{i(k)}, \varpi^*) + \xi(\sigma_{i(k)}, \sigma^*)}{2}\right) \\
< \;& \frac{1}{\beta\Big((\varpi_{i(k)}, \sigma_{i(k)}), (\varpi^*, \sigma^*)\Big)} \phi\left(\frac{\xi(\varpi_{i(k)}, \varpi^*) + \xi(\sigma_{i(k)}, \sigma^*)}{2}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\phi(\xi((\varpi^*, \sigma^*), \mho(\varpi^*, \sigma^*))) &= \lim_{k\to\infty} \phi\Big(\xi\Big(\Big(\varpi_{i(k)+1}, \sigma_{i(k)+1}\Big), \mho(\varpi^*, \sigma^*)\Big)\Big) \\
&\leq \frac{1}{p} \lim_{k\to\infty} \phi\left(\frac{\xi(\varpi_{i(k)}, \varpi^*) + \xi(\sigma_{i(k)}, \sigma^*)}{2}\right) = 0.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}\phi(\xi((\sigma^*,\varpi^*),\mho(\sigma^*,\varpi^*))) &= \lim_{k\to\infty}\phi\Big(\xi\Big(\Big(\sigma_{i(k)+1},\varpi_{i(k)+1}\Big),\mho(\sigma^*,\varpi^*)\Big)\Big)\\ &\leq \frac{1}{q}\lim_{k\to\infty}\phi\left(\frac{\xi(\varpi_{i(k)},\varpi^*)+\xi(\sigma_{i(k)},\sigma^*)}{2}\right)=0.\end{aligned}$$

Therefore, (ϖ^*,σ^*) is a CFP of $\mho$. □

Theorem 8. *The CFP (ϖ^*,σ^*) of $\mho$ in Theorems 6 and 7 is unique if condition (C) is added to the hypotheses of Theorems 6 and 7.*

Proof. From Theorem 6 (resp. Theorem 7), we conclude that the mapping $\mho$ has a CFP, say $(\varpi^*,\sigma^*)\in\Lambda\times\Lambda$. Let $(s^*,t^*)\in\Lambda\times\Lambda$ be another CFP of $\mho$. Then there exists $(\ell_1,\ell_2)\in\Lambda\times\Lambda$ such that

$$\beta((\varpi^*,\sigma^*),(\ell_1,\ell_2))\geq 1 \text{ and } \beta((s^*,t^*),(\ell_1,\ell_2))\geq 1. \tag{21}$$

Since $\mho$ is β-admissible, then for $i\in\mathbb{N}$, we have

$$\beta\Big((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)\Big)\geq 1 \text{ and } \beta\Big((s^*,t^*),\mho^i(\ell_1,\ell_2)\Big)\geq 1.$$

Thus, for $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)\Big)\Big) &\leq \beta\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\phi\Big(\xi\Big(\mho(\varpi^*,\sigma^*),\mho\mho^i(\ell_1,\ell_2)\Big)\Big)\\ &\leq \pi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big)\phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big)\\ &< \phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big).\end{aligned} \tag{22}$$

Therefore, the sequence $\{\phi(\xi((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)))\}$ is nonincearsing, there exists $v\geq 0$ such that $\lim_{i\to\infty}\phi(\xi((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)))=v$. From (22), one can write

$$\frac{\phi(\xi((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)))}{\phi(\xi((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)))}\leq\pi\Big(\phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big)\Big).$$

Letting $i\to+\infty$ in above inequality, we reach

$$\lim_{i\to+\infty}\pi\Big(\phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^{i-1}(\ell_1,\ell_2)\Big)\Big)\Big)=1.$$

Hence,

$$\lim_{i\to\infty}\phi\Big(\xi\Big((\varpi^*,\sigma^*),\mho^i(\ell_1,\ell_2)\Big)\Big)=0,$$

which implies that $\lim_{i\to\infty}\mho^i(\ell_1,\ell_2)=(\varpi^*,\sigma^*)$. Analogously, one can prove that $\lim_{i\to\infty}\mho^i(\ell_1,\ell_2)=(s^*,t^*)$. Thus, we conclude that $(\varpi^*,\sigma^*)=(s^*,t^*)$. □

Corollary 1. *Let $(\Lambda,\preceq)$ be a POS and (Λ,ξ) be a CMS. Assume the mapping $\mho:\Lambda\times\Lambda\to\Lambda$ satisfies the following hypotheses:*

(i) *$\mho$ is extended β-Geraghty contraction.*
(ii) *$\mho$ has a mixed monotone property.*
(iii) *$\mho$ is ${}^{GT}\beta_{AM}$.*
(iv) *There exist $\varpi_0,\sigma_0\in\Lambda$ such that*

$$\beta((\varpi_0,\sigma_0),(\mho(\varpi_0,\sigma_0),\mho(\sigma_0,\varpi_0)))\geq 1 \text{ and } \beta((\sigma_0,\varpi_0),(\mho(\sigma_0,\varpi_0),\mho(\varpi_0,\sigma_0)))\geq 1.$$

(v) $\mho$ *is continuous or the sequences* $\{\varpi_n\}$ *and* $\{\sigma_n\}$ *are* β*-regular.*

If there exist $\varpi_0, \sigma_0 \in \Lambda$ *such that* $\varpi_0 \leq \mho(\varpi_0, \sigma_0)$ *and* $\sigma_0 \geq \mho(\sigma_0, \varpi_0)$*, then* $\mho$ *has a CFP. Moreover, this CFP is unique if the condition (C) is met.*

Proof. The proof follows immediately from Theorems 3–5 if we set $\phi(\tau) = \tau$. □

Corollary 2. *Let* $(\Lambda, \leq)$ *be a POS and* (Λ, ξ) *be a CMS. Assume the mapping* $\mho : \Lambda \times \Lambda \to \Lambda$ *satisfies the the following hypotheses:*

(i) $\mho$ *is* β*-Geraghty contraction.*
(ii) $\mho$ *has a mixed monotone property.*
(iii) $\mho$ *is* ${}^{GT}\beta_{AM}$.
(iv) *There exist* $\varpi_0, \sigma_0 \in \Lambda$ *such that*

$$\beta((\varpi_0, \sigma_0), (\mho(\varpi_0, \sigma_0), \mho(\sigma_0, \varpi_0))) \geq 1 \text{ and } \beta((\sigma_0, \varpi_0), (\mho(\sigma_0, \varpi_0), \mho(\varpi_0, \sigma_0))) \geq 1.$$

(v) $\mho$ *is continuous or the sequences* $\{\varpi_n\}$ *and* $\{\sigma_n\}$ *are* β*-regular.*

If there exist $\varpi_0, \sigma_0 \in \Lambda$ *so that* $\varpi_0 \leq \mho(\varpi_0, \sigma_0)$ *and* $\sigma_0 \geq \mho(\sigma_0, \varpi_0)$*, then* $\mho$ *has a CFP. Moreover, this CFP is unique if the condition* (C) *is met.*

Proof. The proof comes from Theorems 6–8 if we take $\phi(\tau) = \tau$. □

The following example supports Theorem 6.

Example 6. *Let* $\Lambda = [0, \infty)$*. Define* $\xi : \Lambda \times \Lambda \to [0, +\infty)$ *by* $\xi(\varpi, \sigma) = |\varpi - \sigma|$*, and* $\phi : [0, +\infty) \to +[0, +\infty)$ *by* $\phi(\tau) = \frac{\tau}{2}$*. Also, define the mapping* $\mho : \Lambda \times \Lambda \to \Lambda$ *by* $\mho(\varpi, \sigma) = \frac{1}{64}\varpi\sigma$ *and the function* $\beta : \Lambda^2 \times \Lambda^2 \to [0, \infty)$ *by*

$$\beta((\varpi, \sigma), (\varrho, v)) = \begin{cases} \frac{4}{3}, & \text{if } \varpi \geq \sigma \text{ and } \varrho \geq v, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mho$ *is continuous and* ${}^{GT}\beta_{AM}$*. Condition* (iv) *of Theorem 6 is satisfied when* $\varpi_0 = 1$ *and* $\sigma_0 = 0$*. For* $\varpi, \sigma, \varrho, v \in \Lambda$*, we have*

$$\begin{aligned} \xi(\mho(\varpi, \sigma), \mho(\varrho, v)) &= \left| \frac{\varpi\sigma}{64} - \frac{\varrho v}{64} \right| \\ &\leq \frac{1}{64}(|\varpi - \varrho| + |\sigma - v|) = \frac{1}{64}(\xi(\varpi, \varrho) + \xi(\sigma, v)). \end{aligned}$$

Therefore,

$$\begin{aligned} \beta((\varpi, \sigma), (\varrho, v))\phi(\xi(\mho(\varpi, \sigma), \mho(\varrho, v))) &\leq \frac{1}{96}(\xi(\varpi, \varrho) + \xi(\sigma, v)) \\ &\leq \frac{1}{6}\left(\frac{\xi(\varpi, \varrho) + \xi(\sigma, v)}{4}\right)^2 \\ &= \frac{1}{6}\left(\frac{\xi(\varpi, \varrho) + \xi(\sigma, v)}{4}\right)\left(\frac{\xi(\varpi, \varrho) + \xi(\sigma, v)}{4}\right) \\ &= \pi\left(\phi\left(\frac{\xi(\varpi, \varrho) + \xi(\sigma, v)}{2}\right)\right)\phi\left(\frac{\xi(\varpi, \varrho) + \xi(\sigma, v)}{2}\right). \end{aligned}$$

Hence, the condition (20) is satisfied when $\pi(\tau) = \frac{1}{6} < 1$*. Therefore all conditions of Theorem 6 are satisfied and hence* $\mho$ *has a CFP. Here* $(0, 0)$ *is the unique CFP of* $\mho$*.*

5. Solving Coupled Ordinary Differential Equations

This part is dedicated to applying Theorem 6 to discuss the existence of solutions to the following CODEs:

$$\begin{cases} -\frac{d^2\varpi}{d\varsigma} = \Im(\varsigma, \varpi(\varsigma), \sigma(\varsigma)), & \varsigma \in I = [0,1], \\ -\frac{d^2\sigma}{d\varsigma} = \Im(\varsigma, \sigma(\varsigma), \varpi(\varsigma)), & \\ \varpi(0) = \sigma(0) = 0, & \varpi(1) = \sigma(1) = 0, \end{cases} \tag{23}$$

where $\Im : [0,1] \times \mathbb{R} \times \mathbb{R}$ is continuous.

Problem (23) can be written as an integral equation [11] in the form

$$\varpi(\varsigma) = \int_0^1 \aleph(\varsigma, \nu)\Im(v, \varpi(\nu), \sigma(\nu))d\nu, \text{ for all } \varsigma \in I,$$

where R is the Green's function described by

$$\aleph(\varsigma, \nu) = \begin{cases} \varsigma(1-\nu), & 0 \le \varsigma \le \nu \le 1, \\ \nu(1-\varsigma), & 0 \le \nu \le \varsigma \le 1. \end{cases}$$

Let $\Lambda = C(I)$, the space of all continuous functions defined on [0,1].
Define $\xi : \Lambda \times \Lambda \to +\infty$ by

$$\xi(\varpi, \sigma) = \|\varpi - \sigma\|_\infty = \sup_{\varsigma \in I} |\varpi(\varsigma) - \sigma(\varsigma)|, \text{ for all } \varpi, \sigma \in \Lambda.$$

Define a partial order $\le$ on Λ by

$$(\varpi, \sigma) \le (\varrho, v) \Leftrightarrow \varpi \le \varrho \text{ and } v \le \sigma, \text{ for all } \varpi, \sigma, \varrho, v \in \Lambda.$$

Then $(\Lambda, \le)$ is a POS, and the pair (Λ, ξ) is a CMS.

Now, on Problem (23), assume the following conditions:

(P_1) The functions $\Im : [0,1] \times \mathbb{R} \times \mathbb{R}$ and $\aleph : [0,1] \times [0,1] \to \mathbb{R} \to \mathbb{R}$ are continuous such that for all $\varsigma \in I$ and all $\varpi, \sigma, \varpi^*, \sigma^* \in \mathbb{R}$,

$$|\Im(\varsigma, \varpi, \sigma) - \Im(\varsigma, \varpi^*, \sigma^*)| \le \ln\left(\frac{|\varpi - \varpi^*| + |\sigma - \sigma^*|}{2} + 1\right),$$

and $\sup_{\varsigma \in I}\left(\int_0^1 \aleph(\varsigma, \nu)d\nu\right) \le \frac{1}{8}$. Moreover, there exists a function such that $\varkappa : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that $\varkappa((\varpi, \sigma), (\varpi^*, \sigma^*)) \ge 0$ and $\varkappa((\sigma, \varpi), (\sigma^*, \varpi^*)) \ge 0 \ \forall \ \varpi, \sigma, \varpi^*, \sigma^* \in \mathbb{R}$.

(P_2) There are $\varpi_1, \sigma_1 \in \Lambda$ such that for all $\varsigma \in I$

$$\varkappa\left((\varpi_1(\varsigma), \sigma_1(\varsigma)), \int_0^1 \aleph(\varsigma, \nu)\Im(v, \varpi_1(\nu), \sigma_1(\nu))d\nu\right) \ge 0$$

and

$$\varkappa\left((\sigma_1(\varsigma), \varpi_1(\varsigma)), \int_0^1 \aleph(\varsigma, \nu)\Im(v, \sigma_1(\nu), \varpi_1(\nu))d\nu\right) \ge 0.$$

(P_3) For all $\varsigma \in I$ and for $\varpi, \sigma, \varpi^*, \sigma^* \in \Lambda$,

$$\varkappa((\varpi(\varsigma), \sigma(\varsigma)), (\varpi^*(\varsigma), \sigma^*(\varsigma))) \ge 0 \text{ and } \varkappa((\sigma(\varsigma), \varpi(\varsigma)), (\sigma^*(\varsigma), \varpi^*(\varsigma))) \ge 0,$$

implies

$$\varkappa\left(\int_0^1 \aleph(\varsigma,\nu)\Im(v,\varpi(\nu),\sigma(\nu))d\nu, \int_0^1 \aleph(\varsigma,\nu)\Im(v,\varpi^*(\nu),\sigma^*(\nu))d\nu\right) \geq 0$$

and

$$\varkappa\left(\int_0^1 \aleph(\varsigma,\nu)\Im(v,\sigma(\nu),\varpi(\nu))d\nu, \int_0^1 \aleph(\varsigma,\nu)\Im(v,\sigma^*(\nu),\varpi^*(\nu))d\nu\right) \geq 0.$$

(P_4) For any cluster points ϖ and σ of the sequences $\{\varpi_i\}$ and $\{\sigma_i\}$ of points in Λ with $\varkappa((\varpi_i,\sigma_i),(\varpi_{i+1},\sigma_{i+1})) \geq 0$ and $\varkappa((\sigma_i,\varpi_i),(\sigma_{i+1},\varpi_{i+1})) \geq 0$, we have $\liminf_{i\to\infty} \varkappa((\varpi_i,\sigma_i),(\varpi,\sigma)) \geq 0$ and $\liminf_{i\to\infty} \varkappa((\sigma_i,\varpi_i),(\sigma,\varpi)) \geq 0$, respectively.

Now, we present a solution to (23).

Theorem 9. *Under the conditions of* (P_1)–(P_4)*, Problem (23) has at least one solution* $(\widehat{\varpi},\widehat{\sigma}) \in \Lambda \times \Lambda$.

Proof. Define the mapping $\mho : \Lambda \times \Lambda \to \Lambda$ by

$$\mho(\varpi,\sigma)(\varsigma) = \int_0^1 \aleph(\varsigma,\nu)\Im(v,\varpi(\nu),\sigma(\nu))d\nu, \text{ for all } \varsigma \in I.$$

It is known that the CFP of $\mho$ is equivalent to the solution of Problem (23). So we will show that $\mho$ has a CFP.

Now, let $\varpi(\varsigma),\sigma(\varsigma),\varpi^*(\varsigma),\sigma^*(\varsigma) \in \Lambda$ be such that $\varkappa((\varpi(\varsigma),\sigma(\varsigma)),(\varpi^*(\varsigma),\sigma^*(\varsigma))) \geq 0$ for all $\varsigma \in I$. From (P_1), we get

$$\begin{aligned}
\xi(\mho(\varpi,\sigma),\mho(\varpi^*,\sigma^*)) &= |\mho(\varpi,\sigma)(\varsigma) - \mho(\varpi^*,\sigma^*)(\varsigma)| \\
&= \left|\int_0^1 \aleph(\varsigma,\nu)[\Im(v,\varpi(\nu),\sigma(\nu) - \Im(v,\varpi^*(\nu),\sigma^*(\nu)]d\nu\right| \\
&\leq \int_0^1 \aleph(\varsigma,\nu)|\Im(v,\varpi(\nu),\sigma(\nu) - \Im(v,\varpi^*(\nu),\sigma^*(\nu)|d\nu \\
&\leq \int_0^1 \aleph(\varsigma,\nu)\ln\left(\frac{|\varpi(\nu)-\varpi^*(\nu)| + |\sigma(\nu)-\sigma^*(\nu)|}{2} + 1\right)d\nu \\
&\leq \sup_{\varsigma\in I}\int_0^1 \aleph(\varsigma,\nu)d\nu \ln\left(\frac{|\varpi(\nu)-\varpi^*(\nu)| + |\sigma(\nu)-\sigma^*(\nu)|}{2} + 1\right) \\
&\leq \frac{1}{8}\ln\left(\frac{|\varpi(\nu)-\varpi^*(\nu)| + |\sigma(\nu)-\sigma^*(\nu)|}{2} + 1\right) \\
&\leq \ln\left(\frac{|\varpi(\nu)-\varpi^*(\nu)| + |\sigma(\nu)-\sigma^*(\nu)|}{2} + 1\right) \\
&= \ln\left(\frac{\xi(\varpi,\varpi^*) + \xi(\sigma,\sigma^*)}{2} + 1\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \ln(\xi(\mho(\varpi,\sigma),\mho(\varpi^*,\sigma^*))+1) \\
\leq\ & \ln\left(\ln\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}+1\right)+1\right) \\
=\ & \frac{\ln\left(\ln\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}+1\right)+1\right)}{\ln\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}+1\right)} \times \ln\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}+1\right).
\end{aligned}
$$

Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = ln(t+1)$. Then it is clear that $\phi \in \Phi$. Now, define $\pi : [0,+\infty) \to [0,1)$ by

$$
\pi(t) = \begin{cases} 0, & \text{if } t = 0 \\ \frac{\phi(t)}{t}, & \text{if } t \neq 0. \end{cases}
$$

Therefore,

$$
\phi(\xi(\mho(\varpi,\sigma),\mho(\varpi^*,\sigma^*))) \leq \pi\left(\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right)\right)\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right). \tag{24}
$$

Similarly, for all $\varpi,\sigma,\varpi^*,\sigma^* \in \Lambda$ with $\varkappa((\sigma,\varpi),(\sigma^*,\varpi^*)) \geq 0$, we can write

$$
\phi(\xi(\mho(\sigma,\varpi),\mho(\sigma^*,\varpi^*))) \leq \pi\left(\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right)\right)\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right). \tag{25}
$$

Now, define $\beta : \Lambda^2 \times \Lambda^2 \to [0,\infty)$ by

$$
\beta((\sigma,\varpi),(\sigma^*,\varpi^*)) = \begin{cases} 1, & \text{if } \varkappa((\varpi,\sigma),(\varpi^*,\sigma^*)) \geq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

If $\varpi,\sigma,\varpi^*,\sigma^* \in \Lambda$, then (24) implies that

$$
\begin{aligned}
& \beta((\sigma,\varpi),(\sigma^*,\varpi^*))\phi(\xi(\mho(\varpi,\sigma),\mho(\varpi^*,\sigma^*))) \\
\leq\ & \pi\left(\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right)\right)\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right).
\end{aligned}
$$

Similarly, (25) implies that

$$
\begin{aligned}
& \beta((\varpi,\sigma),(\varpi^*,\sigma^*))\phi(\xi(\mho(\sigma,\varpi),\mho(\sigma^*,\varpi^*))) \\
\leq\ & \pi\left(\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right)\right)\phi\left(\frac{\xi(\varpi,\varpi^*)+\xi(\sigma,\sigma^*)}{2}\right).
\end{aligned}
$$

Clearly, $\beta((\varpi,\sigma),(\varpi^*,\sigma^*)) = 1$ and $\beta((\varpi^*,\sigma^*),(\varrho,\upsilon)) = 1$ imply $\beta((\varpi,\sigma),(\varrho,\upsilon)) = 1$, for all $\varpi,\sigma,\varpi^*,\sigma^*,\varrho,\upsilon \in \Lambda$.

If $\beta((\varpi,\sigma),(\varpi^*,\sigma^*)) = 1$ for all $\varpi,\sigma,\varpi^*,\sigma^* \in \Lambda$, then $\varkappa((\varpi,\sigma),(\varpi^*,\sigma^*)) \geq 0$. Using (P_3), we get $\varkappa(\mho(\varpi,\sigma)(\varsigma),\mho(\varpi^*,\sigma^*)(\varsigma)) \geq 0$, and so $\beta(\mho(\varpi,\sigma),\mho(\varpi^*,\sigma^*)) \geq 0$. Thus, $\mho$ is ${}^{GT}\beta_{AM}$.

Analogously, If $\beta((\sigma,\varpi),(\sigma^*,\varpi^*)) = 1$ for all $\varpi,\sigma,\varpi^*,\sigma^* \in \Lambda$, then $\varkappa((\sigma,\varpi),(\sigma^*,\varpi^*)) \geq 0$. So (P_3) implies that $\varkappa(\mho(\sigma,\varpi)(\varsigma),\mho(\sigma^*,\varpi^*)(\varsigma)) \geq 0$, and so $\beta(\mho(\sigma,\varpi),\mho(\sigma^*,\varpi^*)) \geq 0$. Thus, $\mho$ is ${}^{GT}\beta_{AM}$.

From (P_2), we can find $\varpi_1,\sigma_1 \in \Lambda$ such that $\varkappa((\varpi_1,\sigma_1),\mho(\varpi_1,\sigma_1)) \geq 0$ and $\varkappa((\sigma_1,\varpi_1),\mho(\sigma_1,\varpi_1)) \geq 0$. Hypothesis (P_4) completes all requirements of Theorem 6. So, $\mho$ has a CFP in Λ; that is, there exists $(\widehat{\varpi},\widehat{\sigma}) \in \Lambda \times \Lambda$ such that $\widehat{\varpi} = \mho(\widehat{\varpi},\widehat{\sigma})$ and $\widehat{\sigma} = \mho(\widehat{\sigma},\widehat{\varpi})$. Therefore, $(\widehat{\varpi},\widehat{\sigma})$ is a solution of (23). □

6. Conclusions and Future Works

In this paper, a new class of mappings called "generalized β-ϕ-Geraghty contraction-type mappings" has been introduced. Through the use of our new concept, several coupled fixed points have been presented and demonstrated. Also, we supported our results with some examples and an application to coupled ordinary differential equations. Lately, Mlaiki et al. [13] launched a new space, called "controlled metric-type space", and they gave a new version of the Banach contraction theorem. For future work, our main concern will be to initiate a study of the coupled fixed-point results by utilizing the concept of generalized β-ϕ-Geraghty contraction-type mappings and applying the new results to solve some real-life problems.

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Abbreviations

The following abbreviations are used in this manuscript:

FP	Fixed point
BCP	Banach contraction principle
CMS	Complete metric space
CFP	Coupled fixed point
POS	Partially ordered set
CODE	Coupled ordinary differential equation
${}^{GT}\beta_{AM}$	Generalized triangular β-admissible mapping
${}^{E}\beta$-ϕ_{GC}	Extended β-ϕ-Geraghty contraction

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