Article

Fractional Stochastic Evolution Inclusions with Control on the Boundary

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Abstract: Symmetry in systems arises as a result of natural design and provides a pivotal mechanism for crucial system properties. In the field of control theory, scattered research has been carried out concerning the control of group-theoretic symmetric systems. In this manuscript, the principles of stochastic analysis, the fixed-point theorem, fractional calculus, and multivalued map theory are implemented to investigate the null boundary controllability (NBC) of stochastic evolution inclusion (SEI) with the Hilfer fractional derivative (HFD) and the Clarke subdifferential. Moreover, an example is depicted to show the effect of the obtained results.

Keywords: fractional stochastic evolution inclusions; boundary controllability; multivalued map theory

MSC: 26A33; 34G25; 93B05

1. Introduction

In recent decades, one of the areas of great concentration in the scientific community has been fractional differential equations and fractional differential inclusions (see [1–3]). Real-world phenomena, such as population growth, stock prices, the weather-prediction model, and heat conduction in materials with memory, are affected by random influences. As a result of noise, deterministic models frequently change. Naturally, such models must be extended to include stochastic models, in which the relevant parameters are regarded as suitable Brownian motion and stochastic process. Stochastic differential equations, as opposed to deterministic equations, are used to model the majority of issues that arise in real-world settings (see [4–7]). Many systems, such as physical, chemical, and biological systems, exhibit natural symmetry. Stochastic differential equations play an important role in explaining some symmetry phenomena (see [8–10]). Evolution inclusions, and the generalization of evolution equations and inequalities, have been used in different fields (see [11–13]. Stochastic evolution inclusions are a combination of deterministic evolution inclusions and a noise term (see [14–21]). One of the fundamental concepts of contemporary control theory is the idea of a dynamical system being controllable. In general, controllability is the capability of a control system to be directed from an arbitrary initial state to a likewise arbitrary final state through a permitted set of controls. Significant consequences for the behavior of linear and nonlinear dynamical systems are drawn from this idea (see [22–29]). Previously, few researcher's studied the boundary controllability, for example, Kumar et al. [30] explored the sufficient conditions for the boundary controllability of nonlocal impulsive neutral integrodifferential evolution equations by using Sadovskii's fixed-point theorem. Carreno et al. [31] studied the boundary controllability of a cascade system coupling fourth-and second-order parabolic equations. Ahmed et al. [32] established the sufficient conditions for the approximate and null boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional

Brownian motion and Poisson jumps by using the Schauder fixed-point theorem. Lizzy and Balachandran [33] studied the boundary controllability of nonlinear stochastic fractional systems in Hilbert spaces. Ma et al. [34] discussed the boundary controllability of nonlocal fractional neutral integrodifferential evolution systems. To our best knowledge, there are no results on the null boundary controllability of nonlocal SEI with the HFD and the Clarke subdifferential. This work aims to address this gap.

Now, consider nonlocal SEI with the HFD and the Clarke subdifferential where the control is on the boundary as:

$$
\begin{cases}
D_{0+}^{\Im,\mathfrak{u}}\mathfrak{B}(\mathfrak{w}) \in F\mathfrak{B}(\mathfrak{w}) + \sigma(\mathfrak{w},\mathfrak{B}(\mathfrak{w})) + \wp(\mathfrak{w},\mathfrak{B}(\mathfrak{w}))\frac{d\omega(\mathfrak{w})}{d\mathfrak{w}} \\
\quad + \partial Y(\mathfrak{w},\mathfrak{B}(\mathfrak{w})), \ \mathfrak{w} \in \mathfrak{J} = (0,\alpha], \\
\delta\mathfrak{B}(\mathfrak{w}) = \varrho_1 \mathcal{V}(\mathfrak{w}), \ \mathfrak{w} \in \mathfrak{J} = (0,\alpha], \\
I_{0+}^{(1-\mathfrak{u})(1-\Im)}\mathfrak{B}(0) + \mathcal{K}(\mathfrak{B}) = \mathfrak{g}\,\mathfrak{B}_0,
\end{cases}
\tag{1}
$$

where $D_{0+}^{\Im,\mathfrak{u}}$ is the HFD of order $\Im \in [0,1]$, $\mathfrak{u} \in (\frac{1}{2},1)$, and F is the bounded linear operator and the linear operator δ from Ξ into $\mathfrak{D}$, where $\mathfrak{D}$ is separable Hilbert space . ϱ_1 stands for a bounded linear operator from Λ into Ξ, where Λ and Ξ are the Hilbert space. The state $\mathfrak{B}(\cdot)$ takes values in Ξ. Let $\mathbb{A} : \Xi \to \Xi$ be a linear operator defined by $Dom(\mathbb{A}) = \{\mathfrak{B} \in Dom(F); F\mathfrak{B} = 0\}$, $\mathbb{A}\mathfrak{B} = F\mathfrak{B}$, for $\mathfrak{B} \in Dom(\mathbb{A})$.

Let $\{\omega(\mathfrak{w})\}_{\mathfrak{w}\geq 0}$ be the $\mathfrak{D}$-valued Brownian motion with a finite trace nuclear covariance operator $\Theta \geq 0$ defined on a complete probability space $(\Omega, \mathcal{S}, \{\mathcal{S}_{\mathfrak{w}}\}_{\mathfrak{w}\geq 0}, P)$ with a normal filtration $\{\mathcal{S}_{\mathfrak{w}}\}_{\mathfrak{w}\geq 0}$ satisfying that $\mathcal{S}_0$ contains all P-null sets of $\mathcal{S}$. Additionally, $\| \cdot \|$ for $L(\mathfrak{D}, \Xi)$, where $L(\mathfrak{D}, \Xi)$ is the space of all bounded linear operators from $\mathfrak{D}$ into Ξ. $\partial Y(\mathfrak{w}, \mathfrak{B}(\mathfrak{w}))$ denotes Clarke's subdifferential of $Y(\mathfrak{w}, \mathfrak{B}(\mathfrak{w}))$. The control function $\mathcal{V}(\cdot)$ is given in $L^2(\mathfrak{J}, \Lambda)$, the Hilbert space of admissible control functions with a Hilbert space Λ. $\sigma : \mathfrak{J} \times \Xi \to 2^{\Xi}$, a non-empty, bounded, closed, and convex (BCC) multivalued map, and $\wp : \mathfrak{J} \times \Xi \to L_{\Theta}(\mathfrak{D}, \Xi)$ are nonlinear functions and $\mathcal{K} : C(\mathfrak{J}, \Xi) \to \Xi$. Let $L_{\Theta}(\mathfrak{D}, \Xi)$ be the space of all Θ-Hilbert-Schmidt operators from $\mathfrak{D}$ to Ξ.

The main contributions of the current work:

- Nonlocal fractional stochastic differential inclusion with the Clarke subdifferential and control on the boundary is introduced.
- We establish a set of sufficient conditions that demonstrate the null boundary controllability for (1).
- An example is provided to show the effect of the results obtained.

2. Preliminaries

Definition 1 ([35,36]). *The HFD of order $0 \leq \Im \leq 1$ and $0 < \mathfrak{u} < 1$ for a function $\mathbb{G}$ can be defined as*

$$
D_{0+}^{\Im,\mathfrak{u}}\mathbb{G}(\mathfrak{w}) = I_{0+}^{\Im(1-\mathfrak{u})}\frac{d}{d\mathfrak{w}}I_{0+}^{(1-\Im)(1-\mathfrak{u})}\mathbb{G}(\mathfrak{w}),
$$

where

$$
I^{\mathfrak{u}}\mathbb{G} = \frac{1}{\Gamma(\mathfrak{u})}\int_0^{\mathfrak{w}}\frac{\mathbb{G}(\nu)}{(\mathfrak{w}-\nu)^{1-\mathfrak{u}}}d\nu, \ \mathfrak{w} > 0, \ \mathfrak{u} > 0.
$$

Let $\mathfrak{M} := \mathfrak{C}(\mathfrak{J}, L^2(\mathcal{S}, \Xi))$ be the Banach space of all continuous functions $\mathfrak{B}$ from $\mathfrak{J}$ into $L^2(\mathcal{S}, \Xi)$, with $\|\mathfrak{B}\|_{\mathfrak{M}} = \sup_{\mathfrak{w}\in\mathfrak{J}} E\|\mathfrak{w}^{(1-\Im)(1-\mathfrak{u})}\mathfrak{B}(\mathfrak{w})\|^2)^{1/2}$, where $L^2(\mathcal{S}, \Xi) = L^2(\Omega, \mathcal{S}, P, \Xi)$ denotes a Hilbert space of strongly $\mathcal{S}$-measurable, $\mathbb{H}$-valued random variables satisfying $E\|\mathfrak{B}\|^2 < \infty$. $L_{\mathcal{S}}^2(\mathfrak{J}, \Xi)$ will denote the Hilbert space of all random processes $\mathcal{S}_{\mathfrak{w}}$-adapted measurable defined on $\mathfrak{J}$ with values in Ξ with the norm $\|\mathfrak{B}\|_{L_{\mathcal{S}}^2(\mathfrak{J},\Xi)} = (\int_0^{\alpha} E\|\mathfrak{B}(\mathfrak{w})\|_{\Xi}^2)^{1/2} < \infty$.

In the present paper, let $\mathfrak{A}_{\mathfrak{r}} = \{\mathfrak{B} \in \mathfrak{M} : \|\mathfrak{B}\|_{\mathfrak{M}}^2 \leq \mathfrak{r}\}$, where $\mathfrak{r} > 0$.

Definition 2 (see [37])**.** *Let* $Y : \mathcal{O} \to \mathcal{R}$ *be a locally Lipschitz functional on* $\mathcal{O}$*, where* $\mathcal{O}$ *is a Banach space with* $\mathcal{O}^*$ *is the dual space . Clarke's generalized directional derivative of* Y *at* $\iota \in \mathcal{O}$ *in the direction* $\varsigma \in Y$ *is defined by*

$$Y^0(\iota; \varsigma) = \limsup_{\substack{\hbar \to 0^+ \\ \mathfrak{v} \to \iota}} \frac{Y(\mathfrak{v} + \hbar\varsigma) - Y(\mathfrak{v})}{\hbar}.$$

Clarke's generalized gradient of Y *at* $\iota \in \mathcal{O}$*, denoted by* $\partial Y(\iota)$*, is a subset of* $\mathcal{O}^*$ *given by*

$$\partial Y(\iota) = \{\iota^* \in \mathcal{O}^* : Y^0(\iota; \varsigma) \geq \langle \iota^*, \varsigma \rangle, \ \forall \ \varsigma \in \mathcal{O}.\}$$

Definition 3 (see [38])**.** *A Family* $\mathcal{M}$ *in* $\mathfrak{M}$ *is called equicontinuous if for every* $\epsilon > 0$ *there is a* $\iota = \iota(\epsilon) > 0$*, such that* $|\Phi(\mathfrak{w}_1) - \Phi(\mathfrak{w}_2)| < \epsilon \ \forall \ \mathfrak{w}_1, \mathfrak{w}_2 \in \mathfrak{J}$ *with* $|\mathfrak{w}_1 - \mathfrak{w}_2| < \iota$ *and all* $\Phi \in \mathcal{M}$*.*

The following hypotheses are necessary to prove the main results.
$(A1)$ $Dom(F) \subset Dom(\delta)$ and the restriction of δ to $Dom(F)$ is continuous relative to graph norm of $Dom(F)$.
$(A2)$ $\exists$ a linear continuous operator $\varrho : \Lambda \to \Xi$ such that for all $\mathcal{V} \in \Lambda$ we have $\varrho\mathcal{V} \in Dom(F)$, $\delta(\varrho\mathcal{V}) = \varrho_1\mathcal{V}$ and $E\|\varrho\mathcal{V}\|^2 \leq C_1 E\|\varrho_1\mathcal{V}\|^2$, where C_1 is a constant.
$(A3)$ $\mathbb{A}$ is the infinitesimal generator of compact semigroup of bounded operator $\{\aleph(\mathfrak{w}), \mathfrak{w} \geq 0\}$ in Ξ and there exists $\Pi > 0$ such that $\sup_{\mathfrak{w} \in \mathfrak{J}} \|\aleph(\mathfrak{w})\| \leq \Pi$.
$(A4)$ $\forall \ \mathfrak{w} \in (0, \alpha]$ and $\mathcal{V} \in \Lambda$, $\aleph(\mathfrak{w})\varrho\mathcal{V} \in Dom(\mathbb{A})$. Moreover, $\exists \ \Pi_1 > 0$ such that $\|\mathbb{A}\aleph(\mathfrak{w})\| \leq \Pi_1$.
$(A5)$ $\sigma : \mathfrak{J} \times \Xi \to 2^\Xi$ is locally Lipschitz continuous (LLC), $\forall \ \mathfrak{w} \in \mathfrak{J}$, $\mathfrak{B}, \mathfrak{B}_1, \mathfrak{B}_2 \in \Xi$, $\exists \ C_2 > 0$ such that

$$E\|\sigma(\mathfrak{w}, \mathfrak{B}_1) - \sigma(\mathfrak{w}, \mathfrak{B}_2)\|^2 \leq C_2(E\|\mathfrak{B}_1 - \mathfrak{B}_2\|^2, \ E\|\sigma(\mathfrak{w}, \mathfrak{B})\|^2 \leq C_2(1 + E\|\mathfrak{B}\|^2).$$

$(A6)$ $\wp : \mathfrak{J} \times \Xi \to L_\Theta(\mathfrak{D}, \Xi)$ is LLC, $\forall \ \mathfrak{w} \in \mathfrak{J}$, $\mathfrak{B}, \mathfrak{B}_1, \mathfrak{B}_2 \in \Xi$, $\exists \ C_3 > 0$ such that

$$E\|\wp(\mathfrak{w}, \mathfrak{B}_1) - \wp(\mathfrak{w}, \mathfrak{B}_2)\|_\Theta^2 \leq C_3(E\|\mathfrak{B}_1 - \mathfrak{B}_2\|^2, \ E\|\wp(\mathfrak{w}, \mathfrak{B})\|_\Theta^2 \leq C_3(1 + E\|\mathfrak{B}\|^2).$$

$(A7)$ $Y : \mathfrak{J} \times \Xi \to \mathcal{R}$ satisfies the following:
(I) $Y(\cdot, \mathfrak{B}) : \mathfrak{J} \to \mathcal{R}$ is measurable $\forall \ \mathfrak{B} \in \Xi$,
(II) $Y(\mathfrak{w}, \cdot) : \Xi \to \mathcal{R}$ is LLC for a.e. $\mathfrak{w} \in I$,
(III) $\exists$ a function $\vartheta \in L^1(\mathfrak{J}, \mathcal{R}^+)$ and a constant $C_4 > 0$ satisfying

$$E\|\partial Y(\mathfrak{w}, \mathfrak{B})\|^2 = \sup\{E\|\mathfrak{N}(\mathfrak{w})\|^2 : \mathfrak{N}(\mathfrak{w}) \in \partial Y(\mathfrak{w}, \mathfrak{B})\} \leq \vartheta(\mathfrak{w}) + C_4 E\|\mathfrak{B}\|^2,$$

for all $\mathfrak{B} \in \Xi$ a.e. $\mathfrak{w} \in \mathfrak{J}$ and $\mathfrak{B} \in \Xi$.
$(A8)$ $\mathcal{K} : C(\mathfrak{J}, \Xi) \to \Xi$ is continuous, for any $\mathfrak{B}, \mathfrak{B}_1, \mathfrak{B}_2 \in C(\mathfrak{J}, \Xi) \ \exists \ C_5 > 0$ such that

$$E\|\mathcal{K}(\mathfrak{B}_1) - \mathcal{K}(\mathfrak{B}_2)\|^2 \leq C_5 E\|\mathfrak{B}_1 - \mathfrak{B}_2\|^2, \ \ E\|\mathcal{K}(\mathfrak{B})\|^2 \leq C_5(1 + E\|\mathfrak{B}\|^2).$$

Let $\mathfrak{B}(\mathfrak{w})$ is the solution of (1). Then, we define
$\mathfrak{x}(\mathfrak{w}) = \mathfrak{B}(\mathfrak{w}) - \varrho\mathcal{V}(\mathfrak{w})$. We see that, from our hypotheses, $\mathfrak{x}(\mathfrak{w}) \in Dom(\mathbb{A})$. Hence, (1) can be expressed in terms of $\mathbb{A}$ and ϱ in the form:

$$\begin{cases} D_{0+}^{\mathfrak{I}, \mathfrak{u}} \mathfrak{x}(\mathfrak{w}) \in \mathbb{A}\mathfrak{x}(\mathfrak{w}) + F\varrho\mathcal{V}(\mathfrak{w}) - \varrho D_{0+}^{\mathfrak{I}, \mathfrak{u}} \mathcal{V}(\mathfrak{w}) \\ + \sigma(\mathfrak{w}, \mathfrak{B}(\mathfrak{w})) + \wp(\mathfrak{w}, \mathfrak{B}(\mathfrak{w})) \frac{d\omega(\mathfrak{w})}{d\mathfrak{w}} + \partial Y(\mathfrak{w}, \mathfrak{B}(\mathfrak{w})), \ \mathfrak{w} \in \mathfrak{J} = (0, \alpha], \\ I_{0+}^{(1-\mathfrak{I})(1-\mathfrak{u})} [\mathfrak{x}(0) + \varrho\mathcal{V}(0)] = I_{0+}^{(1-\mathfrak{I})(1-\mathfrak{u})} \mathfrak{B}(0) = \mathfrak{B}_0 - \mathcal{K}(\mathfrak{B}). \end{cases}$$

Hence, the integral inclusion of (1) is given by

$$
\begin{aligned}
\mathfrak{B}(\mathfrak{w}) \ \in \ & \frac{\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})}{\Gamma(\Im + \mathfrak{u} - \Im\mathfrak{u})} \mathfrak{w}^{(\Im-1)(1-\mathfrak{u})} + \frac{1}{\Gamma(\mathfrak{u})} \int_0^{\mathfrak{w}} (\mathfrak{w} - \nu)^{\mathfrak{u}-1} \mathbb{A}\mathfrak{B}(\nu) d\nu \\
& + \frac{1}{\Gamma(\mathfrak{u})} \int_0^{\mathfrak{w}} (\mathfrak{w} - \nu)^{\mathfrak{u}-1} [F - \mathbb{A}] \varrho \mathcal{V}(\nu) d\nu + \frac{1}{\Gamma(\mathfrak{u})} \int_0^{\mathfrak{w}} (\mathfrak{w} - \nu)^{\mathfrak{u}-1} \sigma(\nu, \mathfrak{B}(\nu)) d\nu \\
& + \frac{1}{\Gamma(\mathfrak{u})} \int_0^{\mathfrak{w}} (\mathfrak{w} - \nu)^{\mathfrak{u}-1} \wp(\nu, \mathfrak{B}(\nu)) d\omega(\nu) \\
& + \frac{1}{\Gamma(\mathfrak{u})} \int_0^{\mathfrak{w}} (\mathfrak{w} - \nu)^{\mathfrak{u}-1} \partial Y(\nu, \mathfrak{B}(\nu)) d\nu.
\end{aligned}
\tag{2}
$$

Lemma 1 (see [32]). *If the integral inclusion (2) holds, then the mild solution of (1) is given by*

$$
\begin{aligned}
\mathfrak{B}(\mathfrak{w}) \ = \ & \aleph_{\Im,\mathfrak{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] + \int_0^{\mathfrak{w}} [P_{\mathfrak{u}}(\mathfrak{w} - \nu)F - \mathbb{A}P_{\mathfrak{u}}(\mathfrak{w} - \nu)] \varrho \mathcal{V}(\nu) d\nu \\
& + \int_0^{\mathfrak{w}} P_{\mathfrak{u}}(\mathfrak{w} - \nu)\sigma(\nu, \mathfrak{B}(\nu)) d\nu + \int_0^{\mathfrak{w}} P_{\mathfrak{u}}(\mathfrak{w} - \nu)\mathfrak{N}(\nu) d\nu \\
& + \int_0^{\mathfrak{w}} P_{\mathfrak{u}}(\mathfrak{w} - \nu)\wp(\nu, \mathfrak{B}(\nu)) d\omega(\nu), \ \ \mathfrak{w} \in \mathfrak{J}
\end{aligned}
$$

where

$$
\aleph_{\Im,\mathfrak{u}}(\mathfrak{w}) = I_{0+}^{\Im(1-\mathfrak{u})} P_{\mathfrak{u}}(\mathfrak{w}), \quad P_{\mathfrak{u}}(\mathfrak{w}) = \mathfrak{w}^{\mathfrak{u}-1} T_{\mathfrak{u}}(\mathfrak{w}), \quad T_{\mathfrak{u}}(\mathfrak{w}) = \int_0^{\infty} \mathfrak{u} \nu \Psi_{\mathfrak{u}}(\nu) \aleph(\mathfrak{w}^{\mathfrak{u}}\nu) d\nu,
$$

with

$$
\Psi_{\mathfrak{u}}(\nu) = \sum_{n=1}^{\infty} \frac{(-\nu)^{n-1}}{(n-1)!\Gamma(1 - n\mathfrak{u})}, \nu \in (0, \infty).
$$

Lemma 2 (see [39]). *The operators $\aleph_{\Im,\mathfrak{u}}$ and $P_{\mathfrak{u}}$ satisfy the following:*
(i) $\{P_{\mathfrak{u}}(\mathfrak{w}) : \mathfrak{w} > 0\}$ is continuous in the uniform operator topology.
(ii) $\aleph_{\Im,\mathfrak{u}}(\mathfrak{w})$ and $P_{\mathfrak{u}}(\mathfrak{w})$ are linear bounded operators, and

$$
\|P_{\mathfrak{u}}(\mathfrak{w})\mathfrak{B}\| \leq \frac{\Pi \mathfrak{w}^{\mathfrak{u}-1}}{\Gamma(\mathfrak{u})} \|\mathfrak{B}\|, \ \|\aleph_{\Im,\mathfrak{u}}(\mathfrak{w})\mathfrak{B}\| \leq \frac{\Pi \mathfrak{w}^{(\Im-1)(1-\mathfrak{u})}}{\Gamma(\Im(1-\mathfrak{u}) + \mathfrak{u})} \|\mathfrak{B}\|, \ \mathfrak{w} > 0.
$$

(iii) $P_{\mathfrak{u}}(\mathfrak{w})$, $\aleph_{\Im,\mathfrak{u}}(\mathfrak{w})$ are strongly continuous, for $\mathfrak{w} > 0$.

Lemma 3 (see [32]). $\|\mathbb{A}P_{\mathfrak{u}}(\mathfrak{w})\mathfrak{B}\| \leq \frac{\Pi_1 \mathfrak{w}^{\mathfrak{u}-1}}{\Gamma(\mathfrak{u})} \|\mathfrak{B}\|$, *if $(A4)$ is satisfied.*

Now, we define an operator $\mathfrak{F} : L_{\mathcal{S}}^2(\mathfrak{J}, \Xi) \to 2^{L_{\mathcal{S}}^2(\mathfrak{J},\Xi)}$ as follows:
$\mathfrak{F}(\mathfrak{B}) = \{\mathfrak{N} \in L_{\mathcal{S}}^2(\mathfrak{J}, \Xi) : \mathfrak{N}(\mathfrak{w}) \in \partial Y(\mathfrak{w}, \mathfrak{B}(\mathfrak{w})) \ a.e. \ \mathfrak{w} \in \mathfrak{J} \ for \ \mathfrak{B} \in L_{\mathcal{S}}^2(\mathfrak{J}, \Xi)\}.$

Lemma 4 (see [37]). $\forall \mathfrak{B} \in L_{\mathcal{S}}^2(\mathfrak{J}, \Xi)$, *the set $\mathfrak{F}(\mathfrak{B})$ has nonempty, convex, and weakly compact values, provided that $(A7)$ is realized.*

Lemma 5 (see [37]). *The operator $\mathfrak{F}$ satisfies: if $\mathfrak{B}_n \to \mathfrak{B}$ in $L_{\mathcal{S}}^2(\mathfrak{J}, \Xi)$, $\varphi_n \to \varphi$ weakly in $L_{\mathcal{S}}^2(\mathfrak{J}, \Xi)$ and $\varphi_n \in \mathfrak{F}(\mathfrak{B}_n)$, then $\varphi \in \mathfrak{F}(\mathfrak{B})$, provided that $(A7)$ is satisfied.*

Theorem 1 (see [40]). *Assume that $\mathfrak{Q}$ is a locally convex Banach space and $\mathcal{M}_\varepsilon : \mathfrak{Q} \to 2^{\mathfrak{Q}}$ is a compact convex-valued (CCV), upper semicontinuous multi-valued map such that there a closed neighbourhood $\mathfrak{L}$ of 0 exists for which $\mathcal{M}_\varepsilon(\mathfrak{L})$ is a relatively compact set. If the set $\Psi = \{\mathfrak{B} \in \mathfrak{Q} : \varsigma \mathfrak{B} \in \mathcal{M}_\varepsilon(\mathfrak{B})$ for some $\varsigma > 1\}$ is bounded, then $\mathcal{M}_\varepsilon$ has a fixed point.*

3. Main Result

To investigate the NBC for (1), we consider linear SEI with the HFD and the control on the boundary

$$
\begin{cases}
D_{0+}^{\Im,\mathfrak{u}}\zeta(\mathfrak{w}) \in F\zeta(\mathfrak{w}) + \sigma(\mathfrak{w}) + \wp(\mathfrak{w})\frac{d\omega(\mathfrak{w})}{d\mathfrak{w}}, & \mathfrak{w} \in \mathfrak{J}, \\
\delta\zeta(\mathfrak{w}) = \varrho_1 \mathcal{V}(\mathfrak{w}), & \mathfrak{w} \in \mathfrak{J}, \\
I_{0+}^{(1-\mathfrak{u})(1-\Im)}\zeta(0) = \zeta_0,
\end{cases}
\tag{3}
$$

associated with the system (1).
Consider

$$
L_0^\alpha \mathcal{V} = \int_0^\alpha [P_{\mathfrak{u}}(\alpha - \nu)F - \mathbb{A}P_{\mathfrak{u}}(\alpha - \nu)]\varrho\mathcal{V}(\nu)d\nu \; : L_2(\mathfrak{J},\Lambda) \to \Xi,
$$

where $L_0^\alpha \mathcal{V}$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(\mathfrak{J},\Lambda)/ker(L_0^\alpha)$, and

$$
N_0^\alpha(\zeta,\sigma,\wp) = \aleph_{\Im,\mathfrak{u}}(\mathfrak{w})\zeta + \int_0^\alpha P_{\mathfrak{u}}(\alpha - \nu)\sigma(\nu)d\nu + \int_0^\alpha P_{\mathfrak{u}}(\alpha - \nu)\wp(\nu)d\omega(\nu) : \Xi \times L_2(\mathfrak{J},\Lambda) \to \Xi.
$$

Definition 4 (see [41]). *(3) is said to is exactly null controllable on $\mathfrak{J}$ if $ImL_0^\alpha \supset ImN_0^\alpha$ or $\exists$ a $\gamma > 0$ such that $\|(L_0^\alpha)^*\zeta\|^2 \geq \gamma\|(N_0^\alpha)^*\zeta\|^2$ for all $\zeta \in \Xi$.*

Lemma 6 (see [42]). *Suppose that (3) is exactly null controllable on $\mathfrak{J}$. Hence, $(L_0)^{-1}N_0^\alpha : \Xi \times L_2(\mathfrak{J},\Xi) \to L_2(\mathfrak{J},\Lambda)$ is bounded and the control*

$$
\mathcal{V}(\mathfrak{w}) = -(L_0)^{-1}\left[\aleph_{\Im,\mathfrak{u}}(\mathfrak{w})\zeta_0 + \int_0^\alpha P_{\mathfrak{u}}(\alpha - \nu)\sigma(\nu)d\nu + \int_0^b P_{\mathfrak{u}}(\alpha - \nu)\wp(\nu)d\omega(\nu)\right](\mathfrak{w})
$$

transfers (3) from ζ_0 to 0, where L_0 is the restriction of L_0^α to $[ker\, L_0^\alpha]^\perp$.

Definition 5 (see [42]). *The problem (1) is said to be exact null controllable on the interval $\mathfrak{J}$ if a stochastic control $\mathcal{V} \in L_2(\mathfrak{J},\Lambda)$ exists such that the solution $\mathfrak{B}(\mathfrak{w})$ of (1) satisfies $\mathfrak{B}(\alpha) = 0$.*
To prove the null boundary controllability, we need the following hypothesis:
(A9) The fractional linear system (3) is exactly null controllable on $\mathfrak{J}$.

Theorem 2. *If (A1)–(A9) are satisfied, then (1) is exactly null controllable on $\mathfrak{J}$ provided that*

$$
\begin{aligned}
\Re_2 &= \left\{ \frac{25C_5\Pi^2\alpha^{2(\Im-1)(1-\mathfrak{u})}}{\Gamma^2(\Im(1-\mathfrak{u})+\mathfrak{u})} + \frac{25\Pi^2\alpha^{2\mathfrak{u}-1}}{(2\mathfrak{u}-1)\Gamma^2(\mathfrak{u})}\left[(C_2 + Tr(\Theta)C_3) + C_4\alpha\right] \right\} \\
&\quad \times \left\{ 1 + \frac{25\|\varrho\|^2\|(L_0)^{-1}\|^2\alpha^{2\mathfrak{u}-1}[\|F\|^2\Pi^2 + \Pi_1^2]}{(2\mathfrak{u}-1)\Gamma^2(\mathfrak{u})} \right\} < 1.
\end{aligned}
$$

Proof. Consider the map $\mathcal{M}_\varepsilon : \mathfrak{M} \to 2^{\mathfrak{M}}$ as follows:

$$
\mathcal{M}_\varepsilon(\mathfrak{B}) = \left\{
\begin{array}{l}
\mathfrak{U} \in \mathfrak{M} : \mathfrak{U}(\mathfrak{w}) = \aleph_{\Im,\mathfrak{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] \\
+ \int_0^{\mathfrak{w}}[P_{\mathfrak{u}}(\mathfrak{w}-\nu)F - \mathbb{A}P_{\mathfrak{u}}(\mathfrak{w}-\nu)]\varrho\mathcal{V}(\nu)d\nu \\
+ \int_0^{\mathfrak{w}} P_{\mathfrak{u}}(\mathfrak{w}-\nu)\sigma(\nu,\mathfrak{B}(\nu))d\nu + \int_0^{\mathfrak{w}} P_{\mathfrak{u}}(\mathfrak{w}-\nu)\mathfrak{N}(\nu)d\nu \\
+ \int_0^{\mathfrak{w}} P_{\mathfrak{u}}(\mathfrak{w}-\nu)\wp(\nu,\mathfrak{B}(\nu))d\omega(\nu), \mathfrak{N} \in F(\mathfrak{B})
\end{array}
\right\}
$$

where

$$
\begin{aligned}
\mathcal{V}(\mathfrak{w}) &= -(L_0)^{-1}\left[\aleph_{\Im,\mathfrak{u}}(\alpha)[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] + \int_0^\alpha P_{\mathfrak{u}}(\alpha - \nu)\sigma(\nu,\mathfrak{B}(\nu))d\nu \right. \\
&\quad \left. + \int_0^\alpha P_{\mathfrak{u}}(\alpha - \nu)\mathfrak{N}(\nu)d\nu + \int_0^\alpha P_{\mathfrak{u}}(\alpha - \nu)\wp(\nu,\mathfrak{B}(\nu))d\omega(\nu)\right](\mathfrak{w}).
\end{aligned}
$$

Now, we demonstrate that $\mathcal{M}_\varepsilon$ has a fixed point, so we subdivided the proof into six steps.

S1: $\forall\, \mathfrak{B} \in \mathfrak{M}$, $\mathcal{M}_\varepsilon(\mathfrak{B})$ are nonempty, convex, and weakly compact values.

Lemma 4 can be used to see that $\mathcal{M}_\varepsilon(\mathfrak{B})$ has nonempty and weakly compact values. Furthermore, $\mathfrak{F}(\mathfrak{B})$ has convex values; so that if κ_1, $\kappa_2 \in \mathfrak{F}(\mathfrak{B})$ then $\ell\kappa_1 + (1-\ell)\kappa_2 \in \mathfrak{F}(\mathfrak{B})$ $\forall\, \ell \in (0,1)$, which implies that $\mathcal{M}_\varepsilon(\mathfrak{B})$ is convex.

S2: $\mathcal{M}_\varepsilon$ is bounded on a bounded subset of $\mathfrak{M}$.

Clearly, $\mathfrak{A}_\mathfrak{r}$ is a BCC set of $\mathfrak{M}$.

We can prove that $E\|\mathfrak{U}(\mathfrak{w})\|^2 \leq \tau$, $\tau > 0 \ \forall\, \mathfrak{U} \in \mathcal{M}_\varepsilon(\mathfrak{B}), \mathfrak{B} \in \mathfrak{A}_\mathfrak{r}$.

If $\Phi \in \mathcal{M}_\varepsilon(\mathfrak{B})$, then $\exists$ a $\mathfrak{N} \in \mathfrak{F}(\mathfrak{B})$ such that

$$
\begin{aligned}
\Phi(\mathfrak{w}) \;=\;& \aleph_{\Im,\mathrm{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] + \int_0^{\mathfrak{w}} [P_\mathrm{u}(\mathfrak{w} - \nu)F - \mathbb{A}P_\mathrm{u}(\mathfrak{w} - \nu)]\varrho\mathcal{V}(\nu)d\nu \\
& + \int_0^{\mathfrak{w}} P_\mathrm{u}(\mathfrak{w} - \nu)\sigma(\nu, \mathfrak{B}(\nu))d\nu + \int_0^{\mathfrak{w}} P_\mathrm{u}(\mathfrak{w} - \nu)\mathfrak{N}(\nu)d\nu \\
& + \int_0^{\mathfrak{w}} P_\mathrm{u}(\mathfrak{w} - \nu)\wp(\nu, \mathfrak{B}(\nu))d\omega(\nu) \quad \mathfrak{w} \in \mathfrak{J}.
\end{aligned} \tag{4}
$$

Then,

$$
\begin{aligned}
\|\Phi(\mathfrak{w})\|_{\mathfrak{M}}^2 \;\leq\;& 25 \sup_{\mathfrak{w} \in \mathfrak{J}} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})}\Big\{E\|\aleph_{\Im,\mathrm{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})]\|^2 \\
& + E\|\int_0^{\mathfrak{w}} [P_\mathrm{u}(\mathfrak{w} - \nu)F - \mathbb{A}P_\mathrm{u}(\mathfrak{w} - \nu)]\varrho\mathcal{V}(\nu)d\nu\|^2 \\
& + E\|\int_0^{\mathfrak{w}} P_\mathrm{u}(\mathfrak{w} - \nu)\sigma(\nu, \mathfrak{B}(\nu))d\nu\|^2 \\
& + E\|\int_0^{\mathfrak{w}} P_\mathrm{u}(\mathfrak{w} - \nu)\mathfrak{N}(\nu)d\nu\|^2 + E\|\int_0^{\mathfrak{w}} P_\mathrm{u}(\mathfrak{w} - \nu)\wp(\nu, \mathfrak{B}(\nu))d\omega(\nu)\|^2\Big\} \\
\leq\;& \Bigg\{ \frac{25\Pi^2}{\Gamma^2(\Im(1-\mathrm{u}) + \mathrm{u})}\Big[E\|\mathfrak{B}_0\|^2 + C_5(1 + \mathfrak{r})\Big] \\
& + \frac{25\Pi^2\alpha^{1-2\Im(1-\mathrm{u})}}{(2\mathrm{u}-1)\Gamma^2(\mathrm{u})}\Big[(C_2 + Tr(\Theta)C_3)(1 + \mathfrak{r}) + \|\mathfrak{N}\|_{L^1(\mathfrak{J},R^+)} + C_4\alpha\mathfrak{r}\Big]\Bigg\} \\
& \times \Bigg\{ 1 + \frac{25\|\varrho\|^2\|(L_0)^{-1}\|^2\alpha^{2\mathrm{u}-1}[\|F\|^2\Pi^2 + \Pi_1^2]}{(2\mathrm{u}-1)\Gamma^2(\mathrm{u})}\Bigg\} := \tau.
\end{aligned}
$$

Hence, $\mathcal{M}_\varepsilon(\mathfrak{A}_\mathfrak{r})$ is bounded in $\mathfrak{M}$.

S3: $\{\mathcal{M}_\varepsilon(\mathfrak{B}) : \mathfrak{B} \in \mathfrak{A}_\mathfrak{r}\}$ is equicontinuous.

For any $\mathfrak{B} \in \mathfrak{A}_\mathfrak{r}$, $\Phi \in \mathcal{M}_\varepsilon(\mathfrak{B})$, $\exists$ a $\mathfrak{N} \in \mathfrak{F}(\mathfrak{B})$ such that (4) holds $\forall\, \mathfrak{w} \in \mathfrak{J}$.

For $0 < \mathfrak{w}_1 < \mathfrak{w}_2 < q$, we can obtain

$$
\begin{aligned}
& E\|\Phi(\mathfrak{w}_2) - \Phi(\mathfrak{w}_1)\|_{\mathfrak{M}}^2 \\
& \leq 25E\|\Big(\aleph_{\Im,\mathrm{u}}(\mathfrak{w}_2) - \aleph_{\Im,\mathrm{u}}(\mathfrak{w}_1)\Big)[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})]\|_{\mathfrak{M}}^2 \\
& + 25E\|\int_0^{\mathfrak{w}_2} P_\mathrm{u}(\mathfrak{w}_2 - \nu)\sigma(\nu, \mathfrak{B}(\nu))d\nu - \int_0^{\mathfrak{w}_1} P_\mathrm{u}(\mathfrak{w}_1 - \nu)\sigma(\nu, \mathfrak{B}(\nu))d\nu\|_{\mathfrak{M}}^2 \\
& + 25E\|\int_0^{\mathfrak{w}_2} P_\mathrm{u}(\mathfrak{w}_2 - \nu)\wp(\nu, \mathfrak{B}(\nu))d\omega(\nu) - \int_0^{\mathfrak{w}_1} P_\mathrm{u}(\mathfrak{w}_1 - \nu)\wp(\nu, \mathfrak{B}(\nu))d\omega(\nu)\|_{\mathfrak{M}}^2 \\
& + 25E\|\int_0^{\mathfrak{w}_2} P_\mathrm{u}(\mathfrak{w}_2 - \nu)\mathfrak{N}(\nu)d\nu - \int_0^{\mathfrak{w}_1} P_\mathrm{u}(\mathfrak{w}_1 - \nu)\mathfrak{N}(\nu)d\nu\|_{\mathfrak{M}}^2 \\
& + 25E\|\int_0^{\mathfrak{w}_2} [P_\mathrm{u}(\mathfrak{w}_2 - \nu)F - \mathbb{A}P_\mathrm{u}(\mathfrak{w}_2 - \nu)]\varrho\mathcal{V}(\nu)d\nu \\
& - \int_0^{\mathfrak{w}_1} [P_\mathrm{u}(\mathfrak{w}_1 - \nu)F - \mathbb{A}P_\mathrm{u}(\mathfrak{w}_1 - \nu)]\varrho\mathcal{V}(\nu)d\nu\|_{\mathfrak{M}}^2
\end{aligned}
$$

$$= 25E \left\| \left(\aleph_{\Im,u}(\mathfrak{w}_2) - \aleph_{\Im,u}(\mathfrak{w}_1) \right) [\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] \right\|_{\mathfrak{M}}^2$$

$$+ 25E \left\| \int_{\mathfrak{w}_1}^{\mathfrak{w}_2} P_u(\mathfrak{w}_2 - \nu) \sigma(\nu, \mathfrak{B}(\nu)) d\nu \right.$$

$$+ \int_0^{\mathfrak{w}_1} \left[P_u(\mathfrak{w}_2 - \nu) - P_u(\mathfrak{w}_1 - \nu) \right] \sigma(s, x(s)) d\nu \Big\|_{\mathfrak{M}}^2$$

$$+ 25E \left\| \int_{\mathfrak{w}_1}^{\mathfrak{w}_2} P_u(\mathfrak{w}_2 - \nu) \wp(\nu, \mathfrak{B}(\nu)) d\omega(\nu) \right.$$

$$+ \int_0^{\mathfrak{w}_1} \left[P_u(\mathfrak{w}_2 - \nu) - P_u(\mathfrak{w}_1 - \nu) \right] \wp(\nu, \mathfrak{B}(\nu)) d\omega(\nu) \Big\|_{\mathfrak{M}}^2$$

$$+ 25E \left\| \int_{\mathfrak{w}_1}^{\mathfrak{w}_2} P_u(\mathfrak{w}_2 - \nu) \mathfrak{N}(\nu) d\nu \right.$$

$$+ \int_0^{\mathfrak{w}_1} \left[P_u(\mathfrak{w}_2 - \nu) - P_u(\mathfrak{w}_1 - \nu) \right] \mathfrak{N}(\nu) d\nu \Big\|_{\mathfrak{M}}^2$$

$$+ 25E \left\| \int_{\mathfrak{w}_1}^{\mathfrak{w}_2} [P_u(\mathfrak{w}_2 - \nu) F - \mathbb{A} P_u(\mathfrak{w}_2 - \nu)] \varrho \mathcal{V}(\nu) d\nu \right.$$

$$+ \int_0^{\mathfrak{w}_1} \left[P_u(\mathfrak{w}_2 - \nu) F - \mathbb{A} P_u(\mathfrak{w}_2 - \nu) - P_u(\mathfrak{w}_1 - \nu) F + \mathbb{A} P_u(\mathfrak{w}_1 - \nu) \right] \varrho \mathcal{V}(\nu) d\nu \Big\|_{\mathfrak{M}}^2.$$

From the compactness of $\aleph(\mathfrak{w})(\mathfrak{w} > 0)$, the above inequality tends to zero as $\mathfrak{w}_2 \to \mathfrak{w}_1$. Thus, $\mathcal{M}_\varepsilon(\mathfrak{B})(\mathfrak{w})$ is continuous from the right in $\mathfrak{J}$. Additionally, for $\mathfrak{w}_1 = 0$ and $\mathfrak{w}_2 \in \mathfrak{J}$, we can prove that $E \| \Phi(\mathfrak{w}_2) - \Phi(0) \|_{\mathfrak{M}}^2 \to 0$ as $\mathfrak{w}_2 \to 0$.

As a result, $\{ \mathcal{M}_\varepsilon(\mathfrak{B})(\mathfrak{w}) : \mathfrak{B} \in \mathfrak{A}_\mathfrak{r} \}$ is equicontinuous.

S4: $\mathcal{M}_\varepsilon$ is completely continuous.

We show that the set $\chi(\mathfrak{w}) = \{ \Phi(\mathfrak{w}) : \Phi \in \mathcal{M}_\varepsilon(\mathfrak{A}_\mathfrak{r}) \}$ is relatively compact in $\Xi \ \forall \ \mathfrak{w} \in \mathfrak{J}$, $\mathfrak{r} > 0$.

Undoubtedly, $\chi(0)$ is relatively compact in $\mathfrak{A}_\mathfrak{r}$. Let $0 < \mathfrak{w} \leq \alpha$ be fixed, $0 < \eta < \mathfrak{w}$; for $\mathfrak{B} \in \mathfrak{A}_{\mathfrak{r},,}$ we define

$$
\begin{aligned}
\Phi^{\eta,\jmath}(\mathfrak{w}) = & \frac{u}{\Gamma(\Im(1-u))} \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{\Im(1-u)-1} \nu^{u-1} \Psi_u(v) \aleph(\nu^u v) [\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] dv d\nu \\
& + u \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) [\aleph((\mathfrak{w}-\nu)^u v) F - \mathbb{A}\aleph((\mathfrak{w}-\nu)^u v)] \varrho\mathcal{V}(\nu) dv d\nu \\
& + u \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) \aleph((\mathfrak{w}-\nu)^u v) \sigma(\nu, \mathfrak{B}(\nu)) dv d\nu \\
& + u \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) \aleph((\mathfrak{w}-\nu)^u v) \mathfrak{N}(\nu) dv d\nu \\
& + u \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) \aleph((\mathfrak{w}-\nu)^u v) \wp(\nu, \mathfrak{B}(\nu)) dv d\omega(\nu) \\
= & \frac{u\aleph(\eta^u \jmath)}{\Gamma(\Im(1-u))} \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{\Im(1-u)-1} \nu^{u-1} \Psi_u(v) \aleph(\nu^u v - \eta^u \jmath) [\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] dv d\nu \\
& + u\aleph(\eta^u \jmath) \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) \aleph((\mathfrak{w}-\nu)^u v - \eta^u \jmath) \sigma(\nu, \mathfrak{B}(\nu)) dv d\nu \\
& + u\aleph(\eta^u \jmath) \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) [\aleph((\mathfrak{w}-\nu)^u v - \eta^u \jmath) F \\
& - \mathbb{A}\aleph((\mathfrak{w}-\nu)^u v - \eta^u \jmath)] \varrho\mathcal{V}(\nu) dv d\nu \\
& + u\aleph(\eta^u \jmath) \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) \aleph((\mathfrak{w}-\nu)^u v - \eta^u \jmath) \mathfrak{N}(\nu) dv d\nu \\
& + u\aleph(\eta^u \jmath) \int_0^{\mathfrak{w}-\eta} \int_\jmath^\infty v(\mathfrak{w}-\nu)^{u-1} \Psi_u(v) \aleph((\mathfrak{w}-\nu)^u v - \eta^u \jmath) \wp(\nu, \mathfrak{B}(\nu)) dv d\omega(\nu).
\end{aligned}
$$

Since $\aleph(\eta^{\mathrm{u}}\jmath)$, $\eta^{\mathrm{u}}\jmath > 0$ is a compact operator. Hence, $\chi^{\eta,\jmath}(\mathfrak{w}) = \{\Phi^{\eta,\jmath}(\mathfrak{w}) : \Phi^{\eta,\jmath} \in \mathcal{M}_\varepsilon(\mathfrak{A}_\mathfrak{r})\}$ is relatively compact in Ξ. Furthermore, we have

$$E\|\Phi(\mathfrak{w}) - \Phi^{\eta,\jmath}(\mathfrak{w})\|_{\mathfrak{M}}^2 = \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\|\Phi(\mathfrak{w}) - \Phi^{\eta,\jmath}(\mathfrak{w})\|^2$$

$$\leq \frac{25\mathrm{u}^2}{\Gamma^2(\Im(1-\mathrm{u}))} \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})}$$

$$\times E\| \int_0^{\mathfrak{w}} \int_0^{\jmath} v(\mathfrak{w}-v)^{\Im(1-\mathrm{u})-1} v^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph(v^{\mathrm{u}}v)[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})]dv dv\|^2$$

$$+ \frac{25\mathrm{u}^2}{\Gamma^2(\Im(1-\mathrm{u}))} \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})}$$

$$\times E\| \int_{\mathfrak{w}-\eta}^{\mathfrak{w}} \int_{\jmath}^{\infty} v(\mathfrak{w}-v)^{\Im(1-\mathrm{u})-1} v^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph(v^{\mathrm{u}}v)[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})]dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_0^{\mathfrak{w}} \int_0^{\jmath} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph((\mathfrak{w}-v)^{\mathrm{u}}v)\sigma(v,\mathfrak{B}(v))dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_{\mathfrak{w}-\eta}^{\mathfrak{w}} \int_{\jmath}^{\infty} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph((\mathfrak{w}-v)^{\mathrm{u}}v)\sigma(v,\mathfrak{B}(v))dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_0^{\mathfrak{w}} \int_0^{\jmath} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v)$$

$$\times [\aleph((\mathfrak{w}-v)^{\mathrm{u}}v - \eta^{\mathrm{u}}\jmath)F - \mathbb{A}\aleph((\mathfrak{w}-v)^{\mathrm{u}}v - \eta^{\mathrm{u}}\jmath)]\varrho\mathcal{V}(v)dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_{\mathfrak{w}-\eta}^{\mathfrak{w}} \int_{\jmath}^{\infty} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v)$$

$$\times [\aleph((\mathfrak{w}-v)^{\mathrm{u}}v - \eta^{\mathrm{u}}\jmath)F - \mathbb{A}\aleph((\mathfrak{w}-v)^{\mathrm{u}}v - \eta^{\mathrm{u}}\jmath)]\varrho\mathcal{V}(v)dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_0^{\mathfrak{w}} \int_0^{\jmath} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph((\mathfrak{w}-v)^{\mathrm{u}}v)\mathfrak{N}(v)dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_{\mathfrak{w}-\eta}^{\mathfrak{w}} \int_{\jmath}^{\infty} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph((\mathfrak{w}-v)^{\mathrm{u}}v)\mathfrak{N}(v)dv dv\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_0^{\mathfrak{w}} \int_0^{\jmath} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph((\mathfrak{w}-v)^{\mathrm{u}}v)\wp(v,\mathfrak{B}(v))dv d\omega(v)\|^2$$

$$+ 25\mathrm{u}^2 \sup_{\mathfrak{w}\in I} \mathfrak{w}^{2(1-\Im)(1-\mathrm{u})} E\| \int_{\mathfrak{w}-\eta}^{\mathfrak{w}} \int_{\jmath}^{\infty} v(\mathfrak{w}-v)^{\mathrm{u}-1} \Psi_{\mathrm{u}}(v) \aleph((\mathfrak{w}-v)^{\mathrm{u}}v)\wp(v,\mathfrak{B}(v))dv d\omega(v)\|^2 \to 0,$$

as $\eta \to 0^+$ $\jmath \to 0^+$.

Therefore, the set $\chi(\mathfrak{w})$ is relatively compact in Ξ. From the Arzela–Ascoli theorem and Step 3, we can deduce that $\mathcal{M}_\varepsilon$ is completely continuous.

S5: $\mathcal{M}_\varepsilon$ has a closed graph.

Let $\mathfrak{B}_n \to \mathfrak{B}_*$ in $\mathfrak{M}$, $\Phi_n \in \mathcal{M}_\varepsilon(\mathfrak{B}_n)$, and $\Phi_n \to \Phi_*$ in $\mathfrak{M}$. We will deduce that $\Phi_* \in \mathcal{M}_\varepsilon(\mathfrak{B}_*)$.

Actually, $\Phi_n \in \mathcal{M}_\varepsilon(\mathfrak{B}_n)$ implies that $\exists$ a $\mathfrak{N}_n \in \mathfrak{F}(\mathfrak{B}_n)$ such that

$$\begin{aligned} \Phi_n(\mathfrak{w}) &= \aleph_{\Im,\mathrm{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B}_n)] + \int_0^{\mathfrak{w}} P_{\mathrm{u}}(\mathfrak{w}-v)\sigma(v,\mathfrak{B}_n(v))dv \\ &+ \int_0^{\mathfrak{w}} [P_{\mathrm{u}}(\mathfrak{w}-v)F - \mathbb{A}P_{\mathrm{u}}(\mathfrak{w}-v)]\varrho\mathcal{V}(v)dv + \int_0^{\mathfrak{w}} P_{\mathrm{u}}(\mathfrak{w}-v)\mathfrak{N}_n(v)dv \quad (5) \\ &+ \int_0^{\mathfrak{w}} P_{\mathrm{u}}(\mathfrak{w}-v)\wp(v,\mathfrak{B}_n(v))d\omega(v). \end{aligned}$$

From $(A1)$–$(A8)$, we can deduce that $\{\mathcal{K}(\mathfrak{B}_n),\sigma(\cdot,\mathfrak{B}_n),\mathfrak{N}_n,\wp(\cdot,\mathfrak{B}_n)\}_{n\geq 1} \subseteq \Xi \times \Xi \times L_{\mathcal{S}}^2(\mathfrak{J},\Xi) \times L_\Theta$ is bounded. Hence, we obtain

$$(\mathcal{K}(\mathfrak{B}_n),\sigma(\cdot,\mathfrak{B}_n),\mathfrak{N}_n,\wp(\cdot,\mathfrak{B}_n)) \to (\mathcal{K}(\mathfrak{B}_*),\sigma(\cdot,\mathfrak{B}_*),\mathfrak{N}_*,\wp(\cdot,\mathfrak{B}_*)) \qquad (6)$$

weakly in $\Xi \times \Xi \times L_{\mathcal{S}}^2(\mathfrak{J},\Xi) \times L_\Theta$.

From the compactness of $\aleph(\mathfrak{w})$, (5), and (6), we obtain

$$\Phi_n(\mathfrak{w}) \rightarrow \aleph_{\Im,\mathtt{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B}_*)] + \int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\sigma(\nu,\mathfrak{B}_*(\nu))d\nu$$

$$+ \int_0^{\mathfrak{w}} [P_{\mathtt{u}}(\mathfrak{w}-\nu)F - \mathbb{A}P_{\mathtt{u}}(\mathfrak{w}-\nu)]\varrho\mathcal{V}(\nu)d\nu + \int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\mathfrak{N}_*(\nu)d\nu \quad (7)$$

$$+ \int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\wp(\nu,\mathfrak{B}_*(\nu))d\omega(\nu).$$

Applying that $\Phi_n \to \Phi_*$ in $\mathfrak{M}$ and $\mathfrak{N}_n \in \mathfrak{F}(\mathfrak{B}_n)$. From (7) and Lemma 5, we obtain $\mathfrak{N}_* \in F(\mathfrak{B}_*)$. Therefore, it can demonstrated that $\Phi_* \in \mathcal{M}_\varepsilon(\mathfrak{B}_*)$; then, $\mathcal{M}_\varepsilon$ has a closed graph and $\mathcal{M}_\varepsilon$ is a completely continuous multi-valued map with compact value. Thus, from [28], $\mathcal{M}_\varepsilon$ is upper semicontinuous.

S6: A priori estimate.

From S1–S5, we found that $\mathcal{M}_\varepsilon$ is CCV and upper semicontinuous and $\mathcal{M}_\varepsilon(\mathfrak{A}_{\mathfrak{r}})$ is relatively compact. By Theorem 1, it remains to demonstrated that
$\Psi = \{\mathfrak{B} \in \mathfrak{M} : \varsigma\mathfrak{B} \in \mathcal{M}_\varepsilon, \ \varsigma > 1\}$ is bounded. $\forall \mathfrak{B} \in \Psi, \exists$ a $\mathfrak{N} \in \mathfrak{F}(\mathfrak{B})$ such that

$$\mathfrak{B}(\mathfrak{w}) = \varsigma^{-1}\aleph_{\Im,\mathtt{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})] + \varsigma^{-1}\int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\sigma(\nu,\mathfrak{B}(\nu))d\nu$$

$$+\varsigma^{-1}\int_0^{\mathfrak{w}} [P_{\mathtt{u}}(\mathfrak{w}-\nu)F - \mathbb{A}P_{\mathtt{u}}(\mathfrak{w}-\nu)]\varrho\mathcal{V}(\nu)d\nu + \varsigma^{-1}\int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\mathfrak{N}(\nu)d\nu \quad (8)$$

$$+\varsigma^{-1}\int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\wp(\nu,\mathfrak{B}(\nu))d\omega(\nu).$$

Applying the hypotheses $(A1)$–$(A8)$, we obtain

$$E\|\mathfrak{B}(\mathfrak{w})\|^2 \leq 25\{E\|\aleph_{\Im,\mathtt{u}}(\mathfrak{w})[\mathfrak{B}_0 - \mathcal{K}(\mathfrak{B})]\|^2$$

$$+ E\|\int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\sigma(\nu,\mathfrak{B}(\nu))d\nu\|^2 + E\|\int_0^{\mathfrak{w}} [P_{\mathtt{u}}(\mathfrak{w}-\nu)F - \mathbb{A}P_{\mathtt{u}}(\mathfrak{w}-\nu)]\varrho\mathcal{V}(\nu)d\nu\|^2$$

$$+ E\|\int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\mathfrak{N}(\nu)d\nu\|^2 + E\|\int_0^{\mathfrak{w}} P_{\mathtt{u}}(\mathfrak{w}-\nu)\wp(\nu,\mathfrak{B}(\nu))d\omega(\nu)\|^2\}$$

$$\leq \left\{ \frac{25\Pi^2\alpha^{2(\Im-1)(1-\mathtt{u})}}{\Gamma^2(\Im(1-\mathtt{u})+\mathtt{u})} \left[(E\|\mathfrak{B}_0\|^2 + C_5(1 + E\|\mathfrak{B}(\mathfrak{w})\|^2) \right] \right. \quad (9)$$

$$+ \frac{25\Pi^2\alpha^{2\mathtt{u}-1}}{(2\mathtt{u}-1)\Gamma^2(\mathtt{u})} \left[(C_2 + Tr(\Theta)C_3)(1 + E\|\mathfrak{B}(\mathfrak{w})\|^2) + \|\mathfrak{N}\|_{L^1(\mathfrak{J},R^+)} + C_4\alpha E\|\mathfrak{B}(\mathfrak{w})\|^2 \right] \right\}$$

$$\times \left\{ 1 + \frac{25\Pi^2\|\varrho\|^2\|(L_0)^{-1}\|^2\alpha^{2\mathtt{u}-1}[\|F\|^2 + \Pi_1^2]}{(2\mathtt{u}-1)\Gamma^2(\mathtt{u})} \right\}$$

$$\leq \Re_1 + \Re_2 E\|\mathfrak{B}(\mathfrak{w})\|^2,$$

where

$$\Re_1 = \left\{ \frac{25\Pi^2\alpha^{2(\Im-1)(1-\mathtt{u})}(E\|\mathfrak{B}_0\|^2 + C_5)}{\Gamma^2(\Im(1-\mathtt{u})+\mathtt{u})} + \frac{25\Pi^2\alpha^{2\mathtt{u}-1}}{(2\mathtt{u}-1)\Gamma^2(\mathtt{u})} \left[(C_2 + Tr(\Theta)C_3) + \|\mathfrak{N}\|_{L^1(\mathfrak{J},R^+)} \right] \right\}$$

$$\times \left\{ 1 + \frac{25\Pi^2\|\varrho\|^2\|(L_0)^{-1}\|^2\alpha^{2\mathtt{u}-1}[\|F\|^2 + \Pi_1^2]}{(2\mathtt{u}-1)\Gamma^2(\mathtt{u})} \right\},$$

and

$$\Re_2 = \left\{ \frac{25C_5\Pi^2\alpha^{2(\Im-1)(1-u)}}{\Gamma^2(\Im(1-u)+u)} + \frac{25\Pi^2\alpha^{2u-1}}{(2u-1)\Gamma^2(u)}\left[(C_2+Tr(\Theta)C_3)+C_4\alpha\right] \right\}$$

$$\times \left\{ 1 + \frac{25\Pi^2\|\varrho\|^2\|(L_0)^{-1}\|^2\alpha^{2u-1}[\|F\|^2+\Pi_1^2]}{(2u-1)\Gamma^2(u)} \right\}.$$

Since $\Re_2 < 1$, from (9), we obtain

$$\|\mathfrak{B}\|_{\mathfrak{M}}^2 = \sup_{\mathfrak{w}\in\mathfrak{J}} E\|\mathfrak{w}^{(1-\Im)(1-u)}\mathfrak{B}(\mathfrak{w})\|^2 \leq \Re_1 + \Re_2\|\mathfrak{B}\|_{\mathfrak{M}}^2.$$

Then, $\|\mathfrak{B}\|_{\mathfrak{M}}^2 \leq \frac{\Re_1}{1-\Re_2}$, consequently, Ψ is bounded. By Theorem 1, $\mathcal{M}_\varepsilon$ has a fixed point. Any fixed point of $\mathcal{M}_\varepsilon$ is a mild solution of (1) on $\mathfrak{J}$. Therefore, the inclusion system (1) is exact null controllable on $\mathfrak{J}$. $\square$

4. Application

We consider the stochastic partial differential inclusions with the HFD and Clarke subdifferential via the nonlocal condition:

$$\begin{cases} D_{0+}^{\frac{2}{3},\frac{5}{6}}\mathfrak{B}(\mathfrak{w},\mu) \in \frac{\partial^2}{\partial\mu^2}(\mathfrak{B}(\mathfrak{w},\mu) \\ \quad +0.05e^{\mathfrak{B}(\mathfrak{w},\mu)} + 0.02sin(\mathfrak{B}(\mathfrak{w},\mu))\frac{d\omega(\mathfrak{w})}{d\mathfrak{w}} + \partial Y(\mathfrak{w},\mathfrak{B}(\mathfrak{w},\mu)), \ \mathfrak{w}\in\mathfrak{J}=(0,1], \ \mu\in\mho \\ \mathfrak{B}(\mathfrak{w},\mu) = \mathcal{V}(\mathfrak{w},\mu), \ \mathfrak{w}\in\mathfrak{J}, \ \mu\in\eth \\ I_{0+}^{\frac{1}{18}}\mathfrak{B}(0,\mu) + \sum_{i=1}^p a_i\mathfrak{B}(\mathfrak{w}_i,\mu) = \mathfrak{B}_0(\mu), \ \mu\in\mho, \end{cases} \quad (10)$$

where $\mathfrak{B}(\mathfrak{w},\mu)$ denotes the temperature at the time $\mathfrak{w}\in\mathfrak{J}$, $D_{0+}^{\frac{2}{3},\frac{5}{6}}$ is the HFD of order $\Im = \frac{2}{3}$, $u = \frac{5}{6}$, $0 < \mathfrak{w}_0 < \mathfrak{w}_1 < \ldots < \mathfrak{w}_p < 1$; $\mho$ is an open subset of $\mathcal{R}$ and bounded with ω is a Brownian motion; and $\eth$ is a sufficiently smooth boundary. The functions $\mathfrak{B}(\mathfrak{w})(\mu) = \mathfrak{B}(\mathfrak{w},\mu)$, $0.05e^{\mathfrak{B}(\mathfrak{w},\mu)} = \sigma(\mathfrak{w},\mathfrak{B}(\mathfrak{w},\mu))$, $0.02sin(\mathfrak{B}(\mathfrak{w},\mu)) = \wp(\mathfrak{w},\mathfrak{B}(\mathfrak{w},\mu))$, and $Y(\mathfrak{w},\mathfrak{B}(\mathfrak{w}))(\mu) = Y(\mathfrak{w},\mathfrak{B}(\mathfrak{w},\mu))$.

Suppose $\Xi = \mathfrak{D} = L^2(\mho)$, $\Lambda = L^2(\eth)$, $\varrho_1 = I$, the identity operator, and $F : Dom(F) \subset \Xi \to \Xi$ is given by $F = \frac{\partial^2}{\partial\mu^2}$ with $Dom(F) = \{\mathfrak{B}\in\Xi, \mathfrak{B}, \frac{\partial\mathfrak{B}}{\partial\mu}$ are absolutely continuous, $\frac{\partial^2\mathfrak{B}}{\partial\mu^2}\in L^2(\mho)\}$.

Then, Δ can be written as

$$\Delta\mathfrak{B} = \sum_{n=1}^\infty (-n^2)(\mathfrak{B},\mathfrak{B}_n)\mathfrak{B}_n, \ \mathfrak{B}\in D(\Delta),$$

where $\mathfrak{B}_n(s) = \sqrt{\frac{2}{\pi}}\sin ns$, $n = 1, 2, \ldots$ is the orthonormal base set of eigenvectors of Δ. Moreover, for $\mathfrak{B}\in\Xi$ we have

$$\aleph(\mathfrak{w})\mathfrak{B} = \sum_{n=1}^\infty e^{\frac{-n^2\mathfrak{w}}{1+n^2}}(\mathfrak{B},\mathfrak{B}_n)\mathfrak{B}_n.$$

Clearly, Δ generates a compact semigroup $\{\aleph(\mathfrak{w})\}_{\mathfrak{w}\geq 0}$ on Ξ. Now, (10) can be written in the abstract form of (1), and all of the assumptions of Theorem 2 are verified and

$$\left\{ \frac{25C_5\Pi^2\alpha^{2(\Im-1)(1-u)}}{\Gamma^2(\Im(1-u)+u)} + \frac{25\Pi^2\alpha^{2u-1}}{(2u-1)\Gamma^2(u)}\left[(C_2+Tr(\Theta)C_3)+C_4\alpha\right] \right\}$$

$$\times \left\{ 1 + \frac{25\|\varrho\|^2\|(L_0)^{-1}\|^2\alpha^{2u-1}[\|F\|^2\Pi^2+\Pi_1^2]}{(2u-1)\Gamma^2(u)} \right\} < 1.$$

Thus, (10) is null controllable on $(0, 1]$.

5. Conclusions

The fractional calculus has many diverse and potential applications in all areas of science and engineering. A new control model is presented with the HFD including the continuous stochastic noises and generalized gradient of Clarke's subdifferential. In this paper, we investigated the null boundary controllability of SEI with the HFD and Clarke subdifferential via nonlocal conditions. Our results were obtained with the aid of non-smooth analysis, fractional calculus, the Clarke subdifferential, stochastic analysis, and fixed-point theorems. Finally, an example was provided to illustrate the developed theoretical results. This helps to establish the results numerically with simulation, and one can give an application in the numerical null controllability using the developed result. In the future, we will study the optimal control problem for the Hilfer fractional stochastic differential inclusions with Sobolev-type and Poisson jumps.

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