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Article

On the Splitting Tensor of the Weak f-Contact Structure

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Abstract: A weak f-contact structure, introduced in our recent works, generalizes the classical f-contact structure on a smooth manifold, and its characteristic distribution defines a totally geodesic foliation with flat leaves. We find the splitting tensor of this foliation and use it to show positive definiteness of the Jacobi operators in the characteristic directions and to obtain a topological obstruction (including the Adams number) to the existence of weak f-K-contact manifolds, and prove integral formulas for a compact weak f-contact manifold. Based on applications of the weak f-contact structure in Riemannian contact geometry considered in the article, we expect that this structure will also be fruitful in theoretical physics, e.g., in QFT.

Keywords: weak f-contact manifold; totally geodesic foliation; splitting tensor; Killing vector; ξ-sectional curvature

MSC: 53C15; 53C25; 53D15

1. Introduction

After many results in classical mechanics related to contact transformations, the famous Boothby-Wang fibration of 1958 marked the beginning of the theory of contact manifolds [1]. Important examples of contact manifolds are the principal circle bundles and the tangent sphere bundles, e.g., [2]. The growing interest in Riemannian contact geometry is associated with its important role in differential equations, differential geometry and topology, as well as in physics, e.g., geometrical optics, classical and quantum mechanics, thermodynamics, integrable systems and control theory.

A framed f-structure on a $(2n+s)$-dimensional Riemannian manifold is defined by a (1,1)-tensor f of constant rank $2n$, which satisfies $f^3+f=0$, and linearly independent vector fields $\{\xi_i\}_{1\le i\le s}$ spanning $\ker f$—the characteristic distribution, see [3–10]. This higher dimensional analog of almost contact ($s=1$) and almost complex ($s=0$) structures appears naturally when studying hypersurfaces of almost contact manifolds, see [11], and submanifolds of almost complex manifolds [12]. Moreover, many space-time manifolds can be endowed with framed f-structures, see [13]. The importance of the tensor field f stems from the fact that its existence is equivalent to the reduction of the structure group of the manifold to $U(n)\times O(s)$, see [3].

An interesting case occurs when $\ker f$ is parallelizable or is defined by a homomorphism of an s-dimensional Lie algebra $\mathfrak{g}$ to the Lie algebra of all vector fields on M, i.e., M admits a $\mathfrak{g}$-foliation, see [14]. In the presence of a compatible metric, such $\mathfrak{g}$-foliations are totally geodesic foliations spanned by Killing vector fields, e.g., [15]. For a 1-dimensional Lie algebra, a $\mathfrak{g}$-foliation is generated by a nonvanishing vector field, and we get contact metric manifolds as well as K-contact and Sasakian ones, see [2].

The f-K-contact structure (i.e., an f-contact structure, whose characteristic vector fields generate 1-parameter groups of isometries), see [16], generalizes the $\mathcal{K}$-contact structure of [3] (i.e., $s=1$), both structures can be regarded as intermediate between a framed f-structure and S-structure (Sasaki structure when $s=1$). In [17], conditions are found under which a given compact f-K-contact manifold is a mapping torus of such a manifold of lower dimension. Various symmetries of contact and framed f-manifolds are

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studied, e.g., in [18], and sufficient conditions are considered when an f-contact manifold carries a canonical metric, such as Einstein-type or constant curvature, or admits a local decomposition (splitting), in [4,19,20].

The characteristic distribution $\ker f$ of an f-contact structure defines a totally geodesic foliation. Such foliations have simple extrinsic geometry, i.e., vanishing second fundamental form of the leaves, and appear on (sub)manifolds with degenerate differential forms and curvature-like tensors, e.g., [21]. The splitting tensor (or, co-nullity tensor) of a foliation is a useful tool to study totally geodesic foliations, and to obtain integral formulas, e.g., [21]. Integral formulas are a powerful tool for proving global results in analysis and geometry. There are series of integral formulas for two orthogonal complementary distributions (beginning with Reeb's formula for foliations of codimension one), which involve the Newton transformations of splitting tensor and certain components of the curvature tensor, and are used to solve many problems (see, surveys in [21–23]): (i) the existence and characterizing of foliations, whose leaves have a given geometric property, such as being totally geodesic, totally umbilical or minimal; (ii) prescribing the higher mean curvatures of the leaves of a foliation; (iii) minimizing functionals such as volume and energy defined for tensor fields on a foliated manifold.

In [24,25] the study of "weak" contact structures on a smooth $(2n+s)$-dimensional manifold (i.e., the complex structure on the characteristic distribution is replaced with a nonsingular skew-symmetric tensor) was initiated. These structures generalize an f-contact structure and its satellites, and allow a new look at the Riemannian classical theory and find its new applications, for example, to build manifolds with Killing vector fields and totally geodesic foliations, (well known examples are flows on the sphere, linear flow on the torus, see Figure 1), to produce Riemannian manifolds with positive ξ-sectional curvature, and to investigate Einstein-type metrics.

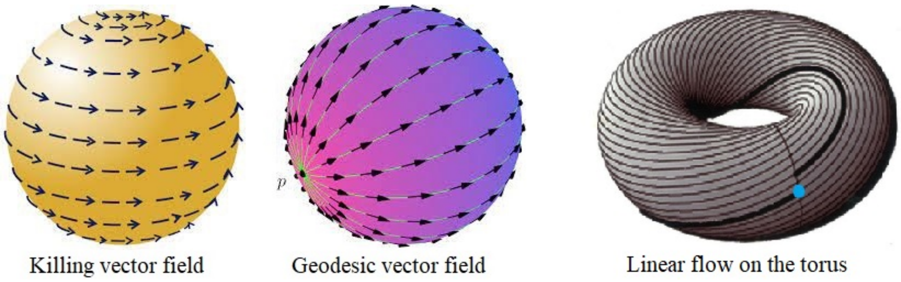

Figure 1. Examples of Killing and geodesic vector fields.

In [25], we retracted weak structures with positive partial Ricci curvature onto the subspace of classical structures of the same type. In [24], we proved that the $\mathcal{S}$-structure is rigid, i.e., our weak $\mathcal{S}$-structure is the $\mathcal{S}$-structure, and a metric weak f-structure with parallel tensor f is the weak $\mathcal{C}$-structure. The characteristic distribution of the weak f-contact structure (and its particular case—the weak f-K-contact structure) is integrable and defines a totally geodesic foliation. The splitting tensor of a totally geodesic foliation is important in studying c-nullity foliations of Riemannian manifolds M (given by $\{X \in TM : R_{X,Y} = c(X \wedge Y)\}$) and relative nullity foliations of Riemannian submanifolds (given by $\{X \in TM : \beta(X,\cdot) = 0\}$, where β is the second fundamental form), e.g., [26], and in integral formulas for foliations and Riemannian almost product manifolds, see [21].

The article continues our study [24,25] of the geometry of weak f-contact manifolds. Our achievement is the generalization of some results on f-contact manifolds to the case of weak f-contact manifolds and demonstration of the usefulness of this weak structure for the study of totally geodesic foliations, Killing vector fields and the corresponding splitting tensors on Riemannian manifolds. The proofs use the properties of new tensors, as well as the constructions required in the classical case.

The article is organized as follows. In Section 2, following the introductory Section 1, we recall some results regarding framed weak f-manifolds. Section 3 contains the main results and consists of three parts. In Section 3.1, we study the geometry of weak f-contact manifolds and show that they are endowed with totally geodesic foliations with flat fibers (Proposition 2). In Section 3.2, we find the splitting tensor for the characteristic foliation of weak f-contact manifolds (Theorem 1), and use it to show positive definiteness of the Jacobi operators in the characteristic directions, or, equivalently, that the ξ-sectional curvature is positive (Theorem 2) and to obtain a topological obstruction, including the Adams number, to the existence of weak f-K-contact manifolds (Theorem 3), and prove integral formulas for compact weak f-contact manifolds (Proposition 4).

2. Preliminaries

Here, we recall the basics of the weak f-contact structure (see [24,25]) as a higher dimensional analog of the weak almost contact structure.

Aframed weak structure on a smooth manifold M^{2n+s} is the set (f, Q, ξ_i, η^i), where f is a $(1,1)$-tensor, Q is a nonsingular $(1,1)$-tensor, ξ_i $(1 \le i \le s)$ are characteristic vector fields, which form the characteristic distribution, and η^i are dual 1-forms, satisfying

$$f^2 = -Q + \sum\nolimits_i \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j, \quad Q\,\xi_i = \xi_i. \tag{1}$$

Thus, $f^3 + fQ = 0$. The forms η^i are linearly independent and determine a smooth $2n$-dimensional contact distribution $\mathcal{D} := \bigcap_i \ker \eta^i$ (the collection of subspaces $\mathcal{D}_m = \{X \in T_m M : \eta^i(X) = 0,\ 1 \le i \le s\}$ for $m \in M$). We assume that $\mathcal{D}$ is f-invariant,

$$fX \in \mathcal{D} \quad (X \in \mathcal{D}), \tag{2}$$

as in the theory of framed f-structure [3,9], where $Q = \mathrm{id}$—the identity map of TM. For a framed weak f-structure on a smooth manifold M, the tensor f has rank $2n$ and

$$f\,\xi_i = 0, \quad \eta^i \circ f = 0, \quad \eta^i \circ Q = \eta^i, \quad [Q, f] = 0;$$

thus, $\mathcal{D} = f(TM)$. By (1) and (2), the distribution $\mathcal{D}$ is invariant for Q: $Q(\mathcal{D}) = \mathcal{D}$.

Let g be a compatible (or, associated) Riemannian metric on $M^{2n+s}(f, Q, \xi_i, \eta^i)$, i.e.,

$$g(fX, fY) = g(X, Q\,Y) - \sum\nolimits_i \eta^i(X)\,\eta^i(Y), \quad X, Y \in \mathfrak{X}_M. \tag{3}$$

Then (f, Q, ξ_i, η^i, g) is called a weak metric structure on M, and $M(f, Q, \xi_i, \eta^i, g)$ is called a weak metricmanifold. Putting $Y = \xi_i$ in (3) and using $Q\,\xi_i = \xi_i$, we get

$$\eta^i(X) = g(X, \xi_i) \quad (i = 1, \ldots, s).$$

In particular, $\ker f$ and $\mathcal{D}$ are additionally orthogonal distributions.

For a weak metric f-structure, the contact distribution $\mathcal{D}$ is nowhere integrable, since for a nonzero $X \perp \ker f$ we get

$$g([X, fX], \xi_i) = 2\,d\eta^i(fX, X) = g(fX, fX) > 0.$$

For a weak metric f-structure, the tensor f is skew-symmetric (the same for a metric f-structure) and Q is self-adjoint, see [25].

Remark 1. *There exists a compatible metric on any framed f-manifold, e.g., [3]. A framed weak f-manifold admits a compatible metric if f in (1)–(2) has a skew-symmetric representation, i.e., for any $x \in M$ there exist a neighborhood $U_x \subset M$ and a frame $\{e_i\}$ on U_x, for which f has a skew-symmetric matrix, see [25]. By (3), we get $g(X, Q\,X) = g(fX, fX) > 0$ for any nonzero vector $X \perp \ker f$; thus, the tensor Q is positive definite.*

A framed (weak) f-structure is called *normal* if the following tensor is zero:

$$N^{(1)}(X,Y) = [f,f](X,Y) + 2\sum\nolimits_i d\eta^i(X,Y)\,\xi_i, \quad X,Y \in \mathfrak{X}_M,$$

where the Nijenhuis torsion $[f,f]$ of f is given by

$$[f,f](X,Y) = f^2[X,Y] + [fX,fY] - f[fX,Y] - f[X,fY], \quad X,Y \in \mathfrak{X}_M, \tag{4}$$

and the exterior derivative of a differential form η^i is given by

$$d\eta^i(X,Y) = \frac{1}{2}\,\{X(\eta^i(Y)) - Y(\eta^i(X)) - \eta^i([X,Y])\}, \quad X,Y \in \mathfrak{X}_M. \tag{5}$$

Recall the following formulas with the Lie derivative $\pounds_Z$ in the Z-direction:

$$(\pounds_Z f)X = [Z,fX] - f[Z,X] \quad (X,Y \in \mathfrak{X}_M), \tag{6}$$
$$(\pounds_{\xi_i}\,g)(X,Y) = \xi_i(g(X,Y)) - g([\xi_i,X],Y) - g(X,[\xi_i,Y]). \tag{7}$$

The following tensors $N^{(2)}_i, N^{(3)}_i$ and $N^{(4)}_{ij}$ are well known, see [3,9] and (5):

$$\begin{aligned}
N^{(2)}_i(X,Y) &= (\pounds_{fX}\,\eta^i)(Y) - (\pounds_{fY}\,\eta^i)(X) = 2\,d\eta^i(fX,Y) - 2\,d\eta^i(fY,X),\\
N^{(3)}_i(X) &= (\pounds_{\xi_i}\,f)X = [\xi_i,fX] - f[\xi_i,X],\\
N^{(4)}_{ij}(X) &= (\pounds_{\xi_i}\,\eta^j)(X) = \xi_i(\eta^j(X)) - \eta^j([\xi_i,X]) = 2\,d\eta^j(\xi_i,X).
\end{aligned}$$

For $s=1$, these tensors reduce to certain tensors on (weak) almost contact manifolds:

$$N^{(2)}(X,Y) = (\pounds_{fX}\,\eta)Y - (\pounds_{fY}\,\eta)X, \quad N^{(3)} = \pounds_\xi\,f, \quad N^{(4)} = \pounds_\xi\,\eta.$$

Remark 2. *Let $M^{2n+s}(f,Q,\xi_i,\eta^i)$ be a framed weak f-manifold. Consider the product manifold $\bar M = M^{2n+s}\times\mathbb{R}^s$, where $\mathbb{R}^s$ is a Euclidean space with a basis $\partial_1,\ldots,\partial_s$, and define tensors $\bar f$ and $\bar Q$ on $\bar M$ putting*

$$\begin{aligned}
\bar f(X, \textstyle\sum_i a^i\partial_i) &= (fX - \textstyle\sum_i a^i\xi_i,\ \sum_j \eta^j(X)\partial_j),\\
\bar Q(X, \textstyle\sum_i a^i\partial_i) &= (QX,\ \textstyle\sum_i a^i\partial_i).
\end{aligned}$$

Hence, $\bar f(X,0) = (fX,0)$, $\bar Q(X,0) = (QX,0)$ for $X \perp \ker f$, $\bar f(\xi_i,0) = (0,\partial_i)$, $\bar Q(\xi_i,0) = (\xi_i,0)$ and $\bar f(0,\partial_i) = (-\xi_i,0)$, $\bar Q(0,\partial_i) = (0,\partial_i)$. Then, it is easy to verify that $\bar f^{\,2} = -\bar Q$. The tensors $N^{(1)}, N^{(2)}_i, N^{(3)}_i$ and $N^{(4)}_{ij}$ appear when we derive the integrability condition $[\bar f,\bar f] = 0$ (vanishing of the Nijenhuis torsion of $\bar f$) and express the normality condition $N^{(1)} = 0$ of a framed weak f-structure on M.

3. Results

In Section 3.1, we introduce the weak f-contact structure and its important case—the weak f-K-contact structure, which generalize the f-contact and f-K-contact structures, and show that the distribution $\ker f$ is integrable and defines a totally geodesic foliation with flat leaves. In Section 3.2, we derive the splitting tensor of such structures and give its applications. In Section 3.3, we study integral formulas involving the splitting tensor on weak f-contact manifolds.

3.1. Geometry of Weak f-Contact Manifolds

The following definition is a copy of the classical definition, for example, [16].

Definition 1. *A weak f-contact structure is a weak metric f-structure satisfying*

$$\Phi = d\eta^1 = \ldots = d\eta^s, \tag{8}$$

(thus, $d\Phi = 0$), where $\Phi(X,Y) = g(X, fY)$ $(X,Y \in \mathfrak{X}_M)$ is called the fundamental 2-form. A weak f-contact structure is called a weak f-K-contact structure if each characteristic vector field ξ_i is Killing, i.e., $\pounds_{\xi_i} g = 0$, see (7). A normal weak f-contact manifold is called a weak $\mathcal{S}$-manifold.

The relationships between the different classes of weak manifolds (considered in this article) can be summarized in the well-known diagram for classical structures:

$$\left|\begin{array}{c}\text{framed weak}\\ f\text{-manifold}\end{array}\right| \xrightarrow{metric} \left|\begin{array}{c}\text{metric weak}\\ f\text{-manifold}\end{array}\right| \xrightarrow{\Phi=d\eta^i} \left|\begin{array}{c}\text{weak}\\ f\text{-contact}\end{array}\right| \xrightarrow{\xi_i\text{-}Killing} \left|\begin{array}{c}\text{weak}\\ f\text{-K-contact}\end{array}\right|.$$

For $s = 1$, we get the following diagram:

$$\left|\begin{array}{c}\text{weak almost}\\ \text{contact}\end{array}\right| \xrightarrow{metric} \left|\begin{array}{c}\text{weak almost}\\ \text{contact metric}\end{array}\right| \xrightarrow{\Phi=d\eta} \left|\begin{array}{c}\text{weak contact}\\ \text{metric}\end{array}\right| \xrightarrow{\xi\text{-}Killing} \left|\begin{array}{c}\text{weak}\\ \text{K-contact}\end{array}\right|.$$

Remark 3.

(a) *In [5], an f-contact manifold was called an almost $\mathcal{S}$-manifold. Analogously, in [24], a weak f-contact manifold was called a weak almost $\mathcal{S}$-manifold. For $s = 1$, a weak (almost) $\mathcal{S}$-manifold is a weak (almost) Sasakian manifold, see [3].*

(b) *For a weak f-contact manifold, the tensors $N^{(2)}_i$ and $N^{(4)}_{ij}$ vanish; moreover, $N^{(3)}_i$ vanishes if and only if ξ_i is a Killing vector field, see ([24], Theorem 2.2).*

The (1,1)-tensor $\widetilde{Q} = Q - \mathrm{id}$ is useful for weak f-contact manifolds. Note that $\widetilde{Q} = 0$ is true for f-contact manifolds. We get $[\widetilde{Q}, f] = 0$ and $\widetilde{Q}\,\xi_i = 0$.

Remark 4. *The Levi-Civita connection ∇ of a Riemannian metric g is given by*

$$\begin{aligned} 2\,g(\nabla_X Y, Z) &= X\,g(Y,Z) + Y\,g(X,Z) - Z\,g(X,Y)\\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X), \end{aligned}$$

and has the properties, for example, [21,27],

$$\begin{aligned} X\,g(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \text{(metric compatible)},\\ [X,Y] &= \nabla_X Y - \nabla_Y X \quad \text{(without torsion)}. \end{aligned}$$

Proposition 1 (see Corollary 2.1 in [24])**.** *For a weak f-contact structure, we get*

$$\begin{aligned} g((\nabla_X f)Y, Z) &= \tfrac{1}{2}\,g(N^{(1)}(Y,Z), fX) + g(fX, fY)\,\overline{\eta}(Z) - g(fX, fZ)\,\overline{\eta}(Y)\\ &+ \tfrac{1}{2}\,N^{(5)}(X,Y,Z), \end{aligned} \tag{9}$$

where $\overline{\eta} = \sum_i \eta^i$, and a skew-symmetric with respect to Y and Z tensor $N^{(5)}(X,Y,Z)$ is given by

$$\begin{aligned} &N^{(5)}(X,Y,Z) = (fZ)\,(g(X, \widetilde{Q}Y)) - (fY)\,(g(X, \widetilde{Q}Z))\\ &+ g([X, fZ], \widetilde{Q}Y) - g([X, fY], \widetilde{Q}Z) + g([Y, fZ] - [Z, fY] - f[Y,Z],\ \widetilde{Q}X). \end{aligned}$$

Only one new tensor $N^{(5)}$ (vanishing at $\widetilde{Q} = 0$), which supplements the sequence of well-known tensors $N^{(i)}$, $i = 1,2,3,4$, is needed to study the weak f-contact structure. For a weak metric f-structure, the tensor $N^{(5)}$ is more complicated, see ([24], Proposition 2.3). For a weak f-contact structure, we find particular values of the tensor $N^{(5)}$:

$$\begin{aligned} N^{(5)}(X,\xi_i,Z) &= -N^{(5)}(X,Z,\xi_i) = g(N_i^{(3)}(Z),\,\widetilde{Q}X),\\ N^{(5)}(\xi_i,Y,Z) &= g([\xi_i,fZ],\widetilde{Q}Y) - g([\xi_i,fY],\widetilde{Q}Z),\\ N^{(5)}(\xi_i,\xi_j,Y) &= N^{(5)}(\xi_i,Y,\xi_j) = 0. \end{aligned} \tag{10}$$

A distribution $\tilde{\mathcal{D}} \subset TM$ is called *totally geodesic* if $\nabla_X Y + \nabla_Y X \in \tilde{\mathcal{D}}$ for any vector fields $X, Y \in \tilde{\mathcal{D}}$ – this is the case when any geodesic of M that is tangent to $\tilde{\mathcal{D}}$ at one point is tangent to $\tilde{\mathcal{D}}$ at all its points, e.g., ([21], Section 1.3.1). An integrable (i.e., tangent to a foliation) and totally geodesic distribution determines a totally geodesic foliation.

The Riemannian curvature tensor is given by the expression, see [27],

$$R_{X,Y}\,Z = \nabla_X\nabla_Y\,Z - \nabla_Y\nabla_X\,Z - \nabla_{[X,Y]}\,Z.$$

Proposition 2. *For the weak f-contact structure, the characteristic distribution* $\ker f$ *is tangent to a totally geodesic foliation with flat leaves (that is $R_{\xi_i,\xi_j}\,\xi_k = 0$).*

Proof. We need to prove the following equality (from which $R_{\xi_i,\xi_j}\,\xi_k = 0$ follows):

$$\nabla_{\xi_i}\,\xi_j = 0,\qquad 1 \le i,j \le s. \tag{11}$$

Taking $X = \xi_i$ in (9) and using $(10)_2$, we get

$$2\,g((\nabla_{\xi_i} f)Y,Z) = g([\xi_i,fZ],\widetilde{Q}Y) - g([\xi_i,fY],\widetilde{Q}Z),\quad 1 \le i \le s. \tag{12}$$

Then, taking $Y = \xi_j$ in (12), we get $f\nabla_{\xi_i}\,\xi_j = 0$, that is $\nabla_{\xi_i}\,\xi_j$ is orthogonal to $\mathcal{D}$. Also,

$$\eta^k([\xi_i,\xi_j]) = -2\,d\eta^k(\xi_i,\xi_j) = -2\,g(\xi_i,f\xi_j) = 0;$$

hence, $[\xi_i,\xi_j] = 0$, i.e., $\nabla_{\xi_i}\,\xi_j = \nabla_{\xi_j}\,\xi_i$. From $g(\xi_j,\xi_k) = \delta_{jk}$, using the covariant derivative with respect to ξ_i and the above equality, we get $\nabla_{\xi_i}\,\xi_j \in \mathcal{D}$. Thus, (11) is true. By (11), the distribution $\ker f$ is integrable (i.e., $[X,Y] = \nabla_X\,Y - \nabla_Y\,X$ belongs to $\mathcal{D}$ for any vector fields X,Y from $\mathcal{D}$) and totally geodesic (i.e., $\nabla_X\,Y + \nabla_Y\,X$ belongs to $\mathcal{D}$ for any vector fields X,Y from $\mathcal{D}$) with flat leaves. $\square$

Remark 5. *(i) Proposition 2 was proved in ([24], Section 2) for some special cases of weak f-contact manifolds. The proof that $\nabla_{\xi_i}\,\xi_j$ is orthogonal to $\mathcal{D}$ requires some calculations with the tensor $N^{(5)}$.*
(ii) By (8) we get $g([X,\xi_i],\xi_k) = 2\,d\eta^k(\xi_i,X) = 2\,\Phi(\xi_i,X) = 0$. Using this and (11) gives the following equality for the weak f-contact structure:

$$g(\nabla_X\,\xi_i,\ \xi_k) = g(\nabla_{\xi_i}X,\xi_k) = -g(\nabla_{\xi_i}\xi_k,X) = 0,\quad X \in TM,\ \ 1 \le i,k \le s. \tag{13}$$

3.2. The Splitting Tensor of a Weak f-Contact Manifold

For a weak f-contact manifold, the *splitting tensor* (or, co-nullity tensor) $C : \ker f \times \mathcal{D} \to \mathcal{D}$ is defined by

$$C_\xi(X) = -P(\nabla_X\,\xi),\quad X \in \mathcal{D},\ \ \xi \in \ker f,\ \ \|\xi\| = 1,$$

where $P : TM \to \mathcal{D}$ is the orthoprojector. Since $\ker f$ defines a totally geodesic foliation, see Proposition 2, then the distribution $\mathcal{D}$ is totally geodesic if and only if C_ξ is skew-symmetric, and $\mathcal{D}$ is integrable if and only if C_ξ is symmetric.

Thus, $C_\xi \equiv 0$ if and only if $\mathcal{D}$ is integrable and defines a totally geodesic foliation; in this case, by de Rham Decomposition Theorem, the manifold splits (is locally the product of Riemannian manifolds defined by distributions $\mathcal{D}$ and $\ker f$), e.g., [21]. The self-adjoint shape operator A_ξ and the skew-symmetric operator T_ξ, on $\mathcal{D}$ are given by

$$A_\xi = (1/2)(C_\xi + C_\xi^*), \quad T_\xi = (1/2)(C_\xi - C_\xi^*), \tag{14}$$

where * is the conjugation of a (1,1)-tensor. Thus, the splitting tensor is decomposed as

$$C_\xi = A_\xi + T_\xi \tag{15}$$

into the sum of skew-symmetric and self-adjoint tensors dual to (1,2)-tensors: the second fundamental form and the integrability tensor, respectively [21]. If $\mathcal{D}$ is integrable (tangent to a foliation), then $C_\xi = A_\xi$, and if $\mathcal{D}$ is totally geodesic, then $C_\xi = T_\xi$.

The tensors $N_i^{(3)}$ are important for f-contact manifolds. Therefore, we define for a weak f-contact manifold the tensor field $h = (h_1, \ldots, h_s)$, where

$$h_i = (1/2)\, N_i^{(3)} = (1/2)\, \pounds_{\xi_i} f.$$

By Remark 3b, $h_i = 0$ if and only if ξ_i is a Killing vector field. Next, we calculate

$$\begin{aligned}(\pounds_{\xi_i} f)X &= \nabla_{\xi_i}(fX) - \nabla_{fX}\,\xi_i - f(\nabla_{\xi_i}X - \nabla_X\,\xi_i) \\ &= (\nabla_{\xi_i} f)X - \nabla_{fX}\,\xi_i + f\nabla_X\,\xi_i.\end{aligned} \tag{16}$$

Taking $X = \xi_i$ in (16) and using $\nabla_{\xi_i}\xi_j = 0$ and $(\nabla_{\xi_i} f)\,\xi_j = 0$, see (12) with $Y = \xi_j$, we get

$$h_i\,\xi_j = 0 \quad (i,j = 1,\ldots,s). \tag{17}$$

Proposition 3. *For a weak f-contact manifold $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$, the tensor h_i and its conjugate h_i^* satisfy*

$$(h_i - h_i^*)X = (1/2)\, N^{(5)}(\xi_i, X, \cdot), \quad X \in \mathfrak{X}_M, \tag{18}$$

$$h_i f + f\,h_i = -(1/2)\,\pounds_{\xi_i}\widetilde{Q}, \tag{19}$$

$$h_i Q - Q\,h_i = (1/2)[\,f, \pounds_{\xi_i}\widetilde{Q}\,]. \tag{20}$$

Proof. The equalities (18) and (19) were proved in [24]. Using (1), (17) and (19),

$$\begin{gathered} h_i Q - Q\,h_i = h_i f^2 - f^2\,h_i = h_i f^2 - f(-h_i f - (1/2)\,\pounds_{\xi_i} Q) \\ = h_i f^2 + (-h_i f - (1/2)\,\pounds_{\xi_i} Q)f + (1/2)\,f\,(\pounds_{\xi_i} Q) = (1/2)\,[\,f\,(\pounds_{\xi_i} Q) - (\pounds_{\xi_i} Q)f\,],\end{gathered}$$

and taking into account that $\pounds_{\xi_i} Q = \pounds_{\xi_i}\widetilde{Q}$, we get (20). □

Proposition 3 generalizes the following well-known properties of f-contact manifolds: the linear operator h_i is self-adjoint and anti-commutes with f. Also, for f-contact manifolds, the equality $\nabla\,\xi_i = -f - fh_i$ holds, see [5]; thus, their splitting tensor is

$$C_{\xi_i} = f + fh_i \quad (i = 1,\ldots,s).$$

Since $h_i = 0$ for f-K-contact manifolds, their splitting tensor has the view $C_{\xi_i} = f$ and is skew-symmetric. Some of these properties are generalized in the following proposition, the proof of which requires some calculations with the tensor $N^{(5)}$.

Theorem 1. *The splitting tensor of a weak f-contact manifold $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$ has the following view:*

$$C_{\xi_i} = f + Q^{-1} f h_i^* \quad (i = 1,\ldots,s). \tag{21}$$

For a weak f-K-contact manifold we get the following skew-symmetric splitting tensor:

$$C_{\xi_i} = f \quad (i = 1,\ldots,s). \tag{22}$$

Proof. From Proposition 1 with $Y = \xi_i$, we find

$$g((\nabla_X f)\xi_i, Z) = \frac{1}{2}\, g(N^{(1)}(\xi_i, Z), fX) - g(fZ, fX) + \frac{1}{2}\, N^{(5)}(X, \xi_i, Z). \tag{23}$$

Note that $\frac{1}{2}\, N^{(5)}(X, \xi_i, Z) = g(h_i Z, \widetilde{Q}X)$, see (10). From (4) with $Y = \xi_i$ and (6), we get

$$[f, f](X, \xi_i) = f^2[X, \xi_i] - f[fX, \xi_i] = f(\pounds_{\xi_i} f)X. \tag{24}$$

Using (3) and (24), we calculate

$$\begin{aligned} g(N^{(1)}(\xi_i, Z), fX) &= g([f, f](\xi_i, Z), fX) = g((\pounds_{\xi_i} f)Z, f^2 X) \\ &= -g((\pounds_{\xi_i} f)Z, QX) + \textstyle\sum_j \eta^j(X)\, \eta^j((\pounds_{\xi_i} f)Z). \end{aligned} \tag{25}$$

Using (16), we get

$$2\, g((\nabla_{\xi_i} f)Z, \xi_j) = N^{(5)}(\xi_i, Z, \xi_j) = 0. \tag{26}$$

From (16) and (26), using (13), we get

$$g((\pounds_{\xi_i} f)X, \xi_j) = -g(\nabla_{fX}\, \xi_i, \xi_j) \overset{(13)}{=} 0. \tag{27}$$

Using $f\,\xi_i = 0$, gives $(\nabla_X f)\,\xi_i = -f\,\nabla_X\,\xi_i$. Thus, combining (23), (25) and (27), we find

$$g(f\,\nabla_X\,\xi_i, Z) = g(X, QZ) + g(h_i Z, X) - \textstyle\sum_j \eta^j(X)\, \eta^j(Z). \tag{28}$$

Replacing Z by fZ in (28) and using (1) and $f\,\xi_i = 0$, we achieve

$$g(Q\,\nabla_X\,\xi_i, Z) = g((fQ + h_i f)Z, X) = -g((fQ + fh_i^*)X, Z),$$

that is $QC_{\xi_i} = fQ + fh_i^*$, see (21). This, under assumption $h_i = 0$, implies (22). □

For a weak f-K-contact manifold, using the property $h_i = h_i^* = 0$ in Proposition 3, we obtain the following equalities for $i = 1, \ldots, s$:

$$\pounds_{\xi_i}\,\widetilde{Q} = 0, \qquad N^{(5)}(\xi_i, \,\cdot\,, \,\cdot\,) = 0. \tag{29}$$

The following corollary generalizes the known properties of f-contact manifolds.

Corollary 1. *Let a weak f-contact manifold $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$ satisfy (29a), then*

$$\operatorname{trace} h_i = 0 \quad (i = 1, \ldots, s);$$

if, in addition, (29b) holds, then the components of C_{ξ_i} in (15) are as follows: $T_{\xi_i} = f$ and $A_{\xi_i} = Q^{-1} f h_i$ for $i = 1, \ldots, s$.

Proof. By the assumptions, from Proposition 3 we get:

$$h_i = h_i^*, \quad h_i f + f h_i = 0, \quad Q\, h_i = h_i Q \quad (i = 1, \ldots, s).$$

If $h_i X = \lambda X$, then using $h_i f = -f h_i$, we get $h_i f X = -\lambda f X$. Thus, if λ is an eigenvalue of h_i, then $-\lambda$ is also an eigenvalue of h_i; hence, trace $h_i = 0$. Since f is skew-symmetric and $Q^{-1} f h_i$ is self-adjoint, by (21) and the equality (29), we prove the claim. □

If a plane contains unit vectors $\xi \in \ker f$ and $X \in \mathcal{D}$, then its sectional curvature is called *ξ-sectional curvature*. The ξ-sectional curvature of an f-contact manifold is the *mixed sectional curvature* of an almost product manifold $(M, g; \ker f, \mathcal{D})$, see [21]. Recall that a Riemannian manifold (M, g) equipped with complementary orthogonal distributions $(\mathcal{D}_1, \mathcal{D}_2)$ is called a *Riemannian almost product manifold*, e.g., [21].

The Jacobi operator R_ξ $(\xi \in \ker f,\ \|\xi\| = 1)$ is defined as $R_\xi : X \to R_{X,\xi}\,\xi$, e.g., [21]. We generalize the property of an f-K-contact manifold that the ξ-sectional curvature is constant and equal to 1, or, equivalently, $R_{\xi_i}(X) = X$ $(X \in \mathcal{D})$, see [17]. Again, the proof requires some calculations with the tensor $N^{(5)}$.

Theorem 2. *For a weak f-K-contact manifold, the ξ-sectional curvature is positive, or, equivalently, the Jacobi operator R_ξ $(\xi \in \ker f,\ \|\xi\| = 1)$ is positive definite on $\mathcal{D}$.*

Proof. By (10), using $N_i^{(3)} = 0$, we get

$$N^{(5)}(\,\cdot\,, \xi_i, \cdot\,) = 0 \quad (i = 1, \ldots, s). \tag{30}$$

By (9) with $Y = \xi_i$, using (30), we get

$$g((\nabla_X f)\,\xi_i, Z) \;=\; g(f^2 X, Z) \quad (i = 1, \ldots, s).$$

Hence, $(\nabla_X f)\,\xi_i = f^2 X$. From this and

$$0 = \nabla_X (f\,\xi_i) = (\nabla_X f)\,\xi_i + f \nabla_X\,\xi_i \quad (i = 1, \ldots, s),$$

we obtain the equality $f \nabla_X\,\xi_i = -f^2 X$. Since f is non-degenerate on $\mathcal{D}$, we get

$$\nabla_X\,\xi_i = -fX \quad (i = 1, \ldots, s). \tag{31}$$

Using (31), we derive some components of the curvature tensor,

$$\begin{aligned} R_{Z,X}\,\xi_i &= \nabla_Z(\nabla_X\,\xi_i) - \nabla_X(\nabla_Z\,\xi_i) - \nabla_{[Z,X]}\,\xi_i) \\ &= \nabla_X(fZ) - \nabla_Z(fX) + f([Z,X]) = (\nabla_X f)Z - (\nabla_Z f)X. \end{aligned} \tag{32}$$

Note that $(\nabla_X \Phi)(Y, Z) = -g((\nabla_X f)Y, Z)$. Using condition $d\Phi = d^2\eta^i = 0$, we get

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y) = 0. \tag{33}$$

From (32), using (33) and skew-symmetry of Φ, we get

$$\begin{aligned} g(R_{\xi_i,X}\,Y, Z) \;&=\; g(R_{Y,Z}\,\xi_i, X) = (\nabla_Z\,\Phi)(X, Y) + (\nabla_Y\,\Phi)(Z, X) \\ &= -(\nabla_X\,\Phi)(Y, Z) = g((\nabla_X f)Y, Z). \end{aligned} \tag{34}$$

By (34) with $Y = \xi_i$, using $f\,\xi_i = 0$ and (31), we find

$$R_{\xi_i,X}\,\xi_i = (\nabla_X f)\,\xi_i = -f \nabla_X\,\xi_i = f^2 X \quad (i = 1, \ldots, s). \tag{35}$$

By (35), $g(R_{\xi_i}(X), X) = g(-f^2 X, X) = g(fX, fX) > 0$ for any $X \neq 0$. $\square$

The *maximal number of point-wise linearly independent vector fields on a sphere* S^{n-1} is denoted by $\rho(n) - 1$. The topological invariant $\rho(n)$, called the *Adams number*, is

$$\rho((\text{odd})\,2^{4b+c}) = 8b + 2^c \;\; \text{for any integers} \;\; b \geq 0,\ 0 \leq c \leq 3,$$

see Table 1, and the inequality $\rho(n) \leq 2\,\log_2 n + 2$ is valid, for example, ([21], Section 1.4.4). There are not many theorems in differential geometry that use $\rho(n)$. Applying the Adams number, we obtain a topological obstruction to the existence of weak f-K-contact manifolds.

Table 1. The number of vector fields on the $(n-1)$-sphere.

$n-1$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
$\rho(n)-1$	1	3	1	7	1	3	1	8	1	3	1	7	1	3	1

Theorem 3. *For a weak f-K-contact manifold $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$ we have $s < \rho(2n)$.*

Proof. For the weak f-contact structure, the following Riccati equation is true, e.g., [21]:

$$\nabla_\xi C_\xi + (C_\xi)^2 + R_\xi = 0 \quad (\xi \in \ker f).$$

Since C_ξ is skew-symmetric for a weak f-K-contact manifold, i.e., $C_\xi = T_\xi$ and $A_\xi = 0$, see (14), and R_ξ is self-adjoined, the Riccati equation splits into two equations on $\mathcal{D}$:

$$\nabla_\xi T_\xi = 0 \text{ (the skew-symmetric part)}, \quad (T_\xi)^2 = -R_\xi \text{ (the self-adjoint part)}.$$

By this and Theorem 2, we get $C_\xi(Y) \neq 0$ for any $\xi \neq 0$ and $Y \neq 0$. Note that a skew-symmetric linear operator T_ξ can only have zero real eigenvalues. Thus, for any point $x \in M$, the following continuous vector fields, $C_{\xi_i}(Y)$, where $\|Y\| = 1$ and $1 \leq i \leq s$, are tangent to the unit sphere $S^{2n-1} \subset (\ker f)_x$. If $s \geq \rho(2n)$, then these vector fields are linearly dependent at some point $\tilde{Y} \in S^{2n-1}$ with weights λ_i, i.e., $\sum_i \lambda_i C_{\xi_i}(\tilde{Y}) = 0$. Then the co-nullity tensor has a real eigenvector $C_\xi(\tilde{Y}) = \lambda\,\tilde{Y}$, where $\xi = \sum_i \lambda_i\,\xi_i \neq 0$ and $\lambda = \langle C_\xi(\tilde{Y}), \tilde{Y}\rangle = 0$, which is a contradiction. Thus, the inequality $s < \rho(2n)$ holds. $\square$

3.3. Integral Formulas on Closed Weak f-Contact Manifolds

The integral formulas we are considering in this section are obtained by applying the Divergence Theorem to the appropriate vector fields.

The *elementary symmetric functions* of any $2n$-by-$2n$ matrix C are the coefficients of t^i in the following equality, e.g., ([21], p. 57): $\sum_{i=0}^{2n} \sigma_i\, t^i = \det(\mathrm{id} + t\, C)$. The *Newton transformations* $\mathrm{T}_r(C)$ of a $2n$-by-$2n$ matrix C are defined as, e.g., [21]:

$$\mathrm{T}_r(C) = \sum\nolimits_{j=0}^{r} (-1)^j \sigma_{r-j}(\xi) C^j = \sigma_r\,\mathrm{id} - \sigma_{r-1} C + \ldots + (-1)^r C^r.$$

For example, $\mathrm{T}_0(C) = \mathrm{id}$ and $\mathrm{T}_{2n}(C) = 0$ (by the Cayley–Hamilton Theorem).

For a weak f-K-contact manifold, by the skew symmetry $C^*_\xi = -C_\xi$ (the distribution $\mathcal{D}$ is totally geodesic), we get $\sum_{i=0}^{2n} \sigma_i(\xi)\, t^i = \sum_{i=0}^{2n} \sigma_i(\xi)\,(-t)^i$ for all $t \in \mathbb{R}$; thus, $\sigma_{2j-1}(\xi) = 0$ $(i > 0)$ and $\sigma_{2j}(\xi)$ of C_ξ are given by $\sum_{j=0}^{n} \sigma_{2j}(\xi)\, t^{2j} = \det(\mathrm{id} + t\, C_\xi)$.

Let $S^\perp = \{\xi \in \ker f : \|\xi\| = 1\}$ be the unit sphere bundle with the Sasaki metric and the volume form $\omega^\perp$. The natural projection $\pi : S^\perp \to M$ is a Riemannian submersion with totally geodesic fibers F—unit spheres $\{S^\perp_q\}_{q\in M}$. Thus, the volume form of $S^\perp$ is decomposed as, see [27],

$$d\,\mathrm{vol}(S^\perp) = d\,\mathrm{vol}(F)\, d\,\mathrm{vol}(M),$$

and the differentiating along M commutes with the integration on the fibers $S^\perp_q$.

Proposition 4. *For the weak f-contact structure (f, Q, ξ_i, η^i, g) on a closed manifold M^{2n+s}, the following integral formulas are true:*

$$\int_{\xi\in S^\perp} \{(r+2)\,\sigma_{r+2}(\xi) - \mathrm{trace}\,(\mathrm{T}_r(C_\xi)\, R_\xi)\}\, d\,\omega^\perp = 0, \quad r \geq 0. \tag{36}$$

Proof. The characteristic distribution of a weak f-contact manifold defines a totally geodesic foliation. Thus, (36) follows from the result of ([28], Corollary 4.1). $\square$

Remark 6. *The integrals over $S^\perp_q$ when $s > 1$ can be reduced to sums. To show this, let $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $y = (y_1, \ldots, y_s)$. Then, see, for example, [28],*

$$I_\lambda := \int_{\|y\|=1} y^\lambda \, d\,\omega_{s-1} = \frac{2}{\Gamma\big(\frac{s}{2} + \frac{1}{2}\sum_{i\le s}\lambda_i\big)} \prod\nolimits_{i\le s} \frac{1}{2}\,(1+(-1)^{\lambda_i})\,\Gamma\Big(\frac{1+\lambda_i}{2}\Big),$$

where $y^\lambda = \prod_{i\le s} y_i^{\lambda_i}$, and Γ is the Gamma function. For example,

$$I_{0,\ldots 0} = \frac{2\,\pi^{s/2}}{\Gamma(s/2)} = \mathrm{Vol}(S^{s-1}_1), \quad I_{2\lambda_1,0,\ldots 0} = 2\,\pi^{\frac{s-1}{2}}\frac{\Gamma(1/2+\lambda_1)}{\Gamma(s/2+\lambda_1)}.$$

For $s = 1$ we integrate in (36) not over $S^\perp$, but over M. The following formula is similar to the result in ([28], Example 5.7) for geodesic vector fields.

Corollary 2. *For the weak contact structure (f, ξ, η, g) on a closed manifold M^{2n+1}, we get*

$$\int_M \big\{(r+2)\,\sigma_{r+2}(\xi) - \mathrm{trace}\,(\mathrm{T}_r(C_\xi)R_\xi)\big\}\,d\,\mathrm{vol} = 0, \quad r \ge 0. \tag{37}$$

Remark 7. *Using the equality $\mathrm{Ric}(\xi,\xi) = \mathrm{trace}\ R_\xi$, we reduce (37) for $r = 0$ to the integral formula $\int_M (2\,\sigma_2(\xi) - \mathrm{Ric}(\xi,\xi))\,d\,\mathrm{vol} = 0$. For a weak K-contact manifold, the integrand in the above formula vanishes. Indeed, by (35), we get $\mathrm{Ric}(\xi,\xi) = \mathrm{trace}\ f^2$. On the other hand, $\mathrm{trace}\ C_\xi = 0$ and $2\,\sigma_2(\xi) = (\mathrm{trace}\ C_\xi)^2 - \mathrm{trace}\ C_\xi^2 = -\,\mathrm{trace}\ f^2$.*

The *mixed scalar curvature* of a Riemannian almost product manifold is the function

$$\mathrm{S}_{\rm mix} = \sum\nolimits_{a,i} g(R_{E_a,\mathcal{E}_i} E_a, \mathcal{E}_i),$$

where $\{\mathcal{E}_i, E_a\}$ is an adapted orthonormal frame, i.e., $\{E_a\} \subset \mathcal{D}_1$ and $\{\mathcal{E}_i\} \subset \mathcal{D}_2$. Let b_i and H_i be the second fundamental form and the mean curvature vector, and T_i be the integrability tensor of the distribution $\mathcal{D}_i$. The following formula [29],

$$\mathrm{S}_{\rm mix} = \mathrm{div}(H_1+H_2) - \|b_1\|^2 - \|b_2\|^2 + \|H_1\|^2 + \|H_2\|^2 + \|T_1\|^2 + \|T_2\|^2 \tag{38}$$

(see also (36) for $r = 0$), has many applications in Riemannian geometry, see [21], and counterparts in Kähler and Sasakian geometries, see [30].

Proposition 5. *For the weak f-contact structure (f, Q, ξ_i, η^i, g) on a closed manifold M^{2n+s} with conditions (29), the following integral formula is true:*

$$\int_M \Big\{\sum\nolimits_i \big(\mathrm{Ric}(\xi_i,\xi_i) + \|Q^{-1}fh_i\|^2 - (\mathrm{trace}\ Q^{-1}fh_i)^2\big) - s\,\|f\|^2\Big\}\,d\,\mathrm{vol} = 0. \tag{39}$$

Proof. According to (38), set $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_2 = \ker f$. Then $b_2 = H_2 = T_2 = 0$ and

$$b_1(X,Y) = \sum\nolimits_i g(Q^{-1}fh_iX,Y)\,\xi_i, \ \ H_1 = \sum\nolimits_i \mathrm{trace}\,(Q^{-1}fh_i)\,\xi_i, \ \ T_1(X,Y) = g(fX,Y)\,\overline{\xi},$$

where $\overline{\xi} = \sum_i \xi_i$. For a weak f-contact manifold we have $\mathrm{S}_{\rm mix} = \sum_i \mathrm{Ric}(\xi_i,\xi_i)$. Thus, (39) is the counterpart of (38) integrated on a closed Riemannian manifold using the Divergence Theorem and Corollary 1. □

4. Conclusions

It was shown that the weak f-contact structure, in particular, its splitting tensor, is a useful tool for studying totally geodesic foliations, Killing vector fields, positiveness of ξ-sectional curvature and other topics of extrinsic geometry of foliations [21]. Some results for f-contact structure have been extended to certain weak structures and can be generalized for such structures with indefinite metrics (see [4] for $Q = \mathrm{id}$). Integral formulas for closed Riemannian manifolds equipped with distributions have counterparts in weak f-contact geometry; the concepts of contact holomorphic distribution and harmonic morphism, see [30], can be applied to weak f-contact manifolds to produce more integral formulas and Bochner-type results. Based on applications of the weak f-contact structure in contact geometry considered in the article, we expect that this structure will also be fruitful in physics, e.g., in QFT. In conclusion, we pose several questions. Is the condition "the ξ-sectional curvature is positive" sufficient for a weak f-contact manifold to be weak f-K-contact? Does a weak f-contact manifold of dimension greater than 3 have some positive ξ-sectional curvature? Is a compact weak f-K-contact Einstein manifold an $\mathcal{S}$-manifold? When is a given weak f-K-contact manifold a mapping torus (see [17]) of a manifold of lower dimension? When does a weak f-contact manifold equipped with a Ricci-type soliton structure carry a canonical (for example, with constant sectional curvature or Einstein-type) metric? One could answer these questions by generalizing some deep results on f-contact manifolds for weak f-contact manifolds.

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