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Certain Results on the Lifts from an LP-Sasakian Manifold to Its Tangent Bundle Associated with a Quarter-Symmetric Metric Connection

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Abstract: The purpose of this study is to examine the complete lifts from the symmetric and concircular symmetric n-dimensional Lorentzian para-Sasakian manifolds (briefly, $(LPS)_n$) to its tangent bundle TM associated with a Riemannian connection D^C and a quarter-symmetric metric connection (QSMC) $\bar{D}^C$.

Keywords: complete lift; vertical lift; tangent bundle; Lorentzian para-Sasakian manifold; mathematical operators; quarter-symmetric metric connection; concircular curvature tensor; partial differential equations

MSC: 53C05; 53C25; 58A30

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Citation: Khan, M.N.I.; Mofarreh, F.; Haseeb, A.; Saxena, M. Certain Results on the Lifts from an LP-Sasakian Manifold to Its Tangent Bundle Associated with a Quarter-Symmetric Metric Connection. *Symmetry* **2023**, *15*, 1553. https://doi.org/10.3390/sym15081553

Academic Editor: Abraham A. Ungar

Received: 17 July 2023
Revised: 2 August 2023
Accepted: 7 August 2023
Published: 8 August 2023

1. Introduction

In 1924, the theory of semi-symmetric linear connection on a differentiable manifold was given by Friedmann and Schouten [1]. Later, Hayden [2] introduced the concept of metric connection with torsion on a Riemannian manifold. Approximately five decades ago, Yano established a relation between the semi-symmetric metric connection and the Levi-Civita connection [3]. As a generalization of semi-symmetric connection, the idea of quarter-symmetric connection was proposed by Golab [4].

A linear connection $\bar{D}$ in a differentiable manifold M (dim $M = n$) is said to be a quarter-symmetric connection [4] if its torsion tensor T is of the type

$$T(\beta_1, \beta_2) = \bar{D}_{\beta_1}\beta_2 - \bar{D}_{\beta_2}\beta_1 - [\beta_1, \beta_2] = \eta(\beta_2)\phi\beta_1 - \eta(\beta_1)\phi\beta_2, \tag{1}$$

where η is a 1-form and ϕ is a tensor field of type (1, 1).

If the connection $\bar{D}$ satisfies the condition $(\bar{D}_{\beta_1} g)(\beta_2, \beta_3) = 0$, for all $\beta_1, \beta_2, \beta_3$ on M, then $\bar{D}$ is said to be a QSMC.

The study of semi-symmetric and quarter-symmetric connections was further developed by many geometers, such as [5–12], among many others.

On the other hand, Matsumoto [13] proposed the idea of LP-Sasakian manifolds in 1989. Subsequently, the same notion was independently introduced by Mihai and Rosca [14] and obtained a number of key results. Numerous geometers worked out on LP-Sasakian manifolds and contributed a number of interesting results. For more details, we refer [15–19] and the references therein.

In differential geometry, the tangent bundles play an important role to investigate the geometrical structures of the manifold and their properties such as integrability conditions, curvature conditions, partial differential equations etc. Yano and Ishihara [20] introduced and studied almost complex structures with some basic properties induced in tangent bundles. Recently, Khan et al. [8] studied the lifts of a QSMC from a Sasakian manifold to its tangent bundle TM. Li et al. [21–31] did a series of theoretic research and development and application of singularity theory and submanifolds theory etc., which also deepens relevant research subjects. For more detail studies about the subject we recommend the papers [32–43] and the reference therein.

In this paper, we investigate the complete lifts from $(LPS)_n$ to its tangent bundle TM associated with a connection D^C and a QSMC $\bar{D}^C$. The following key conclusions are drawn:

- We established the relationship between D^C and $\bar{D}^C$ on TM of an $(LPS)_n$.
- We derived the curvature tensor, the Ricci tensor and the scalar curvature associated with the connection $\bar{D}^C$ on TM of an $(LPS)_n$.
- We proved that the tangent bundle TM of an $(LPS)_n$ is symmetric and ϕ^C-symmetric with respect to (wrt) the connection $\bar{D}^C$ if and only if it is so wrt D^C.
- We proved that the tangent bundle TM of an $(LPS)_n$ is concircular symmetric and concircular ϕ^C-symmetric wrt $\bar{D}^C$ if and only if it is so wrt D^C.
- We proved that the tangent bundle TM of an $(LPS)_n$ is concircular symmetric and concircular ϕ^C-symmetric wrt $\bar{D}^C$ if and only if it is symmetric wrt D^C subject to r^C constant.

Notations: Let $\Im_0^1(M), \Im_1^0(M), \Im_1^1(M)$ be the set of vector fields, the set of 1-forms and the set of tensor fields of type (1,1) in M, respectively. Similarly, we assume that $\Im_0^1(TM)$, $\Im_1^0(TM), \Im_1^1(TM)$ be the set of vector fields, the set of 1-forms and the set of tensor fields of type (1,1) in the tangent bundle TM, respectively.

2. Preliminaries

A manifold M $(dim\ M = n)$, endowed with a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g is an $(LPS)_n$ if [13,14]:

$$\eta(\xi) = -1, \tag{2}$$
$$\phi\beta_1 = \beta_1 + \eta(\beta_1)\xi, \tag{3}$$
$$g(\phi\beta_1, \phi\beta_2) = g(\beta_1, \beta_2) + \eta(\beta_1)\eta(\beta_2), \tag{4}$$
$$g(\beta_1, \xi) = \eta(\beta_1), \tag{5}$$
$$D_{\beta_1}\xi = \phi\beta_1, \tag{6}$$
$$(D_{\beta_1}\phi)\beta_2 = g(\beta_1, \beta_2)\xi + \eta(\beta_2)\beta_1 + 2\eta(\beta_1)\eta(\beta_2)\xi, \tag{7}$$

and

$$\phi\xi = 0,\ \ \eta(\phi\beta_1) = 0,\ rank\phi = n - 1. \tag{8}$$

If we put

$$\Phi(\beta_1, \beta_2) = g(\beta_1, \phi\beta_2), \forall \beta_1, \beta_2 \in \Im_0^1(M), \tag{9}$$

then Φ is a symmetric tensor field of type (2,0). As η is closed, then we infer [13,44]

$$(D_{\beta_1}\eta)(\beta_2) = \Phi(\beta_1, \beta_2),\ \ \Phi(\beta_1, \xi) = 0, \tag{10}$$

for all $\beta_1, \beta_2 \in \Im_0^1(M)$.

In an $(LPS)_n$, we have

$$\begin{aligned} g(R(\beta_1,\beta_2)\beta_3,\xi) &= \eta(R(\beta_1,\beta_2)\beta_3) \\ &= g(\beta_2,\beta_3)\eta(\beta_1) - g(\beta_1,\beta_3)\eta(\beta_2), \quad (11) \\ R(\xi,\beta_1)\beta_2 &= g(\beta_1,\beta_2)\xi - \eta(\beta_2)\beta_1, \quad (12) \\ R(\beta_1,\beta_2)\xi &= \eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2, \quad (13) \\ S(\beta_1,\xi) &= (n-1)\eta(\beta_1), \quad (14) \\ S(\phi\beta_1,\phi\beta_2) &= S(\beta_1,\beta_2) + (n-1)\eta(\beta_1)\eta(\beta_2), \quad (15) \end{aligned}$$

for all $\beta_1, \beta_2, \beta_3 \in \Im_0^1(M)$, where R and S denote the Riemannian curvature tensor and the Ricci tensor of M, respectively.

In an M_n, the curvature tensor R of $\bar{D}$ is given by

$$R(\beta_1,\beta_2)\beta_3 = \bar{D}_{\beta_1}\bar{D}_{\beta_2}\beta_3 - \bar{D}_{\beta_2}\bar{D}_{\beta_1}\beta_3 - \bar{D}_{[\beta_1,\beta_2]}\beta_3.$$

Now let $\{\jmath_1, \jmath_2, \jmath_3 \ldots\ldots, \jmath_n = \xi\}$ be a frame of orthonormal basis of the tangent space at any point of M_n. Then we find S and r as follows:

$$S(\beta_1,\beta_2) = \sum_{i=1}^{n} \epsilon_i g(R(\jmath_i,\beta_1)\beta_2,\jmath_i),$$

$$r = \sum_{i=1}^{n} \epsilon_i S(\jmath_i,\jmath_i),$$

respectively, where r is the scalar curvature of M and $\epsilon_i = g(\jmath_i,\jmath_i) = +1$ or -1.

Example 1. *Let us consider a 3-manifold $M_3 = \{(u,v,w) : u,v,w \in \Re^3, w \neq 0\}$. Let $\jmath_1, \jmath_2, \jmath_3$ be linearly independent vector fields on M_3 given by*

$$\jmath_1 = e^w \frac{\partial}{\partial u}, \;\; \jmath_2 = e^{w-au}\frac{\partial}{\partial v}, \;\; \jmath_3 = \frac{\partial}{\partial w} = \xi,$$

where $a(\neq 0)$ is a constant. Let g be the Lorentzian metric and η be a 1-form on M_3 given by

$$g(\jmath_1,\jmath_2) = g(\jmath_1,\jmath_3) = g(\jmath_2,\jmath_3) = 0, \;\; g(\jmath_1,\jmath_1) = g(\jmath_2,\jmath_2) = 1, \;\; g(\jmath_3,\jmath_3) = -1$$

and

$$\eta(\jmath_3) = g(\jmath_3,\xi), \;\; \jmath_3 \in \Im_0^1(M).$$

Let ϕ be the (1,1) tensor field defined by $\phi\jmath_1 = -\jmath_1, \phi\jmath_2 = -\jmath_2, \phi\jmath_3 = 0$. By using the linearity of ϕ and g, we acquire $\eta(\xi) = -1$, $\phi^2\jmath_1 = \jmath_1 + \eta(\jmath_1)\xi$ and $g(\phi\jmath_1,\phi\jmath_2) = g(\jmath_1,\jmath_2) + \eta(\jmath_1)\eta(\jmath_2)$. Thus for $\jmath_3 = \xi$, the structure (ϕ,ξ,η,g) is a Lorentzian paracontact structure on M_3.

Let D be the Levi-Civita connection wrt the Lorentzian metric g, then we have

$$[\jmath_1,\jmath_2] = -ae^w\jmath_2, \;\; [\jmath_1,\jmath_3] = -\jmath_1, \;\; [\jmath_2,\jmath_3] = -\jmath_2.$$

By using the Koszul's formula for the Lorentzian metric g, we infer

$$\begin{aligned} D_{\jmath_1}\jmath_1 &= -\jmath_3,\; D_{\jmath_1}\jmath_2 = 0,\; D_{\jmath_1}\jmath_3 = -\jmath_1, \quad (16) \\ D_{\jmath_2}\jmath_1 &= ae^w\jmath_2,\; D_{\jmath_2}\jmath_2 = -ae^w\jmath_1 - \jmath_3,\; D_{\jmath_2}\jmath_3 = -\jmath_2, \quad (17) \\ D_{\jmath_3}\jmath_1 &= 0,\; D_{\jmath_3}\jmath_2 = 0,\;\; D_{\jmath_3}\jmath_3 = 0, \quad (18) \\ (D_{\jmath_1}\phi)\jmath_2 &= g(\jmath_1,\jmath_2)\xi + \eta(\jmath_2)\jmath_1 + 2\eta(\jmath_1)\eta(\jmath_2)\xi. \quad (19) \end{aligned}$$

From the above relations, it can be easily seen that for $\jmath_3 = \xi$, (ϕ,ξ,η,g) is an LP-Sasakian structure on M_3. Consequently, $M_3(\phi,\xi,\eta,g)$ is an $(LPS)_3$.

Let $\jmath_1^C, \jmath_2^C, \jmath_3^C$ be the complete lifts and $\jmath_1^V, \jmath_2^V, \jmath_3^V$ be the vertical lifts on TM of $\jmath_1, \jmath_2, \jmath_3$ on M_3. Let g^C be the complete lift of the Lorentzian metric g on TM such that

$$\begin{aligned} g^C(\xi^V, \jmath_3^C) &= (g^C(\xi, \jmath_3))^V = (\eta(\xi))^V, && (20)\\ g^C(\xi^C, \jmath_3^C) &= (g^C(\xi, \jmath_3))^C = (\eta(\xi))^C, && (21)\\ g^C(\jmath_3^C, \jmath_3^C) &= 1,\ g^V(\xi^V, \jmath_3^C) = 0,\ g^V(\jmath_3^V, \jmath_3^V) = 0, && (22)\end{aligned}$$

and so on.

Let ϕ^C and ϕ^V be the complete and vertical lifts of ϕ defined by

$$\phi^V(\jmath_1^V) = -\jmath_1^V,\ \phi^C(\jmath_1^C) = -\jmath_1^C,$$
$$\phi^V(\jmath_2^V) = -\jmath_2^V,\ \phi^C(\jmath_2^C) = -\jmath_2^C,$$
$$\phi^V(\jmath_3^V) = \phi^C(\jmath_3^C) = 0.$$

By using the linearity of ϕ and g, we infer

$$(\phi^2\xi)^C = \xi^C + \eta^V(\xi)\jmath_1^C + \eta^C(\xi)\jmath_3^V, \tag{23}$$

$$\begin{aligned} g^C((\phi\jmath_1)^C, (\phi\jmath_2)^C) &= g^C(\jmath_1^C, \jmath_2^C) + (\eta(\jmath_1))^C(\eta(\jmath_2))^V \\ &+ (\eta(\jmath_1))^V(\eta(\jmath_2))^C. \end{aligned}$$

Thus, for $\jmath_3 = \xi$ in (20), (21) and (23), the structure $(\phi^C, \xi^C, \eta^C, g^C)$ is a Lorentzian paracontact structure on TM and satisfies the relation

$$\begin{aligned} (D_{\jmath_1^C}\phi^C)\jmath_2^C &= g^C(\jmath_1^C, \jmath_2^C)\xi^V + g^C(\jmath_1^V, \jmath_2^C)\xi^C \\ &+ 2\{\eta^C(\jmath_2^C)\jmath_1^V + \eta^V(\jmath_2^C)\jmath_1^C\}. \end{aligned}$$

Then, $(\phi^C, \xi^C, \eta^C, g^C, TM)$ is an $(LPS)_3$.

Definition 1. *An $(LPS)_n$ wrt D is said to be symmetric if [45]*

$$(D_{\beta_4}R)(\beta_1, \beta_2)\beta_3 = 0,$$

for all $\beta_1, \beta_2, \beta_3, \beta_4 \in \Im_0^1(M)$.

Definition 2. *An $(LPS)_n$ wrt D is said to be ϕ-symmetric if [45]*

$$\phi^2(D_{\beta_4}R)(\beta_1, \beta_2)\beta_3 = 0,$$

for all $\beta_1, \beta_2, \beta_3, \beta_4 \in \Im_0^1(M)$.

Definition 3. *An $(LPS)_n$ wrt D is said to be concircular symmetric if [45]*

$$(D_{\beta_4}\bar{C})(\beta_1, \beta_2)\beta_3 = 0,$$

for all $\beta_1, \beta_2, \beta_3, \beta_4 \in \Im_0^1(M)$, where $\bar{C}$ is the concircular curvature tensor given by [45]

$$\bar{C}(\beta_1, \beta_2)\beta_3 = R(\beta_1, \beta_2)\beta_3 - \frac{r}{n(n-1)}[g(\beta_2, \beta_3)\beta_1 - g(\beta_1, \beta_3)\beta_2]. \tag{24}$$

Definition 4. *An $(LPS)_n$ is called concircular ϕ-symmetric if*

$$\phi^2(D_{\beta_4}\bar{C})(\beta_1, \beta_2)\beta_3 = 0,$$

for all $\beta_1, \beta_2, \beta_3, \beta_4 \in \Im_0^1(M)$.

3. Lifts of a QSMC from an $(LPS)_n$ to Its TM

Let $TM = \bigcup_{p\in M} T_pM$ be the tangent bundle over the manifold M, where T_pM denotes the set of all tangent vectors of the manifold M at a point p. Let β_1, η, ϕ and D be a vector field, a 1-form, a tensor field of type (1,1) and an affine connection on the manifold M, respectively. Then, ${\beta_1}^V, \eta^V, \phi^V, D^V$ and ${\beta_1}^C, \eta^C, \phi^C, D^C$ are the vertical and complete lifts of a vector field, a 1-form, a tensor field of type (1,1) and an affine connection, respectively in TM [46,47].

The complete and vertical lifts by mathematical operators are given by

$$\begin{aligned}
\eta^V({\beta_1}^C) &= \eta^C({\beta_1}^V) = \eta(\beta_1)^V, \;\; \eta^C({\beta_1}^C) = \eta(\beta_1)^C, \\
\phi^V {\beta_1}^C &= (\phi\beta_1)^V, \;\; \phi^C {\beta_1}^C = (\phi\beta_1)^C, \\
[\beta_1, \beta_2]^V &= [{\beta_1}^C, {\beta_2}^V] = [{\beta_1}^V, {\beta_2}^C], \;\; [\beta_1, \beta_2]^C = [{\beta_1}^C, {\beta_2}^C], \\
D^C_{{\beta_1}^C} {\beta_2}^C &= (D_{\beta_1}\beta_2)^C, \qquad D^C_{{\beta_1}^C} {\beta_2}^V = (D_{\beta_1}\beta_2)^V.
\end{aligned}$$

Taking the complete lifts of (2)–(7), by mathematical operators we infer

$$\begin{aligned}
\eta^C(\xi^C) &= -1, \\
(\phi^2\beta_1)^C &= {\beta_1}^C + \eta^C({\beta_1}^C)\xi^V + \eta^V({\beta_1}^C)\xi^C, \\
g^C((\phi\beta_1)^C, (\phi\beta_2)^C) &= g^C({\beta_1}^C, {\beta_2}^C) + \eta^C({\beta_1}^C)\eta^V({\beta_2}^C) \\
&+ \eta^V({\beta_1}^C)\eta^V({\beta_2}^C),
\end{aligned} \tag{25}$$

$$\begin{aligned}
g^C({\beta_1}^C, \xi^C) &= \eta^C({\beta_1}^C), \\
D^C_{{\beta_1}^C}\xi^C &= (\phi\beta_1)^C,
\end{aligned} \tag{26}$$

$$\begin{aligned}
(D^C_{{\beta_1}^C}\phi^C){\beta_2}^C &= g^C({\beta_1}^C, {\beta_2}^C)\xi^C + \eta^C({\beta_2}^C){\beta_1}^V + \eta^V({\beta_2}^C){\beta_1}^C \\
&+ 2\{\eta^C({\beta_1}^C)\eta^C({\beta_2}^C)\xi^V + \eta^C({\beta_1}^C)\eta^V({\beta_2}^C)\xi^C \\
&+ \eta^V({\beta_1}^C)\eta^C({\beta_2}^C)\xi^C,
\end{aligned} \tag{27}$$

for all ${\beta_1}^C, {\beta_2}^C \in \Im^1_0(TM)$.

From (8)–(10), we have

$$(\phi\xi)^C = 0, \;\; \eta^C(\phi\beta_1)^C = 0, \tag{28}$$

$$\Phi^C({\beta_1}^C, {\beta_2}^C) = g^C({\beta_1}^C, (\phi\beta_2)^C), \tag{29}$$

$$(D^C_{{\beta_1}^C}\eta^C)({\beta_2}^C) = \Phi^C({\beta_1}^C, {\beta_2}^C), \;\; \Phi^C({\beta_1}^C, \xi^C) = 0 \tag{30}$$

for all ${\beta_1}^C, {\beta_2}^C \in \Im^1_0(TM)$, then $\Phi^C({\beta_1}^C, {\beta_2}^C)$ is a symmetric tensor field.

Now taking the complete lifts of (11)–(15), we have

$$\begin{aligned}
g^C(R^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C, \xi^C) &= \eta^C(R^C({\beta_1}^C, {\beta_2}^C){\beta_3}^C) \\
&= g^C({\beta_2}^C, {\beta_3}^C)\eta^V({\beta_1}^C) + g^C({\beta_2}^V, {\beta_3}^C)\eta^C({\beta_1}^C) \\
&- g^C({\beta_1}^C, {\beta_3}^C)\eta^V({\beta_2}^C) \\
&- g^C({\beta_1}^V, {\beta_3}^C)\eta^C({\beta_2}^C), \\
R^C(\xi^C, {\beta_1}^C){\beta_2}^C &= g^C({\beta_1}^C, {\beta_2}^C)\xi^V + g^C({\beta_1}^V, {\beta_2}^C)\xi^C \\
&- \eta^C({\beta_2}^C){\beta_1}^V - \eta^V({\beta_2}^C){\beta_1}^C, \\
R^C({\beta_1}^C, {\beta_2}^C)\xi^C &= \eta^C({\beta_2}^C){\beta_1}^V + \eta^V({\beta_2}^C){\beta_1}^C \\
&- \eta^C({\beta_1}^C){\beta_2}^V - \eta^V({\beta_1}^C){\beta_2}^C, \\
S^C({\beta_1}^C, \xi^C) &= (n-1)\eta^C({\beta_1}^C),
\end{aligned}$$

$$
\begin{aligned}
S^C((\phi\beta_1)^C,(\phi\beta_2)^C) &= S^C({\beta_1}^C,{\beta_2}^C)+(n-1)\{\eta^C({\beta_1}^C)\eta^V({\beta_2}^C)\\
&+ \eta^V({\beta_1}^C)\eta^V({\beta_2}^C)\},
\end{aligned}
$$

for all ${\beta_1}^C,{\beta_2}^C \in \Im_0^1(TM)$, where R^C and S^C denote the complete lifts on TM of R and S, respectively.

4. An Expression of $\tilde{R}^C$ on TM of an $(LPS)_n$

In this section, we establish the relationship between D^C and $\bar{D}^C$ on TM of an $(LPS)_n$. Moreover, the curvature tensor $\tilde{R}^C$, the Ricci tensor $\tilde{S}^C$ and the scalar curvature $\tilde{r}^C$ associated to $\bar{D}^C$ on TM of an $(LPS)_n$ are derived.

Let M be an almost contact metric manifold with a Riemannian connection D and let TM be its tangent bundle. A linear connection $\bar{D}$ and the tensor H of type (1,1) are related by

$$\bar{D}_{\beta_1}\beta_2 = D_{\beta_1}\beta_2 + H(\beta_1,\beta_2). \tag{31}$$

For the connection $\bar{D}$ to be a QSMC in M, we have [4]

$$2H(\beta_1,\beta_2) = T'(\beta_1,\beta_2)+T'(\beta_2,\beta_1)+T(\beta_1,\beta_2), \tag{32}$$

where

$$g(T'(\beta_1,\beta_2),\beta_3) = g(T(\beta_3,\beta_1),\beta_2). \tag{33}$$

From (1) and (33), it follows that

$$T'(\beta_1,\beta_2) = \eta(\beta_1)\phi\beta_2 - g(\beta_1,\phi\beta_2)\xi. \tag{34}$$

By using (1) and (34) in (32), we obtain

$$H(\beta_1,\beta_2) = \eta(\beta_2)\phi\beta_1 - g(\beta_1,\phi\beta_2)\xi.$$

Thus, a QSMC $\bar{D}$ on an $(LPS)_n$ is expressed as

$$\bar{D}_{\beta_1}\beta_2 = D_{\beta_1}\beta_2 + \eta(\beta_2)\phi\beta_1 - g(\beta_1,\phi\beta_2)\xi.$$

Taking the complete lifts of (1), (31)–(34), we have

$$
\begin{aligned}
T^C({\beta_1}^C,{\beta_2}^C) &= \bar{D}^C_{{\beta_1}^C}{\beta_2}^C - \bar{D}^C_{{\beta_2}^C}{\beta_1}^C - [{\beta_1}^C,{\beta_2}^C]\\
&= \eta^C({\beta_2}^C)(\phi\beta_1)^V + \eta^V({\beta_1}^C)(\phi\beta_2)^C\\
&- \eta^C({\beta_2}^C)(\phi\beta_1)^V - \eta^V({\beta_1}^C)(\phi\beta_2)^C,
\end{aligned} \tag{35}
$$

$$\bar{D}^C_{{\beta_1}^C}{\beta_2}^C = D^C_{{\beta_1}^C}{\beta_2}^C + H^C({\beta_1}^C,{\beta_2}^C),$$

where

$$H^C({\beta_1}^C,{\beta_2}^C) = \frac{1}{2}[T^C({\beta_1}^C,{\beta_2}^C)+T'^C({\beta_1}^C,{\beta_2}^C)+T'^C({\beta_2}^C,{\beta_1}^C)] \tag{36}$$

and

$$g^C(T'^C({\beta_1}^C,{\beta_2}^C),{\beta_3}^C) = g^C(T^C({\beta_3}^C,{\beta_1}^C),{\beta_2}^C) \tag{37}$$

for all $\beta_1,\beta_2,\beta_3 \in \Im_0^1(M)$.

From (35) and (37), we acquire

$$
\begin{aligned}
T'^C({\beta_1}^C,{\beta_2}^C) &= \eta^C({\beta_1}^C)(\phi\beta_2)^V + \eta^V({\beta_1}^C)(\phi\beta_2)^C\\
&- g^C({\beta_1}^C,(\phi\beta_2)^C)\xi^C - g^C({\beta_1}^C,(\phi\beta_2)^V)\xi^C.
\end{aligned} \tag{38}
$$

By using (35) and (38) in (36), we have

$$\begin{aligned} H^C({\beta_1}^C,{\beta_2}^C) &= \eta^C({\beta_2}^C)(\phi\beta_1)^V + \eta^V({\beta_2}^C)(\phi\beta_1)^C \\ &- g^C({\beta_1}^C,(\phi\beta_2)^C)\xi^C - g^C({\beta_1}^C,(\phi\beta_2)^V)\xi^C, \end{aligned}$$

where H^C is the complete lift of H.

Thus, a QSMC $\bar{D}^C$ on TM of an $(LPS)_n$ wrt the Riemannian connection D^C is given by

$$\begin{aligned} \bar{D}^C_{{\beta_1}^C}\beta_2 &= D^C_{{\beta_1}^C}{\beta_2}^C + \eta^C({\beta_2}^C)(\phi\beta_1)^V + \eta^V({\beta_2}^C)(\phi\beta_1)^C \\ &- g^C({\beta_1}^C,(\phi\beta_2)^C)\xi^V - g^C({\beta_1}^V,(\phi\beta_2)^C)\xi^C. \end{aligned} \tag{39}$$

Thus, (39) is the relation between the connections D^C and $\bar{D}^C$ on TM of an $(LPS)_n$. Hence, we state the following theorem:

Theorem 1. *Let $\bar{D}$ be the QSMC on an $(LPS)_n$ and $\bar{D}^C$ be the complete lift of $\bar{D}$ on TM of the manifold. Then, the relation between D^C and $\bar{D}^C$ on TM is given by (39).*

Let $\tilde{R}$ be the curvature tensor wrt $\bar{D}$ on TM of an $(LPS)_n$. Then the curvature tensor $\tilde{R}^C$ wrt $\bar{D}^C$ on TM is defined by

$$\tilde{R}^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C = \bar{D}^C_{{\beta_1}^C}\bar{D}^C_{{\beta_2}^C}{\beta_3}^C - \bar{D}^C_{{\beta_2}^C}\bar{D}^C_{{\beta_1}^C}{\beta_3}^C - \bar{D}^C_{[{\beta_1}^C,{\beta_2}^C]}{\beta_3}^C, \tag{40}$$

where

$$\tilde{R}(\beta_1,\beta_2)\beta_3 = \bar{D}_{\beta_1}\bar{D}_{\beta_2}\beta_3 - \bar{D}_{\beta_2}\bar{D}_{\beta_1}\beta_3 - \bar{D}_{[\beta_1,\beta_2]}\beta_3.$$

From (39) we can easily find

$$\bar{D}^C_{{\beta_1}^C}\bar{D}^C_{{\beta_2}^C}{\beta_3}^C = \bar{D}^C_{{\beta_1}^C}D^C_{{\beta_2}^C}{\beta_3}^C - \bar{D}^C_{{\beta_1}^C}\eta^C({\beta_2}^C)(\phi\beta_3)^V - \bar{D}^C_{{\beta_1}^C}\eta^V({\beta_2}^C)(\phi\beta_3)^C. \tag{41}$$

From (39)–(41), we obtain

$$\begin{aligned} \tilde{R}^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C &= R^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C \\ &+ g^C({\beta_1}^C,(\phi\beta_3)^C)(\phi\beta_2)^V + g^C({\beta_1}^V,(\phi\beta_3)^C)(\phi\beta_2)^C \\ &- g^C({\beta_2}^C,(\phi\beta_3)^C)(\phi\beta_1)^V - g^C({\beta_2}^V,(\phi\beta_3)^C)(\phi\beta_1)^C \\ &+ \eta^V({\beta_1}^C)g^C({\beta_2}^C,{\beta_3}^C)\xi^C \\ &+ \eta^C({\beta_1}^C)g^C({\beta_2}^V,{\beta_3}^C)\xi^C + \eta^C X^C)g^C({\beta_2}^C,{\beta_3}^C)\xi^V \\ &- \eta^V({\beta_2}^C)g^C({\beta_1}^C,{\beta_3}^C)\xi^C - \eta^C({\beta_2}^C)g^C({\beta_1}^V,{\beta_3}^C)\xi^C \\ &- \eta^C Y^C)g^C({\beta_1}^C,{\beta_3}^C)\xi^V - \{\eta^V({\beta_1}^C)\eta^C({\beta_3}^C){\beta_2}^C \\ &+ \eta^C({\beta_1}^C)\eta^V({\beta_3}^C){\beta_2}^C + \eta^C({\beta_1}^C)\eta^C({\beta_3}^C){\beta_2}^V \\ &- \eta^V({\beta_2}^C)\eta^C({\beta_3}^C){\beta_1}^C - \eta^C({\beta_2}^C)\eta^V({\beta_3}^C){\beta_1}^C \\ &- \eta^C({\beta_2}^C)\eta^C({\beta_3}^C){\beta_1}^V\}, \end{aligned} \tag{42}$$

where $R^C({\beta_1}^C,{\beta_2}^C){\beta_3}^C$ is the curvature tensor of D^C. Thus a relation between the curvature tensors of TM associated to $\bar{D}^C$ and D^C is given by (42).

From (42), we obtain the following relation

$$\begin{aligned} \tilde{S}^C({\beta_2}^C,{\beta_3}^C) &= S^C({\beta_2}^C,{\beta_3}^C) - g^C({\beta_2}^C,(\phi\beta_3)^C)\psi^V - g^C({\beta_2}^V,(\phi\beta_3)^C)\psi^C \\ &+ (n-1)\{\eta^C({\beta_2}^C)\eta^V({\beta_3}^C) + \eta^V({\beta_2}^C)\eta^C({\beta_3}^C)\}. \end{aligned} \tag{43}$$

On contracting (43), we lead to

$$\tilde{r}^C = r^C - 2\psi^C\psi^V - (n-1), \quad \psi = trace\phi, \tag{44}$$

$\tilde{r}^C$ and r^C represent the scalar curvatures of $\bar{D}^C$ and D^C, respectively.

5. Symmetry on TM of an $(LPS)_n$ wrt $\bar{D}^C$

An $(LPS)_n$ is called symmetric wrt $\bar{D}$ if [48]

$$(\bar{D}_{\beta_4}\tilde{R})(\beta_1, \beta_2)\beta_3 = 0$$

for all $\beta_1, \beta_2, \beta_3, \beta_4 \in \Im_0^1(M)$.

Using (39), we have

$$\begin{aligned}
(\bar{D}^C_{\beta_4{}^C}\tilde{R}^C)(\beta_1{}^C, \beta_2{}^C)\beta_3{}^C &= ((D_{\beta_4}\tilde{R})(\beta_1, \beta_2)\beta_3)^C \\
&+ \eta^C(\tilde{R}(\beta_1, \beta_2)\beta_3)^C(\phi\beta_4)^V + \eta^V(\tilde{R}(\beta_1, \beta_2)\beta_3)^C(\phi\beta_4)^C \\
&- g^C(\beta_4, \phi\tilde{R}(\beta_1, \beta_2)\beta_3)^C\xi^V - g^C(\beta_4, \phi\tilde{R}(\beta_1, \beta_2)\beta_3)^C\xi^V \\
&- \eta^C(\beta_1{}^C)(\tilde{R})(\phi\beta_4, \beta_2)\beta_3)^V - \eta^V(\beta_1{}^C)(\tilde{R})(\phi\beta_4, \beta_2)\beta_3)^C \\
&- \eta^C(\beta_2{}^C)(\tilde{R})(\beta_1, \phi\beta_4)\beta_3)^V - \eta^V(\beta_2{}^C)(\tilde{R})(\beta_1, \phi\beta_4)\beta_3)^C \\
&- \eta^C(\beta_3{}^C)(\tilde{R}(\beta_1, \beta_2)\phi\beta_4)^V - \eta^V(\beta_3{}^C)(\tilde{R}(\beta_1, \beta_2)\phi\beta_4)^C \\
&+ g^C(\beta_4{}^C, (\phi\beta_1)^C)(\tilde{R}(\xi, \beta_2)\beta_3)^V \\
&+ g^C(\beta_4{}^V, (\phi\beta_1)^C)(\tilde{R}(\xi, \beta_2)\beta_3)^C \\
&+ g^C(\beta_4{}^C, (\phi\beta_2)^C)(\tilde{R}(\beta_1, \xi)\beta_3)^V \\
&+ g^C(\beta_4{}^V, (\phi\beta_2)^C)(\tilde{R}(\beta_1, \xi)\beta_3)^C \\
&+ g^C(\beta_4{}^C, (\phi\beta_3)^C)(\tilde{R}(\beta_1, \beta_2)\xi)^V \\
&+ g^C(\beta_4{}^V, (\phi\beta_3)^C)(\tilde{R}(\beta_1, \beta_2)\xi)^C.
\end{aligned} \tag{45}$$

By differentiating (42) wrt β_4 and using (26), (28) and (30), we lead to

$$(D^C_{\beta_4{}^C}\tilde{R}^C)(\beta_1{}^C, \beta_2{}^C)\beta_3{}^C = ((D_{\beta_4}R)(\beta_1, \beta_2)\beta_3)^C + \Theta^C(\beta_1{}^C, \beta_2{}^C, \beta_3{}^C, \beta_4{}^C), \tag{46}$$

where

$$\begin{aligned}
\Theta^C(\beta_1{}^C, \beta_2{}^C, \beta_3{}^C, \beta_4{}^C) &= -\{\eta^C(\beta_2{}^C)g^C(\beta_4{}^C, \beta_3{}^C)(\phi\beta_1)^V \\
&+ \eta^V(\beta_2{}^C)g^C(\beta_4{}^V, \beta_3{}^C)(\phi\beta_1)^C \\
&+ \eta^V(\beta_2{}^C)g^C(\beta_4{}^C, \beta_3{}^C)(\phi\beta_1)^C \\
&+ \eta^C(\beta_3{}^C)g^C(\beta_2{}^C, \beta_4{}^C)(\phi\beta_1)^V \\
&+ \eta^V(\beta_3{}^C)g^C(\beta_2{}^V, \beta_4{}^C)(\phi\beta_1)^C \\
&+ \eta^V(\beta_3{}^C)g^C(\beta_2{}^C, \beta_4{}^C)(\phi\beta_1)^C \\
&+ 2\{\eta^C(\beta_2{}^C)\eta^C(\beta_3{}^C)\eta^C(\beta_4{}^C)(\phi\beta_1)^V \\
&+ \eta^C(\beta_2{}^C)\eta^C(\beta_3{}^C)\eta^V(\beta_4{}^C)(\phi\beta_1)^C \\
&+ \eta^C(\beta_2{}^C)\eta^V(\beta_3{}^C)\eta^C(\beta_4{}^C)(\phi\beta_1)^C \\
&+ \eta^V(\beta_2{}^C)\eta^C(\beta_3{}^C)\eta^C(\beta_4{}^C)(\phi\beta_1)^C\} \\
&+ \eta^C(\beta_1{}^C)g^C(\beta_4{}^C, \beta_3{}^C)(\phi\beta_2)^V \\
&+ \eta^V(\beta_1{}^C)g^C(\beta_4{}^V, \beta_3{}^C)(\phi\beta_2)^C \\
&+ \eta^V(\beta_1{}^C)g^C(\beta_4{}^C, \beta_3{}^C)(\phi\beta_2)^C \\
&+ \eta^C(\beta_3{}^C)g^C(\beta_2{}^C, \beta_4{}^C)(\phi\beta_2)^V \\
&+ \eta^V(\beta_3{}^C)g^C(\beta_2{}^V, \beta_4{}^C)(\phi\beta_2)^C \\
&+ \eta^V(\beta_3{}^C)g^C(\beta_2{}^C, \beta_4{}^C)(\phi\beta_2)^C \\
&+ 2\{\eta^C(\beta_1{}^C)\eta^C(\beta_3{}^C)\eta^C(\beta_4{}^C)(\phi\beta_2)^V \\
&+ \eta^C(\beta_1{}^C)\eta^C(\beta_3{}^C)\eta^V(\beta_4{}^C)(\phi\beta_2)^C \\
&+ \eta^C(\beta_1{}^C)\eta^V(\beta_3{}^C)\eta^C(\beta_4{}^C)(\phi\beta_2)^C
\end{aligned} \tag{47}$$

$$
\begin{aligned}
&+ \quad \eta^V({\beta_1}^C)\eta^C({\beta_3}^C)\eta^C({\beta_4}^C)(\phi\beta_2)^C \\
&+ \quad g^C({\beta_1}^C,(\phi\beta_3)^C)g^C({\beta_4}^C,{\beta_2}^C)\xi^V \\
&+ \quad g^C({\beta_1}^C,(\phi\beta_3)^C)g^C({\beta_4}^V,{\beta_2}^C)\xi^C \\
&+ \quad g^C({\beta_1}^V,(\phi\beta_3)^C)g^C({\beta_4}^C,{\beta_2}^C)\xi^C \\
&+ \quad g^C({\beta_1}^C,(\phi\beta_3)^C)\eta^C({\beta_2}^C){\beta_4}^V \\
&+ \quad g^C({\beta_1}^C,(\phi\beta_3)^C)\eta^V({\beta_2}^C){\beta_4}^C \\
&+ \quad 2\{g^C({\beta_1}^C,(\phi\beta_3)^C)\eta^C({\beta_4}^C)\eta^C({\beta_4}^C)\xi^V \\
&+ \quad g^C({\beta_1}^C,(\phi\beta_3)^C)\eta^C({\beta_4}^C)\eta^V({\beta_4}^C)\xi^C \\
&+ \quad g^C({\beta_1}^C,(\phi\beta_3)^C)\eta^V({\beta_4}^C)\eta^C({\beta_4}^C)\xi^C \\
&+ \quad g^C({\beta_1}^V,(\phi\beta_3)^C)\eta^C({\beta_4}^C)\eta^C({\beta_4}^C)\xi^C\} \\
&- \quad g^C({\beta_2}^C,(\phi\beta_3)^C)g^C({\beta_4}^C,{\beta_1}^C)\xi^V \\
&- \quad g^C({\beta_2}^C,(\phi\beta_3)^C)g^C({\beta_4}^V,{\beta_1}^C)\xi^C \\
&- \quad g^C({\beta_2}^V,(\phi\beta_3)^C)g^C({\beta_4}^C,{\beta_1}^C)\xi^C \\
&- \quad g^C({\beta_2}^C,(\phi\beta_3)^C)\eta^C({\beta_1}^C){\beta_4}^V \\
&- \quad g^C({\beta_2}^C,(\phi\beta_3)^C)\eta^V({\beta_1}^C){\beta_4}^C \\
&- \quad 2\{g^C({\beta_2}^C,(\phi\beta_3)^C)\eta^C({\beta_4}^C)\eta^C({\beta_4}^C)\xi^V \\
&- \quad g^C({\beta_2}^C,(\phi\beta_3)^C)\eta^C({\beta_4}^C)\eta^V({\beta_4}^C)\xi^C \\
&- \quad g^C({\beta_2}^C,(\phi\beta_3)^C)\eta^V({\beta_4}^C)\eta^C({\beta_4}^C)\xi^C \\
&- \quad g^C({\beta_2}^V,(\phi\beta_3)^C)\eta^C({\beta_4}^C)\eta^C({\beta_4}^C)\xi^C\} \\
&+ \quad g^C({\beta_4}^C,(\phi\beta_1)^C)g^C({\beta_2}^C,{\beta_3}^C)\xi^V \\
&+ \quad g^C({\beta_4}^C,(\phi\beta_1)^C)g^C({\beta_2}^V,{\beta_3}^C)\xi^C \\
&+ \quad g^C({\beta_4}^V,(\phi\beta_1)^C)g^C({\beta_2}^C,{\beta_3}^C)\xi^C \\
&- \quad g^C({\beta_4}^C,(\phi\beta_2)^C)g^C({\beta_1}^C,{\beta_3}^C)\xi^V \\
&- \quad g^C({\beta_4}^C,(\phi\beta_2)^C)g^C({\beta_1}^V,{\beta_3}^C)\xi^C \\
&- \quad g^C({\beta_4}^V,(\phi\beta_2)^C)g^C({\beta_1}^C,{\beta_3}^C)\xi^C \\
&+ \quad \eta^C({\beta_1}^C)g^C({\beta_2}^C,{\beta_3}^C)(\phi\beta_4)^V \\
&+ \quad \eta^C({\beta_1}^C)g^C({\beta_2}^V,{\beta_3}^C)(\phi\beta_4)^C \\
&+ \quad \eta^V({\beta_1}^C)g^C({\beta_2}^C,{\beta_3}^C)(\phi\beta_4)^C \\
&- \quad \eta^C({\beta_2}^C)g^C({\beta_1}^C,{\beta_3}^C)(\phi\beta_4)^V \\
&- \quad \eta^C({\beta_2}^C)g^C({\beta_1}^V,{\beta_3}^C)(\phi\beta_4)^C \\
&- \quad \eta^V({\beta_2}^C)g^C({\beta_1}^C,{\beta_3}^C)(\phi\beta_4)^C \\
&+ \quad g^C({\beta_4}^C,(\phi\beta_2)^C)\eta^C({\beta_3}^C){\beta_1}^V \\
&+ \quad g^C({\beta_4}^C,(\phi\beta_2)^C)\eta^V({\beta_3}^C){\beta_1}^C \\
&+ \quad g^C({\beta_4}^V,(\phi\beta_2)^C)\eta^C({\beta_3}^C){\beta_1}^C \\
&- \quad g^C({\beta_4}^C,(\phi\beta_1)^C)\eta^C({\beta_3}^C){\beta_2}^V \\
&- \quad g^C({\beta_4}^C,(\phi\beta_1)^C)\eta^V({\beta_3}^C){\beta_2}^C \\
&- \quad g^C({\beta_4}^V,(\phi\beta_1)^C)\eta^C({\beta_3}^C){\beta_2}^C \\
&+ \quad g^C({\beta_4}^C,(\phi\beta_3)^C)\eta^C({\beta_2}^C){\beta_1}^V \\
&+ \quad g^C({\beta_4}^C,(\phi\beta_3)^C)\eta^V({\beta_2}^C){\beta_1}^C \\
&+ \quad g^C({\beta_4}^V,(\phi\beta_3)^C)\eta^C({\beta_2}^C){\beta_1}^C \\
&- \quad g^C({\beta_4}^C,(\phi\beta_3)^C)\eta^C({\beta_1}^C){\beta_2}^V \\
&- \quad g^C({\beta_4}^C,(\phi\beta_3)^C)\eta^V({\beta_1}^C){\beta_2}^C \\
&- \quad g^C({\beta_4}^V,(\phi\beta_3)^C)\eta^C({\beta_1}^C){\beta_2}^C.
\end{aligned}
$$

Using (25), (28) and (47) in (45), we infer

$$(\bar{D}^C_{\beta_4{}^C}\tilde{R}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C = (D^C_{\beta_4{}^C}R^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C. \tag{48}$$

Thus, we have the following:

Theorem 2. *The tangent bundle TM of an $(LPS)_n$ is symmetric wrt the connection $\bar{D}^C$ if it is so wrt the connection D^C.*

Corollary 1. *The tangent bundle TM of an $(LPS)_n$ is ϕ-symmetric wrt the connection $\bar{D}^C$ if it is so wrt the connection D^C.*

6. Concircular Symmetry on TM of an $(LPS)_n$ wrt $\bar{D}^C$

An $(LPS)_n$ is called concircular symmetric wrt $\bar{D}$ if [48]

$$(\bar{D}_{\beta_4}\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\beta_3 = 0,$$

for all $\beta_1,\beta_2,\beta_3,\beta_4$, where $\tilde{\bar{\mathcal{C}}}$ is the concircular curvature tensor wrt $\bar{D}$ given by

$$\tilde{\bar{\mathcal{C}}}(\beta_1,\beta_2)\beta_3 = \tilde{R}(\beta_1,\beta_2)\beta_3 - \frac{\tilde{r}}{n(n-1)}[g(\beta_2,\beta_3)\beta_1 - g(\beta_1,\beta_3)\beta_2], \tag{49}$$

where $\tilde{R}$ is the Riemannian curvature tensor and $\tilde{r}$ is the scalar curvature wrt $\bar{D}$.

Taking the complete lift of (49) and using (39), we have

$$\begin{aligned}(\bar{D}^C_{\beta_4{}^C}\tilde{\bar{\mathcal{C}}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= (D^C_{\beta_4{}^C}\tilde{\bar{\mathcal{C}}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C\\ &+ \eta^C(\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\beta_3)^C(\phi\beta_4)^V\\ &+ \eta^V(\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\beta_3)^C(\phi\beta_4)^C\\ &- g^C(\beta_4,\phi\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\beta_3)^C\xi^V\\ &- g^V(\beta_4,\phi\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\beta_3)^C\xi^C\\ &- \eta^C(\beta_1{}^C)(\tilde{\bar{\mathcal{C}}})(\phi\beta_4,\beta_2)\beta_3)^V\\ &- \eta^V(\beta_1{}^C)(\tilde{\bar{\mathcal{C}}})(\phi\beta_4,\beta_2)\beta_3)^C\\ &- \eta^C(\beta_2{}^C)(\tilde{\bar{\mathcal{C}}})(\beta_1,\phi\beta_4)\beta_3)^V\\ &- \eta^V(\beta_2{}^C)(\tilde{\bar{\mathcal{C}}})(\beta_1,\phi\beta_4)\beta_3)^C\\ &- \eta^C(\beta_3{}^C)(\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\phi\beta_4)^V\\ &- \eta^V(\beta_3{}^C)(\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\phi\beta_4)^C\\ &+ g^C(\beta_4{}^C,(\phi\beta_1)^C)((\tilde{\bar{\mathcal{C}}})(\xi,\beta_2)\beta_3)^C\\ &+ g^C(\beta_4{}^C,(\phi\beta_2)^C)((\tilde{\bar{\mathcal{C}}})(\beta_1,\xi)\beta_3)^C\\ &+ g^C(\beta_4{}^C,(\phi\beta_3)^C)((\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\xi)^C.\end{aligned} \tag{50}$$

Now differentiating (49) wrt β_4, we find

$$\begin{aligned}(D^C_{\beta_4^C}\tilde{\bar{\mathcal{C}}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= (D^C_{\beta_4^C}\tilde{R}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C\\ &- \frac{D^C_{\beta_4^C}\tilde{r}^C}{n(n-1)}\{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + g^C(\beta_2{}^V,\beta_3{}^C)\beta_1{}^C\\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - g^C(\beta_1{}^V,\beta_3{}^C)\beta_2{}^C\}.\end{aligned} \tag{51}$$

By using (24), (44) and (47) in (51), we have

$$\begin{aligned}(D^C_{\beta_4{}^C}\tilde{\bar{\mathcal{C}}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= ((D_{\beta_4}\bar{\mathcal{C}})(\beta_1,\beta_2)\beta_3))^C - \Theta^C(\beta_1{}^C,\beta_2{}^C,\beta_3{}^C,\beta_4{}^C)\\ &- \frac{D^C_{\beta_4{}^C}r^C - 2(\psi^C D^C_{\beta_4{}^C}\psi^V + \psi^C D^C_{\beta_4{}^C}\psi^V)}{n(n-1)}.\\ &\quad \{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + g^C(\beta_2{}^V,\beta_3{}^C)\beta_1{}^C\\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - g^C(\beta_1{}^V,\beta_3{}^C)\beta_2{}^C\}.\end{aligned} \tag{52}$$

Now, by using (25), (28) and (52) in (50), we arrive at

$$\begin{aligned}(D^C_{\beta_4{}^C}\bar{\mathcal{C}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= ((D^C_{\beta_4}\tilde{\bar{\mathcal{C}}})(\beta_1,\beta_2)\beta_3))^C\\ &- \frac{D^C_{\beta_4{}^C}r^C - 2(\psi^C D^C_{\beta_4{}^C}\psi^V + \psi^C D^C_{\beta_4{}^C}\psi^V)}{n(n-1)}\\ &\quad \{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + g^C(\beta_2{}^V,\beta_3{}^C)\beta_1{}^C\\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - g^C(\beta_1{}^V,\beta_3{}^C)\beta_2{}^C\}.\end{aligned}$$

Hence, we have the following:

Theorem 3. *The tangent bundle TM of an $(LPS)_n$ is concircular symmetric wrt $\bar{D}^C$ if it is so wrt D^C.*

Corollary 2. *The tangent bundle TM of an $(LPS)_n$ M is concircular ϕ-symmetric wrt $\bar{D}^C$ if it is so wrt D^C.*

By making use of (2), (8) and (52) in (50), it follows that

$$\begin{aligned}(\bar{D}^C_{\beta_4{}^C}\tilde{\bar{\mathcal{C}}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= (D^C_{\beta_4{}^C}R^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C\\ &- \frac{D^C_{\beta_4{}^C}\tilde{r}^C}{n(n-1)}\{g^C(\beta_2{}^C,\beta_3{}^C)\beta_1{}^V + g^C(\beta_2{}^V,\beta_3{}^C)\beta_1{}^C\\ &- g^C(\beta_1{}^C,\beta_3{}^C)\beta_2{}^V - g^C(\beta_1{}^V,\beta_3{}^C)\beta_2{}^C\}.\end{aligned} \tag{53}$$

If r is constant, then (53) takes the form

$$\begin{aligned}(\bar{D}^C_{\beta_4{}^C}\tilde{\bar{\mathcal{C}}}^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C &= (D^C_{\beta_4{}^C}R^C)(\beta_1{}^C,\beta_2{}^C)\beta_3{}^C\\ &+ \frac{2(\psi^C D^C_{\beta_4{}^C}\psi^V + \psi^C D^C_{\beta_4{}^C}\psi^V)}{n(n-1)}.\end{aligned}$$

Hence, we have the following:

Theorem 4. *The tangent bundle TM of an $(LPS)_n$ is concircular symmetric wrt $\bar{D}^C$ if it is symmetric wrt D^C subject to r^C constant.*

Corollary 3. *The tangent bundle TM of an $(LPS)_n$ is concircular ϕ^C-symmetric wrt $\bar{D}^C$ if it is symmetric wrt D^C subject to r^C constant.*

Author Contributions: Conceptualization, M.N.I.K., A.H., F.M. and M.S.; methodology, M.N.I.K., A.H., F.M. and M.S.; investigation, M.N.I.K., A.H., F.M. and M.S.; writing—original draft preparation, M.N.I.K., A.H. and F.M.; writing—review and editing, M.N.I.K., A.H. and M.S. All authors have read and agreed to the published version of the manuscript.

Funding: The author, F.M., expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the editor and anonymous referees for the constructive comments given to improve the quality of the paper. The second author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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