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New Results about Fuzzy Differential Subordinations Associated with Pascal Distribution

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Abstract: Based upon the Pascal distribution series $\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta) := \zeta + \sum\limits_{j=m+1}^{\infty} \binom{j+r-2}{r-1}[1+\lambda(j-1)]q^{j-1}(1-q)^r a_j\zeta^j$, we can obtain a set of fuzzy differential subordinations in this investigation. We also newly obtain class $P_{q,\lambda}^{F,r,m}(\eta)$ of univalent analytic functions defined by the operator $\mathcal{N}_{q,\lambda}^{r,m}$, give certain properties for the class $P_{q,\lambda}^{F,r,m}(\eta)$ and also obtain some applications connected with a special case for the operator. New research directions can be taken on fuzzy differential subordinations associated with symmetry operators.

Keywords: convolution; fuzzy differential subordination; Pascal distribution

MSC: 30C45; 30C50; 30C80

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1. Introduction

Let $\mathcal{H}_m(\varpi)$ represent the class of holomorphic and univalent functions on ϖ such that $\varpi \subset \mathbb{C}$ and let $\mathcal{H}(\varpi)$ denote the class of holomorphic functions on ϖ. The class of holomorphic functions in the open unit disk of the complex plane $\Lambda = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ is denoted in this study by a note $\mathcal{H}(\Lambda)$, with $B_\Lambda = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ standing as the unit disk's boundary. For $m \in \mathbb{N} = \{1, 2, ...\}$, we define

$$\mathcal{H}_m[\gamma] = \left\{\Upsilon \in \mathcal{H}(\Lambda) : \ \Upsilon(\zeta) = \gamma + \sum_{j=m+1}^{\infty} a_j\,\zeta^j, \ \ \zeta \in \Lambda\right\},$$

$$\mathcal{A}_m = \left\{\Upsilon \in \mathcal{H}(\Lambda) : \ \Upsilon(\zeta) = \zeta + \sum_{j=m+1}^{\infty} a_j\,\zeta^j, \ \zeta \in \Lambda\right\} \quad \text{with} \quad \mathcal{A}_1 = \mathcal{A}, \tag{1}$$

and

$$\mathcal{S} = \{\Upsilon \in \mathcal{A}_m : \ \Upsilon \text{ is a univalent function in } \Lambda\}.$$

We denote by

$$\mathcal{C} = \left\{\Upsilon \in \mathcal{A}_m : \Re\left(1 + \frac{\zeta\Upsilon''(\zeta)}{\Upsilon'(\zeta)}\right) > 0, \ \zeta \in \Lambda\right\},$$

which is the set of convex functions on Λ.

Let Υ_1 and Υ_2 be analytic in Λ. Then Υ_1 is subordinate to Υ_2 written as $\Upsilon_1 \prec \Upsilon_2$ if there exists a Schwarz function ϕ, which is analytic in Λ with $\phi(0) = 0$ and $|\phi(\zeta)| < 1$

for all $\zeta \in \Lambda$ such that $\Upsilon_1(\zeta) = \Upsilon_2(\phi(\zeta))$. Furthermore, if the function Υ_2 is univalent in Λ, then we have the following equivalence (see [1,2]):

$$\Upsilon_1(\zeta) \prec \Upsilon_2(\zeta) \Leftrightarrow \Upsilon_1(0) = \Upsilon_2(0) \text{ and } \Upsilon_1(\Lambda) \subset \Upsilon_2(\Lambda).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

Definition 1 ([3])**.** *Assume that $\mathcal{T} \neq \varnothing$ is a Fuzzy subset and $\mathcal{F} : \mathcal{T} \to [0,1]$ is an application. A pair of $(\Lambda, \mathcal{F}_\Lambda)$, where $\mathcal{F}_\Lambda : \mathcal{T} \to [0,1]$, and*

$$\mathcal{R} = \{x \in \mathcal{T} : \ 0 < \mathcal{F}_\Lambda(x) \leq 1\} = \sup(\Lambda, \mathcal{F}_\Lambda),$$

a fuzzy subset. The fuzzy set $(\Lambda, \mathcal{F}_\Lambda)$ is called a function $\mathcal{F}_\Lambda$.

Let $\Upsilon, g \in \mathcal{H}(\varpi)$ be denoted by

$$\Upsilon(\varpi) = \left\{\Upsilon(\zeta) : \ 0 < \mathcal{F}_{\Upsilon(\varpi)}\Upsilon(\zeta) \leq 1, \ \zeta \in \varpi\right\} = \sup\left(\Upsilon(\varpi), \mathcal{F}_{\Upsilon(\varpi)}\right), \tag{2}$$

and

$$g(\varpi) = \left\{g(\zeta) : \ 0 < \mathcal{F}_{g(\varpi)}g(\zeta) \leq 1, \ \zeta \in \varpi\right\} = \sup\left(g(\varpi), \mathcal{F}_{g(\varpi)}\right). \tag{3}$$

Proposition 1 ([4])**.** *(i) If $(\mathcal{B}, \mathcal{F}_\mathcal{B}) = (\mathcal{U}, \mathcal{F}_\mathcal{U})$, then we have $\mathcal{B} = \mathcal{U}$, where $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_\mathcal{B})$ and $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_\mathcal{U})$; (ii) if $(\mathcal{B}, \mathcal{F}_\mathcal{B}) \subseteq (\mathcal{U}, \mathcal{F}_\mathcal{U})$, then we have $\mathcal{B} \subseteq \mathcal{U}$, where $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_\mathcal{B})$ and $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_\mathcal{U})$.*

Definition 2 ([4])**.** *Let $\zeta_0 \in \varpi$ be a fixed point and let the functions $\Upsilon, g \in \mathcal{H}(\varpi)$. The function Υ is said to be fuzzy subordinate to g, and we write $\Upsilon \prec_\mathcal{F} g$ or $\Upsilon(\zeta) \prec_\mathcal{F} g(\zeta)$, which satisfies the following conditions:*

(i) $\Upsilon(\zeta_0) = g(\zeta_0)$;
(ii) $\mathcal{F}_{\Upsilon(\varpi)}\Upsilon(\zeta) \leq \mathcal{F}_{g(\varpi)}g(\zeta), \quad \zeta \in \varpi.$

Proposition 2 ([4])**.** *Assume that $\zeta_0 \in \omega$ is a fixed point and the functions $\Upsilon, \mathrm{g} \in H(\omega)$. If $\Upsilon(\zeta) \prec_\mathcal{F} g(\zeta)$, $\zeta \in \omega$, then*

(i) $\Upsilon(\zeta_0) = g(\zeta_0)$
(ii) $\Upsilon(\omega) \subseteq g(\omega), \ \ \mathcal{F}_{\Upsilon(\omega)}\Upsilon(\zeta) \leq \mathcal{F}_{g(\omega)}g(\zeta), \ \zeta \in \omega,$

where $\Upsilon(\omega)$ and $\mathrm{g}(\omega)$ are defined by (2) and (3), respectively.

Definition 3 ([5])**.** *Assume that $h \in \mathcal{S}$ and $\Phi : \mathbb{C}^3 \times \Lambda \to \mathbb{C}$, $\Phi(\alpha, 0, 0; 0) = h(0) = \alpha$. If p satisfies the requirements of the second-order fuzzy differential subordination and is analytic in Λ, with $p(0) = \alpha$,*

$$\mathcal{F}_{\Phi(\mathbb{C}^3 \times \Lambda)}\Phi\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta\right) \leq \mathcal{F}_{h(\Lambda)}h(\zeta). \tag{4}$$

If q is a fuzzy dominant of the fuzzy differential subordination solutions, then p is said to be a fuzzy solution of the fuzzy differential subordination and satisfies

$$\mathcal{F}_{p(\Lambda)}p(\zeta) \leq \mathcal{F}_{q(\Lambda)}q(\zeta), \quad i.e., \quad p(\zeta) \prec_\mathcal{F} q(\zeta), \ \ \zeta \in \Lambda,$$

for each and every p satisfying (4).

Definition 4. *A fuzzy dominant $\widetilde{q}$ that satisfies*

$$\mathcal{F}_{\widetilde{q}(\Lambda)}\widetilde{q}(\zeta) \leq \mathcal{F}_{q(\Lambda)}q(\zeta),$$

then

$$\widetilde{q}(\zeta) \prec_{\mathcal{F}} q(\zeta), \ \ \zeta \in \Lambda.$$

The fuzzy best dominant of (4) is referred to for all fuzzy dominants.

Assume the function $\Omega \in \mathcal{A}_m$ is given by

$$\Omega(\zeta) := \zeta + \sum_{j=m+1}^{\infty} \psi_j \, \zeta^j, \ \zeta \in \Lambda.$$

The Hadamard (or convolution) product of Υ and Ω is defined as

$$(\Upsilon * \Omega)(\zeta) := \zeta + \sum_{j=m+1}^{\infty} a_j \psi_j \, \zeta^j, \ \zeta \in \Lambda.$$

A variable x is said to have the *Pascal distribution* if it takes the values $0, 1, 2, 3, \ldots$ with the probabilities $(1-q)^r$, $\frac{qr(1-q)^r}{1!}$, $\frac{q^2 r(r+1)(1-q)^r}{2!}$, $\frac{q^3 r(r+1)(r+2)(1-q)^r}{3!}$, $\ldots$, respectively, where q and r are called the parameters, and thus we have the probability formula

$$P(X = k) = \binom{k+r-1}{r-1} q^k (1-q)^r, \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Now we present a power series whose coefficients are Pascal distribution probabilities, i.e.,

$$\mathrm{Q}^r_{q,m}(\zeta) := \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} q^{j-1} (1-q)^r \, \zeta^j, \ \zeta \in \Lambda,$$
$$(m \in \mathbb{N}, \ r \geq 1, \ 0 \leq q \leq 1).$$

We easily determine from the ratio test that the radius of convergence of the above power series is at least $\dfrac{1}{q} \geq 1$; hence, $\mathrm{Q}^r_{q,m} \in \mathcal{A}_m$.

We define the functions

$$\begin{aligned} \mathcal{M}^{r,m}_{q,\lambda}(\zeta) &:= (1-\lambda)\mathrm{Q}^r_{q,m}(\zeta) + \lambda z \left(\mathrm{Q}^r_{q,m}(\zeta)\right)' \\ &= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r \zeta^j, \ \zeta \in \Lambda, \\ &\quad (m \in \mathbb{N}, \ r \geq 1, \ 0 \leq q \leq 1, \ \lambda \geq 0). \end{aligned}$$

El-Deeb and Bulboacă [6] introduced the linear operator $\mathcal{N}^{r,m}_{q,\lambda} : \mathcal{A}_m \to \mathcal{A}_m$ defined by

$$\begin{aligned} \mathcal{N}^{r,m}_{q,\lambda}\Upsilon(\zeta) &:= \mathcal{M}^{r,m}_{q,\lambda}(\zeta) * \Upsilon(\zeta) \\ &= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r a_j \zeta^j, \ \zeta \in \Lambda, \\ &\quad (m \in \mathbb{N}, \ r \geq 1, \ 0 \leq q \leq 1, \ \lambda \geq 0), \end{aligned}$$

where Υ is given by (1), and the symbol "$*$" stands for the *Hadamard (or convolution) product*.

Remark 1. *(i) For $m = 1$, the operator $\mathcal{N}^{r,m}_{q,\lambda}$ reduces to $\mathcal{I}^r_{q,\lambda} := \mathcal{N}^{r,1}_{q,\lambda}$, introduced and studied by El-Deeb et al. [7]; (ii) for $m = 1$ and $\lambda = 0$, the operator Q^r_q reduces to $\mathrm{Q}^r_q := \mathcal{N}^{r,1}_{q,0}$, introduced and studied by El-Deeb et al. [7].*

Using the operator $\mathcal{N}_{q,\lambda}^{r,m}$, we create a class of analytical functions and derive several fuzzy differential subordinations for this class.

Definition 5. *If the function $\Upsilon \in \mathcal{A}$ belongs to the class $P_{q,\lambda}^{F,r,m}(\eta)$ for all $\eta \in [0,1)$ and satisfies the inequality*

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)' > \eta, \quad (\zeta \in \Lambda).$$

2. Preliminary

The following lemmas are needed to show our results.

Lemma 1 ([2]). *Assume that $\digamma \in \mathcal{A}$ and $\mathcal{G}(\zeta) = \frac{1}{\zeta}\int_0^\zeta \digamma(t)dt$, $\zeta \in \Lambda$. If $\Re\left\{1 + \frac{\zeta\digamma''(\zeta)}{\digamma'(\zeta)}\right\} > \frac{-1}{2}$, $\zeta \in \Lambda$, then $\mathcal{G} \in \mathcal{C}$.*

Lemma 2 (Theorem 2.6 in [8]). *If $\digamma$ is a convex function such that $\digamma(0) = \gamma$, $\nu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with $\Re(\nu) \geq 0$. If $p \in \mathcal{H}_m[\gamma]$ such that $p(0) = \gamma$, $\Phi : \mathbb{C}^2 \times \Lambda \to \mathbb{C}$, $\Phi\left(p(\zeta), \zeta p'(\zeta); \zeta\right) = p(\zeta) + \frac{1}{\nu}\zeta p'(\zeta)$ is an analytic function in Λ and*

$$\mathcal{F}_{\Phi(\mathbb{C}^2\times\Lambda)}\left(p(\zeta) + \frac{1}{\nu}\zeta p'(\zeta)\right) \leq \mathcal{F}_{h(\Lambda)}h(\zeta) \ \to p(\zeta) + \frac{1}{\nu}\zeta p'(\zeta) \prec_{\mathcal{F}} h(\zeta),\ \zeta \in \Lambda,$$

then

$$\mathcal{F}_{p(\Lambda)}p(\zeta) \leq \mathcal{F}_{q(\Lambda)}q(\zeta) \leq \mathcal{F}_{h(\Lambda)}h(\zeta) \ \to \ p(\zeta) \prec_{\mathcal{F}} q(\zeta),\ \zeta \in \Lambda,$$

where

$$q(\zeta) = \frac{\nu}{m\,\zeta^{\frac{\nu}{m}}}\int_0^\zeta \psi(t)t^{\frac{\nu}{m}-1}dt,\ \zeta \in \Lambda.$$

The function q is convex, and it is the fuzzy best dominant.

Lemma 3 (Theorem 2.7 in [8]). *Let g be a convex function in Λ and $\digamma(\zeta) = g(\zeta) + m\,\gamma\zeta g'(\zeta)$, where $\zeta \in \Lambda$, $m \in \mathbb{N}$ and $\gamma > 0$, if*

$$p(\zeta) = g(0) + p_m\,\zeta^m + p_{m+1}\,\zeta^{m+1} + \ldots \in \mathcal{H}(\Lambda),$$

and

$$\mathcal{F}_{p(\Lambda)}\left(p(\zeta) + \gamma\zeta p'(\zeta)\right) \leq \mathcal{F}_{\psi(\Lambda)}\psi(\zeta) \ \to \ p(\zeta) + \gamma\zeta p'(\zeta) \prec_{\mathcal{F}} \psi(\zeta),\ \zeta \in \Lambda.$$

Then

$$\mathcal{F}_{p(\Lambda)}(p(\zeta)) \leq \mathcal{F}_{g(\Lambda)}g(\zeta) \ \to \ p(\zeta) \prec_{\mathcal{F}} g(\zeta),\ \zeta \in \Lambda.$$

This result is sharp.

We define the fuzzy differential subordination general theory and its applications (see [9–13]). The method of fuzzy differential subordination is applied in the next section to obtain a set of fuzzy differential subordinations related to the operator $\mathcal{N}_{q,\lambda}^{r,m}$.

3. Main Results

Assume that $\eta \in [0,1)$, $m \in \mathbb{N}$, $r \geq 1$, $0 \leq q \leq 1$, $\lambda \geq 0$ and $\zeta \in \Lambda$ are mentioned throughout this paper.

Theorem 1. *Let k belong to $\mathcal{C}$ in Λ, and $h(\zeta) = k(\zeta) + \frac{1}{\rho+2}\zeta k^{'}(\zeta)$. If $\Upsilon \in P_{q,\lambda}^{F,r,m}(\eta)$ and*

$$\mathcal{G}(\zeta) = \mathcal{I}^{\rho}\Upsilon(\zeta) = \frac{\rho+2}{\zeta^{\rho+1}}\int_{0}^{\zeta} t^{\rho}\Upsilon(t)dt, \tag{5}$$

then

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)^{'}(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)^{'} \leq F_{h(\Lambda)}h(\zeta) \quad \rightarrow \quad \left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)^{'} \prec_{\mathcal{F}} h(\zeta), \tag{6}$$

implies

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}\right)^{'}(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'} \leq F_{k(\Lambda)}k(\zeta) \quad \rightarrow \quad \left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'} \prec_{\mathcal{F}} k(\zeta).$$

Proof. Since

$$\zeta^{\rho+1}\mathcal{G}(\zeta) = (\rho+2)\int_{0}^{\zeta} t^{\rho}\Upsilon(t)dt,$$

by differentiating, we obtain

$$(\rho+1)\mathcal{G}(\zeta) + \zeta\mathcal{G}^{'}(\zeta) = (\rho+2)\Upsilon(\zeta),$$

and

$$(\rho+1)\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta) + \zeta\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'} = (\rho+2)\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta), \tag{7}$$

and also, by differentiating (7), we obtain

$$\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'} + \frac{1}{(\rho+2)}\zeta\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{''} = \left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)^{'}. \tag{8}$$

The fuzzy differential subordination (6) technique is used

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)^{'}(\Lambda)}\left(\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'} + \frac{1}{(\rho+2)}\zeta\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{''}\right) \leq$$
$$F_{h(\Lambda)}\left(k(\zeta) + \frac{1}{(\rho+2)}\zeta k^{'}(\zeta)\right). \tag{9}$$

We denote

$$q(\zeta) = \left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'}, \text{ so } \; q \in \mathcal{H}_1[n]. \tag{10}$$

Putting (10) in (9), we have

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)^{'}(\Lambda)}\left(q(\zeta) + \frac{1}{(\rho+2)}\zeta q^{'}(\zeta)\right) \leq F_{h(\Lambda)}\left(k(\zeta) + \frac{1}{(\rho+2)}\zeta k^{'}(\zeta)\right). \tag{11}$$

Using Lemma (3), we obtain

$$F_{q(\Lambda)}q(\zeta) \leq F_{k(\Lambda)}k(\zeta), \quad \text{i.e.} \quad F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'}(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)^{'} \leq F_{k(\Lambda)}k(\zeta),$$

and therefore, $\left(\mathcal{N}_{q,\lambda}^{r,m}G(\zeta)\right)^{'} \prec_{\mathcal{F}} k(\zeta)$,where k is the fuzzy best dominant. □

Putting $m = 1$ and $\lambda = 0$ in Theorem 1, we obtain the following example since the operator Q_q^r reduces to $\mathrm{Q}_q^r := \mathcal{N}_{q,0}^{r,1}$.

Example 1. *Let k be an element of $\mathcal{C}$ in Λ and $h(\zeta) = k(\zeta) + \frac{1}{\rho+2}\zeta k'(\zeta)$. If $\Upsilon \in P_{q,\lambda}^{F,r,m}(\eta)$ and $\mathcal{G}$ is given by (5), then*

$$F_{(\mathrm{Q}_q^r\Upsilon)'(\Lambda)}\left(\mathrm{Q}_q^r\Upsilon(\zeta)\right)' \leq F_{h(\Lambda)}h(\zeta) \quad \rightarrow \quad \left(\mathrm{Q}_q^r\Upsilon(\zeta)\right)' \prec_{\mathcal{F}} h(\zeta),$$

implies

$$F_{(\mathrm{Q}_q^r\mathcal{G})'(\Lambda)}\left(\mathrm{Q}_q^r\mathcal{G}(\zeta)\right)' \leq F_{k(\Lambda)}k(\zeta) \quad \rightarrow \quad \left(\mathrm{Q}_q^r\mathcal{G}(\zeta)\right)' \prec_{\mathcal{F}} k(\zeta).$$

Theorem 2. *Assume that $h(\zeta) = \frac{1+(2\eta-1)\zeta}{1+\zeta}$, $\eta \in [0,1)$, $\lambda > 0$ and $\mathcal{I}^\rho$ is given by (5), then*

$$\mathcal{I}^\rho\left[P_{q,\lambda}^{F,r,m}(\eta)\right] \subset P_{q,\lambda}^{F,r,m}(\eta^*), \tag{12}$$

where

$$\eta^* = 2\eta - 1 + (\rho+2)(2-2\eta)\int_0^1 \frac{t^{\rho+2}}{t+1}dt. \tag{13}$$

Proof. A function h belongs to $\mathcal{C}$, and we obtain from the hypothesis of Theorem 2 using the same technique as that in the proof of Theorem 1 that

$$F_{q(\Lambda)}\left(q(\zeta) + \frac{1}{(\rho+2)}\zeta q'(\zeta)\right) \leq F_{h(\Lambda)}h(\zeta),$$

where $q(\zeta)$ is defined in (10). By using Lemma 2, we obtain

$$F_{q(\Lambda)}q(\zeta) \leq F_{k(\Lambda)}k(\zeta) \leq F_{h(\Lambda)}h(\zeta),$$

which implies

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)' \leq F_{k(\Lambda)}k(\zeta) \leq F_{h(\Lambda)}h(\zeta),$$

where

$$\begin{aligned} k(\zeta) &= \frac{\rho+2}{\zeta^{\rho+2}}\int_0^\zeta t^{\rho+1}\frac{1+(2\eta-1)t}{1+t}dt \\ &= (2\eta-1) + \frac{(\rho+2)(2-2\eta)}{\zeta^{\rho+2}}\int_0^\zeta \frac{t^{\rho+1}}{1+t}dt \in \mathcal{C}, \end{aligned}$$

where $k(\Lambda)$ is symmetric with respect to the real axis, so we have

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\mathcal{G}(\zeta)\right)' \geq \min_{|\zeta|=1} F_{k(\Lambda)}k(\zeta) = F_{k(\Lambda)}k(1), \tag{14}$$

and $\eta^* = k(1) = 2\eta - 1 + (\rho+2)(2-2\eta)\int_0^1 \frac{t^{\rho+2}}{t+1}dt$. □

Theorem 3. *Assume that k belongs to $\mathcal{C}$ in Λ, that $k(0)=1$, and that $h(\zeta)=k(\zeta)+\zeta k'(\zeta)$. When $\Upsilon\in\mathcal{A}$ and the fuzzy differential subordination is satisfied,*

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\leq F_{h(\Lambda)}h(\zeta)\quad\rightarrow\quad\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\prec_{\mathcal{F}}h(\zeta),\tag{15}$$

holds, then

$$F_{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\Lambda)}\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\leq F_{k(\Lambda)}k(\zeta)\quad\rightarrow\quad\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\prec_{\mathcal{F}}k(\zeta).\tag{16}$$

Proof. Let

$$\begin{aligned}q(\zeta)&=\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}=\frac{\zeta+\sum\limits_{j=m+1}^{\infty}\binom{j+r-2}{r-1}[1+\lambda(j-1)]q^{j-1}(1-q)^r a_j\zeta^j}{\zeta}\\&=1+\sum_{j=m+1}^{\infty}\binom{j+r-2}{r-1}[1+\lambda(j-1)]q^{j-1}(1-q)^r a_j\,\zeta^{j-1},\end{aligned}$$

and we obtain that

$$q(\zeta)+\zeta q'(\zeta)=\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)',$$

so

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\leq F_{h(\Lambda)}h(\zeta)$$

implies

$$F_{q(\Lambda)}\left(q(\zeta)+\zeta q'(\zeta)\right)\leq F_{h(\Lambda)}h(\zeta)=F_{k(\Lambda)}\left(k(\zeta)+\zeta k'(\zeta)\right).$$

Using the Lemma 3, we obtain

$$F_{q(\Lambda)}q(\zeta)\leq F_{k(\Lambda)}k(\zeta)\quad\rightarrow\quad F_{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\Lambda)}\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\leq F_{k(\Lambda)}k(\zeta),$$

and we obtain

$$\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\prec_{\mathcal{F}}k(\zeta).$$

□

Theorem 4. *Consider $h\in\mathcal{H}(\Lambda)$, which satisfies $\Re\left(1+\frac{\zeta h''(\zeta)}{h'(\zeta)}\right)>\frac{-1}{2}$ when $h(0)=1$. If the fuzzy differential subordination*

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\leq F_{h(\Lambda)}h(\zeta)\quad\rightarrow\quad\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\prec_{\mathcal{F}}h(\zeta),\tag{17}$$

then

$$F_{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\Lambda)}\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\leq F_{k(\Lambda)}k(\zeta)\quad i.e.\quad\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\prec_{\mathcal{F}}k(\zeta),\tag{18}$$

where

$$k(\zeta)=\frac{1}{\zeta}\int_0^{\zeta}h(t)dt,$$

the function k is convex, and it is the fuzzy best dominant.

Proof. Let

$$q(\zeta)=\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}=1+\sum_{j=d+1}^{\infty}\frac{[j]_q!}{[\lambda+1]_{q,j-1}}a_j\psi_j\,\zeta^{j-1},\quad q\in\mathcal{H}_1[1],$$

where $\Re\left(1+\frac{\zeta h''(\zeta)}{h'(\zeta)}\right)>\frac{-1}{2}$. From Lemma 1, we have

$$k(\zeta)=\frac{1}{\zeta}\int_0^{\zeta}h(t)dt\in$$

belongs to the class $\mathcal{C}$, which satisfies the fuzzy differential subordination (17). Since

$$k(\zeta)+\zeta k'(\zeta)=h(\zeta),$$

it is the fuzzy best dominant. We have

$$q(\zeta)+\zeta q'(\zeta)=\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)',$$

then (17) becomes

$$F_{q(\Lambda)}\left(q(\zeta)+\zeta q'(\zeta)\right)\le F_{h(\Lambda)}h(\zeta).$$

By using Lemma 3, we obtain

$$F_{q(\Lambda)}q(\zeta)\le F_{k(\Lambda)}k(\zeta),\quad \text{i.e.}\quad F_{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\Lambda)}\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\le F_{k(\Lambda)}k(\zeta),$$

then

$$\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\prec_{\mathcal{F}}k(\zeta).$$

□

Putting $h(\zeta)=\frac{1+(2v-1)\zeta}{1+\zeta}$ in Theorem 4. As a result, we have the following corollary:

Corollary 1. *Let $h=\frac{1+(2v-1)\zeta}{1+\zeta}$ be a convex function in Λ, with $h(0)=1$, $0\le\beta<1$. If $\Upsilon\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon\right)'(\Lambda)}\left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\le F_{h(\Lambda)}\left(\frac{1+(2v-1)\zeta}{1+\zeta}\right),\quad \text{i.e.}\quad \left(\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)\right)'\prec_{\mathcal{F}}\left(\frac{1+(2v-1)\zeta}{1+\zeta}\right),$$

then

$$F_{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\Lambda)}\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\le F_{k(\Lambda)}k(\zeta),\tag{19}$$

then

$$\frac{\mathcal{N}_{q,\lambda}^{r,m}\Upsilon(\zeta)}{\zeta}\prec_{\mathcal{F}}k(\zeta),$$

where

$$k(\zeta)=2v-1+\frac{2(1-v)}{\zeta}\ln(1+\zeta),$$

the function k is convex and it is the fuzzy best dominant.

Putting $m=1$ and $\lambda=0$ in Corollary 1, we obtain the following example.

Example 2. *Let* $h = \frac{1+(2v-1)\zeta}{1+\zeta}$ *be a convex function in* Λ, *with* $h(0) = 1$, $0 \le \beta < 1$. *If* $f \in \mathcal{A}$ *and verifies the fuzzy differential subordination*

$$F_{(\mathrm{Q}_q^r\mathrm{Y})'(\Lambda)}\left(\mathrm{Q}_q^r\mathrm{Y}(\zeta)\right)' \le F_{h(\Lambda)}\left(\frac{1+(2v-1)\zeta}{1+\zeta}\right), \quad i.e. \quad \left(\mathrm{Q}_q^r\mathrm{Y}(\zeta)\right)' \prec_{\mathcal{F}} \left(\frac{1+(2v-1)\zeta}{1+\zeta}\right)$$

then

$$F_{\mathrm{Q}_q^r\mathrm{Y}(\Lambda)}\frac{\mathrm{Q}_q^r\mathrm{Y}(\zeta)}{\zeta} \le F_{k(\Lambda)}k(\zeta) \quad i.e. \quad \frac{\mathrm{Q}_q^r\mathrm{Y}(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta), \tag{20}$$

where

$$k(\zeta) = 2v - 1 + \frac{2(1-v)}{\zeta}\ln(1+\zeta).$$

4. Conclusions

All of the above results provide information about fuzzy differential subordinations for the operator $\mathcal{N}_{q,\lambda}^{r,m}$; we also provide certain properties for the class $P_{q,\lambda}^{F,r,m}(\eta)$ of univalent analytic functions. Using these classes and operators, we can create some simple applications.

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