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Coefficient Inequalities and Fekete–Szegö-Type Problems for Family of Bi-Univalent Functions

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Abstract: In this study, we present a novel family of holomorphic and bi-univalent functions, denoted as $\mathrm{E}_{\Omega}(\eta, \varepsilon; \digamma)$. We establish the coefficient bounds for this family by utilizing the generalized telephone numbers. Additionally, we solve the Fekete–Szegö functional for functions that belong to this family within the open unit disk. Moreover, our results have several consequences.

Keywords: holomorphic; univalent; bi-univalent; Maclaurin series; coefficient inequalities; Fekete–Szegö

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1. Introduction

Consider the class of functions denoted by $\mathcal{A}$, which is represented in the following form:

$$f(\daleth) = \daleth + \sum_{n=2}^{\infty} a_n \daleth^n, \tag{1}$$

The functions in class $\mathcal{A}$ are holomorphic in the open unit disk $\Delta = \{\daleth : |\daleth| < 1\}$. Furthermore, $\mathcal{N}$ denotes the set of all functions in $\mathcal{A}$ that are univalent in Δ. Each function f in the set $\mathcal{N}$ has an inverse denoted as f^{-1}, which is defined by

$$f^{-1}(f(\daleth)) = \daleth \qquad (\daleth \in \Delta)$$

and f applied to the inverse of f for input w yields w, subject to the condition that the absolute value of w is less than the radius $r_0(f)$, where $r_0(f)$ is greater than or equal to $1/4$.

The inverse of function f applied to input w can be expressed as a series expansion starting with the term w, and subsequently involving terms like $-a_2w^2$, $(2a_2^2 - a_3)w^3$, $-(5a_2^3 - 5a_2a_3 + a_4)w^4$, and so on.

A function f is considered to be in the class Ω, or the class of bi-univalent functions in the unit disk Δ if both $f(z)$ and its inverse $f^{-1}(z)$ are univalent in Δ. Lewin (2011) showed that for every function in $\mathbf{\Omega}$ described by Equation (1), the absolute value of a_2 is less than 1.51. Brannan and Clunie (12) further refined Lewin's findings by proposing the hypothesis that $|a_2| \leq \sqrt{2}$. Netanyahu later proved that the maximum absolute value of a_2 is $\frac{4}{3}$, where f belongs to the set $\mathbf{\Omega}$ [1]. The problem of estimating the coefficient $|a_n|$ for $n \in \{4, 5, 6, \cdots\}$ remains unresolved (see [2] for more information). Several researchers have investigated different subfamilies of $\mathbf{\Omega}$ and obtained estimates for the Maclaurin coefficients $|a_2|$ and $|a_3|$ (see [3–6]). Extensive research has been dedicated to the Fekete–Szegö functional, represented as $|a_3 - \kappa a_2^2|$, within the domain of geometric function theory. Its historical significance holds broad recognition. The origins of its development can be traced back to Fekete and Szegö, who employed it to refute the Littlewood–Paley conjecture [7]. Numerous scholars have identified Fekete–Szegö inequalities applicable to diverse function families. Currently, there exists substantial interest among geometric function theory researchers (refer to citations [8–29]).

Consider the two holomorphic functions g and h defined within the open unit disk Δ. We say that g is subordinated to h if there is a holomorphic function w in the same unit disk Δ such that $|w(z)| < 1$, $w(0) = 0$, and the relationship $g(\beth) = h(w(\beth))$ holds.

Moreover, for a function h that is univalent in the unit disk Δ, the inequality $g(\beth) \prec h(\beth)$ holds if and only if $g(0) = h(0)$ and the image of g under the same mapping is contained within the image of h, i.e., $g(\Delta) \subset h(\Delta)$.

The recurrence relation quantifies standard telephone numbers

$$T(\daleth) = T(\daleth - 1) + (\daleth - 1)T(\daleth - 2) \qquad \daleth \geq 2,$$

with initial conditions

$$T(0) = 1 = T(1).$$

For non-negative integers $\iota \geq 0$ and $\daleth \geq 1$, Wloch and Wolowiec-Musial [30] introduced a series of numbers called generalized telephone numbers $T(\iota, \daleth)$, which are defined using a recurrence relation:

$$T(\iota, \daleth) = \iota T(\iota, \daleth - 1) + (\daleth - 1)T(\iota, \daleth - 2),$$

with initial conditions

$$T(\iota, 0) = 1 \quad \text{and} \quad T(\iota, 1) = \iota.$$

In a recent study, Bednarz and Wolowiec-Musial [31] explored an approach to generalize telephone numbers in an accessible manner, considering a new perspective on the concept:

$$T\iota(\daleth) = T\iota(\daleth - 1) + \iota(\daleth - 1)T\iota(\daleth - 2),$$

where $\daleth \geq 2$ and $\iota \geq 1$ with initial conditions

$$T\iota(0) = T\iota(1) = 1.$$

In a recent investigation, Deniz [32] studied the exponential generating function for $T(\iota, \daleth)$ in the following manner:

$$e^{\left(r + \iota \frac{r}{2}\right)} = \sum_{\daleth=0}^{\infty} T\iota(\daleth) \frac{r^{\daleth}}{\daleth!}.$$

Clearly, when $\iota = 1$, we have $T\iota(\daleth) \equiv T(\daleth)$ classical telephone numbers.

Here, with the domain of the open unit disk Δ, we define the function

$$\digamma(\beth) = e^{\left(\beth + \iota \frac{\beth^2}{2}\right)} = 1 + \beth + \frac{1}{2}\beth^2 + \frac{1+\iota}{6}\beth^3 + \frac{1+3\iota}{24}\beth^4 + \cdots. \tag{2}$$

The function $\digamma(\beth)$ is holomorphic in the domain Δ. It has the properties $\digamma(0) = 1$, $\digamma'(0) > 0$, and it maps Δ onto a star-like region centered at 1 and symmetric with respect to the real axis.

Lemma 1 ([33])**.** *Let the function $s_j \in \mathcal{P}$ have the form*

$$s(\beth) = 1 + s_1\beth + s_2\beth^2 + s_3\beth^3 + \cdots \quad (\beth \in \Delta),$$

then $|s_j| \leq 2, \ j \in \mathbb{N}$.

This paper aims to introduce the family of bi-univalent functions, denoted as $\mathrm{E}_{\Omega}(\tau,\varepsilon;\digamma)$, and derive upper bounds for the Maclaurin coefficients $|a_2|$ and $|a_3|$. We also discuss the Fekete–Szegö inequality for this family.

2. The Main Results of Function Class $\mathrm{E}_{\mathbf{\Omega}}(\tau,\varepsilon;\digamma)$

We now present the subsequent subfamilies of holomorphic functions.

Definition 1. *A function $f(\beth)\in\mathbf{\Omega}$ is allegedly in the class $\mathrm{E}_{\mathbf{\Omega}}(\tau,\varepsilon;\digamma)$ if it fulfills the following two subordinations:*

$$(1-\tau)\frac{f(\beth)}{\beth}+\tau(f(\beth))'+\varepsilon\beth(f(\beth))''\prec\digamma(\beth)=:e^{\left(\beth+\iota\frac{\beth^2}{2}\right)}$$

and

$$(1-\tau)\frac{h(w)}{w}+\tau(h(w))'+\varepsilon w(h(w))''\prec\digamma(w)=:e^{\left(w+\iota\frac{w^2}{2}\right)}$$

where $\tau\geq 1$, $\varepsilon\geq 0$, $z,w\ \in\ \Delta$ and $h=f^{-1}$.

For $\tau=1$, the family $\mathrm{E}_{\mathbf{\Omega}}(\tau,\varepsilon;\digamma)$ reduces to the subsequent subfamily.

Definition 2. *A function $f(\beth)\in\mathbf{\Omega}$ is allegedly in the subclass $\mathrm{E}_{\mathbf{\Omega}}(1,\varepsilon;\digamma)$ iff*

$$(f(\beth))'+\varepsilon\beth(f(\beth))''\prec\digamma(\beth)=:e^{\left(\beth+\iota\frac{\beth^2}{2}\right)}$$

and

$$(h(w))'+\varepsilon w(h(w))''\prec\digamma(w)=:e^{\left(w+\iota\frac{w^2}{2}\right)}$$

where z,w in Δ and $h=f^{-1}$.

For $\varepsilon=0$, the family $\mathrm{E}_{\mathbf{\Omega}}(\tau,\varepsilon;\digamma)$ reduces to the subsequent subfamily.

Definition 3. *A function $f(\beth)\in\mathbf{\Omega}$ is allegedly in the subclass $\mathrm{E}_{\mathbf{\Omega}}(\tau,0;\digamma)$ iff*

$$(1-\tau)\frac{f(\beth)}{\beth}+\tau(f(\beth))'\prec\digamma(\beth)=:e^{\left(\beth+\iota\frac{\beth^2}{2}\right)}$$

and

$$(1-\tau)\frac{h(w)}{w}+\tau(h(w))'\prec\digamma(w)=:e^{\left(w+\iota\frac{w^2}{2}\right)}$$

where z,w in Δ and $h=f^{-1}$.

For $\varepsilon=0$, the family $\mathrm{E}_{\mathbf{\Omega}}(1,\varepsilon;\digamma)$ reduces to the subsequent subfamily.

Definition 4. *A function $f(\beth)\in\mathbf{\Omega}$ is allegedly in the subclass $\mathrm{E}_{\mathbf{\Omega}}(1,0;\digamma)$ iff*

$$(f(\beth))'\prec\digamma(\beth)=:e^{\left(\beth+\iota\frac{\beth^2}{2}\right)}$$

and

$$(h(w))'\prec\digamma(w)=:e^{\left(w+\iota\frac{w^2}{2}\right)}$$

where z,w in Δ and $h=f^{-1}$.

Estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the family $\mathrm{E}_{\mathbf{\Omega}}(\tau,\varepsilon;\digamma)$ are provided in the subsequent theorem.

Theorem 1. *If the function $f(z)$ defined by Equation (1) lies within the family denoted as $\mathrm{E}_{\boldsymbol{\Omega}}(\tau,\varepsilon;\digamma)$, then*

$$|a_2| \leq \min\left\{\frac{1}{2\varepsilon+\tau+1}, \frac{2}{\sqrt{|2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2|}}\right\} \tag{3}$$

and

$$|a_3| \leq \min\left\{\frac{\iota+3}{2(6\varepsilon+2\tau+1)}, \frac{1}{6\varepsilon+2\tau+1}+\frac{1}{(2\varepsilon+\tau+1)^2}\right\}. \tag{4}$$

Proof. Suppose $f(\beth)$ belongs to the family $\mathrm{E}_{\boldsymbol{\Omega}}(\tau,\varepsilon;\digamma)$, and let $h=f^{-1}$. Then, there exist the two holomorphic functions ψ and ϑ, mapping from the open unit disk Δ to itself, with the initial conditions $\psi(0)=\vartheta(0)=0$. Additionally, these functions satisfy the following requirements:

$$(1-\tau)\frac{f(\beth)}{\beth}+\tau(f(\beth))'+\varepsilon\beth(f(\beth))''=\digamma(\psi(\beth)), \quad \beth \in \Delta \tag{5}$$

and

$$(1-\tau)\frac{h(w)}{w}+\tau(h(w))'+\varepsilon w(h(w))''=\digamma(\vartheta(w)), \quad w \in \Delta. \tag{6}$$

We define the functions s and t by

$$s(\beth)=\frac{1+\psi(\beth)}{1-\psi(\beth)}=1+s_1\beth+s_2\beth^2+\cdots$$

and

$$t(\beth)=\frac{1+\vartheta(\beth)}{1-\vartheta(\beth)}=1+t_1\beth+t_2\beth^2+\cdots.$$

Then the functions s and t are holomorphic in Δ with $s(0)=t(0)=1$. Since we have $\psi,\vartheta:\Delta\to\Delta$, each of s and t has a positive real part in Δ.

For ψ and ϑ, we have

$$\psi(\beth)=\frac{s(\beth)-1}{s(\beth)+1}=\frac{1}{2}\left[s_1\beth+\left(s_2-\frac{s_1^2}{2}\right)\beth^2\right]+\cdots \quad (\beth\in\Delta) \tag{7}$$

and

$$\vartheta(\beth)=\frac{t(\beth)-1}{t(\beth)+1}=\frac{1}{2}\left[t_1\beth+\left(t_2-\frac{t_1^2}{2}\right)\beth^2\right]+\cdots \quad (\beth\in\Delta). \tag{8}$$

Substituting (7) and (8) into (5) and (6) and applying (2), we have

$$\begin{aligned} &(1-\tau)\frac{f(\beth)}{\beth}+\tau(f(\beth))'+\varepsilon\beth(f(\beth))'' \\ =\;& \digamma(\psi(\beth)) =: e^{\left(\frac{s(\beth)-1}{s(\beth)+1}+\frac{\iota}{2}\left(\frac{s(\beth)-1}{s(\beth)+1}\right)^2\right)} \\ =\;& 1+\frac{1}{2}s_1\beth+\left(\frac{s_2}{2}+\frac{(\iota-1)s_1^2}{8}\right)\beth^2+\cdots \end{aligned} \tag{9}$$

and

$$\begin{aligned} & (1-\tau)\frac{h(w)}{w} + \tau(h(w))' + \varepsilon w(h(w))'' \\ = \quad & F(\vartheta(w)) =: e^{\left(\frac{t(w)-1}{t(w)+1} + \frac{\iota}{2}\left(\frac{t(w)-1}{t(w)+1}\right)^2\right)} \\ = \quad & 1 + \frac{1}{2}t_1 w + \left(\frac{t_2}{2} + \frac{(\iota-1)t_1^2}{8}\right)w^2 + \cdots . \end{aligned} \tag{10}$$

From (9) and (10), we obtain

$$(2\varepsilon+\tau+1)a_2 = \frac{s_1}{2}, \tag{11}$$

$$(6\varepsilon+2\tau+1)a_3 = \frac{s_2}{2} + \frac{(\iota-1)s_1^2}{8}, \tag{12}$$

$$-(2\varepsilon+\tau+1)a_2 = \frac{t_1}{2}, \tag{13}$$

and

$$(6\varepsilon+2\tau+1)(2a_2^2 - a_3) = \frac{t_2}{2} + \frac{(\iota-1)t_1^2}{8}. \tag{14}$$

From (11) and (13), we obtain

$$s_1 = -t_1 \tag{15}$$

and

$$2(2\varepsilon+\tau+1)^2 a_2^2 = \frac{1}{4}(s_1^2 + t_1^2). \tag{16}$$

By adding (12) with (14), we have

$$2(6\varepsilon+2\tau+1)a_2^2 = \frac{1}{2}(s_2+t_2) + \frac{(\iota-1)}{8}(s_1^2+t_1^2). \tag{17}$$

Substituting the value of $s_1^2 + t_1^2$ from (16) in (17), we find that

$$a_2^2 = \frac{s_2+t_2}{2(6\varepsilon+2\tau+1) + (1-\iota)(2\varepsilon+\tau+1)^2}. \tag{18}$$

Then, applying Lemma 1 for (16) and (18), we obtain

$$|a_2| \le \frac{1}{2\varepsilon+\tau+1}, \qquad |a_2| \le \frac{2}{\sqrt{|2(6\varepsilon+2\tau+1) + (1-\iota)(2\varepsilon+\tau+1)^2|}}$$

which provides estimates for the coefficient $|a_2|$.

Next, to find the bound on $|a_3|$, we subtract (14) from (12), and then applying (15), we obtain $s_1^2 = t_1^2$, hence

$$2(6\varepsilon+2\tau+1)\left(a_3 - a_2^2\right) = \frac{1}{2}(s_2 - t_2), \tag{19}$$

then by substituting the value of a_2^2 from (16) into (19), we have

$$a_3 = \frac{s_2 - t_2}{4(6\varepsilon+2\tau+1)} + \frac{s_1^2 + t_1^2}{8(2\varepsilon+\tau+1)^2}.$$

So, we have

$$|a_3| \le \frac{1}{6\varepsilon+2\tau+1} + \frac{1}{(2\varepsilon+\tau+1)^2}.$$

Also, substituting the value of a_2^2 from (17) into (19), we have

$$a_3 = \frac{(s_2 - t_2) + (s_2 + t_2) + \frac{1}{4}(\iota - 1)(s_1^2 + t_1^2)}{4(6\varepsilon + 2\tau + 1)}.$$

So, we have

$$|a_3| \leq \frac{\iota + 3}{2(6\varepsilon + 2\tau + 1)}$$

which provides estimates for the coefficient $|a_3|$. $\square$

By taking $\tau = 1$ in Theorem 1, we have

Corollary 1. *If $f(z)$ is given by (1) and in the subfamily $\mathrm{E}_{\Omega}(1, \varepsilon; \digamma)$, then*

$$|a_2| \leq \min\left\{\frac{1}{2\varepsilon + 2}, \frac{2}{\sqrt{|2(6\varepsilon + 3) + (1 - \iota)(2\varepsilon + 2)^2|}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{\iota + 3}{2(6\varepsilon + 3)}, \frac{1}{6\varepsilon + 3} + \frac{1}{(2\varepsilon + 2)^2}\right\}.$$

Putting $\varepsilon = 0$ in Theorem 1, we have the following.

Corollary 2. *If $f(\sqsupset)$ is given by (1) and in the subfamily $\mathrm{E}_{\Omega}(\tau, 0; \digamma)$, then*

$$|a_2| \leq \min\left\{\frac{1}{\tau + 1}, \frac{2}{\sqrt{|2(2\tau + 1) + (1 - \iota)(\tau + 1)^2|}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{\iota + 3}{2(2\tau + 1)}, \frac{1}{2\tau + 1} + \frac{1}{(\tau + 1)^2}\right\}.$$

Putting $\varepsilon = 0$ in Corollary 1, we have the following.

Corollary 3. *If $f(\sqsupset)$ given by (1) and in the subfamily $\mathrm{E}_{\Omega}(1, 0; \digamma)$,*

$$|a_2| \leq \min\left\{\frac{1}{2}, \frac{2}{\sqrt{|10 - 4\iota|}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{\iota + 3}{6}, \frac{7}{12}\right\}.$$

Now, we provide the Fekete–Szegö functional $\left|a_3 - \kappa a_2^2\right|$ for $f \in \mathrm{E}_{\Omega}(\tau, \varepsilon; \digamma)$.

Theorem 2. *Let $f(\sqsupset)$ be given by (1) and in the family $\mathrm{E}_{\Omega}(\tau, \varepsilon; \digamma)$. Then*

$$\left|a_3 - \kappa a_2^2\right| \leq \begin{cases} \frac{1}{6\varepsilon + 2\tau + 1} & \text{for } 0 \leq |\kappa - 1| \leq \frac{2(6\varepsilon + 2\tau + 1) + (1 - \iota)(2\varepsilon + \tau + 1)^2}{4(6\varepsilon + 2\tau + 1)}, \\ \frac{4|\kappa - 1|}{|2(6\varepsilon + 2\tau + 1) + (1 - \iota)(2\varepsilon + \tau + 1)^2|} & \text{for } |\kappa - 1| \geq \frac{2(6\varepsilon + 2\tau + 1) + (1 - \iota)(2\varepsilon + \tau + 1)^2}{4(6\varepsilon + 2\tau + 1)}. \end{cases}$$

Proof. From (18) and (19), it follows that

$$\begin{aligned} a_3-\kappa a_2^2 &= \frac{s_2-t_2}{4(6\varepsilon+2\tau+1)}+(1-\kappa)a_2^2 \\ &= \frac{s_2-t_2}{4(6\varepsilon+2\tau+1)}+\frac{(1-\kappa)(s_2+t_2)}{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2} \\ &= \left[\frac{1-\kappa}{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2}+\frac{1}{4(6\varepsilon+2\tau+1)}\right]s_2 \\ &\quad +\left[\frac{1-\kappa}{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2}-\frac{1}{4(6\varepsilon+2\tau+1)}\right]t_2. \end{aligned}$$

According to Lemma 1, we obtain

$$\left|a_3-\kappa a_2^2\right| \le \begin{cases} \frac{1}{6\varepsilon+2\tau+1} & \text{for } 0\le \left|\frac{1-\kappa}{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2}\right| \le \frac{1}{4(6\varepsilon+2\tau+1)}, \\ \frac{4|\kappa-1|}{\left|2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2\right|} & \text{for } \left|\frac{1-\kappa}{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2}\right| \ge \frac{1}{4(6\varepsilon+2\tau+1)}. \end{cases}$$

Finally, after some computations, we have

$$\left|a_3-\kappa a_2^2\right| \le \begin{cases} \frac{1}{6\varepsilon+2\tau+1} & \text{for } 0\le |\kappa-1| \le \frac{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2}{4(6\varepsilon+2\tau+1)}, \\ \frac{4|\kappa-1|}{\left|2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2\right|} & \text{for } |\kappa-1| \ge \frac{2(6\varepsilon+2\tau+1)+(1-\iota)(2\varepsilon+\tau+1)^2}{4(6\varepsilon+2\tau+1)}. \end{cases}$$

□

By taking $\tau=1$ in Theorem 2, we have

Corollary 4. *Let $f(\beth)$ be given by (1) and in the subfamily $\mathrm{E}_{\Omega}(1,\varepsilon;\digamma)$. Then*

$$\left|a_3-\kappa a_2^2\right| \le \begin{cases} \frac{1}{3(2\varepsilon+1)} & \text{for } 0\le |\kappa-1| \le \frac{3(2\varepsilon+1)+2(1-\iota)(\varepsilon+1)^2}{6(2\varepsilon+1)}, \\ \frac{2|\kappa-1|}{\left|3(2\varepsilon+1)+2(1-\iota)(\varepsilon+1)^2\right|} & \text{for } |\kappa-1| \ge \frac{3(2\varepsilon+1)+2(1-\iota)(\varepsilon+1)^2}{6(2\varepsilon+1)}. \end{cases}.$$

Putting $\varepsilon=0$ in Theorem 2, we have

Corollary 5. *Let $f(\beth)$ given by (1) and in the subfamily $\mathrm{E}_{\Omega}(\tau,0;\digamma)$. Then*

$$\left|a_3-\kappa a_2^2\right| \le \begin{cases} \frac{1}{2\tau+1} & \text{for } 0\le |\kappa-1| \le \frac{2(2\tau+1)+(1-\iota)(\tau+1)^2}{4(2\tau+1)}, \\ \frac{4|\kappa-1|}{\left|2(2\tau+1)+(1-\iota)(\tau+1)^2\right|} & \text{for } |\kappa-1| \ge \frac{2(2\tau+1)+(1-\iota)(\tau+1)^2}{4(2\tau+1)}. \end{cases}.$$

Putting $\varepsilon=0$ in Corollary 4, we have the following.

Corollary 6. *Let $f(\beth)$ be given by (1) and in the subfamily $\mathrm{E}_{\Omega}(1,\varepsilon;\digamma)$. Then*

$$\left|a_3-\kappa a_2^2\right| \le \begin{cases} \frac{1}{3} & \text{for } 0\le |\kappa-1| \le \frac{5-2\iota}{6}, \\ \frac{2|\kappa-1|}{|5-2\iota|} & \text{for } |\kappa-1| \ge \frac{5-2\iota}{6}. \end{cases}.$$

3. Conclusions

In this research, we introduced a novel category of normalized holomorphic and bi-univalent functions, which we denoted as $\mathrm{E}_{\Omega}(\tau, \varepsilon; F)$. We derived estimations for the magnitudes of the coefficients $|a_2|$ and $|a_3|$ in the Taylor0-Maclaurin series, along with tackling Fekete–Szegö functional problems.

Furthermore, by appropriately configuring the parameters τ and ε, one can determine the outcomes for the specific subclasses $\mathrm{E}_{\Omega}(1, \varepsilon; F)$, $\mathrm{E}_{\Omega}(\tau, 0; F)$, and $\mathrm{E}_{\Omega}(1, 0; F)$ as defined in Definitions (2), (3), and (4), respectively. Employing these classes of holomorphic and bi-univalent functions could serve as an inspiration for researchers seeking to establish estimates for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and delve into Fekete–Szegö functional problems for functions belonging to newly defined subclasses of bi-univalent functions, which are defined based on the telephone number associated with this distribution series.

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