symmetry

Article

Neutrosophic $\mathfrak{N}$-Structures in Semimodules over Semirings

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Abstract: The study of symmetry is a fascinating and unifying subject that connects various areas of mathematics in the twenty-first century. Algebraic structures offer a framework for comprehending the symmetries of geometric objects in pure mathematics. This paper introduces new concepts in algebraic structures, concentrating on semimodules over semirings and analysing the neutrosophic structure in this context. We explore the properties of neutrosophic subsemimodules and neutrosophic ideals after defining them. We discuss, utilizing neutrosophic products, the representations of neutrosophic ideals and subsemimodules, as well as the relationship between neutrosophic products and intersections. Finally, we derive equivalent criteria in terms of neutrosophic structures for a semiring to be fully idempotent.

Keywords: semirings; semimodules; neutrosophic $\mathfrak{N}$-subsemimodules, ideals; regularity

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1. Introduction

The exploration of symmetry is a foundational and captivating topic that unites various disciplines in contemporary mathematics. Algebraic structures provide valuable tools in pure mathematics for understanding the symmetries of geometric objects. For instance, in ring theory, homomorphisms are essential functions that preserve the ring operation. These functions are crucial for studying the symmetries within the context of ring theory. Additionally, the theory of groups, another significant algebraic structure, offers a comprehensive framework for exploring symmetry. Using group theory, various types of symmetries can be examined and analysed. As a result, group theory has become widely employed as an algebraic tool for understanding and characterizing symmetries in diverse contexts. Semirings play an important role in computer science as well as in mathematics. It is advantageous to characterise a ring's properties using modules over the ring. Consequently, semimodules over semirings are common as a generalisation of modules over rings (see [1–4]).

In 1965, Zadeh [5] pioneered the notion of fuzzy sets and their characteristics; since then, a wide range of fields involving uncertainty have made extensive use of fuzzy sets and fuzzy logic, including robotics, machine learning, computer engineering, control theory, business administration, and operational science. However, it has been noted that some situations are still not covered by fuzzy sets, so the idea of interval-valued fuzzy sets was developed in order to capture those situations. While fuzzy set theory is incredibly effective at managing uncertainties resulting from an element's vagueness within a set, it is unable to capture all types of uncertainties found in various real-world physical problems, such as those involving incomplete information.

In [6], Altassan et al. defined the concept of a ω-fuzzy set, ω-fuzzy subring, and ω-fuzzy ideal, where they also looked into various fundamental outcomes of this phenomenon. Furthermore, they developed a quotient ring with respect to this specific fuzzy ideal analogue to the classical quotient ring and proposed the idea of a ω-fuzzy coset. They also established a ω-fuzzy homomorphism between a ω-fuzzy subring of the quotient ring and a ω-fuzzy subring of this ring, and they proved some additional basic theorems of ω-fuzzy homomorphism for these particular fuzzy subrings. Additionally, they described ω-fuzzy homomorphism and ω-fuzzy isomorphism. Numerous authors have investigated fuzzy logic in a number of different structures (see [7–9]).

In [10], Atanassov created intuitionistic fuzzy sets (IFS), a further generalisation of the fuzzy set. Each element in IFS has a non-membership grade attached to it in addition to a membership grade. Additionally, the total of these two grades cannot be greater than or equal to unity. When there is insufficient data available to define imprecision using traditional fuzzy sets, the idea of IFS can be seen as a suitable or alternative approach.

To address the ubiquitous uncertainty, Smarandache [11] proposed neutrosophic sets. In addition to fuzzy sets, they also generalise intuitionistic fuzzy sets. The three characteristics of neutrosophic sets are truth(T) membership functions, falsity(F), and indeterminacy(I). These sets can be used to address the complexities brought about by ambiguous information in a wide range of applications. A neutrosophic set can distinguish between absolute and relative membership functions. Smarandache used these sets for non-traditional analyses such as control theory, decision-making theory, sports decisions (winning/defeating/tie), etc.

In [12], Khan et al. investigated several characteristics of the ϵ-neutrosophic $\mathfrak{N}$-subsemigroup as well as the neutrosophic $\mathfrak{N}$-subsemigroup in a semigroup. In [13], B. Elavarasan et al. investigated various properties of neutrosophic $\mathfrak{N}$-ideals in semigroups. In [14], Muhiuddin et al. defined neutrosophic $\mathfrak{N}$-ideals and neutrosophic $\mathfrak{N}$-interior ideals in ordered semigroups and studied their properties. They also used neutrosophic $\mathfrak{N}$-ideals and neutrosophic $\mathfrak{N}$-interior ideals to describe ordered semigroups.

In [15], Karaaslan obtained some information pertaining to the determinant and adjoint of the interval-valued neutrosophic matrices by defining the determinant and adjoint of interval-valued neutrosophic (IVN) matrices based on the permanent function. In [16], Jun et al. introduced the notion of neutrosophic quadruple BCK/BCI-numbers and studied neutrosophic quadruple BCK/BCI-algebras. In [17], Muhiuddin et al. continued this work by coming up with the idea of implicative neutrosophic quadruple BCK-algebras and looking into some of their properties. In [18], Nagarajan et al. described a way to find the correlation coefficient of neutrosophic sets, which tells us how strong the connections are between variables based on neutrosophic sets. They also talked about the multiple regression method in the context of neutrosophic sets.

In this paper, we investigate neutrosophic structures in semiring modules, the concept of neutrosophic $\mathfrak{N}$-subsemimodules, and neutrosophic $\mathfrak{N}$-ideals over semirings, and establish their various properties. In addition, we investigate the concept of neutrosophic right t-pure ideals in semirings and the relations between neutrosophic t-pure ideals and neutrosophic $\mathfrak{N}$-submodules in semirings. Moreover, we obtain equivalent statements for a semiring that is fully idempotent.

2. Preliminary Definitions of Semirings

In this section, we summarize the preliminary definitions of semirings that are required later in this paper.

Definition 1 ([1])**.** *Let $\mathcal{R}(\neq \emptyset)$, " $+$ " and " $\cdot$ " be two binary operations defined on $\mathcal{R}$. Then $\mathcal{R}$ is called a semiring if it satisfies the below requirements:*

(i) $(\mathcal{R}, +)$ and $(\mathcal{R}, \cdot)$ are commutative semigroups with identity elements 0 and $1(\neq 0)$, respectively.

(ii) $j_1 \cdot (y_1 + d_1) = j_1 \cdot y_1 + j_1 \cdot d_1$ *and* $(j_1 + y_1) \cdot d_1 = j_1 \cdot d_1 + y_1 \cdot d_1$, $\forall j_1, y_1, d_1 \in \mathcal{R}$.
(iii) $\forall d_1 \in \mathcal{R}$, $0 \cdot d_1 = d_1 \cdot 0 = 0$.

Obviously, a ring is a semiring, where each element has an additive inverse. A module over a ring is a vector space over a field generalisation where the corresponding scalars are components of a ring that were selected at random (with identity) and the elements of the modules and rings are multiplied (on the right and/or on the left).

Definition 2 ([1])**.** *Let* $(\mathcal{R}, +, \cdot)$ *be a semiring. A non-empty set* $\mathbb{O}$ *is called a right* $\mathcal{R}$*-semimodule over* $\mathcal{R}$ *if the following are satisfied:*
(i) $(\mathbb{O}, +)$ *is a commutative semigroup with an identity element* 0*, For any* $m_1, m_2 \in \mathbb{O}$ *and* $u_1, u_2 \in \mathcal{R}$*,*
(ii) $m_1 \cdot u_1 \in \mathbb{O}$,
(iii) $(m_1 + m_2) \cdot u_1 = m_1 \cdot u_1 + m_2 \cdot u_1$,
(iv) $m_1 \cdot (u_1 + u_2) = m_1 \cdot u_1 + m_1 \cdot u_2$,
(v) $m_1 \cdot (u_1 \cdot u_2) = (m_1 \cdot u_1) \cdot u_2$,
(vi) $m_1 \cdot 1 = m_1$,
(vii) $0 \cdot u_1 = m_1 \cdot 0 = 0$.
It is denoted by $\mathbb{O}_{\mathcal{R}}$.

A left $\mathcal{R}$-semimodule ${}_{\mathcal{R}}\mathbb{O}$ can be defined in a similar manner. It is obvious that each semiring $\mathcal{R}$ is a right (left) $\mathcal{R}$ semimodule over itself.

Hereafter, a semiring can be represented by $\mathcal{R}$, $\mathbb{O}$ denotes a right $\mathcal{R}$-semimodule over $\mathcal{R}$, and the power set of a set $\mathscr{B}$ can be expressed as $\mathscr{P}(\mathscr{B})$.

Definition 3 ([1])**.** *Let* $\mathbb{O}$ *be a right* $\mathcal{R}$*-semimodule and* $\mathcal{C} \in \mathscr{P}(\mathbb{O})$*. Then* $\mathcal{C}$ *is termed as a subsemimodule of* $\mathbb{O}$ *if* $s_0 + z_0 \in \mathcal{C}$ *and* $s_0 l_0 \in \mathcal{C}$ $\forall s_0, z_0 \in \mathcal{C}$ *and* $l_0 \in \mathcal{R}$*.*

Naturally, $\mathcal{C}$ has evolved into its own $\mathcal{R}$-module, with the same addition and scalar multiplication as $\mathbb{O}$. Clearly, a ring is a semiring, so a left module over a ring $\mathcal{R}$ is a left semimodule over $\mathcal{R}$.

Definition 4. *Let* $\mathcal{C} \in \mathscr{P}(\mathcal{R})$*. If* $\mathcal{C}$ *of* $\mathcal{R}_{\mathcal{R}}({}_{\mathcal{R}}\mathcal{R})$ *is a subsemimodule, then* $\mathcal{C}$ *is termed as a right (left) ideal of* $\mathcal{R}$*.*
If $\mathcal{C}$ *of* $\mathcal{R}$ *is both a right and a left ideal, then it is described as an ideal of* $\mathcal{R}$*.*

Definition 5. *If* $b_1 \in \mathcal{R}$ *satisfies* $b_1 + b_1 = b_1$*, it is known as an additive idempotent.*
If each element b_1 *of* $\mathcal{R}$ *satisfies* $b_1 + b_1 = b_1$*, then* $\mathcal{R}$ *is described as an idempotent semiring.*

3. Preliminary Definitions and Results of Neutrosophic $\mathfrak{N}$-Structure

This portions outlines the basic ideas of neutrosophic $\mathfrak{N}$-structures of $\mathbb{O}$, which are essential for the sequel.

A set $Q(\neq \emptyset)$, $\mathcal{F}(Q, \mathbb{I}^-)$ is the family of functions with negative values from a set Q to $\mathbb{I}^-$. An element $k_1 \in \mathcal{F}(Q, \mathbb{I}^-)$ is known as a $\mathfrak{N}$-function on Q and $\mathfrak{N}$-structure denotes (Q, k_1) of X, where $\mathbb{I}^- = [-1, 0]$.

Definition 6 ([12])**.** *For a set* $Q(\neq \emptyset)$*, a neutrosophic* $\mathfrak{N}$*- structure of* Q *is described as below:*

$$Q_M := \frac{Q}{(T_M, I_M, F_M)} = \left\{ \frac{v_0}{(T_M(v_0), I_M(v_0), F_M(v_0))} \ : \ v_0 \in Q \right\},$$

where T_M *means the negative truth membership function on* Q*,* I_M *means the negative indeterminacy membership function on* Q*, and* F_M *means the negative falsity membership function on* Q*.*

Remark 1. Q_M *satisfies the requirement* $-3 \leq T_M(b_1) + I_M(b_1) + F_M(b_1) \leq 0$ $\forall b_1 \in Q$*.*

Definition 7 ([12]). *Let $Q(\neq \emptyset)$. For any $Q_J := \frac{Q}{(T_J, I_J, F_J)}$ and $Q_M := \frac{Q}{(T_M, I_M, F_M)}$,*

(i) Q_M is defined as a neutrosophic $\mathfrak{N}$-substructure of Q_J, represented by $Q_J \subseteq Q_M$, if it fulfils the below criteria: for any $z_0 \in Q$,

$$T_J(z_0) \geq T_M(z_0), I_J(z_0) \leq I_M(z_0), F_J(z_0) \geq F_M(z_0).$$

If $Q_J \subseteq Q_M$ and $Q_M \subseteq Q_J$, then $Q_J = Q_M$.

(ii) The intersection and union of Q_J and Q_M are neutrosophic $\mathfrak{N}$-structures over Q and are defined as follows:

(a) $Q_J \cap Q_M = Q_{J \cap M} = (Q; T_{J\cap M}, I_{J\cap M}, F_{J\cap M})$, where

$$\begin{aligned}(T_J \cap T_M)(h_0) =& T_{J\cap M}(h_0) = T_J(h_0) \vee T_M(h_0),\\ (I_J \cap I_M)(h_0) =& I_{J\cap M}(h_0) = I_J(h_0) \wedge I_M(h_0),\\ (F_J \cap F_M)(h_0) =& F_{J\cap M}(h_0) = F_J(h_0) \vee F_M(h_0) \; for\ any\ h_0 \in Q.\end{aligned}$$

(b) $Q_J \cup Q_M = Q_{J \cup M} = (Q; T_{J\cup M}, I_{J\cup M}, F_{J\cup M})$, where

$$\begin{aligned}(T_J \cup T_M)(y_0) =& T_{J\cup M}(y_0) = T_J(y_0) \wedge T_M(y_0),\\ (I_J \cup I_M)(y_0) =& I_{J\cup M}(y_0) = I_J(y_0) \vee I_M(y_0),\\ (F_J \cup F_M)(y_0) =& F_{J\cup M}(y_0) = F_J(y_0) \wedge F_M(y_0) \; for\ any\ y_0 \in Q.\end{aligned}$$

Definition 8. *For $V \subseteq Q \neq \emptyset$, consider the neutrosophic $\mathfrak{N}$-structure*

$$\chi_V(Q_D) := \frac{Q}{(\chi_V(T)_D, \chi_V(I)_D, \chi_V(F)_D)},$$

where, for any $r_0 \in Q$,

$$\chi_V(T)_D : Q \to \mathbb{I}^-,\ r_0 \to \begin{cases} -1 & if\ r_0 \in V \\ 0 & if\ r_0 \notin V, \end{cases}$$

$$\chi_V(I)_D : Q \to \mathbb{I}^-,\ r_0 \to \begin{cases} 0 & if\ r_0 \in V \\ -1 & if\ r_0 \notin V, \end{cases}$$

$$\chi_V(F)_D : Q \to \mathbb{I}^-,\ r_0 \to \begin{cases} -1 & if\ r_0 \in V \\ 0 & if\ r_0 \notin V, \end{cases}$$

which is described as the characteristic neutrosophic $\mathfrak{N}$-structure of V over Q.

Definition 9 ([12]). *For a nonempty set Q, let $Q_N := \frac{Q}{(T_N, I_N, F_N)}$ and $\vartheta, \varphi, \nu \in \mathbb{I}^-$ with $-3 \leq \vartheta + \varphi + \nu \leq 0$. Consider the following sets:*

$T_N^{\vartheta} = \{c_1 \in Q \mid T_N(c_1) \leq \vartheta\}, I_N^{\varphi} = \{c_1 \in Q \mid I_N(c_1) \geq \varphi\}, F_N^{\nu} = \{c_1 \in Q | F_N(c_1) \leq \nu\}$. Then the set $Q_N(\vartheta, \varphi, \nu) = \{c_1 \in Q | T_N(c_1) \leq \vartheta, I_N(c_1) \geq \varphi, F_N(c_1) \leq \nu\}$ is known as a $(\vartheta, \varphi, \nu)$-level set of Q_N. Note that $Q_N(\vartheta, \varphi, \nu) = T_N^{\vartheta} \cap I_N^{\varphi} \cap F_N^{\nu}$.

Definition 10. *Let $\mathbb{O}_K := \frac{\mathbb{O}}{(T_K, I_K, F_K)}$ and $\mathbb{O}_P := \frac{\mathbb{O}}{(T_P, I_P, F_P)}$ be neutrosophic $\mathfrak{N}$-structures in $\mathbb{O}$. Then:*

(i) The neutrosophic $\mathfrak{N}$-sum of $\mathbb{O}_K$ and $\mathbb{O}_P$ is described as a neutrosophic $\mathfrak{N}$-structure of $\mathbb{O}$, $\mathbb{O}_K \oplus \mathbb{O}_P := \frac{\mathbb{O}}{(T_{K+P}, I_{K+P}, F_{K+P})} = \left\{ \frac{v_0}{(T_{K+P}(v_0), I_{K+P}(v_0), F_{K+P}(v_0))} \mid v_0 \in \mathbb{O} \right\}$, where

$$(T_K + T_P)(v_0) = T_{K+P}(v_0) = \begin{cases} \bigwedge\limits_{v_0=s_0+c_0} \{T_K(s_0) \vee T_P(c_0)\} & if\ \exists s_0, c_0 \in \mathbb{O} \ : \ v_0 = s_0 + c_0 \\ 0 & otherwise, \end{cases}$$

$$(I_K + I_P)(v_0) = I_{K+P}(v_0) = \begin{cases} \bigvee\limits_{v_0=s_0+c_0} \{I_K(s_0) \wedge I_P(c_0)\} & if\ \exists s_0, c_0 \in \mathbb{O} \ : \ v_0 = s_0 + c_0 \\ -1 & otherwise, \end{cases}$$

$$(F_K + F_P)(v_0) = F_{K+P}(v_0) = \begin{cases} \bigwedge\limits_{v_0=s_0+c_0} \{F_K(s_0) \vee F_P(c_0)\} & if\ \exists s_0, c_0 \in \mathbb{O} \ : \ v_0 = s_0 + c_0 \\ 0 & otherwise. \end{cases}$$

For $v_0 \in \mathbb{O}$, the element $\frac{v_0}{(T_{K+P}(v_0), I_{K+P}(v_0), F_{K+P}(v_0))}$ is simply denoted by $(\mathbb{O}_K \oplus \mathbb{O}_P)(v_0) = (T_{K+P}(v_0), I_{K+P}(v_0), F_{K+P}(v_0))$.

(ii) The neutrosophic $\mathfrak{N}$-product of $\mathbb{O}_K$ and $\mathbb{O}_P$ is described to be a neutrosophic $\mathfrak{N}$-structure of $\mathbb{O}$, $\mathbb{O}_K \odot \mathbb{O}_P := \frac{\mathbb{O}}{(T_{K\circ P}, I_{K\circ P}, F_{K\circ P})} = \left\{ \frac{k}{(T_{K\circ P}(k), I_{K\circ P}(k), F_{K\circ P}(k))} \mid k \in \mathbb{O} \right\}$, where

$$(T_K \circ T_P)(k) = T_{K\circ P}(k) = \begin{cases} \bigwedge\limits_{k=st} \{T_K(s) \vee T_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R} \ : \ k = st \\ 0 & otherwise, \end{cases}$$

$$(I_K \circ I_P)(k) = I_{K\circ P}(k) = \begin{cases} \bigvee\limits_{k=st} \{I_K(s) \wedge I_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R} \ : \ k = st \\ -1 & otherwise, \end{cases}$$

$$(F_K \circ F_P)(k) = F_{K\circ P}(k) = \begin{cases} \bigwedge\limits_{k=st} \{F_K(s) \vee F_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R} \ : \ k = st \\ 0 & otherwise. \end{cases}$$

For $k \in \mathbb{O}$, the element $\frac{k}{(T_{K\circ P}(k), I_{K\circ P}(k), F_{K\circ P}(k))}$ is simply denoted by $(\mathbb{O}_K \odot \mathbb{O}_P)(k) = (T_{K\circ P}(k), I_{K\circ P}(k), F_{K\circ P}(k))$.

4. Main Results

The neutrosophic $\mathfrak{N}$-subsemimodule is defined and its various properties are examined in this section. Additionally, we define and examine the notion of neutrosophic right t-pure ideals in semirings as well as the connections between neutrosophic t-pure ideals and neutrosophic $\mathfrak{N}$-submodules in semirings.

Definition 11. *A neutrosophic $\mathfrak{N}$-structure $\mathbb{O}_N := \frac{\mathbb{O}}{(T_N, I_N, F_N)}$ of $\mathbb{O}$ is defined as a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$ if it satisfies the following:*

(i) $(\forall\ y_0, f_0 \in \mathbb{O}) \begin{pmatrix} T_N(y_0 + f_0) \le T_N(y_0) \vee T_N(f_0) \\ I_N(y_0 + f_0) \ge I_N(y_0) \wedge I_N(f_0) \\ F_N(y_0 + f_0) \le F_N(y_0) \vee F_N(f_0) \end{pmatrix}$.

(ii) $(\forall\ k_0 \in \mathbb{O}; s_0 \in \mathcal{R}) \begin{pmatrix} T_N(k_0 s_0) \le T_N(k_0) \\ I_N(k_0 s_0) \ge I_N(k_0) \\ F_N(k_0 s_0) \le F_N(k_0) \end{pmatrix}$.

It is clear that, for any *neutrosophic $\mathfrak{N}$-subsemimodule* $\mathbb{O}_N$ of $\mathbb{O}$, we obtain

$$(\forall l \in \mathbb{O}) \begin{pmatrix} T_N(0) \le T_M(l) \\ I_N(0) \ge I_M(l) \\ F_N(0) \le F_M(l) \end{pmatrix}.$$

Definition 12. *If $\mathcal{R}_N$, a neutrosophic $\mathfrak{N}$-structure of $\mathcal{R}$, is a neutrosophic $\mathfrak{N}$-subsemimodule of a right $\mathcal{R}$-semimodule $\mathcal{R}_{\mathcal{R}}$, then $\mathcal{R}_N$ is referred to as a neutrosophic $\mathfrak{N}$-right ideal of $\mathcal{R}$.*

If $\mathcal{R}_N$ is a neutrosophic $\mathfrak{N}$-subsemimodule of a left $\mathcal{R}$-semimodule ${}_{\mathcal{R}}\mathcal{R}$, then $\mathcal{R}_N$ is referred as a neutrosophic $\mathfrak{N}$-left ideal of $\mathcal{R}$.

$\mathcal{R}_N$ of $\mathcal{R}$ is defined as a neutrosophic $\mathfrak{N}$-ideal if it is both a neutrosophic $\mathfrak{N}$-right and a neutrosophic $\mathfrak{N}$-left ideal of $\mathcal{R}$.

Example 1. *Let $\mathbb{O}$ be the set of all non-zero negative integers. Then, with respect to usual addition "$+$" and multiplication "$*$", $(\mathbb{O}, +)$ is a commutative semigroup with an identity element 0 and $(\mathbb{O}, +, *)$ and $(2\mathbb{O}, +, *)$ are semirings. Clearly, $\mathbb{O}$ is a right $2\mathbb{O}$-semimodule over $2\mathbb{O}$ and $\mathbb{O}$ is a right $\mathbb{O}$-semimodule over $\mathbb{O}$. Define a neutrosophic $\mathfrak{N}$- structure $\mathbb{O}_N := \frac{\mathbb{O}}{(T_N, I_N, F_N)}$, where, for any $y_0 \in \mathbb{O}$,*

$$T_N(y_0) = F_N(y_0) = \begin{cases} -1 & if \ \ y_0 = 0 \\ -1 + \frac{1}{n} & otherwise \end{cases}; \ I_N(y_0) = \begin{cases} 0 & if \ \ y_0 = 0 \\ -\frac{1}{n} & otherwise. \end{cases}$$

It is then easy to verify that $\mathbb{O}_N$ is a neutrosophic $2\mathbb{O}$-subsemimodule of $\mathbb{O}$ and $\mathbb{O}_N$ is a neutrosophic $\mathbb{O}_N$ ideal of $\mathbb{O}$.

Theorem 1. *Let $\mathbb{O}_N := \frac{\mathbb{O}}{(T_N, I_N, F_N)}$. Then the following criteria are equivalent:*
(i) For any $\varrho, \lambda, \nu \in \mathbb{I}^-$, $\mathbb{O}_N(\varrho, \lambda, \nu)(\neq \phi)$ is a subsemimodule of $\mathbb{O}$;
(ii) $\mathbb{O}_N$ of $\mathbb{O}$ is a neutrosophic $\mathfrak{N}$-subsemimodule.

Proof. $(i) \Rightarrow (ii)$ Let $c, z \in \mathbb{O}$. Then $T_N(c) = q_1, F_N(c) = r_1, I_N(c) = t_1$ and $T_N(z) = q_2, F_N(z) = r_2, I_N(z) = t_2$, for some $q_1, q_2, t_1, t_2, r_1, r_2 \in \mathbb{I}^-$.

If $q = max\{q_1, q_2\}; t = min\{t_1, t_2\}$ and $r = max\{r_1, r_2\}$, then $T_N(c) \leq q, I_N(c) \geq t, F_N(c) \leq r$ and $T_N(z) \leq q, I_N(z) \geq t, F_N(z) \leq r$, so $c, z \in \mathbb{O}_N(q, t, r)$. Since $\mathbb{O}_N(q, t, r)$ is a subsemimodule of $\mathbb{O}$, we obtain $c + z \in \mathbb{O}_N(q, t, r)$, which implies $T_N(c + z) \leq q = T_N(c) \vee T_N(z)$, $I_N(c + z) \geq t = I_N(c) \wedge I_N(z)$, $F_N(c + z) \leq r = F_N(c) \vee F_N(z)$.

In addition, for $r \in \mathcal{R}$, we have $cr \in \mathbb{O}_N(q_1, t_1, r_1)$, which implies $T_N(cr) \leq q_1 = T_N(c), I_N(cr) \geq t_1 = I_N(c), F_N(cr) \geq r_1 = F_N(c)$. Therefore, $\mathbb{O}_N$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$.

$(ii) \Rightarrow (i)$ For $\varrho, \lambda, \nu \in \mathbb{I}^-$, let $q, z \in \mathbb{O}_N(\varrho, \lambda, \nu)$. Then, $T_N(q + z) \leq T_N(q) \vee T_N(z) \leq \varrho, ; I_N(q + z) \geq I_N(q) \wedge I_N(z) \geq \lambda$ and $F_N(q + z) \leq F_N(q) \vee F_N(z) \leq \nu$, which imply $q + z \in \mathbb{O}_N(\varrho, \lambda, \nu)$.

In addition, for $r \in \mathcal{R}$, $T_N(qr) \leq T_N(q) \leq \varrho$, $I_N(qr) \geq I_N(q) \geq \lambda$, and $F_N(qr) \leq F_N(q) \leq \nu$ imply that $qr \in \mathbb{O}_N(\varrho, \lambda, \nu)$. Therefore, $\mathbb{O}_N(\varrho, \lambda, \nu)$ is a subsemimodule of $\mathbb{O}$. $\square$

Remark 2. *Based on the equivalent conditions of the above Theorem 1, we have the following succeeding Corollary as an outcome of Theorem 1.*

Corollary 1. *For $\emptyset \neq \mathbb{D} \subseteq \mathbb{O}$, a neutrosophic $\mathfrak{N}$- structure $\mathbb{D}_N := \frac{\mathbb{D}}{(T_N, I_N, F_N)}$ of $\mathbb{D}$ is characterized as below: For $g_1, l_1, \omega_1, t_1, s_1, v_1 \in \mathbb{I}^-$,*

$$T_N(y_0) := \begin{cases} g_1 & if \ \ y_0 \in \mathbb{D} \\ l_1 & otherwise \end{cases}; \quad I_N(y_0) := \begin{cases} \omega_1 & if \ \ y_0 \in \mathbb{D} \\ t_1 & otherwise, \end{cases}; \quad F_N(y_0) := \begin{cases} s_1 & if \ \ y_0 \in \mathbb{D} \\ v_1 & otherwise, \end{cases}$$

where $g_1 < l_1; \omega_1 > t_1$ and $s_1 < v_1$ in $\mathbb{I}^-$, the listed below statements are equivalent:
(i) $\mathbb{D}$ of $\mathbb{O}$ is a subsemimodule;
(ii) $\mathbb{D}_N$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$.

Proof. $(i) \Rightarrow (ii)$ For $y_0, f_0 \in \mathbb{O}; s_0 \in \mathcal{R}$. If $y_0 + f_0 \in \mathbb{D}$, then $T_N(y_0 + f_0) = g_1 \leq T_N(y_0) \vee T_N(f_0), I_N(y_0 + f_0) = w_1 \geq I_N(y_0) \wedge I_N(f_0), F_N(y_0 + f_0) = s_1 \leq F_N(y_0) \vee F_N(f_0)$. Otherwise, $y_0 + f_0 \notin \mathbb{D}$. Then, $y_0 \notin \mathbb{D}$ or $f_0 \notin \mathbb{D}$, which implies $T_N(y_0 + f_0) = l_1 =$

$T_N(y_0) \vee T_N(f_0), I_N(y_0+f_0) = t_1 = I_N(y_0) \wedge I_N(f_0), F_N(y_0+f_0) = v_1 = F_N(y_0) \vee F_N(f_0)$. For $s_0 \in \mathcal{R}$, if $y_0s_0 \in \mathbb{D}$, then $T_N(y_0s_0) = g_1 \leq T_N(y_0), I_N(y_0s_0) = w_1 \geq I_N(y_0), F_N(y_0s_0) = s_1 \leq F_N(y_0)$. Otherwise, $y_0s_0 \notin \mathbb{D}$. Then, $y_0 \notin \mathbb{D}$, which implies $T_N(y_0s_0) = l_1 = T_N(y_0), I_N(y_0s_0) = t_1 = I_N(y_0), F_N(y_0s_0) = v_1 = F_N(y_0)$. Therefore, $\mathbb{D}_N$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$.

$(ii) \Rightarrow (i)$ If $\mathbb{D}_N$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$, then, by Theorem 1, $\mathbb{D}_N(g_1, w_1, s_1) = \mathbb{D}$ is a subsemimodule of $\mathbb{O}$. □

Remark 3. *If we take $g_1 = t_1 = s_1 = -1$ and $l_1 = w_1 = v_1 = 0$ in Corollary 1, then we obtain the following Corollary:*

Corollary 2. *For $\emptyset \neq K \subseteq \mathbb{O}$ and $\mathbb{O}_N := \frac{\mathbb{O}}{(T_N, I_N, F_N)}$, the listed below statements are equivalent:*
(i) $\chi_K(\mathbb{O}_N)$ of $\mathbb{O}$ is a neutrosophic $\mathfrak{N}$-subsemimodule;
(ii) K of $\mathbb{O}$ is a subsemimodule.

Next, we prove the following result:

Theorem 2. *Let $\mathbb{O}_K := \frac{\mathbb{O}}{(T_K, I_K, F_K)}$ and $\mathbb{O}_P := \frac{\mathbb{O}}{(T_P, I_P, F_P)}$ be neutrosophic $\mathfrak{N}$-structures in $\mathbb{O}$. If $\mathbb{O}_K$ and $\mathbb{O}_P$ are neutrosophic $\mathfrak{N}$-subsemimodules of $\mathbb{O}$, then $\mathbb{O}_K \oplus \mathbb{O}_P$ is also a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$.*

Proof. Let $z, x \in \mathbb{O}$. Then, for $d, d', q, q' \in \mathbb{O}$, we have

$$
\begin{aligned}
T_{K+P}(x) \vee T_{K+P}(z) &= \left[\bigwedge_{x=d+q} \{T_K(d) \vee T_P(q)\}\right] \vee \left[\bigwedge_{z=d'+q'} \{T_K(d') \vee T_P(q')\}\right] \\
&= \bigwedge_{\substack{x=d+q \\ z=d'+q'}} \{\{T_K(d) \vee T_P(q)\} \vee \{T_K(d') \vee T_P(q')\}\} \\
&= \bigwedge_{\substack{x=d+q \\ z=d'+q'}} \{T_K(d) \vee T_K(d') \vee T_P(q) \vee T_P(q')\} \\
&\geqslant \bigwedge_{x+z=(d+d')+(q+q')} \{T_K(d+d') \vee T_P(q+q')\} = T_{K+P}(x+z),
\end{aligned}
$$

$$
\begin{aligned}
I_{K+P}(x) \wedge I_{K+P}(z) &= \left[\bigvee_{x=d+q} \{I_K(d) \wedge I_P(q)\}\right] \wedge \left[\bigvee_{z=d'+q'} \{I_K(d') \wedge I_P(q')\}\right] \\
&= \bigvee_{\substack{x=d+q \\ z=d'+q'}} \{\{I_K(d) \wedge I_P(q)\} \wedge \{I_K(d') \wedge I_P(q')\}\} \\
&= \bigvee_{\substack{x=d+q \\ z=d'+q'}} \{I_K(d) \wedge I_K(d') \wedge I_P(q) \wedge I_P(q')\} \\
&\leqslant \bigvee_{x+z=(d+d')+(q+q')} \{I_K(d+d') \wedge I_P(q+q')\} = I_{K+P}(x+z),
\end{aligned}
$$

$$\begin{aligned}F_{K+P}(x)\vee F_{K+P}(z) &= \left[\bigwedge_{x=d+q}\{F_K(d)\vee F_P(q)\}\right]\vee\left[\bigwedge_{z=d'+q'}\{F_K(d')\vee F_P(q')\}\right]\\ &= \bigwedge_{\substack{x=d+q\\ z=d'+q'}}\{\{F_K(d)\vee F_P(q)\}\vee\{F_K(d')\vee F_P(q')\}\}\\ &= \bigwedge_{\substack{x=d+q\\ z=d'+q'}}\{F_K(d)\vee F_K(d')\vee F_P(q)\vee F_P(q')\}\\ &\geqslant \bigwedge_{x+z=(d+d')+(q+q')}\{F_K(d+d')\vee F_P(q+q')\} = F_{K+P}(x+z).\end{aligned}$$

For $z\in\mathbb{O}$ and $r\in\mathcal{R}$, we obtain

$$\begin{aligned}T_{K+P}(z) &= \bigwedge_{z=q+w}\{T_K(q)\vee T_P(w)\}\\ &\geqslant \bigwedge_{zr=qr+wr}\{T_K(qr)\vee T_P(wr)\}\geqslant\bigwedge_{zr=q'+w'}\{T_K(q')\vee T_P(w')\}=T_{K+P}(zr),\\ I_{K+P}(z) &= \bigvee_{z=q+w}\{I_K(q)\wedge I_P(w)\}\\ &\leqslant \bigvee_{zr=qr+wr}\{I_K(qr)\wedge I_P(wr)\}\leqslant\bigvee_{zr=q'+w'}\{I_K(q')\wedge I_P(w')\}=I_{K+P}(zr),\end{aligned}$$

$$\begin{aligned}F_{K+P}(z) &= \bigwedge_{z=q+w}\{F_K(q)\vee F_P(w)\}\\ &\geqslant \bigwedge_{zr=qr+wr}\{F_K(qr)\vee F_P(wr)\}\geqslant\bigwedge_{zr=q'+w'}\{F_K(q')\vee F_P(w')\}=F_{K+P}(zr).\end{aligned}$$

Therefore, $\mathbb{O}_K\oplus\mathbb{O}_P$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$. □

Theorem 3. *Let $\mathbb{O}_K := \frac{\mathbb{O}}{(T_K,I_K,F_K)}$ be a neutrosophic $\mathfrak{N}$-structure in $\mathbb{O}$. If $\mathcal{R}_P := \frac{\mathcal{R}}{(T_P,I_P,F_P)}$ is a neutrosophic $\mathfrak{N}$-right ideal of $\mathcal{R}$, then $\mathbb{O}_K\odot\mathcal{R}_P$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$.*

Proof. For $z,x\in\mathbb{O}$. If $\exists w,w'\in\mathbb{O}$ and $q,q'\in\mathcal{R}: x=wq$ and $z=w'q'$, then

$$\begin{aligned}T_{K\circ P}(x)\vee T_{K\circ P}(z) &= \left[\bigwedge_{x=wq}\{T_K(w)\vee T_P(q)\}\right]\vee\left[\bigwedge_{z=w'q'}\{T_K(w')\vee T_P(q')\}\right]\\ &= \bigwedge_{\substack{x=wq\\ z=w'q'}}\{\{T_K(w)\vee T_P(q)\}\vee\{T_K(w')\vee T_P(q')\}\}\\ &= \bigwedge_{\substack{x=wq\\ z=w'q'}}\{T_K(w)\vee T_K(w')\vee T_P(q)\vee T_P(q')\}\\ &\geqslant \bigwedge_{x+z=(ww')+(qq')}\{T_K(ww')\vee T_P(qq')\}=T_{K\circ P}(x+z),\end{aligned}$$

$$\begin{aligned}
I_{K\circ P}(x) \wedge I_{K\circ P}(z) &= \left[\bigvee_{x=wq}\{I_K(w) \wedge I_P(q)\}\right] \wedge \left[\bigvee_{z=w'q'}\{I_K(w') \wedge I_P(q')\}\right] \\
&= \bigvee_{\substack{x=wq \\ z=w'q'}} \{\{I_K(w) \wedge I_P(q)\} \wedge \{I_K(w') \wedge I_P(q')\}\} \\
&= \bigvee_{\substack{x=wq \\ z=w'q'}} \{I_K(w) \wedge I_K(w') \wedge I_P(q) \wedge I_P(q')\} \\
&\leqslant \bigvee_{x+z=(ww')+(qq')} \{I_K(ww') \wedge I_P(qq')\} = I_{K\circ P}(x+z), \\
F_{K\circ P}(x) \vee F_{K\circ P}(z) &= \left[\bigwedge_{x=wq}\{F_K(w) \vee F_P(q)\}\right] \vee \left[\bigwedge_{z=w'q'}\{F_K(w') \vee F_P(q')\}\right] \\
&= \bigwedge_{\substack{x=wq \\ z=w'q'}} \{\{F_K(w) \vee F_P(q)\} \vee \{F_K(w') \vee F_P(q')\}\} \\
&= \bigwedge_{\substack{x=wq \\ z=w'q'}} \{F_K(w) \vee F_K(w') \vee F_P(q) \vee F_P(q')\} \\
&\geqslant \bigwedge_{x+z=(ww')+(qq')} \{F_K(ww') \vee F_P(qq')\} = F_{K\circ P}(x+z).
\end{aligned}$$

For $z \in \mathbb{O}$ and $r \in \mathcal{R}$, we obtain

$$\begin{aligned}
T_{K\circ P}(z) &= \bigwedge_{z=qw} \{T_K(q) \vee T_P(w)\} \\
&\geqslant \bigwedge_{zr=q(wr)} \{T_K(q) \vee T_P(wr)\} \geqslant \bigwedge_{zr=q'w'} \{T_K(q') \vee T_P(w')\} = T_{K\circ P}(zr),
\end{aligned}$$

$$\begin{aligned}
I_{K\circ P}(z) &= \bigvee_{z=qw} \{I_K(q) \wedge I_P(w)\} \\
&\leqslant \bigvee_{zr=qwr} \{I_K(q) \wedge I_P(wr)\} \leqslant \bigvee_{zr=q'w'} \{I_K(q') \wedge I_P(w')\} = I_{K\circ P}(zr),
\end{aligned}$$

$$\begin{aligned}
F_{K\circ P}(z) &= \bigwedge_{z=qw} \{F_K(q) \vee F_P(w)\} \\
&\geqslant \bigwedge_{zr=qwr} \{F_K(q) \vee F_P(wr)\} \geqslant \bigwedge_{zr=q'w'} \{F_K(q') \vee F_P(w')\} = F_{K\circ P}(zr).
\end{aligned}$$

Therefore, $\mathbb{O}_K \odot \mathcal{R}_P$ is a neutrosophic $\mathfrak{N}$-subsemimodule of $\mathbb{O}$. □

Corollary 3. *If $\mathcal{R}_K$ and $\mathcal{R}_P$ are neutrosophic $\mathfrak{N}$-ideals in $\mathcal{R}$, then $\mathcal{R}_K \oplus \mathcal{R}_P$ and $\mathcal{R}_K \odot \mathcal{R}_P$ are neutrosophic $\mathfrak{N}$-ideals in $\mathcal{R}$.*

Definition 13. *The neutrosophic $\mathfrak{N}$-product of $\mathcal{R}_K$ and $\mathcal{R}_P$ is described to be a neutrosophic $\mathfrak{N}$-structure of $\mathcal{R}$, $\mathcal{R}_K \odot \mathcal{R}_P := \frac{\mathcal{R}}{(T_{K\circ P}, I_{K\circ P}, F_{K\circ P})} = \left\{ \frac{k}{(T_{K\circ P}(k), I_{K\circ P}(k), F_{K\circ P}(k))} \mid k \in \mathcal{R} \right\}$, where*

$$(T_K \circ T_P)(k) = T_{K\circ P}(k) = \begin{cases} \bigwedge_{k=st} \{T_K(s) \vee T_P(t)\} & if\ \exists s,t \in \mathcal{R} \ : \ k = st \\ 0 & otherwise, \end{cases}$$

$$(I_K \circ I_P)(k) = I_{K\circ P}(k) = \begin{cases} \bigvee_{k=st} \{I_K(s) \wedge I_P(t)\} & if\ \exists s,t \in \mathcal{R}\ :\ k = st \\ -1 & otherwise, \end{cases}$$

$$(F_K \circ F_P)(k) = F_{K\circ P}(k) = \begin{cases} \bigwedge_{k=st} \{F_K(s) \vee F_P(t)\} & if\ \exists s,t \in \mathcal{R}\ :\ k = st \\ 0 & otherwise. \end{cases}$$

Theorem 4. *Let $\mathcal{R}_{\mathcal{Z}} := \frac{\mathcal{R}}{(T_{\mathcal{Z}}, I_{\mathcal{Z}}, F_{\mathcal{Z}})}$. Then, for any nonempty subsets J, D of $\mathcal{R}$, the following statements hold:*
(i) $\chi_J(\mathcal{R}_{\mathcal{Z}}) \cap \chi_D(\mathcal{R}_{\mathcal{Z}}) = \chi_{J\cap D}(\mathcal{R}_{\mathcal{Z}})$;
(ii) $\chi_J(\mathcal{R}_{\mathcal{Z}}) \odot \chi_D(\mathcal{R}_{\mathcal{Z}}) = \chi_{JD}(\mathcal{R}_{\mathcal{Z}})$.

Proof. (i) Let $v_1 \in \mathcal{R}$. If $v_1 \in J \cap D$, then $(\chi_J(T)_{\mathcal{Z}} \cap \chi_D(T)_{\mathcal{Z}})(v_1) = \chi_J(T)_{\mathcal{Z}}(v_1) \vee \chi_D(T)_{\mathcal{Z}}(v_1) = -1 = \chi_{J\cap D}(T)_{\mathcal{Z}}(v_1)$, $(\chi_J(I)_{\mathcal{Z}} \cap \chi_D(I)_{\mathcal{Z}})(v_1) = \chi_J(I)_{\mathcal{Z}}(v_1) \wedge \chi_D(I)_{\mathcal{Z}}(v_1) = 0 = \chi_{J\cap D}(I)_{\mathcal{Z}}(v_1)$ and $(\chi_J(F)_{\mathcal{Z}} \cap \chi_D(F)_{\mathcal{Z}})(v_1) = \chi_J(F)_{\mathcal{Z}}(v_1) \vee \chi_D(F)_{\mathcal{Z}}(v_1) = -1 = \chi_{J\cap D}(F)_{\mathcal{Z}}(v_1)$.

If $v_1 \notin J \cap D$, then $(\chi_J(T)_{\mathcal{Z}} \cap \chi_D(T)_{\mathcal{Z}})(v_1) = \chi_J(T)_{\mathcal{Z}}(v_1) \vee \chi_D(T)_{\mathcal{Z}}(v_1) = 0 = \chi_{J\cap D}(T)_{\mathcal{Z}}(v_1)$, $(\chi_J(I)_{\mathcal{Z}} \cap \chi_D(I)_{\mathcal{Z}})(v_1) = \chi_J(I)_{\mathcal{Z}}(v_1) \wedge \chi_D(I)_{\mathcal{Z}}(v_1) = -1 = \chi_{J\cap D}(I)_{\mathcal{Z}}(v_1)$ and $(\chi_J(F)_{\mathcal{Z}} \cap \chi_D(F)_{\mathcal{Z}})(v_1) = \chi_J(F)_{\mathcal{Z}}(v_1) \vee \chi_D(F)_{\mathcal{Z}}(v_1) = 0 = \chi_{J\cap D}(F)_{\mathcal{Z}}(v_1)$.

Therefore, $\chi_J(\mathcal{R}_{\mathcal{Z}}) \cap \chi_D(\mathcal{R}_{\mathcal{Z}}) = \chi_{J\cap D}(\mathcal{R}_{\mathcal{Z}})$.

(ii) Let $v_1 \in \mathcal{R}$. If $v_1 = c_1 d_1$ for some $c_1 \in J$ and $d_1 \in D$; then we have

$$\begin{aligned}
(\chi_J(T)_{\mathcal{Z}} \circ \chi_D(T)_{\mathcal{Z}})(v_1) &= \bigwedge_{v_1 = s_1 l_1} \{\chi_J(T)_{\mathcal{Z}}(s_1) \vee \chi_J(T)_{\mathcal{Z}}(l_1)\} \\
&\leq \chi_J(T)_{\mathcal{Z}}(c_1) \vee \chi_J(T)_{\mathcal{Z}}(d_1) = -1 = \chi_{JD}(T)_{\mathcal{Z}}(v_1), \\
(\chi_J(I)_{\mathcal{Z}} \circ \chi_D(I)_{\mathcal{Z}})(v_1) &= \bigvee_{v_1 = s_1 l_1} \{\chi_J(I)_{\mathcal{Z}}(s_1) \wedge \chi_J(I)_{\mathcal{Z}}(l_1)\} \\
&\geq \chi_J(I)_{\mathcal{Z}}(c_1) \vee \chi_J(I)_{\mathcal{Z}}(d_1) = 0 = \chi_{JD}(I)_{\mathcal{Z}}(v_1), \\
(\chi_J(F)_{\mathcal{Z}} \circ \chi_D(F)_{\mathcal{Z}})(v_1) &= \bigwedge_{v_1 = s_1 l_1} \{\chi_J(F)_{\mathcal{Z}}(s_1) \vee \chi_J(F)_{\mathcal{Z}}(l_1)\} \\
&\leq \chi_J(F)_{\mathcal{Z}}(c_1) \vee \chi_J(F)_{\mathcal{Z}}(d_1) = -1 = \chi_{JD}(F)_{\mathcal{Z}}(v_1).
\end{aligned}$$

If $v_1 \neq c_1 d_1$ for any $c_1 \in J$ and $d_1 \in D$, then we have

$$\begin{aligned}
(\chi_J(T)_{\mathcal{Z}} \circ \chi_D(T)_{\mathcal{Z}})(v_1) &= \bigwedge_{v_1 = s_1 l_1} \{\chi_J(T)_{\mathcal{Z}}(s_1) \vee \chi_J(T)_{\mathcal{Z}}(l_1)\} = 0 = \chi_{JD}(T)_{\mathcal{Z}}(v_1), \\
(\chi_J(I)_{\mathcal{Z}} \circ \chi_D(I)_{\mathcal{Z}})(v_1) &= \bigvee_{v_1 = s_1 l_1} \{\chi_J(I)_{\mathcal{Z}}(s_1) \wedge \chi_J(I)_{\mathcal{Z}}(l_1)\} = -1 = \chi_{JD}(I)_{\mathcal{Z}}(v_1), \\
(\chi_J(F)_{\mathcal{Z}} \circ \chi_D(F)_{\mathcal{Z}})(v_1) &= \bigwedge_{v_1 = s_1 l_1} \{\chi_J(F)_{\mathcal{Z}}(s_1) \vee \chi_J(F)_{\mathcal{Z}}(l_1)\} = 0 = \chi_{JD}(F)_{\mathcal{Z}}(v_1).
\end{aligned}$$

Therefore, $\chi_J(\mathcal{R}_{\mathcal{Z}}) \odot \chi_D(\mathcal{R}_{\mathcal{Z}}) = \chi_{JD}(\mathcal{R}_{\mathcal{Z}})$. □

The equivalent condition for a non-empty subset of $\mathcal{R}$ to be an ideal of $\mathcal{R}$ is given below.

Theorem 5. *Let $\mathcal{R}_{\mathcal{H}} := \frac{\mathcal{R}}{(T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}})}$. Then, for any subset $C(\neq \emptyset)$ of $\mathcal{R}$, the below criteria are equivalent:*
(i) C of $\mathcal{R}$ is a left (right) ideal;
(ii) $\chi_C(\mathcal{R}_{\mathcal{H}})$ of $\mathcal{R}$ is a neutrosophic $\mathfrak{N}$-left (right) ideal.

Proof. $(i) \Rightarrow (ii)$ Let $z, x \in \mathcal{R}$. If $x \in C$ and $z \in C$, then $xz \in C$, so $\chi_C(T)_{\mathcal{H}}(xz) = -1 = \chi_C(T)_{\mathcal{H}}(x) \vee \chi_C(T)_{\mathcal{H}}(z)$, $\chi_C(I)_{\mathcal{H}}(xz) = 0 = \chi_C(I)_{\mathcal{H}}(x) \wedge \chi_C(I)_{\mathcal{H}}(z)$ and $\chi_C(F)_{\mathcal{H}}(xz) =$

$-1 = \chi_C(F)_{\mathcal{H}}(x) \vee \chi_C(F)_{\mathcal{H}}(z)$. If $x \notin C$ or $z \notin C$, then $\chi_C(T)_{\mathcal{H}}(xz) \leq 0 = \chi_C(T)_{\mathcal{H}}(x) \vee \chi_C(T)_{\mathcal{H}}(z)$, $\chi_C(I)_{\mathcal{H}}(xz) \geq -1 = \chi_C(I)_{\mathcal{H}}(x) \wedge \chi_C(I)_{\mathcal{H}}(z)$ and $\chi_C(F)_{\mathcal{H}}(xz) \leq 0 = \chi_C(F)_{\mathcal{H}}(x) \vee \chi_C(F)_{\mathcal{H}}(z)$.

If $x \in C$, then $xz \in C$, which implies $\chi_C(T)_{\mathcal{H}}(xz) = -1 \leq \chi_C(T)_{\mathcal{H}}(z)$, $\chi_C(I)_{\mathcal{H}}(xz) = 0 \geq \chi_C(I)_{\mathcal{H}}(z)$ and $\chi_C(F)_{\mathcal{H}}(xz) = -1 \leq \chi_C(F)_{\mathcal{H}}(z)$. Therefore, $\chi_C(\mathcal{R}_{\mathcal{H}})$ is a neutrosophic $\mathfrak{N}$-left ideal of $\mathcal{R}$.

$(ii) \Rightarrow (i)$ Let $l, z \in C$ and $y \in \mathcal{R}$. Then $\chi_C(T)_{\mathcal{H}}(l+z) \leq \chi_C(T)_{\mathcal{H}}(l) \vee \chi_C(T)_{\mathcal{H}}(z) = -1, \chi_C(I)_{\mathcal{H}}(l+z) \geq \chi_C(I)_{\mathcal{H}}(l) \wedge \chi_C(I)_{\mathcal{H}}(z) = 0$ and $\chi_C(F)_{\mathcal{H}}(l+z) \leq \chi_C(F)_{\mathcal{H}}(l) \vee \chi_C(F)_{\mathcal{H}}(z) = -1$, which imply $l+z \in C$.

In addition, $\chi_C(T)_{\mathcal{H}}(ly) \leq \chi_C(T)_{\mathcal{H}}(y) = -1, \chi_C(I)_{\mathcal{H}}(ly) \geq \chi_C(I)_{\mathcal{H}}(y) = 0$ and $\chi_C(F)_{\mathcal{H}}(ly) \leq \chi_C(F)_{\mathcal{H}}(y) = -1$, which imply $ly \in C$. Therefore, C of $\mathcal{R}$ is a left ideal. □

Definition 14 ([1])**.** *In $\mathcal{R}$, an ideal P is known as a right t-pure if for $t \in P$, $\exists\, d \in P\ : t = td$.*

Theorem 6 ([1])**.** *If Y of $\mathcal{R}$ is a two-sided ideal, then the below criteria are equivalent:*
(i) for any right ideal G of $\mathcal{R}$, $G \cap Y = GY$;
(ii) Y is right t-pure.

Definition 15. *A subsemimodule N of $\mathbb{O}$ is said to be pure in $\mathbb{O}$ if, for any ideal I of $\mathcal{R}$, $N \cap \mathbb{O}I = NI$. If $\mathbb{O}$ is described as normal, then each subsemimodule of $\mathbb{O}$ is pure in $\mathbb{O}$.*

Definition 16. *A neutrosophic $\mathfrak{N}$-right ideal $\mathcal{R}_K := \frac{\mathcal{R}}{(T_K, I_K, F_K)}$ is described as a neutrosophic right t-pure $\mathfrak{N}$-ideal in $\mathcal{R}$ if $\mathcal{R}_P \cap \mathcal{R}_K = \mathcal{R}_P \odot \mathcal{R}_K$ for every neutrosophic $\mathfrak{N}$ -right ideal $\mathcal{R}_P$ in $\mathcal{R}$.*

Below is the equivalent condition for an ideal of $\mathcal{R}$ to be a right t-pure ideal of $\mathcal{R}$.

Theorem 7. *Let $\mathcal{R}_K := \frac{\mathcal{R}}{(T_K, I_K, F_K)}$ and C be an ideal of $\mathcal{R}$. Then, the below criteria are equivalent:*
(i) $\chi_C(\mathcal{R}_K)$ of $\mathcal{R}$ is a neutrosophic right t-pure $\mathfrak{N}$-ideal;
(ii) C of $\mathcal{R}$ is a right t-pure ideal.

Proof. $(i) \Rightarrow (ii)$ By Theorem 5, C is a right ideal of $\mathcal{R}$. For any right ideal D of $\mathcal{R}$, we have $\chi_C(\mathcal{R}_K) \cap \chi_D(\mathcal{R}_K) = \chi_C(\mathcal{R}_K) \odot \chi_D(\mathcal{R}_K)$. By Theorem 4, we have $\chi_{C \cap D}(\mathcal{R}_K) = \chi_{CD}(\mathcal{R}_K)$, which implies that $C \cap D = CD$; therefore, C is a right t-pure ideal of $\mathcal{R}$.

$(ii) \Rightarrow (i)$ By Theorem 5, $\chi_C(\mathcal{R}_K)$ is a neutrosophic right $\mathfrak{N}$-ideal of $\mathcal{R}$.
Let $\mathcal{R}_P := \frac{\mathcal{R}}{(T_P, I_P, F_P)}$ be a neutrosophic $\mathfrak{N}$ -right ideal in $\mathcal{R}$.
Now, we show that $\mathcal{R}_P \cap \chi_C(\mathcal{R}_K) = \mathcal{R}_P \odot \chi_C(\mathcal{R}_K)$.
Let $x \in \mathcal{R}$. Then,

$$
\begin{aligned}
(T_P \circ \chi_C(T)_K)(w) &= \bigwedge_{w=st} \{T_P(s) \vee \chi_C(T)_K(t)\} \\
&\geq \bigwedge_{w=st} \{T_P(st) \vee \chi_C(T)_K(st)\} = T_P(x) \vee \chi_C(T)_K(x), \\
(I_P \circ \chi_C(I)_K)(w) &= \bigvee_{w=st} \{I_P(s) \wedge \chi_C(I)_K(t)\} \\
&\leq \bigvee_{w=st} \{I_P(st) \wedge \chi_C(I)_K(st)\} = I_P(x) \wedge \chi_C(I)_K(x), \\
(F_P \circ \chi_C(F)_K)(w) &= \bigwedge_{w=st} \{F_P(s) \vee \chi_C(F)_K(t)\} \\
&\geq \bigwedge_{w=st} \{F_P(st) \vee \chi_C(F)_K(st)\} = F_P(x) \vee \chi_C(F)_K(x).
\end{aligned}
$$

Therefore, $\mathcal{R}_P \cap \chi_C(\mathcal{R}_K) \supseteq \mathcal{R}_P \odot \chi_C(\mathcal{R}_K)$.
Let $h \in \mathcal{R}$. If $h \notin C$, then

$$(T_P \cap \chi_C(T)_K)(h) = T_P(h) \vee \chi_C(T)_K(h) = 0 \geq (T_P \dot{\chi}_C(T)_K)(h),$$
$$(I_P \cap \chi_C(I)_K)(h) = I_P(h) \wedge \chi_C(I)_K(h) = -1 \leq (I_P \dot{\chi}_C(I)_K)(h),$$
$$(F_P \cap \chi_C(F)_K)(h) = F_P(h) \vee \chi_C(F)_K(h) = 0 \geq (F_P \dot{\chi}_C(F)_K)(h).$$

If $h \in C$, then $\exists r \in C : h = hr$. Now,

$$\begin{aligned}(T_P \cap \chi_C(T)_K)(h) &= T_P(h) \vee \chi_C(T)_K(h) \\ &= T_P(h) \vee \chi_C(T)_K(hr) \geq \bigwedge_{h=ad} \{T_P(a) \vee \chi_C(T)_K(d)\} = (T_P \dot{\chi}_C(T)_K)(h), \\ (I_P \cap \chi_C(I)_K)(h) &= I_P(h) \wedge \chi_C(I)_K(h) \\ &= I_P(h) \wedge \chi_C(I)_K(hr) \leq \bigvee_{h=ad} \{I_P(a) \wedge \chi_C(I)_K(d)\} = (I_P \dot{\chi}_C(I)_K)(h), \\ (F_P \cap \chi_C(F)_K)(h) &= F_P(h) \vee \chi_C(F)_K(h) \\ &= F_P(h) \vee \chi_C(F)_K(hr) \geq \bigwedge_{h=ad} \{F_P(a) \vee \chi_C(F)_K(d)\} = (F_P \dot{\chi}_C(F)_K)(h).\end{aligned}$$

Thus, $\mathcal{R}_P \cap \chi_C(\mathcal{R}_K) \subseteq \mathcal{R}_P \odot \chi_C(\mathcal{R}_K)$ and hence $\mathcal{R}_P \cap \chi_C(\mathcal{R}_K) = \mathcal{R}_P \odot \chi_C(\mathcal{R}_K)$. □

Definition 17. *An ideal I of $\mathcal{R}$ is idempotent if $I^2 = I$. If every ideal of $\mathcal{R}$ is idempotent, then $\mathcal{R}$ is termed as fully idempotent. A neutrosophic $\mathfrak{N}$-structure $\mathcal{R}_K$ of $\mathcal{R}$ is called idempotent if $\mathcal{R}_K \odot \mathcal{R}_K = \mathcal{R}_K$.*

Definition 18. *A semiring $\mathcal{R}$ is termed as regular if, for $j \in \mathcal{R}$, $\exists z \in \mathcal{R} : j = jzj$. Clearly, every regular semiring is fully idempotent.*

Theorem 8. *For $\mathcal{R}$, the conditions listed below are equivalent:*
(i) $\mathcal{R}$ is fully idempotent;
(ii) every neutrosophic $\mathfrak{N}$-ideal in $\mathcal{R}$ is idempotent;
(iii) for every neutrosophic $\mathfrak{N}$-ideals $\mathcal{R}_{\mathscr{W}}$ and $\mathcal{R}_{\mathscr{B}}$ in $\mathcal{R}$, $\mathcal{R}_{\mathscr{W}} \cap \mathcal{R}_{\mathscr{B}} = \mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{B}}$.
If $\mathcal{R}$ is commutative, then the above criteria are equivalent to
(iv) $\mathcal{R}$ is regular.

Proof. $(i) \Rightarrow (ii)$ Let $\mathcal{R}_{\mathscr{W}} := \frac{\mathcal{R}}{(T_{\mathscr{W}}, I_{\mathscr{W}}, F_{\mathscr{W}})}$ of $\mathcal{R}$ be a neutrosophic $\mathfrak{N}$-ideal. Then, for any $f \in \mathcal{R}$, we obtain $(T_{\mathscr{W}} \circ T_{\mathscr{W}})(f) = \bigwedge_{f=ad} \{T_{\mathscr{W}}(a) \vee T_{\mathscr{W}}(d)\} \geq \bigwedge_{f=ad} \{T_{\mathscr{W}}(ad) \vee T_{\mathscr{W}}(ad)\} = \bigwedge_{f=ad} \{T_{\mathscr{W}}(f) \vee T_{\mathscr{W}}(f)\} = T_{\mathscr{W}}(f)$, $(I_{\mathscr{W}} \circ I_{\mathscr{W}})(f) = \bigvee_{f=ad} \{I_{\mathscr{W}}(a) \wedge I_{\mathscr{W}}(d)\} \leq \bigvee_{f=ad} \{I_{\mathscr{W}}(ad) \wedge I_{\mathscr{W}}(ad)\} = \bigvee_{f=ad} \{I_{\mathscr{W}}(f) \wedge I_{\mathscr{W}}(f)\} = I_{\mathscr{W}}(f)$ and $(F_{\mathscr{W}} \circ F_{\mathscr{W}})(f) = \bigwedge_{f=ad} \{F_{\mathscr{W}}(a) \vee F_{\mathscr{W}}(d)\} \geq \bigwedge_{f=ad} \{F_{\mathscr{W}}(ad) \vee F_{\mathscr{W}}(ad)\} = \bigwedge_{f=ad} \{F_{\mathscr{W}}(f) \vee F_{\mathscr{W}}(f)\} = F_{\mathscr{W}}(f)$. Therefore, $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{W}} \subseteq \mathcal{R}_{\mathscr{W}}$.

Since $\mathcal{R}$ is fully idempotent, we have $f \in < f > = < f >^2 = \mathcal{R}f\mathcal{R}\mathcal{R}f\mathcal{R}$, so $\exists p_1, f_1, p_2, f_2 \in \mathcal{R} : f = p_1 f f_1 p_2 f f_2$. Now,

$$\begin{aligned} T_{\mathscr{W}}(f) &= T_{\mathscr{W}}(f) \vee T_{\mathscr{W}}(f) \geq T_{\mathscr{W}}(p_1 f f_1) \vee T_{\mathscr{W}}(p_2 f f_2) \\ &\geq \bigwedge_{f=p_1 f f_1 p_2 f f_2} \{T_{\mathscr{W}}(p_1 f f_1) \vee T_{\mathscr{W}}(p_2 f f_2)\} \geq \bigwedge_{f=st} \{T_{\mathscr{W}}(s) \vee T_{\mathscr{W}}(t)\} = (T_{\mathscr{W}} \circ T_{\mathscr{W}})(f), \\ I_{\mathscr{W}}(f) &= I_{\mathscr{W}}(f) \wedge I_{\mathscr{W}}(f) \leq I_{\mathscr{W}}(p_1 f f_1) \wedge I_{\mathscr{W}}(p_2 f f_2) \\ &\leq \bigvee_{f=p_1 f f_1 p_2 f f_2} \{I_{\mathscr{W}}(p_1 f f_1) \wedge I_{\mathscr{W}}(p_2 f f_2)\} \leq \bigvee_{f=st} \{I_{\mathscr{W}}(s) \wedge I_{\mathscr{W}}(t)\} = (I_{\mathscr{W}} \circ I_{\mathscr{W}})(f),\end{aligned}$$

$$
\begin{aligned}
F_{\mathscr{W}}(f) &= F_{\mathscr{W}}(f) \vee F_{\mathscr{W}}(f) \geq F_{\mathscr{W}}(p_1 f f_1) \vee F_{\mathscr{W}}(p_2 f f_2) \\
&\geq \bigwedge_{f = p_1 f f_1 p_2 f f_2} \{F_{\mathscr{W}}(p_1 f f_1) \vee F_{\mathscr{W}}(p_2 f f_2)\} \geq \bigwedge_{f=st} \{F_{\mathscr{W}}(s) \vee F_{\mathscr{W}}(t)\} = (F_{\mathscr{W}} \circ F_{\mathscr{W}})(f).
\end{aligned}
$$

Thus, $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{W}} \supseteq \mathcal{R}_{\mathscr{W}}$ and hence $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{W}} = \mathcal{R}_{\mathscr{W}}$.

$(ii) \Rightarrow (i)$ If $\mathcal{R}_{\mathscr{W}}$ is a neutrosophic $\mathfrak{N}$-ideal of $\mathcal{R}$ and C is an ideal of $\mathcal{R}$, then $\chi_C(\mathcal{R}_{\mathscr{W}})$ is a neutrosophic $\mathfrak{N}$-left ideal of $\mathcal{R}$, so $\chi_C(\mathcal{R}_{\mathscr{W}}) \odot \chi_C(\mathcal{R}_{\mathscr{W}}) = \chi_C(\mathcal{R}_{\mathscr{W}})$ implies that $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{W}} = \mathcal{R}_{\mathscr{W}}$.

$(i) \Rightarrow (iii)$ Let $\mathcal{R}_{\mathscr{W}}$ and $\mathcal{R}_{\mathscr{B}}$ be any two neutrosophic $\mathfrak{N}$-ideals of $\mathcal{R}$ and $b \in \mathcal{R}$. Then,

$$
\begin{aligned}
(T_{\mathscr{W}} \circ T_{\mathscr{B}})(b) &= \bigwedge_{b=st} \{T_{\mathscr{W}}(s) \vee T_{\mathscr{B}}(t)\} \geq \bigwedge_{b=st} \{T_{\mathscr{W}}(st) \vee T_{\mathscr{B}}(st)\} \\
&= \bigwedge_{b=st} \{T_{\mathscr{W}}(b) \vee T_{\mathscr{B}}(b)\} = T_{\mathscr{W}}(b) \vee T_{\mathscr{B}}(b) = (T_{\mathscr{W}} \cap T_{\mathscr{B}})(b), \\
(I_{\mathscr{W}} \circ I_{\mathscr{B}})(b) &= \bigvee_{b=st} \{I_{\mathscr{W}}(s) \wedge I_{\mathscr{B}}(t)\} \leq \bigvee_{b=st} \{I_{\mathscr{W}}(st) \wedge I_{\mathscr{B}}(st)\} \\
&= \bigvee_{b=st} \{I_{\mathscr{W}}(b) \wedge I_{\mathscr{B}}(b)\} = I_{\mathscr{W}}(b) \wedge I_{\mathscr{B}}(b) = (I_{\mathscr{W}} \cup I_{\mathscr{B}})(b), \\
(F_{\mathscr{W}} \circ F_{\mathscr{B}})(b) &= \bigwedge_{b=st} \{F_{\mathscr{W}}(s) \vee F_{\mathscr{B}}(t)\} \geq \bigwedge_{b=st} \{F_{\mathscr{W}}(st) \vee F_{\mathscr{B}}(st)\} \\
&= \bigwedge_{b=st} \{F_{\mathscr{W}}(b) \vee F_{\mathscr{B}}(b)\} = F_{\mathscr{W}}(b) \vee F_{\mathscr{B}}(b) = (F_{\mathscr{W}} \cap F_{\mathscr{B}})(b).
\end{aligned}
$$

Therefore, $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{B}} \subseteq \mathcal{R}_{\mathscr{W}} \cap \mathcal{R}_{\mathscr{B}}$.

Since $\mathcal{R}$ is fully idempotent, we have $< b >=< b >^2$ for any $b \in \mathcal{R}$. In the first section of this Theorem's proof, we mentioned that we obtain

$$
\begin{aligned}
(T_{\mathscr{W}} \cap T_{\mathscr{B}})(b) &= T_{\mathscr{W}}(b) \vee T_{\mathscr{B}}(b) \geq \bigwedge_{b=st} \{T_{\mathscr{W}}(s) \vee T_{\mathscr{B}}(t)\} = (T_{\mathscr{W}} \circ T_{\mathscr{B}})(b), \\
(I_{\mathscr{W}} \cup I_{\mathscr{B}})(b) &= I_{\mathscr{W}}(b) \wedge I_{\mathscr{B}}(b) \leq \bigvee_{b=st} \{I_{\mathscr{W}}(s) \wedge I_{\mathscr{B}}(t)\} = (I_{\mathscr{W}} \circ I_{\mathscr{B}})(b), \\
(F_{\mathscr{W}} \cap F_{\mathscr{B}})(b) &= F_{\mathscr{W}}(b) \vee F_{\mathscr{B}}(b) \geq \bigwedge_{b=st} \{F_{\mathscr{W}}(s) \vee F_{\mathscr{B}}(t)\} = (F_{\mathscr{W}} \circ F_{\mathscr{B}})(b).
\end{aligned}
$$

Thus, $\mathcal{R}_{\mathscr{W}} \cap \mathcal{R}_{\mathscr{B}} \subseteq \mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{B}}$ and hence $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{B}} = \mathcal{R}_{\mathscr{W}} \cap \mathcal{R}_{\mathscr{B}}$.

$(iii) \Rightarrow (i)$ If $\mathcal{R}_{\mathscr{W}}$ of $\mathcal{R}$ is a neutrosophic $\mathfrak{N}$-ideal, then $\mathcal{R}_{\mathscr{W}} \odot \mathcal{R}_{\mathscr{W}} = \mathcal{R}_{\mathscr{W}} \cap \mathcal{R}_{\mathscr{W}} = \mathcal{R}_{\mathscr{W}}$. As $(iii) \iff (i)$ and $(ii) \iff (i)$, we obtain $(i) \iff (ii) \iff (iii)$. If $\mathcal{R}$ is commutative, then it is simple to obtain $(iv) \iff (i)$. □

Theorem 9. *For any neutrosophic $\mathfrak{N}$-subsemimodule $\mathbb{O}_K$ of $\mathbb{O}$ and neutrosophic $\mathfrak{N}$- ideal $\mathcal{R}_I$ in $\mathcal{R}$, if $\mathcal{R}$ is regular, then for any $x \in \mathbb{O}$, we have*

$$
(T_K \circ T_P)(x) = T_{K \circ P}(x) = \begin{cases} \bigwedge_{x=st} \{T_K(st) \vee T_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R} : x = st \\ 0 & otherwise, \end{cases}
$$

$$
(I_K \circ I_P)(x) = I_{K \circ P}(x) = \begin{cases} \bigvee_{x=st} \{I_K(st) \wedge I_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R} : x = st \\ -1 & otherwise, \end{cases}
$$

$$
(F_K \circ F_P)(x) = F_{K \circ P}(x) = \begin{cases} \bigwedge_{x=st} \{F_K(st) \vee F_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R} : x = st \\ 0 & otherwise. \end{cases}
$$

Proof. Let $x \in \mathbb{O}$. Then, by Definition, we have

$$(T_K \circ T_P)(x) = T_{K \circ P}(x) = \begin{cases} \bigwedge\limits_{x=st} \{T_K(s) \vee T_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R}\ :\ x = st \\ 0 & otherwise, \end{cases}$$

$$(I_K \circ I_P)(x) = I_{K \circ P}(x) = \begin{cases} \bigvee\limits_{x=st} \{I_K(s) \wedge I_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R}\ :\ x = st \\ -1 & otherwise, \end{cases}$$

$$(F_K \circ F_P)(x) = F_{K \circ P}(x) = \begin{cases} \bigwedge\limits_{x=st} \{F_K(s) \vee F_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R}\ :\ x = st \\ 0 & otherwise. \end{cases}$$

Since $\mathcal{R}$ is regular, for $u \in \mathcal{R}, \exists t_1, t_2 \in \mathcal{R} : u = ut_1ut_2$. Clearly, $T_K(zu) \geq T_K(zut_1) \geq T_K(zut_1ut_2) = T_K(zu)$, $I_K(zu) \leq I_K(zut_1) \leq I_K(zut_1ut_2) = I_K(zu)$, $F_K(zu) \geq F_K(zut_1) \geq F_K(zut_1ut_2) = F_K(zu)$. In addition, $T_P(u) \geq T_P(ut_1) \geq T_P(ut_1ut_2) = T_P(u)$, $I_P(u) \leq I_P(ut_1) \leq I_P(ut_1ut_2) = I_P(u)$, $F_P(u) \geq F_P(ut_1) \geq F_P(ut_1ut_2) = F_P(u)$.

Now,

$$\begin{aligned} (T_K \circ T_P)(x) = T_{K \circ P}(x) &\geq \bigwedge_{x=zu} \{T_K(zu) \vee T_P(u)\} \\ &= \bigwedge_{x=zu} \{T_K(zut_1) \vee T_P(ut_1)\} \\ &\geq \bigwedge_{x=ys} \{T_K(y) \vee T_P(s)\} = T_{K \circ P}(x) = (T_K \circ T_P)(x), \\ (I_K \circ I_P)(x) = I_{K \circ P}(x) &\leq \bigvee_{x=zu} \{I_K(zu) \wedge I_P(u)\} \\ &= \bigvee_{x=zu} \{I_K(zut_1) \wedge I_P(ut_1)\} \\ &\leq \bigvee_{x=ys} \{I_K(y) \wedge I_P(s)\} = I_{K \circ P}(x) = (I_K \circ I_P)(x), \\ (F_K \circ F_P)(x) = F_{K \circ P}(x) &\geq \bigwedge_{x=zu} \{F_K(zu) \vee F_P(u)\} \\ &= \bigwedge_{x=zu} \{F_K(zut_1) \vee F_P(ut_1)\} \\ &\geq \bigwedge_{x=ys} \{F_K(y) \vee F_P(s)\} = F_{K \circ P}(x) = (F_K \circ F_P)(x). \end{aligned}$$

Therefore,

$$(T_K \circ T_P)(x) = T_{K \circ P}(x) = \begin{cases} \bigwedge\limits_{x=st} \{T_K(st) \vee T_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R}\ :\ x = st \\ 0 & otherwise, \end{cases}$$

$$(I_K \circ I_P)(x) = I_{K \circ P}(x) = \begin{cases} \bigvee\limits_{x=st} \{I_K(st) \wedge I_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R}\ :\ x = st \\ -1 & otherwise, \end{cases}$$

$$(F_K \circ F_P)(x) = F_{K \circ P}(x) = \begin{cases} \bigwedge\limits_{x=st} \{F_K(st) \vee F_P(t)\} & if\ \exists s \in \mathbb{O}, t \in \mathcal{R}\ :\ x = st \\ 0 & otherwise. \end{cases}$$

□

5. Conclusions

Algebraic structures are significant in mathematics, having a broad impact in diverse fields such as theoretical physics, computer science, control engineering, information science, coding theory, and topological spaces, among others. Symmetry is a crucial and aesthetically pleasing concept that connects several domains in modern mathematics, with algebraic structures providing a valuable apparatus in pure mathematics for understanding the symmetries of geometric entities. We have obtained the neutrosophic structures for semiring modules; the idea of neutrosophic $\mathfrak{N}$-subsemimodules and neutrosophic $\mathfrak{N}$-ideals was established in this study, and some of their characteristics were discussed. In addition, we looked into the idea of neutrosophic right t-pure ideals in a semiring and the many connections between neutrosophic t-pure ideals and neutrosophic $\mathfrak{N}$-submodules in a semiring. Moreover, we have obtained equivalent statements for a semiring to be fully idempotent. Using the ideas and findings of this paper, it is possible to define the concept of neutrosophic $\mathfrak{N}$-prime(resp., semi) ideals and derive their various properties and equivalent conditions for a neutrosophic $\mathfrak{N}$-ideal to be a neutrosophic $\mathfrak{N}$-prime (resp., semi) ideal in a neutrosophic $\mathfrak{N}$-subsemimodule.

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