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Article

Coset Group Construction of Multidimensional Number Systems

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Abstract: Extensions of real numbers in more than two dimensions, in particular quaternions and octonions, are finding applications in physics due to the fact that they naturally capture symmetries of physical systems. However, in the conventional mathematical construction of complex and multicomplex numbers multiplication rules are postulated instead of being derived from a general principle. A more transparent and systematic approach is proposed here based on the concept of coset product from group theory. It is shown that extensions of real numbers in two or more dimensions follow naturally from the closure property of finite coset groups adding insight into the utility of multidimensional number systems in describing symmetries in nature.

Keywords: complex numbers; quaternions; representations

1. Introduction

While the utility of the familiar complex numbers in physics and applied sciences is not questionable, one important question still stands: where do they come from? What are complex numbers fundamentally? This question becomes even more important considering that complex numbers in more than two dimensions such as quaternions and octonions are gaining renewed interest in physics. This manuscript demonstrates that multidimensional numbers systems are representations of small group symmetries.

Although measurements in the laboratory produce real numbers, complex numbers in two dimensions are essential elements of mathematical descriptions of physical systems. Another example of multidimensional number systems are the quaternions [1]. These are extensions of real numbers in four dimensions constructed using the basis $\{1, i, j, k\}$ with multiplication rules $i^2 = j^2 = k^2 = ijk = -1$ [2]. Just like the set of complex numbers, the set of quaternions is closed under both addition and multiplication. Due to their particular group structure, quaternions can be used in physics for representations of the Lorentz group and in general for transformations involving 4-vectors [3,4]. Quaternions are also receiving attention in quantum mechanics due to a direct relationship with Pauli matrices and Pauli's group [5–10].

Other variations of quaternions have been postulated and used in physics, such as split-quaternions [11,12] and bicomplex numbers which are obtained from a 4-dimensional basis like the quaternions but with different multiplication rules [13]. Similarly, different kinds of 2-dimensional numbers can be constructed by modifying the multiplication rule. For example, split-complex numbers correspond to $i^2 = 1$ (also called hyperbolic numbers) and dual complex numbers correspond to $i^2 = 0$ [13–15].

The emerging role of multidimensional numbers systems such as quaternions (4D) and octonions (8D) in physics suggests that other extensions of real numbers could possibly be useful for describing symmetries in nature [16]. Traditionally, multidimensional number systems are constructed as algebras over the field of reals using a set of elements (called basis) and by postulating multiplication rules for basis elements [13,17,18]. However, it is not immediately clear what extensions are possible and how multiplication rules are discovered. A more transparent and systematic approach is proposed here based on the concept of coset product from group theory. As opposed to groups and subgroups, the concept of coset is less known outside mathematics. (However cosets are more common than recognized; e.g., parallel planes in 3D are in fact cosets improperly termed subgroups or subspaces.) Briefly, cosets are shifted replicas of a given subgroup forming a set that is closed under coset product. It is this property of closure of the coset group that gives rise naturally to multidimensional numbers simply by virtue of their group properties. In this respect, the coset approach is an alternative to the established algebra over fields construction with the advantage that multiplication rules follow directly from group closure instead of being postulated. A complete classification of multidimensional numbers systems is obtained in this systematic way. In what follows, the general coset construction is presented and then applied to coset groups of order $n = 2$, 3, and 4. Multiplication rules and general matrix representations are derived for each number system corresponding to these small groups.

For additional reading on multidimensional number systems, references [18–20] review hypercomplex numbers in general, reference [16] discusses quaternions and octonions, and reference [21] presents a detailed analysis of a selection of multidimensional numbers including geometrical representations and functional analysis. For a historical perspective on the early development of quaternions, readers are directed to the original paper by Hamilton [2] and to references [20,22–27].

2. General Construction of Coset Extensions of Reals

Consider that there exists a set of elements g_i outside of the set of real numbers $\mathbb{R}$ but compatible with operations in $\mathbb{R}$. Using these elements, we can construct a collection $\mathbb{G}$ of cosets of $\mathbb{R}$,

$$\mathbb{G} = \{...g_i\mathbb{R}, g_ig_j\mathbb{R}, g_ig_jg_k\mathbb{R}, ...\} \tag{1}$$

where $g_i\mathbb{R}$ represents the set obtained by multiplying each real number by g_i. In $g_i\mathbb{R}$, all real numbers have been "shifted" or scaled by a factor of g_i, in $g_ig_j\mathbb{R}$ by a factor of g_ig_j and so forth. These shifted sets are called cosets. The set $\mathbb{G}$ is then the set of cosets of $\mathbb{R}$ and we require that $\mathbb{G}$ is finite and closed under coset product, namely if $g_i\mathbb{R} \in \mathbb{G}$ and $g_j\mathbb{R} \in \mathbb{G}$, then $g_ig_j\mathbb{R} \in \mathbb{G}$ for all elements g_i and g_j. (Technically, $\mathbb{G}$ is a quotient group modulo $\mathbb{R}$ of a larger group.)

By restricting the number n of distinct elements in $\mathbb{G}$, we generate various extensions of real numbers in which multiplication rules follow from the structure of the coset group. We will treat explicitly the simplest cases in which $\mathbb{G}$ is one of the small groups with $n = 2$, 3, or 4 elements. For $n = 2$ and $n = 3$ the only possible group structure is cyclic, while for $n = 4$ there are two distinct groups, one cyclic and the other called the Klein four-group. By setting $\mathbb{G}$ to be one of these groups, we obtain extensions of $\mathbb{R}$ in a systematic way. Multiplication rules follow naturally from the closure of the coset group showing explicitly the underlying symmetry of each number system.

3. Case $n = 2$

In this simplest case, we require that $\mathbb{G}$ has exactly two distinct elements, $\mathbb{G} = \{\mathbb{R}, g\mathbb{R}\}$ and is closed under coset product. Since this construction generates the well know complex numbers, the coset construction is presented here in detail in order to illustrate the procedure using a familiar number system.

Imposing that $\mathbb{G} = \{\mathbb{R}, g\mathbb{R}\}$ is closed requires that each element in Equation (1) must be either $\mathbb{R}$ or $g\mathbb{R}$. Assuming that $g^{-1}\mathbb{R} = \mathbb{R}$ and multiplying by g, we obtain $gg^{-1}\mathbb{R} = g\mathbb{R}$ which gives $\mathbb{R} = g\mathbb{R}$. This contradicts the requirement that $\mathbb{G}$ has two distinct elements. It follows that $g^{-1}\mathbb{R} = g\mathbb{R}$ and therefore $g^2\mathbb{R} = (g\mathbb{R})(g\mathbb{R}) = (g\mathbb{R})(g^{-1}\mathbb{R}) = (gg^{-1})\mathbb{R} = \mathbb{R}$. We have obtained the important result that g^2 is a real number while g itself is not. We can then set $g^2 = \alpha$, where α is a real number, not necessarily positive. Generalized complex numbers are obtained by generating a set that is closed under addition and multiplication of elements within cosets. Closure under addition generates the set

$$\{z = a + gb | a, b \in \mathbb{R}\}$$

for which addition and multiplication rules follow directly from the properties of g:

$$z_1 + z_2 = (a_1 + a_2) + g(b_1 + b_2) \tag{2}$$

$$z_1 z_2 = (a_1 a_2 + \alpha b_1 b_2) + g(a_1 b_2 + a_2 b_1) \tag{3}$$

For $\alpha = -1$, the set just generated is the field of complex numbers, $\mathbb{C}$ as expected.

As typically done, Equation (3) can be written in matrix form,

$$(a_1 a_2 + \alpha b_1 b_2, a_1 b_2 + a_2 b_1) = (a_1, b_1) \begin{pmatrix} a_2 & b_2 \\ \alpha b_2 & a_2 \end{pmatrix} \tag{4}$$

which gives the general matrix form for complex numbers,

$$z = \begin{pmatrix} a & b \\ \alpha b & a \end{pmatrix} = a\mathbf{1} + b\mathbf{I} \tag{5}$$

where $\mathbf{1}$ is the identity matrix and

$$\mathbf{I} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \tag{6}$$

Although α can assume any real value, it is generally sufficient to consider the particular values $-1, 0$, and 1. This is because we have $\alpha\mathbb{R} = \mathbb{R}$ for any real value of α rendering the overall (positive) scale irrelevant for most purposes. The choice $\alpha = -1$ gives the usual complex numbers, $\alpha = 1$ the split-complex (hyperbolic) numbers and $\alpha = 0$ the dual complex numbers [13].

As it is known, the number z has an inverse if $\det(z) = a^2 - \alpha b^2$ is non zero, in which case the inverse element is $(a - gb)/\det(z)$. Split complex numbers do not form a field since multiplicative inverses do not exist for $a = \pm b$. Similarly, dual complex numbers do not have inverses for $a = 0$ (pure imaginary numbers). Also note that the transpose of the matrix in Equation (5) is an equivalent representation of a complex number. In general, the matrix representations that we obtain below are determined up to a transposition.

4. Case $n = 3$

In this case the coset group $\mathbb{G}$ is a cyclic group of order 3 with $\mathbb{G} = \{\mathbb{R}, g\mathbb{R}, g^{-1}\mathbb{R}\}$. The cosets $g\mathbb{R}$ and $g^{-1}\mathbb{R}$ are distinct, with $g^2\mathbb{R} = g^{-1}\mathbb{R}$, giving $g^{-2}\mathbb{R} = g\mathbb{R}$, and $g^3\mathbb{R} = g^{-3}\mathbb{R} = \mathbb{R}$. By choosing elements $i \in g\mathbb{R}$ and $j \in g^{-1}\mathbb{R}$, we have the rules:

$$\begin{aligned} ij &\in \mathbb{R} \\ ji &\in \mathbb{R} \\ i^2 &\in j\mathbb{R} \\ j^2 &\in i\mathbb{R} \end{aligned} \tag{7}$$

from which it follows that $i^3 \in \mathbb{R}$ and $j^3 \in \mathbb{R}$. The above mean that there exist three real numbers α, β, and γ such that

$$\begin{aligned} ij &= ji = \alpha \\ i^2 &= \beta j \\ j^2 &= \gamma i \end{aligned} \tag{8}$$

Note that i and j commute because $g\mathbb{R}$ and $g^{-1}\mathbb{R}$ are inverse elements of each other in the coset group. (Commutativity can also be seen by setting $ij = \alpha$, $ji = \alpha'$ and by evaluating the product iji which gives $\alpha i = i\alpha'$ and therefore $\alpha' = \alpha$.) Furthermore, parameters α, β, and γ are not independent. It can be shown that $\alpha = \beta\gamma$, for example by considering the product $i^2 j = \beta j^2$ which gives $\alpha i = \beta\gamma i$. Table 1 lists possible assignments for parameters α, β, and γ, with $\alpha = ij$ chosen to designate the signature of the number system.

Table 1. Assignments for 3D extensions of $\mathbb{R}$.

α	β	γ	$i^3 = \alpha\beta$	$j^3 = \alpha\gamma$
1	1	1	1	1
	-1	-1	-1	-1
0	1	0	0	0
	0	0	0	0
	-1	0	0	0
-1	1	-1	-1	1

As shown in Table 1, there are 6 distinct choices (of 0, -1, $+1$) for parameters α, β and γ. Some alternative assignments are equivalent and are not shown. For example, the assignment $\beta = 1, \gamma = -1$ is equivalent to $\beta = -1, \gamma = 1$ because i and j can be switched.

Multiplying two generic 3D numbers of the form $a + bi + cj$, and setting the result in matrix form (as done above for complex numbers), gives the general matrix representation for 3D numbers as

$$z = \begin{pmatrix} a & b & c \\ \alpha c & a & \beta b \\ \alpha b & \gamma c & a \end{pmatrix} = a\mathbf{1} + b\mathbf{I} + c\mathbf{J} \tag{9}$$

where $\mathbf{1}$ is the identity matrix and

$$\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \beta \\ \alpha & 0 & 0 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix} \tag{10}$$

Since the general matrix form in Equation (9) is invariant under matrix multiplication, the form of the determinant, $det(z) = a^3 + \alpha\beta b^3 + \alpha\gamma c^3 - 3\alpha abc$ is invariant as well for all $\alpha = \beta\gamma$.

5. Case $n = 4$, Cyclic

The cyclic structure for $n = 4$ gives the following form for $\mathbb{G}$,

$$\mathbb{G} = \{\mathbb{R}, g\mathbb{R}, g^{-1}\mathbb{R}, g^2\mathbb{R} = g^{-2}\mathbb{R}\} \tag{11}$$

in which $g^3\mathbb{R} = g^{-1}\mathbb{R}$, $g^{-3}\mathbb{R} = g\mathbb{R}$, $g^4\mathbb{R} = g^{-4}\mathbb{R} = \mathbb{R}$. With elements i, j, k from $g\mathbb{R}$, $g^{-1}\mathbb{R}$, and $g^2\mathbb{R}$, respectively, coset products within this group give the following multiplication rules:

$$\begin{array}{lll} ij = ji = \alpha & jk = \beta i & ki = \gamma j \\ i^2 = \delta k & j^2 = \epsilon k & k^2 = \varphi \end{array} \tag{12}$$

where Greek symbols indicate real coefficients. The six scaling parameters are not independent since it can be shown that

$$\varphi = \beta\gamma \tag{13}$$

$$\beta\delta = \gamma\epsilon = \alpha \tag{14}$$

This leaves four independent scales to be chosen as combinations of -1, 0, and 1. Table 2 gives possible assignment values with $\alpha = ij$ chosen as the signature of the number system.

The general matrix form for cyclic 4D numbers is

$$z = \begin{pmatrix} a & b & c & d \\ \alpha c & a & \gamma d & \delta b \\ \alpha b & \beta d & a & \epsilon c \\ \beta\gamma d & \beta c & \gamma b & a \end{pmatrix} = a\mathbf{1} + b\mathbf{I} + c\mathbf{J} + d\mathbf{K} \tag{15}$$

with **1** being the unit matrix and

$$\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & \beta & 0 & 0 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \gamma & 0 \\ 0 & \beta & 0 & 0 \\ \beta\gamma & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

Table 2. Assignments for 4D cyclic extensions of $\mathbb{R}$. The last column indicates particular assignments considered in reference [21].

α	β	γ	δ	ϵ	φ	
1	1	1	1	1	1	Polar
	1	−1	1	−1	−1	
	−1	1	−1	1	−1	
	−1	−1	−1	−1	1	
0	$\beta\delta = 0, \gamma\epsilon = 0$				$\beta\gamma$	
−1	1	1	−1	−1	1	
	1	−1	−1	1	−1	
	−1	1	1	−1	−1	Planar
	−1	−1	1	1	1	

6. Case $n = 4$, Klein Group

In this case, the coset group has the form $\mathbb{G} = \{\mathbb{R}, g_1\mathbb{R} = g_1^{-1}\mathbb{R}, g_2\mathbb{R} = g_2^{-1}\mathbb{R}, g_3\mathbb{R} = g_3^{-1}\mathbb{R}\}$, where $g_3 = g_1g_2$. Elements i, j, k can then be chosen from $g_1\mathbb{R}$, $g_2\mathbb{R}$, and $g_3\mathbb{R}$, respectively. With Greek symbols indicating scaling coefficients, coset products within this group give the following multiplication rules:

$$\begin{array}{lll} i^2 = \alpha & j^2 = \beta & k^2 = \gamma \\ jk = \alpha' i & ki = \beta' j & ij = \gamma' k \\ kj = \alpha'' i & ik = \beta'' j & ji = \gamma'' k \end{array}$$

The 9 scaling parameters are obviously not independent. Constraints are derived from products such as ijk, $(ij)(ji)$, and $(ij)(jk)$. We obtain:

$$\begin{aligned} \alpha\beta &= \gamma\gamma'\gamma'' \\ \beta\gamma &= \alpha\alpha'\alpha'' \\ \gamma\alpha &= \beta\beta'\beta'' \\ \alpha\alpha' &= \beta\beta' = \gamma\gamma' = \alpha''\beta''\gamma'' \\ \alpha\alpha'' &= \beta\beta'' = \gamma\gamma'' = \alpha'\beta'\gamma' \end{aligned} \tag{17}$$

An alternative way of expressing the constraints is

$$\begin{aligned} \alpha'\beta' &= \alpha''\beta'' \\ \alpha'\beta'' &= \alpha''\beta' = \gamma \end{aligned} \tag{18}$$

and all corresponding cyclic permutations. With notations $\rho = \alpha\beta\gamma$, $\rho' = \alpha'\beta'\gamma'$, and $\rho'' = \alpha''\beta''\gamma''$, it can be shown that $\rho\rho'\rho'' = \rho''^4 = \rho'^4 = \rho^2 \geq 0$. It means that the three products ρ, ρ', and ρ'' are either all equal to zero or all different than zero. Note that ρ' and ρ'' are products of basis elements: $\rho'' = ijk = jki = kij$ and $\rho' = ikj = kji = jik$. One convenient assignment procedure is to choose α, β, γ, and ρ''. Then primed parameters are determined from $\alpha\alpha' = \beta\beta' = \gamma\gamma' = \rho''$ and double-primed parameters from $\alpha\alpha'' = \beta\beta'' = \gamma\gamma'' = \alpha'\beta'\gamma' = \rho'$. In this parameterization, Hamilton's quaternions correspond to $\alpha = \beta = \gamma = \rho'' = -1$. This and other possible non-zero assignments are summarized in Table 3. The last column indicates particular assignments corresponding to common quaternionic number systems: quaternions (Q), split-quaternions (S), bicomplex numbers (B), and to 4D numbers analyzed in [21], namely hyperbolic (H) and circular (C). The products ρ, ρ', ρ'' can be considered as signatures for each assignment and can provide a quick consistency check. Also note that commutativity requires that corresponding primed and double-primed parameters are equal, which is possible if and only if $\rho > 0$. There are four commutative cases in Table 3 including the bicomplex (B), hyperbolic (H), and circular (C) numbers, and four non-commutative cases including the quaternions (Q) and split-quaternions (S).

Table 3. Assignments for 4D Klein extensions of $\mathbb{R}$. Particular number systems are indicated in the last column: hyperbolic (H), split-quaternions (S, called pseudoquaternions in Reference [13]), bicomplex (B), quaternions (Q), and circular (C).

α	β	γ	ρ	α'	β'	γ'	ρ'	α''	β''	γ''	ρ''	
1	1	1	1	1	1	1	1	1	1	1	1	H
−1	1	1	−1	−1	1	1	−1	1	−1	−1		S
−1	−1	1	1	−1	−1	1	1	−1	−1	1		B
−1	−1	−1	−1	−1	−1	−1	−1	1	1	1		
−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	Q
1	−1	−1	1	−1	1	1	−1	−1	1	1		C
1	1	−1	−1	−1	−1	1	1	1	1	−1		
1	1	1	1	−1	−1	−1	−1	−1	−1	−1		

If we allow any of the parameters to be zero, then automatically we need to satisfy the constraint $\rho = \rho' = \rho'' = 0$. These assignments give rise to degenerate quaternions (page 24 of [13]). Assignment options for zero-valued parameters are summarized in Table 4 and particular assignments are indicated in the last column as follows: degenerate pseudoquaternions (DPQ), doubly degenerated quaternions (DDQ), and degenerate quaternions (DQ) [13].

The general matrix form for Klein 4D numbers is

$$z = \begin{pmatrix} a & b & c & d \\ \alpha b & a & \beta'' d & \gamma' c \\ \beta c & \alpha' d & a & \gamma'' b \\ \gamma d & \alpha'' c & \beta' b & a \end{pmatrix} = a\mathbf{1} + b\mathbf{I} + c\mathbf{J} + d\mathbf{K} \tag{19}$$

with

$$\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma'' \\ 0 & 0 & \beta' & 0 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma' \\ \beta & 0 & 0 & 0 \\ 0 & \alpha'' & 0 & 0 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \beta'' & 0 \\ 0 & \alpha' & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix} \quad (20)$$

Note that matrices corresponding to cyclic and Klein coset groups in Equations (15) and (19) have different forms since the two groups have different structures. Therefore number systems from the two categories will have significantly different properties. For example, references [21] compares and gives a detailed analysis of four different kinds of commutative 4D numbers called polar, planar, hyperbolic, and circular. The first two belong to cyclic cosets and the last two correspond to Klein cosets, as indicated in Tables 2 and 3.

Table 4. Assignments for Klein 4D extensions of $\mathbb{R}$ for $\rho = \rho' = \rho'' = 0$. Particular number systems are indicated in the last column: degenerate pseudoquaternions (DPQ), doubly degenerated quaternions (DDQ), and degenerate quaternions (DQ).

α	β	γ	ρ	α'	β'	γ'	ρ'	α''	β''	γ''	ρ''	
1	0	0	0	0	1	1	0	0	1	1	0	
				0	−1	1		0	1	−1		DPQ
				0	−1	−1		0	−1	−1		
0	0	0	0	0	0	±1	0	0	0	0	0	
				0	0	0		0	0	±1		
				0	0	±1		0	0	±1		DDQ
				0	0	0		0	0	0		
−1	0	0	0	0	1	1	0	0	−1	−1	0	DQ
				0	−1	1		0	−1	1		
				0	−1	−1		0	1	1		

7. 2D Extensions over the Field of Complex Numbers

The multiplication rules obtained above apply to extensions of complex numbers as well. In this case, coefficients and multiplication parameters can assume complex values. As an example, the coset group $\mathbb{G} = \{\mathbb{C}, g\mathbb{C}\}$ can be used to generate bicomplex numbers. (This is a particular case of Cayley-Dickson construction of algebras, also called Dickson doubling, e.g., in [16].) The simplest construction using Equation (5) is the block matrix,

$$z = \begin{pmatrix} A & B \\ \alpha B & A \end{pmatrix} \quad (21)$$

where A, and B, and α are now complex numbers. Representing A and B as 2×2 real matrices, we obtain

$$z = \left(\begin{array}{cc|cc} a & b & c & d \\ \alpha b & a & \alpha d & c \\ \hline \alpha c & \alpha d & a & b \\ \alpha^2 d & \alpha c & \alpha b & a \end{array}\right) = a\mathbf{1} + b\mathbf{I} + c\mathbf{J} + d\mathbf{K} \tag{22}$$

with

$$\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

Comparison with the general matrix form of Klein 4D numbers in Equation (20) gives the assignments shown in Table 5.

Table 5. Correspondence between 4D Klein extension over reals and 2D extension over complex. The last three rows show particular real assignments corresponding to $\alpha = -1$ (bicomplex, B), $\alpha = 0$ (dual, D), and $\alpha = 1$ (hyperbolic, H).

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\rho}$	$\boldsymbol{\alpha'}$	$\boldsymbol{\beta'}$	$\boldsymbol{\gamma'}$	$\boldsymbol{\rho'}$	$\boldsymbol{\alpha''}$	$\boldsymbol{\beta''}$	$\boldsymbol{\gamma''}$	$\boldsymbol{\rho''}$	
$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha^2}$	$\boldsymbol{\alpha^4}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	**1**	$\boldsymbol{\alpha^2}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	**1**	$\boldsymbol{\alpha^2}$	
−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	B
0	0	0	0	0	0	1	0	0	0	1	0	D
1	1	1	1	1	1	1	1	1	1	1	1	H

8. Conclusions

Multidimensional numbers systems are representations of abstract coset groups. A group theoretic approach allows for a natural construction of multidimensional number systems for which multiplication rules follow directly from the known structure of coset groups. For each number of dimensions, the set of possible extensions is determined by the structure of small groups exhausting all possibilities for multiplication rules of basis elements.

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Conflicts of Interest

The author declares no conflict of interest.

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