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e-Distance in Menger PGM Spaces with an Application

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Abstract: The main purpose of the present paper is to define the concept of an e-distance (as a generalization of r-distance) on a Menger PGM space and to introduce some of its properties. Moreover, some coupled fixed point results, in terms of this distance on a complete PGM space, are proved. To support our definitions and main results, several examples and an application are considered.

Keywords: e-distance; Menger PGM space; coupled fixed point

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1. Introduction and Preliminaries

In 1942, Menger [1] introduced Menger probabilistic metric spaces as an extension of metric spaces. After that, Sehgal and Bharucha-Reid [2,3] studied some fixed point results for different classes of probabilistic contractions (also, see and references in the citation). Moreover, in 2009, Saadati et al. [4] introduced the concept of r-distance on this space.

Throughout this paper, the set of all Menger distance distribution functions are denoted by D^+.

Definition 1 ([5], page 1). *A binary mapping $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is called t-norm if the following propertied are held:*

(a) *$\mathcal{T}$ is commutative and associative;*
(b) *$\mathcal{T}$ is continuous;*
(c) *$\mathcal{T}(a,1) = a$ if $a \in [0,1]$;*
(d) *$\mathcal{T}(a,b) \leq \mathcal{T}(c,d)$ if $a \leq c$ and $b \leq d$ for every $a,b,c,d \in [0,1]$.*

Definition 2 ([4]). *A t-norm $\mathcal{T}$ is called an H-type I if for $\epsilon \in (0,1)$, there exist $\delta \in (0,1)$ so that $\mathcal{T}^m(1-\delta,...,1-\delta) > 1-\epsilon$ for each $m \in \mathbb{N}$, where $\mathcal{T}^m$ recursively defined by $\mathcal{T}^1 = \mathcal{T}$ and $\mathcal{T}^m(t_1,t_2,...,t_{m+1}) = \mathcal{T}(\mathcal{T}^{m-1}(t_1,t_2,...,t_m),t_{m+1})$ for $m = 2,3,\cdots$ and $t_i \in [0,1]$.*

All t-norms in the sequel are from class of H-type I.

From another point of view, Mustafa and Sims [6] defined G-metric spaces as another extension of metric spaces, analyzed the structure of this space, and continued the theory of fixed point in such spaces. In 2014, Zhou et al. [7], by combining Menger PM-spaces and G-metric spaces, defined Menger probabilistic generalized metric space (shortly, Menger PGM space). Other researchers extended several fixed point theorems in [8–10] and references contained therein.

Definition 3 ([7]). *Assume that $\mathcal{X}$ is a nonempty set, $\mathcal{T}$ is a continuous t-norm and $G : \mathcal{X}^3 \to D^+$ is a mapping satisfying the following properties for all $x,y,z,a \in \mathcal{X}$ and $s,t > 0$:*

(PG1) $G_{x,y,z}(t) = 1$ *if and only if* $x = y = z$*;*
(PG2) $G_{x,x,y}(t) \geq G_{x,y,z}(t)$*, where* $z \neq y$*;*
(PG3) $G_{x,y,z}(t) = G_{x,z,y}(t) = G_{y,x,z}(t) = \cdots$*;*
(PG4) $G_{x,y,z}(t+s) \geq \mathcal{T}(G_{x,a,a}(s), G_{a,y,z}(t))$*.*

Then $(\mathcal{X}, G, \mathcal{T})$ *is named a Menger PGM space.*

For the definitions of convergent, completeness, closedness and some theorems by regarding these concepts in such spaces, one can see [7]. In 2004, Ran and Reurings [11] discussed on fixed point results for comparable elements of a metric space $(\mathcal{X}, d)$ provided with a partial order. Then, Bhaskar and Lakshmikantham [12] presented several fixed point results for a mapping having mixed monotone property in such spaces (see [13,14]).

Definition 4 ([12])**.** *Consider a ordered set* $(\mathcal{X}, \preceq)$ *and a mapping* $F : \mathcal{X}^2 \to \mathcal{X}$*. The mapping* F *is told to be have mixed monotone property if*

$$\begin{aligned} x_1 \preceq x_2 \text{ implies that } F(x_1, y) \preceq F(x_2, y) \qquad \forall x_1, x_2 \in \mathcal{X}, \\ y_1 \preceq y_2 \text{ implies that } F(x, y_1) \succeq F(x, y_2) \qquad \forall y_1, y_2 \in \mathcal{X}. \end{aligned}$$

for every $x, y \in \mathcal{X}$*.*

Here we introduce an e-distance on Menger PGM spaces and some of its properties. Then we obtain some coupled fixed point results in the quasi-ordered version of such spaces. The subject of the paper offers novelties compared to the related background literature since a new distance in Menger spaces is defined while some of its properties are revisited and extended.

2. Main Results

Here, we consider an e-distance on a Menger PGM space, which is an extension of r-distance introduced by Saadati et al. [4].

Definition 5. *Consider a Menger PGM space* $(\mathcal{X}, G, \mathcal{T})$*. Then the function* $g : \mathcal{X}^3 \times [0, \infty] \to [0, 1]$ *is called an e-distance, if for all* $x, y, z, a \in \mathcal{X}$ *and* $s, t \geq 0$ *the following are held:*

(r1) $g_{x,y,z}(t+s) \geq \mathcal{T}(g_{x,a,a}(s), g_{a,y,z}(t))$*;*
(r2) $g_{x,y,.}(t)$ *and* $g_{x,.,y}(t)$ *are continuous;*
(r3) for each $\epsilon > 0$*, there exists* $\delta > 0$ *provided that* $g_{a,y,z}(t) \geq 1 - \delta$ *and* $g_{x,a,a}(s) \geq 1 - \delta$ *conclude that* $G_{x,y,z}(t+s) \geq 1 - \epsilon$*.*

Lemma 1. *Each Menger PGM is an e-distance on* $\mathcal{X}$*.*

Proof. Clearly, (r1) and (r2) are true. Only, we prove that (r3) is true. Assume $\epsilon > 0$ and select $\delta > 0$ so that $\mathcal{T}(1-\delta, 1-\delta) \geq 1 - \epsilon$. Then, for $G_{a,y,z}(t) \geq 1 - \delta$ and $G_{x,a,a}(s) \geq 1 - \delta$, we get

$$G_{x,y,z}(t+s) \geq \mathcal{T}(G_{a,y,z}(t), G_{x,a,a}(s)) \geq \mathcal{T}(1-\delta, 1-\delta) \geq 1 - \epsilon.$$

□

Example 1. *Assume* $(\mathcal{X}, G, \mathcal{T})$ *is a Menger PGM space. Define a function* $g : \mathcal{X}^3 \times [0, \infty] \to [0, 1]$ *by* $g_{x,y,z}(t) = 1 - c$ *for each* $x, y, z \in \mathcal{X}$ *and* $t > 0$ *with* $c \in (0, 1)$*. Then* g *is an e-distance.*

Lemma 2. *Consider a Menger PGM space with a continuous mapping* A *on* $\mathcal{X}$ *and a function* $g : \mathcal{X}^3 \times [0, \infty] \to [0, 1]$ *by* $g_{x,y,z}(t) = \min\{G_{x,y,z}(t), G_{Ax,Ay,Az}(t)\}$ *for each* $x, y, z \in \mathcal{X}$ *and* $t > 0$*. Then* g *is an e-distance on* $\mathcal{X}$*.*

Proof. The condition (r2) is clearly established. To prove (r1), consider $x, y, z, a \in \mathcal{X}$ and $t, s > 0$. Then, we have two following cases:

Case 1: if $G_{x,y,z}(t) = \min\{G_{x,y,z}(t), G_{Ax,Ay,Az}(t)\}$, then

$$\begin{aligned}
g_{x,y,z}(t+s) &= G_{x,y,z}(t+s) \\
&\geq \mathcal{T}(G_{x,a,a}(t), G_{a,y,z}(s)) \\
&\geq \mathcal{T}(\min\{G_{x,a,a}(t), G_{Ax,Aa,Aa}(t)\}, \min\{G_{a,y,z}(s), G_{Aa,Ay,Az}(s)\}) \\
&\geq \mathcal{T}(g_{x,a,a}(t), g_{a,y,z}(s)).
\end{aligned}$$

Case 2: if $G_{Ax,Ay,Az}(t) = \min\{G_{x,y,z}(t), G_{Ax,Ay,Az}(t)\}$, then

$$\begin{aligned}
g_{x,y,z}(t+s) &= G_{Ax,Ay,Az}(t+s) \\
&\geq \mathcal{T}(G_{Ax,Aa,Aa}(t), G_{Aa,Ay,Az}(s)) \\
&\geq \mathcal{T}(\min\{G_{x,a,a}(t), G_{Ax,Aa,Aa}(t)\}, \min\{G_{a,y,z}(s), G_{Aa,Ay,Az}(s)\}) \\
&\geq \mathcal{T}(g_{x,a,a}(t), g_{a,y,z}(s)).
\end{aligned}$$

Therefore, (r1) is established. Now, assume $\epsilon > 0$ and select $\delta > 0$ so that $\mathcal{T}(1-\delta, 1-\delta) \geq 1-\epsilon$. Using $g_{x,a,a}(t) \geq 1-\delta$ and $g_{a,y,z}(s) \geq 1-\delta$, we get

$$\begin{aligned}
\min\{G_{x,a,a}(t), G_{Ax,Aa,Aa}(t)\} = g_{x,a,a}(t) \geq 1-\delta, \\
\min\{G_{a,y,z}(s), G_{Aa,Ay,Az}(s)\} = g_{a,y,z}(s) \geq 1-\delta,
\end{aligned}$$

which induces that

$$\begin{aligned}
G_{x,y,z}(t+s) &\geq \mathcal{T}(G_{x,a,a}(t), G_{a,y,z}(s)) \\
&\geq \mathcal{T}(\min\{G_{x,a,a}(t), G_{Ax,Aa,Aa}(t)\}, \min\{G_{a,y,z}(s), G_{Aa,Ay,Az}(s)\}) \\
&= \mathcal{T}(g_{x,a,a}(t), g_{a,y,z}(s)) \geq \mathcal{T}(1-\delta, 1-\delta) \geq 1-\epsilon.
\end{aligned}$$

Thus, (r3) is established. This completes the proof. □

Lemma 3. *Consider an e-distance g on $(\mathcal{X}, G, \mathcal{T})$ with two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two non-negative sequences converging to 0. Then for $x, y, z \in \mathcal{X}$ and $t, s > 0$ the following assertions are established:*

(i) *$g_{z,y,x_n}(t) \geq 1-\alpha_n$ and $g_{x,x_n,x_n}(t) \geq 1-\beta_n$ for any $n \in \mathbb{N}$ imply $x = y = z$. Specially, $g_{x,a,a}(t) = 1$ and $g_{a,y,z}(s) = 1$ imply $x = y = z$;*
(ii) *$g_{y_n,x_n,x_n}(t) \geq 1-\alpha_n$ and $g_{x_n,y_m,z}(t) \geq 1-\beta_n$ for all $m > n$ with $m, n \in \mathbb{N}$ imply $G_{y_n,y_m,z}(t+s) \to 1$ as $n \to \infty$;*
(iii) *let $g_{x_n,x_m,x_l}(t) \geq 1-\alpha_n$ for all $n, m, l \in \mathbb{N}$, where $l > m > n$. Then $\{x_n\}$ is a Cauchy sequence;*
(iv) *let $g_{y,y,x_l}(t) \geq 1-\alpha_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. To prove (ii), assume $\epsilon > 0$. By applying the definition of e-distance, there exists $\delta > 0$ so that $g_{a,y,z}(t) \geq 1-\delta$ and $g_{x,a,a}(s) \geq 1-\delta$ induce $G_{x,y,z}(t+s) \geq 1-\epsilon$. Select $n_0 \in \mathbb{N}$ provided that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for each $n \geq n_0$. Then $g_{y_n,x_n,x_n}(t) \geq 1-\alpha_n \geq 1-\delta$ and $g_{x_n,y_m,z}(t) \geq 1-\beta_n \geq 1-\delta$ for any $n \geq n_0$ and hence $G_{y_n,y_m,z}(t+s) \geq 1-\epsilon$. Therefore, $\{y_n\}$ converges to z. Now, using (ii), (i) is established. To prove (iii), assume $\epsilon > 0$. Similar to the proof of (ii), select $\delta > 0$ and $n_0 \in \mathbb{N}$. Then, for all $n, m, l \geq n_0 + 1$, we get $g_{x_n,x_{n_0},x_{n_0}}(t) \geq 1-\alpha_{n_0} \geq 1-\delta$ and $g_{x_{n_0},x_l,x_m}(t) \geq 1-\alpha_{n_0} \geq 1-\delta$. Therefore, $G_{x_n,x_m,x_l}(t) \geq 1-\epsilon$. Hence, $\{x_n\}$ is a Cauchy sequence. Now, it follows from (iii) that (iv) is true. □

Lemma 4. *Consider an e-distance g on $(\mathcal{X}, G, \mathcal{T})$. Suppose that $E_{\lambda,g} : \mathcal{X}^3 \to \mathbb{R}^+ \cup \{0\}$ is introduced by $E_{\lambda,g}(x,y,z) = \inf\{t > 0 : g_{x,y,z}(t) > 1-\lambda\}$ for any $x, y, z \in \mathcal{X}$ and $\lambda \in (0,1)$. Then*

(1) *for all $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ so that*

$$E_{\mu,g}(x_1,x_1,x_n) \leq E_{\lambda,g}(x_1,x_1,x_2) + E_{\lambda,g}(x_2,x_2,x_3) + \cdots + E_{\lambda,g}(x_{n-1},x_{n-1},x_n)$$

for each $x_1,\cdots,x_n \in \mathcal{X}$;

(2) *for every sequence $\{x_n\}$ in $\mathcal{X}$, $g_{x_n,x,x}(t) \to 1$ iff $E_{\lambda,g}(x_n,x,x) \to 0$. Further, the sequence $\{x_n\}$ is Cauchy w.r.t. g iff it is Cauchy with $E_{\lambda,g}$.*

Proof.

(1) For every $\mu \in (0,1)$, we can gain $\lambda \in (0,1)$ provided that $\mathcal{T}^{n-1}(1-\lambda,...,1-\lambda) \geq 1-\mu$. Now, for every $\delta > 0$, we have

$$\begin{aligned}
& g_{x_1,x_1,x_n}(E_{\lambda,g}(x_1,x_1,x_2) + E_{\lambda,g}(x_2,x_2,x_3) + \cdots + E_{\lambda,g}(x_{n-1},x_{n-1},x_n) + n\delta) \\
& \geq \mathcal{T}^{n-1}(g_{x_1,x_1,x_2}(E_{\lambda,g}(x_1,x_1,x_2)+\delta), g_{x_2,x_2,x_3}(E_{\lambda,g}(x_2,x_2,x_3)+\delta) \\
& \quad ,\cdots, g_{x_{n-1},x_{n-1},x_n}(E_{\lambda,g}(x_{n-1},x_{n-1},x_n)+\delta)) \\
& \geq \mathcal{T}^{n-1}(1-\lambda,...,1-\lambda) \geq 1-\mu
\end{aligned}$$

which induces that

$$E_{\mu,g}(x_1,x_1,x_n) \leq E_{\lambda,g}(x_1,x_1,x_2) + E_{\lambda,g}(x_2,x_2,x_3) + \cdots + E_{\lambda,g}(x_{n-1},x_{n-1},x_n) + n\delta.$$

Since $\delta > 0$ is optional, we obtain

$$E_{\mu,g}(x_1,x_1,x_n) \leq E_{\lambda,g}(x_1,x_1,x_2) + E_{\lambda,g}(x_2,x_2,x_3) + \cdots + E_{\lambda,g}(x_{n-1},x_{n-1},x_n).$$

(2) Note that $g_{x_n,x,x}(\eta) \to 1-\lambda$ as $n \to \infty$ iff $E_{\lambda,g}(x_n,x,x) < \eta$ for each $n \in \mathbb{N}$ and $\eta > 0$. □

In the sequel, we establish some coupled fixed point theorems by regarding an e-distance on a quasi-ordered complete PGM space.

Theorem 1. *Let $(\mathcal{X}, G, \mathcal{T}, \preceq)$ be a quasi-ordered complete Menger PGM space with $\mathcal{T}$ of Hadzić-type I, g be an e-distance and $f : \mathcal{X}^2 \to \mathcal{X}$ be a mapping having the mixed monotone property on $\mathcal{X}$. Assume that there exists a $k \in [0,1)$ such that*

$$g_{f(x,y),f(u,v),f(w,z)}(t) \geq \frac{1}{2}(g_{x,u,w}(\frac{t}{k}) + g_{y,v,z}(\frac{t}{k})) \tag{1}$$

for all $x,y,z,u,v,w \in \mathcal{X}$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$ and

$$\sup\{\mathcal{T}(g_{x,y,z}(t), g_{x,y,f(x,y)}(t)) : x,y \in \mathcal{X}\} < 1. \tag{2}$$

for all $z \in \mathcal{X}$, where $z \neq f(z,q)$ for all $q \in \mathcal{X}$. If there exist $x_0, y_0 \in \mathcal{X}$ so that $x_0 \preceq f(x_0,y_0)$ and $y_0 \succeq f(y_0,x_0)$, then f have a coupled fixed point in $\mathcal{X}^2$.

Proof. Since there exist $x_0, y_0 \in \mathcal{X}$ with $x_0 \preceq f(x_0,y_0)$ and $y_0 \succeq f(y_0,x_0)$, and f has the mixed monotone property, we can construct Bhaskar-Lakshmikantham type iterative as follow:

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n+1} \preceq \cdots \quad , \quad y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_{n+1} \succeq \cdots$$

for all $n \geq 0$, where

$$\begin{aligned}
x_{n+1} &= f^{n+1}(x_0,y_0) = f(f^n(x_0,y_0), f^n(y_0,x_0)), \\
y_{n+1} &= f^{n+1}(y_0,x_0) = f(f^n(y_0,x_0), f^n(x_0,y_0)).
\end{aligned}$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then f has a coupled fixed point. Otherwise, assume $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for each $n \geq 0$; that is, either $x_{n+1} = f(x_n, y_n) \neq x_n$ or $y_{n+1} = f(y_n, x_n) \neq y_n$. Now, by induction and (1), we obtain

$$\begin{aligned} g_{x_n,x_n,x_{n+1}}(t) &\geq \frac{1}{2}(g_{x_0,x_0,x_1}(\frac{t}{k^n}) + g_{y_0,y_0,y_1}(\frac{t}{k^n})), \\ g_{y_n,y_n,y_{n+1}}(t) &\geq \frac{1}{2}(g_{y_0,y_0,y_1}(\frac{t}{k^n}) + g_{x_0,x_0,x_1}(\frac{t}{k^n})), \end{aligned}$$

for each $n \geq 0$ which induces that $g_{x_n,x_n,x_{n+1}}(t) \geq \frac{1}{2}g_{x_0,x_0,x_1}(\frac{t}{k^n})$ and $g_{y_n,y_n,y_{n+1}}(t) \geq \frac{1}{2}g_{y_0,y_0,y_1}(\frac{t}{k^n})$. Therefore,

$$\begin{aligned} E_{\lambda,g}(x_n, x_n, x_{n+1}) &= \inf\{t > 0 : g_{x_n,x_n,x_{n+1}}(t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 : \frac{1}{2}g_{x_0,x_0,x_1}(\frac{t}{k^n}) > 1 - \lambda\} \\ &= 2k^n E_{\lambda,g}(x_0, x_0, x_1). \end{aligned}$$

Thus, for $m > n$ and $\lambda \in (0,1)$, there exists $\gamma \in (0,1)$ so that

$$E_{\lambda,g}(x_n, x_n, x_m) \leq E_{\gamma,g}(x_n, x_n, x_{n+1}) + \cdots + E_{\gamma,g}(x_{m-1}, x_{m-1}, x_m) \leq \frac{2k^n}{1-k} E_{\gamma,g}(x_0, x_0, x_1).$$

Now, there exists $n_0 \in \mathbb{N}$ so that for each $n > n_0$, $E_{\lambda,g}(x_n, x_n, x_m) \to 0$. By Lemmas 3 and 4, $\{x_n\}$ is a Cauchy sequence. Thus, using Lemma 4 (ii), there exit $n_1 \in \mathbb{N}$ and a sequence $\delta_n \to 0$ so that $g_{x_n,x_n,x_m}(t) \geq 1 - \delta_n$ for $n \geq \max\{n_0, n_1\}$. Since $\mathcal{X}$ is complete, $\{x_n\}$ converges to a point $p \in \mathcal{X}$. Similarly, $\{y_n\}$ is convergent to a point $q \in \mathcal{X}$. By (r2), we obtain $g_{x_n,x_n,p}(t) = \lim_{m\to\infty} g_{x_n,x_n,x_m}(t) \geq 1 - \delta_n$ for $n \geq \max\{n_0, n_1\}$. Moreover, we get $g_{x_n,x_{n+1},x_{n+1}}(t) \geq 1 - \delta_n$. Now, we show that f has a coupled fixed point. Let $p \neq f(p,q)$. Then, by (2), we obtain

$$\begin{aligned} 1 &> \sup\{\mathcal{T}(g_{x,y,p}(t), g_{x,y,f(x,y)}(t)) : x, y \in \mathcal{X}\} \\ &\geq \sup\{\mathcal{T}(g_{x_n,x_n,p}(t), g_{x_n,x_{n+1},x_{n+1}}(t)) : n \in \mathbb{N}\} \\ &\geq \sup\{\mathcal{T}(1-\delta_n, 1-\delta_n) : n \in \mathbb{N}\} = 1, \end{aligned}$$

which is a contradiction. Consequently, we get $p = f(p,q)$. Similarly, we obtain $f(q,p) = q$. Here, the proof ends. □

Theorem 2. *Assume the assumptions of Theorem 1 are held and consider the continuity of f instead of relation (2). Then f has a coupled fixed point.*

Proof. As in the proof of Theorem 1, construct $\{x_n\}$ and $\{y_n\}$, where $x_n \to p$, $y_n \to q$, $x_{n+1} = f(x_n, y_n)$. Now, by the continuity of f and by taking the limit as $n \to \infty$, we get $f(p,q) = p$. Analogously, we can obtain $f(q,p) = q$. Therefore, (p,q) is a coupled fixed point of f. □

Example 2. *Assume that $\mathcal{X} = [0,\infty)$, "$\preceq$" is a quasi-ordered on $\mathcal{X}$ and $\mathcal{T}(a,b) = min\{a,b\}$. Define a constant function $f : \mathcal{X}^2 \to \mathcal{X}$ by $f(a,b) = p$ and $G : \mathcal{X}^3 \to D^+$ by $G_{x,y,z}(t) = \frac{t}{t+G^*(x,y,z)}$ with $G^*(x,y,z) = |x-y| + |x-z| + |y-z|$ for each $x,y,z \in \mathcal{X}$. Clearly, G satisfies (PG1)-(PG4). Consider $g_{x,y,z}(t) = 1 - c$, where $c \in (0,1)$. Then g is an e-distance on $\mathcal{X}$. Clearly, for all $x,y,z,u,v,w \in \mathcal{X}$ and for any $t > 0$, we have $g_{f(x,y),f(u,v),f(w,z)}(t) \geq \frac{1}{2}(g_{x,u,w}(\frac{t}{k}) + g_{y,v,z}(\frac{t}{k}))$. Moreover, there exist $x_0 = 0$ and $y_0 = 1$ so that $0 = x_0 \preceq f(x_0, y_0)$ and $1 = y_0 \succeq f(y_0, x_0) = 1$. Therefore, all of the hypothesis of Theorem 2 are held. Clearly, (p,p) is a coupled fixed point the function f.*

3. Application

Consider the following system of integral equations:

$$\begin{cases} x(t) = \int_a^b M(t,s)K(s,x(s),y(s))ds, \\ y(t) = \int_a^b M(t,s)K(s,y(s),x(s))ds, \end{cases} \tag{3}$$

for all $t \in I = [a,b]$, where $b > a$, $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Let $C(I,\mathbb{R})$ be the Banach space of every real continuous functions on I with $||x||_\infty = \max_{t\in I} |x(t)|$ for all $x \in C(I,\mathbb{R})$ and $C(I \times I \times C(I,\mathbb{R}), \mathbb{R})$ be the space of every continuous functions on $I \times I \times C(I,\mathbb{R})$. Define a mapping $G : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to D^+$ by $G_{x,y,z}(t) = \chi(\frac{t}{2} - (\|x-y\|_\infty + \|x-z\|_\infty + \|y-z\|_\infty))$ for all $x,y,z \in C(I,\mathbb{R})$ and $t > 0$, where

$$\chi(t) = \begin{cases} 0 & if \quad t \leq 0 \\ 1 & if \quad t > 0 \end{cases}$$

Then, $(C(I,\mathbb{R}), G, \mathcal{T})$ with $\mathcal{T}(a,b) = \min\{a,b\}$ is a complete Menger PGM space ([7]). Consider an e-distance on $\mathcal{X}$ by $g_{x,y,z}(t) = \min\{G_{x,y,z}(t), G_{Ax,Ay,Az}(t)\}$, where $A : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ and $Ax = \frac{x}{2}$. Moreover, we define the relation " $\preceq$ " on $C(I,\mathbb{R})$ by $x \preceq y \Leftrightarrow ||x||_\infty \leq ||y||_\infty$ for all $x,y \in C(I,\mathbb{R})$. Clearly the relation " $\preceq$ " is a quasi-order relation on $C(I,\mathbb{R})$ and $(C(I,\mathbb{R}), G, \mathcal{T}, \preceq)$ is a quasi-ordered complete PGM space.

Theorem 3. *Let $(C(I,\mathbb{R}), G, \mathcal{T}, \preceq)$ be a quasi-ordered complete Menger PGM space and $f : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to C(I,\mathbb{R})$ be a operator defined by $f(x,y)(t) = \int_a^b M(t,s)K(s,x(s),y(s))ds$, where $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are two operators. Assume the following properties are held:*

(i) $||K||_\infty = \sup_{s\in I,\, x,y\in C(I,\mathbb{R})} |K(s,x(s),y(s))| < \infty$;

(ii) *for every $x,y \in C(I,\mathbb{R})$ and every $t,s \in I$, we have*

$$||K(s,x(s),y(s)) - K(s,u(s),v(s))||_\infty \leq \frac{1}{4}(\max|x(s)-u(s)| + \max|y(s)-v(s)|);$$

(iii) $\max_{t\in I} \int_a^b M(t,s)ds < 1$.

Then, the system (3) have a solution in $C(I,\mathbb{R}) \times C(I,\mathbb{R})$.

Proof. For all $x,y \in C(I,\mathbb{R})$, let $\|x-y\|_\infty = \max_{t\in I}(|x(t)-y(t)|)$. Then, for all $x,y,z,u,v,w \in C(I,\mathbb{R})$, we have

$$\begin{aligned} \|f(x,y) - f(u,v)\|_\infty &\leq \max_{t\in I} \int_a^b M(t,s)|K(s,x(s)y(s)) - K(s,u(s),v(s))|ds \\ &\leq \max(\frac{1}{4}(|x(s)-u(s)| + |y(s)-v(s)|)) \max_{t\in I} \int_a^b M(t,s)ds \\ &\leq \max(\frac{1}{4}(|x(s)-u(s)| + |y(s)-v(s)|)). \end{aligned}$$

We consider two following cases:

Case 1. Let

$$\begin{aligned} g_{f(x,y),f(u,v),f(w,z)}(t) &= \min\{G_{f(x,y),f(u,v),f(w,z)}(t), G_{Af(x,y),Af(u,v),Af(w,z)}(t)\} \\ &= G_{f(x,y),f(u,v),f(w,z)}(t). \end{aligned}$$

Then, we obtain

$$\begin{aligned}
g_{f(x,y),f(u,v),f(w,z)}(t) &= G_{f(x,y),f(u,v),f(w,z)}(t)\\
&= \chi(\frac{t}{2} - (\|f(x,y) - f(u,v)\|_\infty + \|f(x,y) - f(w,z)\|_\infty + \|f(u,v) - f(w,z)\|_\infty))\\
&\geq \chi(\frac{t}{2} - (\max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|))\\
&\quad + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|))\\
&\quad + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|))))\\
&= \chi(t - \frac{1}{2}(\max((|x(s) - u(s)| + |y(s) - v(s)|))\\
&\quad + \max((|x(s) - w(s)| + |y(s) - z(s)|)) + \max((|u(s) - w(s)| + |v(s) - z(s)|))))\\
&\geq \frac{1}{2}(\chi(t - (\max(|x(s) - u(s)| + |x(s) - w(s)| + |u(s) - w(s)|)))\\
&\quad + \chi(t - (\max(|y(s) - v(s)| + |y(s) - z(s)| + |v(s) - z(s)|))))\\
&= \frac{1}{2}(G_{x,u,w}(2t) + G_{y,v,z}(2t)) \geq \frac{1}{2}(g_{x,u,w}(2t) + g_{y,v,z}(2t)).
\end{aligned}$$

Case 2. Let

$$\begin{aligned}
g_{f(x,y),f(u,v),f(w,z)}(t) &= \min\{G_{f(x,y),f(u,v),f(w,z)}(t), G_{Af(x,y),Af(u,v),Af(w,z)}(t)\}\\
&= G_{Af(x,y),Af(u,v),Af(w,z)}(t).
\end{aligned}$$

By $Ax = \frac{x}{2}$, we have

$$\begin{aligned}
g_{f(x,y),f(u,v),f(w,z)}(t) &= G_{Af(x,y),Af(u,v),Af(w,z)}(t)\\
&= \chi(\frac{t}{2} - \frac{1}{2}(\|f(x,y) - f(u,v)\|_\infty + \|f(x,y) - f(w,z)\|_\infty + \|f(u,v) - f(w,z)\|_\infty))\\
&\geq \chi(\frac{t}{2} - (\|f(x,y) - f(u,v)\|_\infty + \|f(x,y) - f(w,z)\|_\infty + \|f(u,v) - f(w,z)\|_\infty))\\
&\geq \chi(\frac{t}{2} - (\max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|))\\
&\quad + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|))\\
&\quad + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|))))\\
&= \chi(t - \frac{1}{2}(\max((|x(s) - u(s)| + |y(s) - v(s)|))\\
&\quad + \max((|x(s) - w(s)| + |y(s) - z(s)|)) + \max((|u(s) - w(s)| + |v(s) - z(s)|))))\\
&\geq \frac{1}{2}(\chi(t - (\max(|x(s) - u(s)| + |x(s) - w(s)| + |u(s) - w(s)|)))\\
&\quad + \chi(t - (\max(|y(s) - v(s)| + |y(s) - z(s)| + |v(s) - z(s)|))))\\
&= \frac{1}{2}(G_{x,u,w}(2t) + G_{y,v,z}(2t)) \geq \frac{1}{2}(g_{x,u,w}(2t) + g_{y,v,z}(2t))
\end{aligned}$$

for all $x, y, z, u, v, w \in C(I, \mathbb{R})$. Therefore, by Theorem 2 with $k = \frac{1}{2}$ for all $x, y, z, u, v, w \in C(I, \mathbb{R})$ and $t > 0$, we deduce that the operator f has a coupled fixed point which is the solution of the system of the integral equations. □

4. Conclusions

The new concept of e-distance, which is a generalization of r-distance in PGM space has been introduced. Moreover, some of properties of e-distance have been discussed. In addition, we obtained several new coupled fixed point results. Ultimately, to illustrate the usability of the main theorem, the existence of a solution for a system of integral equations is proved.

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