axioms

Article

Remarkable Classes of Almost 3-Contact Metric Manifolds

Giulia Dileo

Department of Mathematics, University of Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy; giulia.dileo@uniba.it

Abstract: We introduce a new class of almost 3-contact metric manifolds, called 3-$(0,\delta)$-Sasaki manifolds. We show fundamental geometric properties of these manifolds, analyzing analogies and differences with the known classes of 3-(α,δ)-Sasaki ($\alpha \neq 0$) and 3-δ-cosymplectic manifolds.

Keywords: almost 3-contact metric manifold; 3-Sasaki; 3-cosymplectic; 3-(α,δ)-Sasaki; 3-δ- cosymplectic; metric connection with skew torsion

MSC: 53C15; 53C25; 53B05

Citation: Dileo, G. Remarkable Classes of Almost 3-Contact Metric Manifolds. *Axioms* **2021**, *10*, 8. https://doi.org/10.3390/axioms10010008

Received: 9 December 2020
Accepted: 7 January 2021
Published: 11 January 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction

An almost 3-contact metric manifold is a $(4n+3)$-dimensional differentiable manifold M endowed with three almost contact metric structures (φ,ξ_i,η_i,g), $i=1,2,3$, sharing the same Riemannian metric g and satisfying suitable compatibility conditions, equivalent to the existence of a sphere of almost contact metric structures. In the recent paper [1], new classes of almost 3-contact metric manifolds were introduced and studied. The first remarkable class is given by 3-(α,δ)-*Sasaki manifolds* defined as almost 3-contact metric manifolds $(M,\varphi_i,\xi_i,\eta_i,g)$ such that

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha-\delta)\eta_j\wedge\eta_k, \qquad \alpha\in\mathbb{R}^*, \delta\in\mathbb{R}, \tag{1}$$

for every even permutation (i,j,k) of $(1,2,3)$. This is a generalization of 3-Sasaki manifolds, which correspond to the values $\alpha=\delta=1$. A second class introduced in [1] is given by 3-δ-*cosymplectic manifolds* defined by the conditions

$$d\eta_i = -2\delta\eta_j\wedge\eta_k, \qquad d\Phi_i = 0, \qquad \delta\in\mathbb{R},$$

generalizing 3-cosymplectic manifolds which correspond to the value $\delta=0$.

In the present paper we will introduce a third class of almost 3-contact metric manifolds, which is in fact a second (and alternative) generalization of 3-cosymplectic manifolds. We will consider almost 3-contact metric manifolds whose structure tensor fields satisfy

$$d\eta_i = -2\delta\eta_j\wedge\eta_k, \qquad d\Phi_i = -2\delta(\eta_j\wedge\Phi_k - \eta_k\wedge\Phi_j), \qquad \delta\in\mathbb{R} \tag{2}$$

for every even permutation (i,j,k) of $(1,2,3)$. When $\delta=0$ we recover a 3-cosymplectic manifold. We will call these manifolds 3-$(0,\delta)$-*Sasaki manifolds*. The choice of name is due to the fact that for a 3-(α,δ)-Sasaki manifold, Equation (1) implies

$$d\Phi_i = 2(\alpha-\delta)(\eta_j\wedge\Phi_k - \eta_k\wedge\Phi_j), \tag{3}$$

so that the two equations in (2) formally correspond to (1) and (3) with $\alpha=0$, although in this case the second equation is no more a consequence of the first one. In fact the two conditions in (2) are not completely independent (see Remark 1). Examples of 3-$(0,\delta)$-Sasaki structures can be defined on the semidirect products $SO(3)\ltimes\mathbb{R}^{4n}$. The structure on these Lie groups was introduced in [2] as an example of *canonical abelian* almost 3-contact metric structure. It is also shown in [2] that the Lie group $\mathrm{SO}(3) \ltimes \mathbb{R}^{4n}$ admits co-compact discrete subgroups, so that the corresponding compact quotients admit almost 3-contact metric structures of the same type.

One can show that for all the above three classes of manifolds, 3-(α,δ)-Sasaki, 3-δ-cosymplectic, and 3-$(0,\delta)$-Sasaki manifolds, the structure is hypernormal, the characteristic vector fields ξ_i, $i = 1,2,3$, are Killing and they span an integrable distribution, called vertical, with totally geodesic leaves. Nevertheless, there are remarkable geometric differences between the three classes. In the 3-(α,δ)-Sasaki case the 1-forms η_i are all contact forms, i.e., $\eta_i \wedge (d\eta_i)^n \neq 0$ everywhere on M, while for the other two classes, the horizontal distribution defined by $\eta_i = 0$, $i = 1,2,3$, is integrable. Both 3-δ-cosymplectic manifolds and 3-$(0,\delta)$-Sasaki manifolds are locally isometric to the Riemannian product of a 3-dimensional Lie group, tangent to the vertical distribution, and a $4n$-dimensional manifold tangent to the horizontal distribution. The Lie group is either isomorphic to $\mathrm{SO}(3)$ or flat depending on wether $\delta \neq 0$ or $\delta = 0$. Each horizontal leaf is endowed with a hyper-Kähler structure. The difference between 3-δ-cosymplectic and 3-$(0,\delta)$-Sasaki manifolds lies in the projectability of the structure tensor fields φ_i, $i = 1,2,3$, with respect to the vertical foliation. They are always projectable for 3-δ-cosymplectic manifolds, but not for 3-$(0,\delta)$-Sasaki manifolds with $\delta \neq 0$. In this case one can project a transverse quaternionic structure, as it happens for 3-(α,δ)-Sasaki manifolds. Finally, for the three classes of manifolds, we analyze the existence of a canonical metric connection with totally skew-symmetric torsion.

2. Almost Contact and Almost 3-Contact Metric Manifolds

An *almost contact manifold* is a smooth manifold M of dimension $2n+1$, endowed with a structure (φ,ξ,η), where φ is a $(1,1)$-tensor field, ξ a vector field, and η a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

implying that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and φ has rank $2n$. The tangent bundle of M splits as $TM = \mathcal{H} \oplus \langle \xi \rangle$, where $\mathcal{H}$ is the $2n$-dimensional distribution defined by $\mathcal{H} = \mathrm{Im}(\varphi) = \mathrm{Ker}(\eta)$. The vector field ξ is called the *characteristic* or *Reeb vector field*.

On the product manifold $M \times \mathbb{R}$ one can define an almost complex structure J by $J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$, where X is a vector field tangent to M, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$ function on $M \times \mathbb{R}$. If J is integrable, the almost contact structure is said to be *normal* and this is equivalent to the vanishing of the tensor field $N_\varphi := [\varphi,\varphi] + d\eta \otimes \xi$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ [3]. More precisely, for any vector fields X and Y, N_φ is given by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + d\eta(X,Y)\xi. \tag{4}$$

It is known that any almost contact manifold admits a compatible metric, that is a Riemannian metric g such that $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ for every $X, Y \in \mathfrak{X}(M)$. Then $\eta = g(\cdot,\xi)$ and $\mathcal{H} = \langle \xi \rangle^\perp$. The manifold (M,φ,ξ,η,g) is called an *almost contact metric manifold*. The associated fundamental 2-form is defined by $\Phi(X,Y) = g(X,\varphi Y)$.

We recall some remarkable classes of almost contact metric manifolds.

- An *α-contact metric manifold* is defined as an almost contact metric manifold such that

$$d\eta = 2\alpha\Phi, \quad \alpha \in \mathbb{R}^*,$$

  When $\alpha = 1$, it is called a *contact metric manifold*; the 1-form η is a *contact form*, that is $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. An *α-Sasaki manifold* is a normal α-contact metric manifold, and again such a manifold with $\alpha = 1$ is called a *Sasaki manifold*.
- An *almost cosymplectic manifold* is defined as an almost contact metric manifold such that

$$d\eta = 0, \quad d\Phi = 0;$$

if furhermore the structure is normal, M is called a *cosymplectic manifold*. It is worth remarking that some authors call these manifolds *almost coKähler* and *coKähler*, respectively ([4]).

- A *quasi-Sasaki manifold* is a normal almost contact metric manifold with closed 2-form Φ. This class includes both α-Sasaki and cosymplectic manifolds. The Reeb vector field of a quasi-Sasaki manifold is always Killing.

Both α-Sasaki manifolds and cosymplectic manifolds can be characterized by means of the Levi-Civita connection ∇^g. Indeed, one can show that an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is α-Sasaki if and only if

$$(\nabla^g_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(X)Y) \qquad \forall X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold is cosymplectic if and only if $\nabla^g \varphi = 0$; further, this is equivalent to requiring the manifold to be locally isometric to the Riemannian product of a real line (tangent to the Reeb vector field) and a Kähler manifold.

For a comprehensive introduction to almost contact metric manifolds we refer to [3]. For Sasaki geometry, we also recommend the monograph [5]; the survey [4] covers fundamental properties and recent results on cosymplectic geometry.

An *almost 3-contact manifold* is a differentiable manifold M of dimension $4n+3$ endowed with three almost contact structures $(\varphi_i, \xi_i, \eta_i)$, $i = 1,2,3$, satisfying the following relations,

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j,$$
$$\xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \quad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i,$$

for any even permutation (i,j,k) of $(1,2,3)$ ([3]). The tangent bundle of M splits as $TM = \mathcal{H} \oplus \mathcal{V}$, where

$$\mathcal{H} := \bigcap_{i=1}^{3} \mathrm{Ker}(\eta_i), \qquad \mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle.$$

In particular, $\mathcal{H}$ has rank $4n$. We call any vector belonging to the distribution $\mathcal{H}$ *horizontal* and any vector belonging to the distribution $\mathcal{V}$ *vertical*. The manifold is said to be *hypernormal* if each almost contact structure (ϕ_i, ξ_i, η_i) is normal. In [6] it was proved that if two of the almost contact structures are normal, then so is the third.

The existence of an almost 3-contact structure is equivalent to the existence of a sphere $\{(\varphi_x, \xi_x, \eta_x)\}_{x \in S^2}$ of almost contact structures such that

$$\varphi_x \circ \varphi_y - \eta_y \otimes \xi_x = \varphi_{x \times y} - (x \cdot y)\, I, \qquad \varphi_x \xi_y = \xi_{x \times y}, \qquad \eta_x \circ \varphi_y = \eta_{x \times y},$$

for every $x, y \in S^2$, where $\cdot$ and $\times$ denote the standard inner product and cross product on $\mathbb{R}^3$. In fact, if the structure is hypernormal, then every structure in the sphere is normal ([7]).

Any almost 3-contact manifold admits a Riemannian metric g which is compatible with each of the three structures. Then M is said to be an *almost* 3*-contact metric manifold* with structure $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1,2,3$. For ease of notation, we will denote an almost 3-contact metric manifold by $(M, \varphi_i, \xi_i, \eta_i, g)$, omitting $i = 1,2,3$. The subbundles $\mathcal{H}$ and $\mathcal{V}$ are orthogonal with respect to g and the three Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal. In fact, the structure group of the tangent bundle is reducible to $\mathrm{Sp}(n) \times \{1\}$ [8].

Given an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$, an *$\mathcal{H}$-homothetic deformation* is defined by

$$\eta_i' = c\eta_i, \qquad \xi_i' = \frac{1}{c}\xi_i, \qquad \varphi_i' = \varphi_i, \qquad g' = ag + b\sum_{i=1}^{3} \eta_i \otimes \eta_i, \tag{5}$$

where a, b, c are real numbers such that $a > 0$, $c^2 = a + b > 0$, ensuring that $(\varphi'_i, \xi'_i, \eta'_i, g')$ is an almost 3-contact metric structure. In particular, the fundamental 2-forms Φ_i and Φ'_i associated to the structures are related by

$$\Phi'_i = a\Phi_i - b\eta_j \wedge \eta_k, \tag{6}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$.

An almost 3-contact metric manifold is called

- *3-α-Sasaki*, with $\alpha \in \mathbb{R}^*$, if $(\varphi_i, \xi_i, \eta_i, g)$ is α-Sasaki for all $i = 1, 2, 3$, i.e. the structure is hypernormal and
$$d\eta_i = 2\alpha\Phi_i, \qquad i = 1, 2, 3; \tag{7}$$
when $\alpha = 1$, it is a 3-*Sasaki manifold*;
- 3-*cosymplectic* if $(\varphi_i, \xi_i, \eta_i, g)$ is cosymplectic for all $i = 1, 2, 3$, i.e. the structure is hypernormal and
$$d\eta_i = 0, \quad d\Phi_i = 0, \qquad i = 1, 2, 3; \tag{8}$$
- 3-*quasi-Sasaki manifold* if each structure $(\varphi_i, \xi_i, \eta_i, g)$ is quasi-Sasaki; this class includes both 3-α-Sasaki and 3-cosymplectic manifolds.

These classes were deeply investigated by various authors. See [5,9,10] and references therein for 3-Sasakian geometry, the papers [7,11,12] for 3-cosymplectic manifolds, and [13,14] for 3-quasi-Sasaki manifolds.

In fact, both for 3-Sasaki and 3-cosymplectic manifolds, the hypernormality is consequence of the structure Equations (7) and (8) respectively. This was proved by Kashiwada in [15] for 3-Sasaki manifolds, and in ([16], Theorem 4.13) for 3-cosymplectic manifolds.

In [1] the new classes of 3-(α, δ)-Sasaki manifolds and 3-δ-cosymplectic manifolds were introduced, generalizing the classes of 3-α-Sasaki and 3-cosymplectic manifolds, respectively. We will review the definitions and the basic properties of these manifolds in the next section. For both these two classes the hypernormality is a consequence of the defining structure equations for the manifolds, thus generalizing the analogous results for 3-Sasaki and 3-cosymplectic manifolds. This is obtained by using the following Lemma:

Lemma 1 ([1])**.** *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. Then the following formula holds $\forall X, Y, Z \in \mathfrak{X}(M)$:*

$$\begin{aligned} g(N_{\varphi_i}(X, Y), Z) \ &= \\ = -d\Phi_j(X, Y, \varphi_j Z) &+ d\Phi_j(\varphi_i X, \varphi_i Y, \varphi_j Z) + d\Phi_k(X, \varphi_i Y, \varphi_j Z) + d\Phi_k(\varphi_i X, Y, \varphi_j Z) \\ &- \eta_i(X)[d\eta_j(\varphi_i Y, \varphi_j Z) + d\eta_k(Y, \varphi_j Z)] + \eta_i(Y)[d\eta_j(\varphi_i X, \varphi_j Z) + d\eta_k(X, \varphi_j Z)] \\ &+ \eta_j(Z)[d\eta_j(X, Y) - d\eta_j(\varphi_i X, \varphi_i Y)] - \eta_j(Z)[d\eta_k(X, \varphi_i Y) + d\eta_k(\varphi_i X, Y)]. \end{aligned} \tag{9}$$

In the following we will be concerned with various classes of almost 3-contact metric manifolds where the three Reeb vector fields are all Killing. In this case one can show that there exists a function $\delta \in \mathcal{C}^\infty(M)$ such that

$$\eta_r([\xi_s, \xi_t]) = 2\delta\epsilon_{rst}, \qquad r, s, t = 1, 2, 3$$

where ϵ_{rst} is the totally skew-symmetric symbol, or equivalently $d\eta_r(\xi_s, \xi_t) = -2\delta\epsilon_{rst}$. We call δ a *Reeb commutator function*, we refer to [1] for more information on this notion.

3. 3-(α, δ)-Sasaki Manifolds and 3-δ-Cosymplectic Manifolds

This section is a short review of 3-(α, δ)-Sasaki manifolds and 3-δ-cosymplectic manifolds. These were discussed in detail in [1,17].

Definition 1. *An almost* 3*-contact metric manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *is called a* 3-(α, δ)-Sasaki manifold *if it satisfies*

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$$

for every even permutation (i, j, k) *of* $(1, 2, 3)$*, where* $\alpha \neq 0$ *and* δ *are real constants.*

When $\alpha = \delta = 1$, we have a 3-contact metric manifold, and hence a 3-Sasaki manifold by Kashiwada's theorem [15]. In the following, when considering 3-(α, δ)-Sasaki manifolds we will always mean $\alpha \neq 0$. As an immediate consequences of the definition one obtains the following properties:

1. Each ξ_i is an infinitesimal automorphism of the distribution $\mathcal{H}$, i. e.

$$d\eta_r(X, \xi_s) = 0 \qquad X \in \Gamma(\mathcal{H}),\ r, s = 1, 2, 3;$$

2. The constant δ is the Reeb commutator function;
3. The differentials $d\Phi_i$ are given by

$$d\Phi_i = 2(\delta - \alpha)(\eta_k \wedge \Phi_j - \eta_j \wedge \Phi_k).$$

Applying Lemma 1 one shows the following

Theorem 1 ([1], Theorem 2.2.1)**.** *Any* 3-(α, δ)*-Sasaki manifold is hypernormal.*

In particular, a 3-(α, δ)-Sasaki manifold with $\alpha = \delta$ is 3-α-Sasaki. It can be also shown that the vertical distribution of any 3-(α, δ)-Sasaki manifold is integrable with totally geodesic leaves and each Reeb vector field ξ_i is Killing.

We can distinguish three main classes of 3-(α, δ)-Sasaki manifolds. A 3-(α, δ)-Sasaki manifold is called *degenerate* if $\delta = 0$ and *non-degenerate* otherwise. Quaternionic Heisenberg groups are examples of degenerate 3-(α, δ)-Sasaki manifolds (see ([1], Example 2.3.2)). Considering an $\mathcal{H}$-homothetic deformation of a 3-(α, δ)-Sasaki structure, as in (5), one can verify that the obtained structure $(\varphi', \xi', \eta', g')$ is a 3-(α', δ')-Sasaki with

$$\alpha' = \alpha\frac{c}{a}, \qquad \delta' = \frac{\delta}{c}.$$

In particular, $\mathcal{H}$-homothetic deformations preserve the class of degenerate manifolds. In the nondegenerate case, one sees immediately that $\alpha'\delta'$ has the same sign as $\alpha\delta$. This justifies the distinction between *positive* 3-(α, δ)-Sasaki manifolds, with $\alpha\delta > 0$, and *negative* 3-(α, δ)-Sasaki manifolds, with $\alpha\delta < 0$. In fact, it can be shown that a 3-(α, δ)-Sasaki manifold is positive if and only if it is $\mathcal{H}$-homothetic to a 3-Sasaki manifold, and negative if and only if it is $\mathcal{H}$-homothetic to a 3-(α', δ')-Sasaki manifold with $\alpha' = -1$, $\delta' = 1$.

Examples of negative 3-(α, δ)-Sasaki manifolds can be obtained in the following way. It is known that quaternionic Kähler (not hyper-Kähler) manifolds with negative scalar curvature admit a canonically associated principal SO(3)-bundle $P(M)$ which is endowed with a *negative* 3*-Sasaki structure* [18,19]. This is a 3-structure $(\varphi_i, \xi_i, \eta_i, \tilde{g})$, $i = 1, 2, 3$, where $(\varphi_i, \xi_i, \eta_i)$ is a normal almost 3-contact structure, and $\tilde{g}$ is a compatible semi-Riemannian metric, with signature $(3, 4n)$, where $4n$ is the dimension of the base space, and $d\eta_i(X, Y) = 2\tilde{g}(X, \varphi_i Y)$. Then, one can define the Riemannian metric

$$g = -\tilde{g} + 2\sum_{i=1}^{3} \eta_i \otimes \eta_i,$$

which is compatible with the structure $(\varphi_i, \xi_i, \eta_i)$, and satisfies $d\eta_i = -2\Phi_i - 4\eta_j \wedge \eta_k$, where $\Phi_i(X, Y) = g(X, \varphi_i Y)$ (see also [19]). Therefore $(\varphi_i, \xi_i, \eta_i, g)$ is a 3-(α, δ)-Sasaki structure with $\alpha = -1$ and $\delta = 1$.

The following Theorem regarding the transverse geometry with respect to the vertical foliation of a 3-(α, δ)-Sasaki manifold is proved in [17]:

Theorem 2. *Any 3-(α, δ)-Sasaki manifold M admits a locally defined Riemannian submersion $\pi\colon M \to N$ along its horizontal distribution $\mathcal{H}$ such that N carries a quaternionic Kähler structure given by*

$$\check{\varphi}_i = \pi_* \circ \varphi_i \circ s_*, \quad i = 1,2,3,$$

where $s\colon U \to M$ is any local smooth section of the Riemannian submersion. The covariant derivatives of the almost complex structures $\check{\varphi}_i$ are given by

$$\nabla^{g_N}_X \check{\varphi}_i = 2\delta(\check{\eta}_k(X)\check{\varphi}_j - \check{\eta}_j(X)\check{\varphi}_k)$$

where $\check{\eta}_i(X) = \eta_i(s_* X) \circ s$ for $i = 1,2,3$. The scalar curvature of the base space N is $16n(n+2)\alpha\delta$.

The Riemannian Ricci tensor of any 3-(α, δ)-Sasaki manifold is computed in [1]:

$$\mathrm{Ric}^g \;=\; 2\alpha\big(2\delta(n+2) - 3\alpha\big)g + 2(\alpha-\delta)\big((2n+3)\alpha - \delta\big)\sum_{i=1}^{3} \eta_i \otimes \eta_i. \tag{10}$$

In particular, a 3-(α, δ)-Sasaki manifold is Riemannian Einstein if and only if $\delta = \alpha$, in which case the structure is 3-α-Sasaki, or $\delta = (2n+3)\alpha$.

Notice that, by Theorem 2, a non-degenerate 3-(α, δ)-Sasaki manifold locally fibers over a quaternionic Kähler space of positive or negative scalar curvature, according to $\alpha\delta > 0$ or $\alpha\delta < 0$, respectively. In [17] a systematic study of homogeneous non-degenerate 3-(α, δ)-Sasaki manifolds has been carried out, obtaining a complete classification in the positive case, where the base space of the homogeneous fibration turns out to be a symmetric Wolf space. In the case $\alpha\delta < 0$, one can provide a general construction of homogeneous 3-(α, δ)-Sasaki manifolds fibering over nonsymmetric Alekseevsky spaces.

We recall now the definition and some basic facts on 3-δ-cosymplectic manifolds.

Definition 2. *A* 3-δ-cosymplectic manifold *is an almost 3-contact metric manifold satisfying*

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \qquad d\Phi_i = 0,$$

for some $\delta \in \mathbb{R}$ and for every even permutation (i, j, k) of $(1, 2, 3)$.

When $\delta = 0$, the fact that the forms η_i and Φ_i are all closed implies that the structure is hypernormal ([16], Theorem 4.13). In fact this immediately follows from (9). Therefore, a 3-δ-cosymplectic manifold with $\delta = 0$ is 3-cosymplectic. In particular, it is 3-quasi-Sasaki and the Reeb vector fields are all Killing. The local structure of these manifolds is described by the following:

Proposition 1 ([12])**.** *Any 3-cosymplectic manifold of dimension $4n+3$ is locally the Riemannian product of a hyper-Kähler manifold of dimension $4n$ and a 3-dimensional flat abelian Lie group.*

As a consequence, since every hyper-Kähler manifold is Ricci flat, even the Riemannian Ricci tensor of any 3-cosymplectic manifold vanishes.

As regards 3-δ-cosymplectic manifolds with $\delta \neq 0$, even in this case one can show that the structure is hypernormal, the Reeb vector fields are Killing, and the manifold locally decomposes as a Riemannian product [1]. In particular,

Proposition 2. *Any 3-δ-cosymplectic manifold with $\delta \neq 0$ is locally the Riemannian product of a hyper-Kähler manifold and a 3-dimensional Lie group isomorphic to* $\mathrm{SO}(3)$*, with constant curvature δ^2. Consequently, the Riemannian Ricci tensor is $\mathrm{Ric}^g = 2\delta^2 \sum_{i=1}^3 \eta_i \otimes \eta_i$.*

In both cases, i.e., $\delta = 0$ or $\delta \neq 0$, the hyper-Kähler manifold is tangent to the horizontal distribution, while the 3-dimensional Lie group is tangent to the vertical distribution. In fact, examples of these manifolds can be obtained taking Riemannian products $N \times G$, where (N, J_i, h), $i = 1, 2, 3$, is a hyper-Kähler manifold, and G is a 3-dimensional Lie group, which is either abelian, or isomorphic to SO(3). If ξ_1, ξ_2, ξ_3 are generators of the Lie algebra $\mathfrak{g}$ of G, satisfying $[\xi_i, \xi_j] = 2\delta\xi_k$, $\delta \in \mathbb{R}$, then one can define in a natural manner an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$ on the product $N \times G$, setting

$$\varphi_i|_{TN} = J_i, \quad \varphi_i\xi_i = 0, \quad \varphi_i\xi_j = \xi_k, \quad \varphi_i\xi_k = -\xi_j,$$

$$\eta_i|_{TN} = 0, \quad \eta_i(\xi_i) = 1, \qquad \eta_i(\xi_j) = \eta_i(\xi_k) = 0,$$

and g the product metric of h and the left invariant Riemannian metric on G with respect to which ξ_1, ξ_2, ξ_3 are an orthonormal basis of $\mathfrak{g}$.

For a comparison with the class of 3-$(0, \delta)$-Sasaki manifolds, which will be introduced in the next section, it is worth remarking that for a 3-δ-cosymplectic manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ the Lie derivatives of the structure tensor fields φ_i, $i = 1, 2, 3$ with respect to the Reeb vector fields are given by

$$\mathcal{L}_{\xi_i}\varphi_i = 0, \qquad \mathcal{L}_{\xi_i}\varphi_j = 2\delta(\eta_i \otimes \xi_j - \eta_j \otimes \xi_i) = -\mathcal{L}_{\xi_j}\varphi_i \tag{11}$$

for every $i, j = 1, 2, 3$. Indeed, in a 3-δ-cosymplectic manifold the Levi-Civita connection satisfies ([1], Proposition 2.1.1):

$$\nabla^g_{\xi_i}\varphi_i = 0,$$

$$(\nabla^g_{\xi_i}\varphi_j)X = \delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) = -(\nabla^g_{\xi_j}\varphi_i)X,$$

$$\nabla^g_X\xi_i = \delta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),$$

where (i, j, k) is an even permutation of $(1, 2, 3)$ and $X \in \mathfrak{X}(M)$. Therefore,

$$\begin{aligned}
(\mathcal{L}_{\xi_i}\varphi_i)X &= [\xi_i, \varphi_i X] - \varphi_i[\xi_i, X] \\
&= \nabla^g_{\xi_i}(\varphi_i X) - \nabla^g_{\varphi_i X}\xi_i - \varphi_i(\nabla^g_{\xi_i}X) + \varphi_i(\nabla^g_X\xi_i) \\
&= (\nabla^g_{\xi_i}\varphi_i)X - \nabla^g_{\varphi_i X}\xi_i + \varphi_i(\nabla^g_X\xi_i) \\
&= -\delta(\eta_k(\varphi_i X)\xi_j - \eta_j(\varphi_i X)\xi_k) + \delta(\eta_k(X)\varphi_i\xi_j - \eta_j(X)\varphi_i\xi_k) = 0.
\end{aligned}$$

In the same way,

$$\begin{aligned}
(\mathcal{L}_{\xi_i}\varphi_j)X &= (\nabla^g_{\xi_i}\varphi_j)X - \nabla^g_{\varphi_j X}\xi_i + \varphi_j(\nabla^g_X\xi_i) \\
&= \delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) - \delta\,\eta_k(\varphi_j X)\xi_j - \delta\,\eta_j(X)\varphi_j\xi_k \\
&= 2\delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) = -(\mathcal{L}_{\xi_j}\varphi_i)X.
\end{aligned}$$

4. 3-$(0, \delta)$-Sasaki Manifolds

In this section we introduce the class of 3-$(0, \delta)$-Sasaki manifolds.

Definition 3. *An almost* 3*-contact metric manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *will be called* 3-$(0, \delta)$-Sasaki manifold *if*

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \qquad d\Phi_i = -2\delta(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j) \tag{12}$$

for every even permutation (i, j, k) *of* $(1, 2, 3)$*, and for some real constant* $\delta \in \mathbb{R}$*.*

In particular, the structure is not 3-quasi-Sasaki when $\delta \neq 0$, and we have the following basic properties for a 3-$(0, \delta)$-Sasaki manifold:

1. The horizontal distribution $\mathcal{H}$ is integrable;

2. Each ξ_i is an infinitesimal automorphism of the distribution $\mathcal{H}$, i. e.

$$d\eta_r(X, \xi_s) = 0 \qquad X \in \Gamma(\mathcal{H}),\ r, s = 1, 2, 3;$$

3. The constant δ is the Reeb commutator function.

Remark 1. *In case $\delta \neq 0$, the two equations in (12) are not completely independent. Indeed, if one assumes $d\Phi_i = -2\gamma(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j)$, $\gamma \in \mathbb{R}^*$, differentiating this equation, and combining with $d\eta_i = -2\delta\eta_j \wedge \eta_k$, a straightforward computation gives $\gamma = \delta$. Thus, there is no freedom for the choice of constant in the second equation.*

If $(\varphi_i, \xi_i, \eta_i, g)$ is a 3-$(0, \delta)$-Sasaki structure, applying an $\mathcal{H}$-homothetic deformation as in (5), an easy computation using (6) shows that the new structure $(\varphi_i', \eta_i', \xi_i', g')$ is again 3-$(0, \delta')$-Sasaki, with $\delta' = \frac{\delta}{c}$.

Example 1. *Consider the abelian Lie algebra $\mathbb{R}^{4n}$ spanned by vectors v_r, v_{n+r}, v_{2n+r}, v_{3n+r}, $r = 1, \dots, n$, and endowed with the hypercomplex structure $\{J_1, J_2, J_3\}$ defined by*

$$J_i(v_r) = v_{in+r}, \quad J_i(v_{in+r}) = -v_r, \quad J_i(v_{jn+r}) = v_{kn+r}, \quad J_i(v_{kn+r}) = -v_{jn+r},$$

for every even permutation (i, j, k) of $(1, 2, 3)$. Let us consider also the Lie algebra $\mathfrak{so}(3)$ spanned by ξ_1, ξ_2, ξ_3 with Lie brackets given by $[\xi_i, \xi_j] = 2\delta\xi_k$, $\delta \neq 0$. Let ρ be the representation of $\mathfrak{so}(3)$ on $\mathbb{R}^{4n}$ given by

$$\rho : \mathfrak{so}(3) \to \mathfrak{gl}(4n, \mathbb{R}), \qquad \rho(\xi_i) = \delta J_i, \quad i = 1, 2, 3.$$

On the Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \ltimes_\rho \mathbb{R}^{4n}$ on can define in a natural way an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$, with

$$\varphi_i|_{\mathbb{R}^{4n}} = J_i, \qquad \varphi(\xi_i) = 0, \qquad \varphi_i(\xi_j) = \xi_k = -\varphi_j(\xi_k),$$

$$\eta_i|_{\mathbb{R}^{4n}} = 0, \qquad \eta_i(\xi_i) = 1, \qquad \eta_i(\xi_j) = \eta_i(\xi_k) = 0,$$

and where g is the inner product such that the vectors ξ_i, v_l, $i = 1, 2, 3$, $l = 1, \dots, 4n$ are orthonormal. In particular, the non zero brackets on $\mathfrak{g}$ are given by

$$[\xi_i, \xi_j] = 2\delta\xi_k, \qquad [\xi_i, X] = \delta\varphi_i(X),\ X \in \mathbb{R}^{4n}.$$

The representation $\rho : \mathfrak{so}(3) \to \mathfrak{gl}(4n, \mathbb{R})$ can be integrated to a representation $\tilde{\rho} : \mathrm{SO}(3) \to \mathrm{GL}(4n, \mathbb{R})$. Therefore, identifying $\mathbb{R}^{4n}$ with $\mathbb{H}^n$ in a natural way, the simply connected Lie group $G = \mathrm{SO}(3) \ltimes_{\tilde{\rho}} \mathbb{H}^n$, with Lie algebra $\mathfrak{g}$, admits a left invariant almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$. One can easily verify that this structure satisfies (12).

Remark 2. *For more details on the above example we refer to [2], where $\mathfrak{g}$ is described as a remarkable example of a Lie algebra endowed with an* abelian almost 3-contact metric structure. *In fact, the structure defined on $\mathfrak{g}$ belongs to the class of* canonical abelian *structures, so that the Lie group G admits a unique metric connection with totally skew symmetric torsion ∇ such that*

$$\nabla_X \varphi_i = 2\delta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k)$$

for every vector field X and for every even permutation (i, j, k) of $(1, 2, 3)$. The torsion of the canonical connection ∇ is $T = 2\delta\eta_1 \wedge \eta_2 \wedge \eta_3$, which satisfies $\nabla T = 0$.

It is also shown in [2] that the Lie group G admits co-compact discrete subgroups, so that the corresponding compact quotients admit almost 3-contact metric structures of the same type.

Proposition 3. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-$(0, \delta)$-Sasaki manifold. Then the structure is hypernormal.*

Proof. In order to compute the tensor fields N_{φ_i}, we apply Lemma 1. We always denote by X, Y, Z horizontal vector fields and by (i, j, k) an even permutation of $(1, 2, 3)$.

Being $d\Phi_i(X, Y, Z) = 0$, then $N_{\varphi_i}(X, Y, Z) = 0$ for every $i = 1, 2, 3$. Furthermore, since the horizontal distribution is integrable, by the definition of the tensor field N_{φ_i} (see (4)), one has $N_{\varphi_i}(X, Y, \xi_r) = 0$ for all $r = 1, 2, 3$. Notice that, since

$$\xi_i \lrcorner \Phi_i = 0, \qquad \xi_j \lrcorner \Phi_i = -\eta_k, \qquad \xi_k \lrcorner \Phi_i = \eta_j,$$

from the second equation in (12), we have,

$$\xi_i \lrcorner d\Phi_i = 0, \quad \xi_j \lrcorner d\Phi_i = -2\delta(\Phi_k + \eta_{ij}), \quad \xi_k \lrcorner d\Phi_i = 2\delta(\Phi_j + \eta_{ki}). \tag{13}$$

Therefore, form Lemma 1, applying (12) and (13), we have

$$\begin{aligned}
N_{\varphi_i}(X, \xi_i, Z) &= -d\Phi_j(X, \xi_i, \varphi_j Z) + d\Phi_k(\varphi_i X, \xi_i, \varphi_j Z) + d\eta_j(\varphi_i X, \varphi_j Z) + d\eta_k(X, \varphi_j Z) \\
&= -2\delta\Phi_k(\varphi_j Z, X) - 2\delta\Phi_j(\varphi_j Z, \varphi_i X) \\
&= 2\delta\Phi_j(\varphi_i X, \varphi_j Z) + 2\delta\Phi_k(X, \varphi_j Z) = -2\delta g(\varphi_i X, Z) - 2\delta g(X, \varphi_i Z) = 0, \\
N_{\varphi_i}(X, \xi_j, Z) &= d\Phi_j(\varphi_i X, \xi_k, \varphi_j Z) + d\Phi_k(\varphi_i X, \xi_j, \varphi_j Z) \\
&= -2\delta\Phi_i(\varphi_j Z, \varphi_i X) + 2\delta\Phi_i(\varphi_j Z, \varphi_i X) = 0, \\
N_{\varphi_i}(X, \xi_k, Z) &= -d\Phi_j(X, \xi_k, \varphi_j Z) - d\Phi_k(X, \xi_j, \varphi_j Z) \\
&= 2\delta\Phi_i(\varphi_j Z, X) - 2\delta\Phi_i(\varphi_j Z, X) = 0.
\end{aligned}$$

Equations (13) implies $d\Phi_r(X, \xi_s, \xi_t) = 0$ for every $r, s, t = 1, 2, 3$ and $X \in \Gamma(\mathcal{H})$. Taking also into account that $d\eta_r(X, \xi_s) = 0$, we deduce from (9) that

$$N_{\varphi_r}(X, \xi_s, \xi_t) = N_{\varphi_r}(\xi_s, \xi_t, X) = 0.$$

Finally, (9) implies together with $d\eta_r(\xi_s, \xi_t) = -2\delta\epsilon_{rst}$ that

$$N_{\varphi_i}(\xi_i, \xi_j, \xi_k) = N_{\varphi_i}(\xi_i, \xi_k, \xi_j) = N_{\varphi_i}(\xi_j, \xi_k, \xi_i) = 0,$$

completing the proof that M is hypernormal. □

Proposition 4. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-$(0, \delta)$-Sasaki manifold. Then the Levi-Civita connection satisfies for all $X, Y \in \mathfrak{X}(M)$ and any cyclic permutation (i, j, k) of $(1, 2, 3)$:*

$$\begin{aligned}
(\nabla^g_X \varphi_i)Y = 2\delta \left[\eta_k(X)\varphi_j Y - \eta_j(X)\varphi_k Y\right] \\
- \delta \left[\eta_j(X)\eta_j(Y) + \eta_k(X)\eta_k(Y)\right]\xi_i + \delta\, \eta_i(Y) \left[\eta_j(X)\xi_j + \eta_k(X)\xi_k\right]
\end{aligned} \tag{14}$$

and

$$\nabla^g_X \xi_i = \delta\, (\eta_k(X)\xi_j - \eta_j(X)\xi_k), \tag{15}$$

$$\nabla^g_{\xi_i} \xi_i = 0, \quad \nabla^g_{\xi_i} \xi_j = -\nabla^g_{\xi_j} \xi_i = \delta\xi_k. \tag{16}$$

In particular, each ξ_i is a Killing vector field.

Proof. Since the structure is hypernormal, by ([3], Lemma 6.1), the Levi-Civita connection satisfies

$$\begin{aligned}
2g((\nabla^g_X \varphi_i)Y, Z) = d\Phi_i(X, \varphi_i Y, \varphi_i Z) - d\Phi_i(X, Y, Z) \\
+ d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y).
\end{aligned} \tag{17}$$

Further, an easy computation (see [1]) shows that for every cyclic permutation (i, j, k) of $(1, 2, 3)$,

$$\Phi_j(\varphi_i X, \varphi_i Y) = -\Phi_j(X, Y) - (\eta_k \wedge \eta_i)(X, Y),$$
$$\Phi_k(\varphi_i X, \varphi_i Y) = -\Phi_k(X, Y) - (\eta_i \wedge \eta_j)(X, Y),$$
$$\Phi_j(\varphi_i X, Y) = -\Phi_k(X, Y) - \eta_i(X)\eta_j(Y),$$
$$\Phi_k(\varphi_i X, Y) = \Phi_j(X, Y) - \eta_i(X)\eta_k(Y).$$

Then, using the second equation in (12) and the above equations, we have

$$\begin{aligned}
&d\Phi_i(X, \varphi_i Y, \varphi_i Z) = \\
&= -2\delta\, [\eta_j(X)\Phi_k(\varphi_i Y, \varphi_i Z) + \eta_j(\varphi_i Y)\Phi_k(\varphi_i Z, X) + \eta_j(\varphi_i Z)\Phi_k(X, \varphi_i Y) \\
&\quad - \eta_k(X)\Phi_j(\varphi_i Y, \varphi_i Z) - \eta_k(\varphi_i Y)\Phi_j(\varphi_i Z, X) - \eta_k(\varphi_i Z)\Phi_j(X, \varphi_i Y)] \\
&= -2\delta\, [-\eta_j(X)\Phi_k(Y, Z) - \eta_j(X)(\eta_i \wedge \eta_j)(Y, Z) \\
&\quad - \eta_k(Y)\Phi_j(Z, X) + \eta_k(Y)\eta_i(Z)\eta_k(X) + \eta_k(Z)\Phi_j(Y, X) - \eta_k(Z)\eta_i(Y)\eta_k(X) \\
&\quad + \eta_k(X)\Phi_j(Y, Z) + \eta_k(X)(\eta_k \wedge \eta_i)(Y, Z) \\
&\quad + \eta_j(Y)\Phi_k(Z, X) + \eta_j(Y)\eta_i(Z)\eta_j(X) - \eta_j(Z)\Phi_k(Y, X) - \eta_j(Z)\eta_i(Y)\eta_j(X)] \\
&= d\Phi_i(X, Y, Z) + 4\delta\, [\eta_j(X)\Phi_k(Y, Z) - \eta_k(X)\Phi_j(Y, Z)] \\
&\quad + 4\delta\, \eta_j(X)[\eta_i(Y)\eta_j(Z) - \eta_j(Y)\eta_i(Z)] \\
&\quad + 4\delta\, \eta_k(X)[\eta_i(Y)\eta_k(Z) - \eta_k(Y)\eta_i(Z)].
\end{aligned}$$

On the other hand, again using the first equation in (12), we obtain

$$\begin{aligned}
&d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y) = \\
&= -2\delta(\eta_j \wedge \eta_k)(\varphi_i Y, X)\eta_i(Z) + 2\delta(\eta_j \wedge \eta_k)(\varphi_i Z, X)\eta_i(Y) \\
&= -2\delta\, \eta_i(Z)[-\eta_k(Y)\eta_k(X) - \eta_j(X)\eta_j(Y)] + 2\delta\, \eta_i(Y)[-\eta_k(Z)\eta_k(X) - \eta_j(X)\eta_j(Z)].
\end{aligned}$$

Inserting the above computations in (17), we conclude that

$$\begin{aligned}
g((\nabla^g_X \varphi_i)Y, Z) = 2\delta[\eta_k(X)g(\varphi_j Y, Z) - \eta_j(X)g(\varphi_k Y, Z)] \\
- \delta\eta_i(Z)[\eta_k(Y)\eta_k(X) + \eta_j(X)\eta_j(Y)] + \delta\eta_i(Y)[\eta_k(Z)\eta_k(X) + \eta_j(X)\eta_j(Z)]
\end{aligned}$$

which implies (14). As regards the proof (15), applying (14) for $Y = \xi_i$, we get

$$(\nabla^g_X \varphi_i)\xi_i = -\delta(\eta_j(X)\xi_j + \eta_k(X)\xi_k).$$

Applying φ_i on both hand-sides, we obtain (15). Equations (16) are immediate consequences of (15). Furthermore, we also get

$$g(\nabla^g_X \xi_i, Y) = -\delta(\eta_j \wedge \eta_k)(X, Y)$$

for every $X, Y \in \mathfrak{X}(M)$. Since $\nabla^g \xi_i$ is skew-symmetric, ξ_i is Killing. □

Corollary 1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-$(0, \delta)$-Sasaki manifold. Then for every even permutation (i, j, k) of $(1, 2, 3)$ we have*

$$\mathcal{L}_{\xi_i}\varphi_i = 0, \qquad \mathcal{L}_{\xi_i}\varphi_j = -\mathcal{L}_{\xi_j}\varphi_i = 2\delta\varphi_k. \tag{18}$$

Proof. For the first Lie derivative, notice that by (14) we have $\nabla^g_{\xi_i}\varphi_i = 0$. Then, applying also (15), for every vector field X we have

$$\begin{aligned}
(\mathcal{L}_{\xi_i}\varphi_i)X &= (\nabla^g_{\xi_i}\varphi_i)X - \nabla^g_{\varphi_i X}\xi_i + \varphi_i(\nabla^g_X \xi_i) \\
&= -\delta(\eta_k(\varphi_i X)\xi_j - \eta_j(\varphi_i X)\xi_k) + \delta(\eta_k(X)\varphi_i\xi_j - \eta_j(X)\varphi_i\xi_k) = 0.
\end{aligned}$$

Now, using (14) for the covariant derivative $\nabla^g \varphi_j$, for every vector field Y, we have

$$(\nabla^g_{\xi_i} \varphi_j)Y = 2\delta\, \varphi_k Y - \delta\big(\eta_i(Y)\xi_j - \eta_j(Y)\xi_i\big).$$

Therefore, applying also (15), we get

$$\begin{aligned}(\mathcal{L}_{\xi_i}\varphi_j)X &= (\nabla^g_{\xi_i}\varphi_j)X - \nabla^g_{\varphi_j X}\xi_i + \varphi_j(\nabla^g_X \xi_i)\\ &= 2\delta\, \varphi_k X - \delta\big(\eta_i(X)\xi_j - \eta_j(X)\xi_i\big) - \delta\, \eta_k(\varphi_j X)\xi_j - \delta\, \eta_j(X)\varphi_j\xi_k\\ &= 2\delta\varphi_k X.\end{aligned}$$

Analogously, $\mathcal{L}_{\xi_j}\varphi_i = -2\delta\varphi_k$. □

Theorem 3. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-$(0, \delta)$-Sasaki manifold. Then both the horizontal and the vertical distribution are integrable with totally geodesic leaves. Each leaf of the vertical distribution is locally isomorphic to the Lie group* SO(3)*, with constant sectional curvature δ^2; each leaf of the horizontal distribution is endowed with a hyper-Kähler structure. Consequently, the Riemannian Ricci tensor of M is given by*

$$\mathrm{Ric}^g = 2\delta^2 \sum_{i=1}^{3} \eta_i \otimes \eta_i. \tag{19}$$

Proof. We already know that the horizontal distribution $\mathcal{H}$ is integrable. From (15), for every $X, Y \in \Gamma(\mathcal{H})$ and $i = 1, 2, 3$, we have

$$g(\nabla^g_X Y, \xi_i) = -g(\nabla^g_X \xi_i, Y) = 0,$$

so that the distribution $\mathcal{H}$ has totally geodesic leaves. Furthermore, Equation (16) implies that the vertical distribution $\mathcal{V}$ is also integrable with totally geodesic leaves. In particular $[\xi_i, \xi_j] = 2\delta\xi_k$ for an even permutation (i, j, k) of $(1, 2, 3)$, so that the leaves of $\mathcal{V}$ are locally isomorphic to the Lie group SO(3) and have constant sectional curvature δ^2. On each leaf of the horizontal distribution $\mathcal{H}$ one can consider the almost hyper-Hermitian structure defined by $(J_i := \varphi_i|_H, g)$, which turns out to be hyper-Kähler due to (14). Consequently, M is locally the Riemannnian product of 3-dimensional sphere of curvature δ^2 and a $4n$-dimensional manifold M', which is endowed with a hyper-Kähler structure. Since any hyper-Kähler manifold is Ricci flat, we obtain that the Riemannian Ricci tensor of M is given by (19). □

Remark 3. *From Theorem 3 it follows that any 3-$(0, \delta)$-Sasaki manifold is locally isometric to the Riemannnian product of 3-dimensional sphere and a $4n$-dimensional manifold M', which is endowed with a hyper-Kähler structure. We recall that 3-δ-cosymplectic manifolds are also locally isometric to the Riemannian product of a 3-dimensional sphere of constant curvature δ^2 and a hyper-Kähler manifold. Nevertheless, there is a difference between the two geometries. Looking at the transverse geometry of the foliation defined by the vertical distribution $\mathcal{V}$, in both cases the Riemannian metric g is projectable, being the vector fields ξ_i, $i = 1, 2, 3$, all Killing. In the case of 3-δ-cosymplectic manifolds, each tensor field φ_i is also projectable, as by (11), the Lie derivatives with respect to the Reeb vector fields satisfy $(\mathcal{L}_{\xi_i}\varphi_j)X = 0$ for every $i, j = 1, 2, 3$ and for every horizontal vector field X. In the case of 3-$(0, \delta)$-Sasaki manifolds, owing to (18), the tensor fields are not projectable. Nevertheless, taking into account the horizontal parts $\Phi_i^{\mathcal{H}} := \Phi_i + \eta_j \wedge \eta_k$ of the fundamental 2-forms Φ_i, one can verify that horizontal 4-form*

$$\Phi_1^{\mathcal{H}} \wedge \Phi_1^{\mathcal{H}} + \Phi_2^{\mathcal{H}} \wedge \Phi_2^{\mathcal{H}} + \Phi_3^{\mathcal{H}} \wedge \Phi_3^{\mathcal{H}}$$

is projectable and defines a transversal quaternionic structure, which turns out to be locally hyper-Kähler.

5. Connections with Totally Skew-Symmetric Torsion

In this section we will show that every 3-$(0,\delta)$-Sasaki manifold is *canonical* in the sense of the definition given in [1], thus admitting a special metric connection with totally skew-symmetric torsion, called canonical. Recall that a metric connection ∇ with torsion T on a Riemannian manifold (M,g) is said to have *totally skew-symmetric torsion*, or *skew torsion* for short, if the $(0,3)$-tensor field T defined by $T(X,Y,Z) := g(T(X,Y),Z)$ is a 3-form. The relation between ∇ and the Levi-Civita connection ∇^g is then given by

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2}T(X,Y).$$

For more details we refer to [20]. We recall now the definition and the characterization of canonical almost 3-contact metric manifolds.

Definition 4 ([1])**.** *An almost* 3*-contact metric manifold* $(M,\varphi_i,\xi_i,\eta_i,g)$ *is called* canonical *if the following conditions are satisfied:*

(i) *each* N_{φ_i} *is totally skew-symmetric on* $\mathcal{H}$*,*
(ii) *each* ξ_i *is a Killing vector field,*
(iii) *for any* $X,Y,Z \in \Gamma(\mathcal{H})$ *and any* $i,j = 1,2,3,$

$$N_{\varphi_i}(X,Y,Z) - d\Phi_i(\varphi_i X,\varphi_i Y,\varphi_i Z) = N_{\varphi_j}(X,Y,Z) - d\Phi_j(\varphi_j X,\varphi_j Y,\varphi_j Z),$$

(iv) *M admits a Reeb Killing function* $\beta \in C^\infty(M)$*, that is the tensor fields* A_{ij} *defined on* $\mathcal{H}$ *by*

$$A_{ij}(X,Y) \;:=\; g((\mathcal{L}_{\xi_j}\varphi_i)X,Y) + d\eta_j(X,\varphi_i Y) + d\eta_j(\varphi_i X,Y),$$

satisfy

$$A_{ii}(X,Y) = 0, \qquad A_{ij}(X,Y) = -A_{ji}(X,Y) = \beta\Phi_k(X,Y),$$

for every $X,Y \in \Gamma(\mathcal{H})$ *and every even permutation* (i,j,k) *of* $(1,2,3)$*.*

Here N_{φ_i} also denotes the $(0,3)$-tensor field defined by $N_{\varphi_i}(X,Y,Z) := g(N_{\varphi_i}(X,Y),Z)$ and we say that N_{φ_i} is totally skew-symmetric on $\mathcal{H}$ if the $(0,3)$-tensor is a 3-form on $\mathcal{H}$.

Theorem 4 ([1])**.** *An almost* 3*-contact metric manifold* $(M,\varphi_i,\xi_i,\eta_i,g)$ *is canonical, with Reeb Killing function* β*, if and only if it admits a metric connection* ∇ *with skew torsion such that*

$$\nabla_X\varphi_i \;=\; \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k)$$

for every vector field X on M and for every even permutation (i,j,k) *of* $(1,2,3)$*. If such a connection* ∇ *exists, it is unique and its torsion is given by*

$$\begin{aligned} T(X,Y,Z) &= N_{\varphi_i}(X,Y,Z) - d\Phi_i(\varphi_i X,\varphi_i Y,\varphi_i Z), \\ T(X,Y,\xi_i) &= d\eta_i(X,Y), \\ T(X,\xi_i,\xi_j) &= -g([\xi_i,\xi_j],X), \\ T(\xi_1,\xi_2,\xi_3) &= 2(\beta-\delta), \end{aligned}$$

for every $X,Y,Z \in \Gamma(\mathcal{H})$*, and* $i,j = 1,2,3$*, and where* δ *is the Reeb commutator function.*

The connection ∇ is called the *canonical connection* of M, and also satisfies

$$\nabla_X\xi_i \;=\; \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k), \qquad \nabla_X\eta_i \;=\; \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k) \tag{20}$$

for every vector field X on M. Therefore, when $\beta = 0$ the canonical connection parallelizes all the structure tensor fields, in which case we call the almost 3-contact metric manifold *parallel*.

Both 3-(α, δ)-Sasaki manifolds and 3-δ-cosymplectic manifolds turn out to be canonical. In particular,

Theorem 5 ([1]). *Every* 3-(α, δ)*-Sasaki manifold is a canonical almost* 3*-contact metric manifold, with constant Reeb Killing function* $\beta = 2(\delta - 2\alpha)$*. The torsion* T *of the canonical connection* ∇ *is given by*

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta - \alpha)\, \eta_{123} = 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} + 2(\delta - 4\alpha)\, \eta_{123}$$

and satisfies $\nabla T = 0$*.*

We denote by η_{123} the 3-form $\eta_1 \wedge \eta_2 \wedge \eta_3$. From the above theorem, it follows that any 3-(α, δ)-Sasaki manifold is a parallel canonical manifold if and only if $\delta = 2\alpha$, in which case the 3-(α, δ)-Sasaki structure is positive ($\alpha\delta > 0$).

Regarding 3-δ-cosymplectic manifolds, we have:

Proposition 5 ([1]). *Any* 3-δ*-cosymplectic manifold is a parallel canonical almost* 3*-contact metric manifold. The torsion of the canonical connection is given by*

$$T = -2\delta\, \eta_{123}.$$

For the class of 3-$(0, \delta)$-Sasaki manifolds, we have the following

Proposition 6. *Every* 3-$(0, \delta)$*-Sasaki manifold is a canonical almost* 3*-contact metric manifold, with constant Reeb Killing function* $\beta = 2\delta$*. The torsion* T *of the canonical connection* ∇ *is given by*

$$T = 2\delta\, \eta_{123},$$

which satisfies $\nabla T = 0$*.*

Proof. Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-$(0, \delta)$-Sasaki manifold. We showed that the structure is hypernormal and the Reeb vector fields are Killing. Furthermore, by the second equation in (12), $d\Phi_i(X, Y, Z) = 0$ for every $X, Y, Z \in \Gamma(\mathcal{H})$. Therefore, conditions (i), (ii) and (iii) in Definition 4 are easily verified. As regards condition (iv), applying the first equation in (4) and Corollary 1, for every $X, Y \in \Gamma(\mathcal{H})$ we have

$$A_{ii}(X, Y) = 0, \qquad A_{ij}(X, Y) = -A_{ji}(X, Y) = 2\delta\Phi_k(X, Y).$$

Hence, the structure is canonical with Reeb commutator function $\beta = 2\delta$. Now, by Theorem 4, taking also into account the fact that the vertical distribution is integrable, the only non-vanishing term of the canonical connection is $T(\xi_1, \xi_2, \xi_3) = 2\delta$, so that $T = 2\delta\, \eta_{123}$. Although the structure is not parallel when $\delta \neq 0$, the torsion satisfies $\nabla T = 0$, as by (20), the 3-form η_{123} is parallel with respect to ∇. □

The above result generalizes the result obtained in [2] for the Lie group described in Example 1 (see also Remark 2).

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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