axioms

Article

Global Stability of a Lotka-Volterra Competition-Diffusion-Advection System with Different Positive Diffusion Distributions

Lili Chen, Shilei Lin and Yanfeng Zhao *

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China; cll2119@sdust.edu.cm (L.C.); linshilei2021@126.com (S.L.)
* Correspondence: zhaoyf1101@sdust.edu.cm

Abstract: In this paper, the problem of a Lotka–Volterra competition–diffusion–advection system between two competing biological organisms in a spatially heterogeneous environments is investigated. When two biological organisms are competing for different fundamental resources, and their advection and diffusion strategies follow different positive diffusion distributions, the functions of specific competition ability are variable. By virtue of the Lyapunov functional method, we discuss the global stability of a non-homogeneous steady-state. Furthermore, the global stability result is also obtained when one of the two organisms has no diffusion ability and is not affected by advection.

Keywords: competition-diffusion-advection; steady-state solution; spatially heterogeneous; global stability

MSC: 35K51; 35B09; 35B35; 92D25

Citation: Chen, L.; Lin, S.; Zhao, Y. Global Stability of a Lotka-Volterra Competition-Diffusion-Advection System with Different Positive Diffusion Distributions. *Axioms* **2021**, *10*, 166. https://doi.org/10.3390/axioms10030166

Academic Editor: Wei-Shih Du

Received: 22 June 2021
Accepted: 26 July 2021
Published: 28 July 2021

1. Introduction

For researchers from the fields of biology and mathematics, advancing the exploration of dynamic systems is a long-term challenge (see [1–3]). The competitive system of two diffusive organisms is often used to simulate population dynamics in biomathematics; for an example, see [1,2,4]. The key to spatial heterogeneity has been discussed in a lot of work, such as [2,5] and its references. In 2020, by proposing a new Lyapunov functional, Ni et al. [6] first studied and proved the global stability of a diffusive, competitive two-organism system, and then extended it to multiple organisms.

Since various methods in the reaction–diffusion–convection system cannot continue to work well, the global dynamics is far from being fully understood. In competitive diffusion advection systems, some progress has been made in [7–11]. Li et al. introduced the weighted Lyapunov functional related to the advection term to study global stability results in 2020 (see [12]), and studied the stability and bifurcation analysis of the model with the time delay term in 2021 (see [11]). Similarly, in 2021, Ma et al. described the overlapping characteristics of bifurcation solutions and studied the influence of advection on the stability of bifurcation solutions. Their results showed that the advection term may change its stability (see [13]). In 2021, Zhou et al. studied the global dynamics of a parabolic system using the competition coefficient (see [14]).

Motivated by the efforts of the aforementioned papers, we will investigate the global stability of a non-homogeneous steady-state solution of a Lotka–Volterra model between two organisms in heterogeneous environments, where two competing organisms have different intrinsic growth rates, advection and diffusion strategies, and follow different positive diffusion distributions.

Hence, we discuss the following advection system:

$$
\begin{cases}
U_t = \nabla\cdot[\mu_1(x)\nabla(\frac{U}{\rho_1(x)}) - R_1(x)\frac{U}{\rho_1(x)}\nabla B_1(x)] + U[\lambda_1(x) - \varpi_{11}(x)U - \varpi_{12}(x)V], & \\
 & in\ \Omega\times\mathbb{R}^+, \\
V_t = \nabla\cdot[\mu_2(x)\nabla(\frac{V}{\rho_2(x)}) - R_2(x)\frac{V}{\rho_2(x)}\nabla B_2(x)] + V[\lambda_2(x) - \varpi_{21}(x)U - \varpi_{22}(x)V], & \\
 & in\ \Omega\times\mathbb{R}^+, \\
\mu_1(x)\frac{\partial}{\partial n}(\frac{U}{\rho_1}) - R_1(x)\frac{U}{\rho_1}\frac{\partial B_1(x)}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\
\mu_2(x)\frac{\partial}{\partial n}(\frac{V}{\rho_2}) - R_2(x)\frac{V}{\rho_2}\frac{\partial B_2(x)}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\
U(x,0) = U_0(x) \geq, \not\equiv 0, V(x,0) = V_0(x) \geq, \not\equiv 0, & in\ \Omega,
\end{cases} \tag{1}
$$

Here, $U(x,t)$ and $V(x,t)$ are the population densities of biological organisms, location $x \in \Omega$, time $t > 0$, which are supposed to be nonnegative. $\mu_1(x), \mu_2(x) > 0$ correspond to the dispersal rates of two competing biological organisms, respectively. $R_1(x), R_2(x) > 0$ correspond to the advection rates of two competing biological organisms, and $B_1(x), B_2(x) \in C^2(\overline{\Omega})$ are the nonconstant functions and represent the advective directions. Two bounded functions $\lambda_1(x)$ and $\lambda_2(x)$ are the intrinsic growth rates of competing organisms , $\rho_1(x)$, $\rho_2(x) \in C^2(\overline{\Omega})$ are two positive diffusion distributions, respectively. $\varpi_{ij}(x) > 0$, $i = 1,2, j = 1,2$ show the strength of competition ability. The spatial habitat $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 \leq N \in \mathbb{Z}$; n denotes the outward unit normal vector on the boundary $\partial\Omega$. No one can enter or leave the habitat boundary.

The following are our basic assumptions:

Hypothesis 1. $0 < \mu_i(x), R_i(x) \in C^{1+\varrho}(\overline{\Omega})$, $0 < \lambda_i(x), \varpi_{ij}(x) \in C^{\varrho}(\overline{\Omega})$, $\varrho \in (0,1)$.

Hypothesis 2. $\frac{\mu_1(x)}{R_1(x)} =: c_1 > 0$, $\frac{\mu_2(x)}{R_2(x)} =: c_2 > 0$, $x \in \overline{\Omega}$, *where* c_1 *and* c_2 *are constants.*

To simplify the calculation, by letting $u = e^{-c_1B_1(x)}\frac{U}{\rho_1(x)}$, $v = e^{-c_2B_2(x)}\frac{V}{\rho_2(x)}$, the system (1) converts into the following coupled system

$$
\begin{cases}
u_t = \dfrac{e^{-c_1B_1(x)}}{\rho_1(x)}\nabla[\mu_1(x)e^{c_1B_1(x)}\nabla u] + u[\lambda_1(x) - \varpi_{11}(x)ue^{c_1B_1(x)}\rho_1(x) & \\
\qquad -\varpi_{12}(x)ve^{c_2B_2(x)}\rho_2(x)], & in\ \Omega\times\mathbb{R}^+, \\
v_t = \dfrac{e^{-c_2B_2(x)}}{\rho_2(x)}\nabla[\mu_2(x)e^{c_2B_2(x)}\nabla v] + v[\lambda_2(x) - \varpi_{21}(x)ue^{c_1B_1(x)}\rho_1(x) & \\
\qquad -\varpi_{22}(x)ve^{c_2B_2(x)}\rho_2(x)], & in\ \Omega\times\mathbb{R}^+, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\
u(x,0) = e^{-c_1B_1(x)}\frac{U_0(x)}{\rho_1(x)} \geq, \not\equiv 0, v(x,0) = e^{-c_2B_2(x)}\frac{V_0(x)}{\rho_2(x)} \geq, \not\equiv 0, & in\ \Omega,
\end{cases} \tag{2}
$$

when $c_1 = c_2 = 0, \rho_1(x) = \rho_2(x) = 1$, the model (2) has been studied in Ni et al. [6]. $c_1 = c_2, B_1(x) = B_2(x), \rho_1(x) = \rho_2(x) = 1$, the model (2) has been studied in Li et al. [12].

The rest of this article is arranged as follows. In Section 2, we carry out some preparatory work and give four lemmas, where some related properties of the system (1) are deduced from the properties of a single organism model (4). Using the Lyapunov functional method, we will provide and prove our main results in Section 3. In Section 4, one example is given to explain our conclusions.

2. Preliminaries

In order to describe our main results, we present the following uniform estimates for the parabolic equation:

$$\begin{cases} w_t = \varpi_{ij}(x)D_{ij}w + \beta_j(x)D_jw + \lambda(x)w + H(x,t,w), & in\ \Omega \times \mathbb{R}^+, \\ \frac{\partial w}{\partial n} = 0, & on\ \partial\Omega \times \mathbb{R}^+, \\ w(x,0) = w_0(x) \geq, \not\equiv 0, & in\ \Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is bounded and $\partial\Omega \in C^{2+\varrho}(\varrho \in (0,1))$ is a smooth boundary. The initial condition $w_0(x) \in W^{2,p}(\Omega), p > 1 + \frac{N}{2}$.

Setting the following assumptions:

(A_1) Let $\varpi_{ij}, \beta_j, \lambda \in C(\overline{\Omega}), \chi_1, \chi_2 > 0$, such that

$$\chi_1|y|^2 \leq \sum_{1 \leq i,j \leq N} \varpi_{ij}(x)y_iy_j \leq \chi_2|y|^2, |\beta_j(x)|, |\lambda(x)| \leq \chi_2, for\ all\ x \in \Omega, y \in R^N.$$

(A_2) Let $\Lambda > 0$ be a constant, such that

$$\|\varpi_{ij}\|_{C^{\varrho}(\overline{\Omega})}, \|\beta_j\|_{C^{\varrho}(\overline{\Omega})}, \|\lambda\|_{C^{\varrho}(\overline{\Omega})} \leq \Lambda.$$

(A_3) $H \in L^{\infty}(\Omega \times [0,\infty) \times [\tau_1, \tau_2])$ for some $\tau_1 < \tau_2$ and there is $\Lambda(\tau_1, \tau_2) > 0$ such that

$$|H(x,t,w_1) – H(x,t,w_2)| \leq \Lambda(\tau_1,\tau_2)|w_1 - w_2|, for\ all\ (x,t) \in \Omega \times [0,\infty), w_1, w_2 \in [\tau_1, \tau_2],$$

and there exists $\Lambda > 0$, satisfying

$$|H(x_1,t_1,w) - H(x_2,t_2,w)| \leq \Lambda(|x_1 - x_2|^{\varrho} + |t_1 - t_2|^{\frac{\varrho}{2}})\ for\ all (x_1,t_1), (x_2,t_2) \in \overline{\Omega} \times [d, d+3], u \in [\tau_1, \tau_2], d \geq 0.$$

The following lemma (see [15,16]) is the boundedness result of the solution $w(x,t)$ in (3).

Lemma 1. *Let $w(x,t)$ be a solution of (3) with $\tau_1 < w < \tau_2, \tau_1, \tau_2 \in \mathbb{R}$. Suppose that $f, \varpi_{ij}, \beta_j, \lambda$ satisfy the assumptions $(A_1) - (A_3)$, then for any $\kappa \geq 1$, there is a constant $\Lambda(\kappa) > 0$ such that*

$$\max_{x \in \overline{\Omega}} \|w_t(x,\cdot)\|_{C^{\frac{\varrho}{2}}([\kappa,+\infty))} + \max_{t \geq \kappa} \|w_t(\cdot,t)\|_{C(\overline{\Omega})} + \max_{t \geq \kappa} \|w(\cdot,t)\|_{C^{2+\varrho}(\overline{\Omega})} \leq \Lambda(\kappa).$$

In the proof of global stability, the following calculus theory and integral inequality are very important. For details, see [6,17].

Lemma 2 ([17])**.** *Let $\beta, \lambda > 0$ be constants, $\varphi(t) \geq 0$ in $[\beta, \infty)$. Assume that $\phi \in C^1([\beta,\infty))$ has lower bound, $\phi'(t) \leq -\lambda\varphi(t)$ in $[\beta,\infty)$. If one of the following alternatives holds:*

- *$\varphi \in C^1([\beta,\infty))$ and $\varphi'(t) \leq P$ in $[\beta,\infty)$ for $P > 0$,*
- *$\varphi \in C^{\varrho}([\beta,\infty))$ and $\|\varphi\|_{C^{\varrho}([\beta,\infty))} \leq P$ for $0 < m < 1$ and $P > 0$,*

where P and m are constants, then $\lim_{t \to \infty} \varphi(t) = 0$.

Lemma 3 ([6])**.** *Let $\alpha, \alpha^* \in C^2(\overline{\Omega})$ with $\alpha, \alpha^* > 0$ and $m \in C^1(\overline{\Omega})$, $b \in C^2(\overline{\Omega})$ with $m, b \geq 0$, α, α^*, m, b are functions. If the following conditions holds:*

- *$q \geq 1$ is a constant, the function $h \in C^{0,1}(\partial\Omega \times [0,\infty))$, $x \in \partial\Omega$, $\frac{h(x,K)}{K}$ is a non-increasing function for $K \in [0,\infty)$,*
- *$\frac{\partial(b(x)\alpha)}{\partial\nu} = h(x,\alpha)$, $\frac{\partial(b(x)\alpha^*)}{\partial\nu} = h(x,\alpha^*)$ on $\partial\Omega$,*

then

$$\begin{aligned}&\int_\Omega \frac{b(x)\alpha^*[\alpha^q-\alpha^{*q}]}{\alpha^q}\left(\nabla\{m(x)\nabla[b(x)\alpha]\}-\frac{\alpha}{\alpha^*}\nabla\{m(x)\nabla[b(x)\alpha^*]\}\right)\mathrm{d}x\\ \le& -\int_\Omega qmb^2\alpha^2\left(\frac{\alpha^*}{\alpha}\right)^{q-1}\left|\nabla\frac{\alpha^*}{\alpha}\right|^2\mathrm{d}x\le 0.\end{aligned} \tag{4}$$

Next, we consider the following scalar evolution eqution

$$\begin{cases} u_t=\dfrac{e^{-cB(x)}}{\rho(x)}\nabla[\mu(x)e^{cB(x)}\nabla u]+u[\lambda(x)-\varpi(x)ue^{cB(x)}\rho(x)], & in\ \Omega\times\mathbb{R}^+,\\ \frac{\partial u}{\partial n}=0, & on\ \partial\Omega\times\mathbb{R}^+,\\ u(x,0)=e^{-cB(x)}\frac{U_0(x)}{\rho(x)}\ge,\not\equiv 0, & in\ \Omega,\end{cases} \tag{5}$$

where $\mu(x), c, \varpi(x), \lambda(x)$ satisty

$$0<\mu(x), R(x)\in C^{1+\varrho}(\overline{\Omega}), 0<\lambda(x),\varpi(x)\in C^{\varrho}(\overline{\Omega}), \varrho\in(0,1), \frac{\mu(x)}{R(x)}=c, \textit{where } c \textit{ is a constant}. \tag{6}$$

Now we see the following useful lemma.

Lemma 4 ([1])**.** *Assume that* $0<\mu(x),\lambda(x),\rho(x),\varpi(x)$ *on* $\overline{\Omega}$*, then the elliptic problem:*

$$\begin{cases} \dfrac{e^{-cB(x)}}{\rho(x)}\nabla[\mu(x)e^{cB(x)}\nabla u]+u[\lambda(x)-\varpi(x)ue^{cB(x)}\rho(x)]=0, & in\ \Omega,\\ \frac{\partial u}{\partial n}=0, & on\ \partial\Omega,\end{cases} \tag{7}$$

has a unique positive solution, denoted by u_θ*.*

3. Main Results

In this section, firstly, by utilizing the Lyapunov function method, the global stability of the model (5) is obtained, and we can see that the non-constant steady-state for (5) is equivalent to the solution u_θ of (7).

Theorem 1. *Assume that* $u_0(x)\gneqq 0$*. If* $\mu,\rho,c,\lambda,\varpi$ *satisfy* (6)*, then Equation (5) has a unique solution* $u(x,t)>0$ *with* $\lim\limits_{t\to\infty}u(x,t)=u_\theta$ *in* $C^2(\Omega)$*.*

Proof. According to the upper–lower solutions method [1,18], we obtain (5) with a unique solution $u(x,t)>0$. Let M be a upper solution of (5), we have $0<u(x,t)<M$, $(x,t)\in\overline{\Omega}\times(0,\infty)$.

By applying Lemma 1, we can obtain that there exists a constant $\Lambda>0$ such that

$$\max_{t\ge1}\|u_t(\cdot,t)\|_{C(\overline{\Omega})}+\max_{t\ge1}\|u(\cdot,t)\|_{C^{2+\varrho}(\overline{\Omega})}\le\Lambda. \tag{8}$$

Then, define a function $\Phi:[0,\infty)\to\mathbb{R}$ by

$$\Phi(t)=\int_\Omega \rho u_\theta e^{cB}\left(u-u_\theta-u_\theta\ln\frac{u}{u_\theta}\right)\mathrm{d}x. \tag{9}$$

Then, $\Phi(t) \geq 0$, $t \geq 0$. By (2) and (4), we have

$$\begin{aligned}\Phi'(t) &= \int_\Omega \rho u_\theta e^{cB}(1-\frac{u_\theta}{u})u_t\,\mathrm{d}x \\ &= \int_\Omega \rho u_\theta e^{cB}(1-\frac{u_\theta}{u})[\frac{e^{-cB}}{\rho}\nabla(\mu e^{cB}\nabla u) + u(\lambda - \varpi u e^{cB}\rho)]\,\mathrm{d}x \\ &= \int_\Omega \rho u_\theta e^{cB}(1-\frac{u_\theta}{u})[\frac{e^{-cB}}{\rho}\nabla(\mu e^{cB}\nabla u) - \frac{ue^{-cB}}{u_\theta\rho}\nabla(\mu e^{cB}\nabla u_\theta)]\,\mathrm{d}x \\ &\quad + \int_\Omega \rho u_\theta e^{cB}(1-\frac{u_\theta}{u})[u(\lambda - \varpi u e^{cB}\rho) - \frac{u}{u_\theta}u_\theta(\lambda - \varpi u_\theta e^{cB}\rho)]\,\mathrm{d}x \\ &\leq -\int_\Omega \mu e^{cB}u^2|\nabla\frac{u_\theta}{u}|^2\,\mathrm{d}x - \int_\Omega \rho^2 u_\theta e^{2cB}\varpi(u-u_\theta)^2\,\mathrm{d}x.\end{aligned} \tag{10}$$

We get

$$\Phi'(t) \leq -\int_\Omega \rho^2 u_\theta e^{2cB}\varpi(u-u_\theta)^2\,\mathrm{d}x =: -\varphi(t) \leq 0. \tag{11}$$

By virtue of (8), we get $|\varphi'(t)| \leq \Lambda$ in $[1,\infty)$ for some $\Lambda > 0$. From Lemma 2, it follows that

$$\lim_{t\to\infty}\varphi(t) = \lim_{t\to\infty}\int_\Omega \rho^2 u_\theta e^{2cB}\varpi(u-u_\theta)^2\,\mathrm{d}x = 0. \tag{12}$$

Applying (8) again, $\{u(\cdot,t) : t \geq 1\}$ is relatively compact in $C^2(\overline{\Omega})$. It can be found that there exists some function $u_\infty(x) \in C^2(\overline{\Omega})$ such that

$$\|u(\cdot,t_s) - u_\infty\|_{C^2(\overline{\Omega})} \to 0 \textit{ as } t_s \to \infty. \tag{13}$$

Combining with (12), we get $u_\infty(x) = u_\theta(x)$ where $x \in \Omega$. Hence, we deduce

$$\lim_{t\to\infty} u(x,t) = u_\theta(x) \textit{ in } C^2(\overline{\Omega}).$$

□

In addition, taking advantage of Lyapunov function method, the global stability results of (2) are obtained.

Theorem 2. *Suppose that $u_0(x), v_0(x) \geq, \not\equiv 0$, (H_1) and (H_2) hold, the system (2) admits a non-homogeneous steady-state $(\widetilde{u}_\theta(x), \widetilde{v}_\theta(x)) > 0$ and there exists*

$$\eta_1 > 0, \eta_2 > 0 \textit{ such that } \eta_1 \leq \frac{\widetilde{u}_\theta(x)}{\widetilde{v}_\theta(x)} \leq \eta_2,\ x \in \overline{\Omega}. \tag{14}$$

Suppose that

$$\sqrt{\frac{\eta_2}{\eta_1}} < \min_{\overline{\Omega}} \frac{\varpi_{11}\varpi_{22}}{\varpi_{12}\varpi_{21}}. \tag{15}$$

Then, the system (2) admits a solution $(u(x,t), v(x,t))$ that satisfies

$$\lim_{t\to\infty} u(x,t) = \widetilde{u}_\theta(x), \lim_{t\to\infty} v(x,t) = \widetilde{v}_\theta(x) \textit{ in } C^2(\overline{\Omega}).$$

Proof. Assume that the inequality (15) holds, let $\Phi : [0,+\infty) \to \mathbb{R}$ defined by

$$\Phi(t) = \int_\Omega \rho_1\widetilde{u}_\theta e^{c_1B_1}(u - \widetilde{u}_\theta - \widetilde{u}_\theta \ln\frac{u}{\widetilde{u}_\theta})\,\mathrm{d}x + \int_\Omega \xi\rho_2\widetilde{v}_\theta e^{c_2B_2}(v - \widetilde{v}_\theta - \widetilde{v}_\theta \ln\frac{v}{\widetilde{v}_\theta})\,\mathrm{d}x, \tag{16}$$

where $0 < \xi(x) := \dfrac{\varpi_{12}\sqrt{\eta_1\eta_2}}{\varpi_{21}}$. Clearly, $\Phi(t) \geq 0$. By (2) and (4), we have

$$\begin{aligned}
\Phi'(t) =& \int_\Omega [\rho_1 \widetilde{u}_\theta e^{c_1B_1}(1-\frac{\widetilde{u}_\theta}{u})u_t + \xi\rho_2\widetilde{v}_\theta e^{c_2B_2}(1-\frac{\widetilde{v}_\theta}{v})v_t]\,\mathrm{d}x \\
=& \int_\Omega \rho_1\widetilde{u}_\theta e^{c_1B_1}(1-\frac{\widetilde{u}_\theta}{u})[\frac{e^{-c_1B_1}}{\rho_1}\nabla(\mu_1 e^{c_1B_1}\nabla u) + u(\lambda_1 - \varpi_{11}ue^{c_1B_1}\rho_1 - \varpi_{12}ve^{c_2B_2}\rho_2)]\,\mathrm{d}x \\
&+ \int_\Omega \xi\rho_2\widetilde{v}_\theta e^{c_2B_2}(1-\frac{\widetilde{v}_\theta}{v})[\frac{e^{-c_2B_2}}{\rho_2}\nabla(\mu_2 e^{c_2B_2}\nabla v) + v(\lambda_2 - \varpi_{21}ue^{c_1B_1}\rho_1 - \varpi_{22}ve^{c_2B_2}\rho_2)]\,\mathrm{d}x \\
=& \int_\Omega \rho_1\widetilde{u}_\theta e^{c_1B_1}(1-\frac{\widetilde{u}_\theta}{u})[\frac{e^{-c_1B_1}}{\rho_1}\nabla(\mu_1 e^{c_1B_1}\nabla u) - \frac{ue^{-c_1B_1}}{\widetilde{u}_\theta\rho_1}\nabla(\mu_1 e^{c_1B_1}\nabla\widetilde{u}_\theta)]\,\mathrm{d}x \\
&+ \int_\Omega \xi\rho_2\widetilde{v}_\theta e^{c_2B_2}(1-\frac{\widetilde{v}_\theta}{v})[\frac{e^{-c_2B_2}}{\rho_2}\nabla(\mu_2 e^{c_2B_2}\nabla v) - \frac{ve^{-c_2B_2}}{\widetilde{v}_\theta\rho_2}\nabla(\mu_2 e^{c_2B_2}\nabla\widetilde{v}_\theta)]\,\mathrm{d}x \\
&+ \int_\Omega \rho_1\widetilde{u}_\theta e^{c_1B_1}(1-\frac{\widetilde{u}_\theta}{u})u(\lambda_1 - \varpi_{11}ue^{c_1B_1}\rho_1 - \varpi_{12}ve^{c_2B_2}\rho_2)\,\mathrm{d}x \\
&- \int_\Omega \rho_1\widetilde{u}_\theta e^{c_1B_1}(1-\frac{\widetilde{u}_\theta}{u})\frac{u}{\widetilde{u}_\theta}\widetilde{u}_\theta(\lambda_1 - \varpi_{11}\widetilde{u}_\theta e^{c_1B_1}\rho_1 - \varpi_{12}\widetilde{v}_\theta e^{c_2B_2}\rho_2)]\,\mathrm{d}x \\
&+ \int_\Omega \xi\rho_2\widetilde{v}_\theta e^{c_2B_2}(1-\frac{\widetilde{v}_\theta}{v})v(\lambda_2 - \varpi_{21}ue^{c_1B_1}\rho_1 - \varpi_{22}ve^{c_2B_2}\rho_2)\,\mathrm{d}x \\
&- \int_\Omega \xi\rho_2\widetilde{v}_\theta e^{c_2B_2}(1-\frac{\widetilde{v}_\theta}{v})\frac{v}{\widetilde{v}_\theta}\widetilde{v}_\theta(\lambda_2 - \varpi_{21}\widetilde{u}_\theta e^{c_1B_1}\rho_1 - \varpi_{22}\widetilde{v}_\theta e^{c_2B_2}\rho_2)\,\mathrm{d}x \\
\le& -\int_\Omega \mu_1 e^{c_1B_1}u^2|\nabla\frac{\widetilde{u}_\theta}{u}|^2\,\mathrm{d}x - \int_\Omega \mu_2 e^{c_2B_2}v^2|\nabla\frac{\widetilde{v}_\theta}{v}|^2\,\mathrm{d}x - \int_\Omega \rho_1^2\widetilde{u}_\theta e^{2c_1B_1}\varpi_{11}(u-\widetilde{u}_\theta)^2\,\mathrm{d}x \\
&- \int_\Omega \rho_1\rho_2 e^{c_1B_1+c_2B_2}(\varpi_{12}\widetilde{u}_\theta + \xi\varpi_{21}\widetilde{v}_\theta)(u-\widetilde{u}_\theta)(v-\widetilde{v}_\theta)\,\mathrm{d}x \\
&- \int_\Omega \xi\rho_2^2\widetilde{v}_\theta e^{2c_2B_2}\varpi_{22}(v-\widetilde{v}_\theta)^2\,\mathrm{d}x.
\end{aligned} \tag{17}$$

Note that (14) and (15) give rise to

$$\begin{aligned}
&2\sqrt{\rho_1^2\widetilde{u}_\theta e^{2c_1B_1}\varpi_{11}\xi\rho_2^2\widetilde{v}_\theta e^{2c_2B_2}\varpi_{22}} - \rho_1\rho_2 e^{c_1B_1+c_2B_2}(\varpi_{12}\widetilde{u}_\theta + \xi\varpi_{21}\widetilde{v}_\theta) \\
=&2\rho_1\rho_2 e^{c_1B_1+c_2B_2}\sqrt{\xi\widetilde{u}_\theta\widetilde{v}_\theta\varpi_{11}\varpi_{22}} - \rho_1\rho_2 e^{c_1B_1+c_2B_2}(\varpi_{12}\widetilde{u}_\theta + \xi\varpi_{21}\widetilde{v}_\theta) \\
=&\rho_1\rho_2 e^{c_1B_1+c_2B_2}(2\sqrt{\xi\widetilde{u}_\theta\widetilde{v}_\theta}\sqrt{\varpi_{11}\varpi_{22}} - \sqrt{\xi\widetilde{u}_\theta\widetilde{v}_\theta}(\varpi_{12}\sqrt{\frac{\widetilde{u}_\theta}{\xi\widetilde{v}_\theta}} + \varpi_{21}\sqrt{\frac{\xi\widetilde{v}_\theta}{\widetilde{u}_\theta}})) \\
=&\rho_1\rho_2 e^{c_1B_1+c_2B_2}\sqrt{\xi\widetilde{u}_\theta\widetilde{v}_\theta}(2\sqrt{\varpi_{11}\varpi_{22}} - (\varpi_{12}\sqrt{\frac{\eta_2}{\xi}} + \varpi_{21}\sqrt{\frac{\xi}{\eta_1}})) \\
\ge&\rho_1\rho_2 e^{c_1B_1+c_2B_2}\sqrt{\xi\widetilde{u}_\theta\widetilde{v}_\theta}(2\sqrt{\varpi_{11}\varpi_{22}} - 2\sqrt{\varpi_{12}\varpi_{21}\sqrt{\frac{\eta_2}{\eta_1}}}) \\
>&0.
\end{aligned}$$

Choosing $0 < \varepsilon \ll 1$, we have

$$2\sqrt{\rho_1^2\widetilde{u}_\theta e^{2c_1B_1}(\varpi_{11}-\varepsilon)\xi\rho_2^2\widetilde{v}_\theta e^{2c_2B_2}(\varpi_{22}-\varepsilon)} - \rho_1\rho_2 e^{c_1B_1+c_2B_2}(\varpi_{12}\widetilde{u}_\theta + \xi\varpi_{21}\widetilde{v}_\theta) > 0.$$

Combining with (17), we can deduce

$$\Phi'(t) \le -\int_\Omega [\rho_1^2\widetilde{u}_\theta e^{2c_1B_1}\varepsilon(u-\widetilde{u}_\theta)^2 + \xi\rho_2^2\widetilde{v}_\theta e^{2c_2B_2}\varepsilon(v-\widetilde{v}_\theta)^2]\,\mathrm{d}x =: -\varphi(t) \le 0.$$

From (13), it follows that

$$\lim_{t\to\infty} u(x,t) = \widetilde{u}_\theta(x),\ \lim_{t\to\infty} v(x,t) = \widetilde{v}_\theta(x)\ in\ C^2(\overline{\Omega}).$$

□

Finally, we consider that if one of the two organisms has no diffusion ability and is not affected by advection, the Lyapunov function method can also deduce the following global stability results in (2).

Theorem 3. *If $u_0, v_0 \in C(\overline{\Omega})$ satisfy $u_0(x) \geq, \not\equiv 0$ and $v_0(x) > 0$ on $\overline{\Omega}$. Let $\frac{\mu_1(x)}{R_1(x)} =: c_1, \mu_2(x) = R_2(x) = 0$ for $x \in \overline{\Omega}$, and*

$$\varpi_{12}(x)\varpi_{21}(x) < \varpi_{11}(x)\varpi_{22}(x), x \in \overline{\Omega}. \tag{18}$$

(i) If

$$\varpi_{22}(x)\lambda_1(x) - \varpi_{12}(x)\lambda_2(x) > 0, \forall x \in \overline{\Omega}, \tag{19}$$

and

$$\min_{\overline{\Omega}} \frac{\lambda_2(x)}{\rho_1(x)\varpi_{21}(x)e^{c_1 B_1(x)}} > \max_{\overline{\Omega}} \frac{\varpi_{22}(x)\lambda_1(x) - \varpi_{12}(x)\lambda_2(x)}{\rho_1(x)e^{c_1 B_1(x)}(\varpi_{11}(x)\varpi_{22}(x) - \varpi_{12}(x)\varpi_{21}(x))}, \tag{20}$$

then there is a unique non-homogeneous steady-state $(\widetilde{u}_\theta(x), \widetilde{v}_\theta(x)) > 0$ for the model (2) such that

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (\widetilde{u}_\theta(x), \widetilde{v}_\theta(x)) \text{ in } C^1(\overline{\Omega}) \times L^2(\Omega).$$

(ii) If

$$\frac{\lambda_2(x)}{\rho_1(x)\varpi_{21}(x)e^{c_1 B_1(x)}} \leq \widetilde{u}_\theta(x), x \in \overline{\Omega}, \tag{21}$$

then there exists a semi-trivial steady-state $(\widetilde{u}_\theta(x), 0)$ for the model (2) such that

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (\widetilde{u}_\theta(x), 0) \text{ in } C^1(\overline{\Omega}) \times L^2(\overline{\Omega}).$$

(iii) Let

$$\frac{\varpi_{22}(x)}{\varpi_{12}(x)} \leq \frac{\lambda_2(x)}{\lambda_1(x)}, x \in \overline{\Omega}, \tag{22}$$

then the model (2) has a semi-trivial steady-state $(0, \widetilde{v}_\theta(x))$,

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (0, \widetilde{v}_\theta(x)) \text{ in } C^1(\overline{\Omega}) \times L^2(\overline{\Omega}),$$

where $\widetilde{v}_\theta(x) = \dfrac{\lambda_2(x)}{\rho_2(x)\varpi_{22}(x)e^{c_2 B_2(x)}}$.

Proof. (i) When $\mu_2(x) = R_2(x) = 0, x \in \overline{\Omega}$, $(\widetilde{u}_\theta(x), \widetilde{v}_\theta(x))$ of the model (2) satisfies

$$\begin{cases} \dfrac{e^{-c_1 B_1(x)}}{\rho_1(x)} \nabla[\mu_1(x)e^{c_1 B_1(x)}\nabla u] + u[\lambda_1(x) - \frac{\varpi_{12}(x)}{\varpi_{22}(x)}\lambda_2(x) - \rho_1(x)ue^{c_1 B_1(x)} & \\ (\varpi_{11}(x) - \frac{\varpi_{12}(x)\varpi_{21}(x)}{\varpi_{22}(x)})] = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{23}$$

and $\widetilde{v}_\theta = \dfrac{\lambda_2 - \varpi_{21}\rho_1\widetilde{u}_\theta e^{c_1 B_1}}{\varpi_{22}\rho_2 e^{c_2 B_2}}$.

If (18) and (19) hold, we see $\mu_1, \lambda_1 - \frac{\varpi_{12}}{\varpi_{22}}\lambda_2, \rho_1 e^{-c_1 B_1}(\varpi_{11} - \frac{\varpi_{12}\varpi_{21}}{\varpi_{22}}) > 0$, then by Lemma 4, the problem (23) has a unique solution $\widetilde{u}_\theta(x) > 0$. By using the maximum principle in elliptic equation, we infer

$$\widetilde{u}_\theta < \max_{\overline{\Omega}} \frac{\varpi_{22}\lambda_1 - \varpi_{12}\lambda_2}{\rho_1 e^{c_1 B_1}(\varpi_{11}\varpi_{22} - \varpi_{12}\varpi_{21})}.$$

According to (20), we can get $\widetilde{v}_\theta = \dfrac{\lambda_2 - \varpi_{21}\rho_1\widetilde{u}_\theta e^{c_1 B_1}}{\varpi_{22}\rho_2 e^{c_2 B_2}} > 0$, hence there exists a unique steady-state for (2), $(\widetilde{u}_\theta(x), \widetilde{v}_\theta(x)) > 0$.

Let us define a function $\Phi : [0, \infty) \to \mathbb{R}$,

$$\Phi(t) = \int_\Omega \rho_1 \widetilde{u_\theta} e^{c_1 B_1} (u - \widetilde{u_\theta} - \widetilde{u_\theta} \ln \frac{u}{\widetilde{u_\theta}}) \, \mathrm{d}x + \int_\Omega \xi \rho_2 e^{c_2 B_2} (v - \widetilde{v_\theta} - \widetilde{v_\theta} \ln \frac{v}{\widetilde{v_\theta}}) \, \mathrm{d}x,$$

where $\xi(x) = \frac{\varpi_{12}(x)\widetilde{u_\theta}(x)}{\varpi_{21}(x)} > 0$. Clearly, $\Phi(t) \geqslant 0$. From (2) and (4), we get

$$\begin{aligned}
\Phi'(t) &= \int_\Omega [\rho_1 \widetilde{u_\theta} e^{c_1 B_1} (1 - \frac{\widetilde{u_\theta}}{u}) u_t + \xi \rho_2 e^{c_2 B_2} (1 - \frac{\widetilde{v_\theta}}{v}) v_t] \, \mathrm{d}x \\
&= \int_\Omega \rho_1 \widetilde{u_\theta} e^{c_1 B_1} (1 - \frac{\widetilde{u_\theta}}{u}) [\frac{e^{-c_1 B_1}}{\rho_1} \nabla(\mu_1 e^{c_1 B_1} \nabla u) \\
&\quad + u(\lambda_1 - \varpi_{11} u e^{c_1 B_1} \rho_1 - \varpi_{12} v e^{c_2 B_2} \rho_2)] \, \mathrm{d}x \\
&\quad + \int_\Omega \xi \rho_2 \widetilde{v_\theta} e^{c_2 B_2} (1 - \frac{\widetilde{v_\theta}}{v}) [v(\lambda_2 - \varpi_{21} u e^{c_1 B_1} \rho_1 - \varpi_{22} v e^{c_2 B_2} \rho_2)] \, \mathrm{d}x \\
&= \int_\Omega \rho_1 \widetilde{u_\theta} e^{c_1 B_1} (1 - \frac{\widetilde{u_\theta}}{u}) [\frac{e^{-c_1 B_1}}{\rho_1} \nabla(\mu_1 e^{c_1 B_1} \nabla u) - \frac{u e^{-c_1 B_1}}{\widetilde{u_\theta} \rho_1} \nabla(\mu_1 e^{c_1 B_1} \nabla \widetilde{u_\theta})] \, \mathrm{d}x \\
&\quad + \int_\Omega \rho_1 \widetilde{u_\theta} e^{c_1 B_1} (1 - \frac{\widetilde{u_\theta}}{u}) u(\lambda_1 - \varpi_{11} u e^{c_1 B_1} \rho_1 - \varpi_{12} v e^{c_2 B_2} \rho_2) \, \mathrm{d}x \\
&\quad - \int_\Omega \rho_1 \widetilde{u_\theta} e^{c_1 B_1} (1 - \frac{\widetilde{u_\theta}}{u}) \frac{u}{\widetilde{u_\theta}} \widetilde{u_\theta} (\lambda_1 - \varpi_{11} \widetilde{u_\theta} e^{c_1 B_1} \rho_1 - \varpi_{12} \widetilde{v_\theta} e^{c_2 B_2} \rho_2)] \, \mathrm{d}x \\
&\quad + \int_\Omega \xi \rho_2 e^{c_2 B_2} (1 - \frac{\widetilde{v_\theta}}{v}) v(\lambda_2 - \varpi_{21} u e^{c_1 B_1} \rho_1 - \varpi_{22} v e^{c_2 B_2} \rho_2) \, \mathrm{d}x \\
&\quad - \int_\Omega \xi \rho_2 e^{c_2 B_2} (1 - \frac{\widetilde{v_\theta}}{v}) \frac{v}{\widetilde{v_\theta}} \widetilde{v_\theta} (\lambda_2 - \varpi_{21} \widetilde{u_\theta} e^{c_1 B_1} \rho_1 - \varpi_{22} \widetilde{v_\theta} e^{c_2 B_2} \rho_2) \, \mathrm{d}x \\
&\leq - \int_\Omega \mu_1 e^{c_1 B_1} u^2 |\nabla \frac{\widetilde{u_\theta}}{u}|^2 \, \mathrm{d}x - \int_\Omega \rho_1^2 \widetilde{u_\theta} e^{2c_1 B_1} \varpi_{11} (u - \widetilde{u_\theta})^2 \, \mathrm{d}x \\
&\quad - \int_\Omega \rho_1 \rho_2 e^{c_1 B_1 + c_2 B_2} (\varpi_{12} \widetilde{u_\theta} + \xi \varpi_{21})(u - \widetilde{u_\theta})(v - \widetilde{v_\theta}) \, \mathrm{d}x \\
&\quad - \int_\Omega \xi \rho_2^2 e^{2c_2 B_2} \varpi_{22} (v - \widetilde{v_\theta})^2 \, \mathrm{d}x.
\end{aligned} \tag{24}$$

We can choose $0 < \varepsilon \ll 1$ and use (18), such that

$$2\sqrt{\rho_1^2 \widetilde{u_\theta} e^{2c_1 B_1} (\varpi_{11} - \varepsilon) \xi \rho_2^2 e^{2c_2 B_2} (\varpi_{22} - \varepsilon)} - \rho_1 \rho_2 e^{c_1 B_1 + c_2 B_2} (\varpi_{12} \widetilde{u_\theta} + \xi \varpi_{21}) > 0.$$

Combining this with (24), we can deduce

$$\Phi'(t) \leq - \int_\Omega [\rho_1^2 \widetilde{u_\theta} e^{2c_1 B_1} \varepsilon (u - \widetilde{u_\theta})^2 + \xi \rho_2^2 e^{2c_2 B_2} \varepsilon (v - \widetilde{v_\theta})^2] \, \mathrm{d}x =: -\varphi(t) \leq 0.$$

Applying the Lemma 1 and Sobolev embedding theorem, we deduce that u and v are bounded in $\Omega \times [0, \infty)$ and there is a constant $\Lambda > 0$ such that

$$\max_{t \geq 1} \|u(\cdot, t)\|_{C^{1+\varrho}(\overline{\Omega})} \leq \Lambda \; for \; some \; 0 < \varrho < 1.$$

Combining with (2) and $|\varphi'(t)| < \Lambda_1$ in $[1, \infty)$ for some $\Lambda_1 > 0$, and making use of Lemma 2, we get $\lim_{t \to \infty} \varphi(t) = 0$ and we deduce that

$$\lim_{t \to \infty} u(x, t) = \widetilde{u_\theta}(x), \lim_{t \to \infty} v(x, t) = \widetilde{v_\theta}(x) \; in \; L^2(\Omega).$$

Applying Theorem 2, we get $\lim_{t \to \infty} u(x, t) = \widetilde{u_\theta}(x)$ in $C^1(\overline{\Omega})$.

(ii) Let's define a function $\Phi : [0, \infty) \to \mathbb{R}$,

$$\Phi(t) = \int_\Omega \rho_1 \widetilde{u}_\theta e^{c_1 B_1}(u - \widetilde{u}_\theta - \widetilde{u}_\theta \ln \frac{u}{\widetilde{u}_\theta})\,\mathrm{d}x + \int_\Omega \xi \rho_2 e^{c_2 B_2} v\,\mathrm{d}x,$$

where $\xi(x) = \frac{\varpi_{12}(x)\widetilde{u}_\theta(x)}{\varpi_{21}(x)} > 0$. From (4) and (21), we have

$$\begin{aligned}
\Phi'(t) = &\int_\Omega \rho_1 \widetilde{u}_\theta e^{c_1 B_1}(1 - \frac{\widetilde{u}_\theta}{u})[\frac{e^{-c_1 B_1}}{\rho_1}\nabla(\mu_1 e^{c_1 B_1}\nabla u) - \frac{u e^{-c_1 B_1}}{\widetilde{u}_\theta \rho_1}\nabla(\mu_1 e^{c_1 B_1}\nabla \widetilde{u}_\theta)]\,\mathrm{d}x \\
&+ \int_\Omega \rho_1 \widetilde{u}_\theta e^{c_1 B_1}(1 - \frac{\widetilde{u}_\theta}{u})u(\lambda_1 - \varpi_{11} u e^{c_1 B_1}\rho_1 - \varpi_{12} v e^{c_2 B_2}\rho_2)\,\mathrm{d}x \\
&- \int_\Omega \rho_1 \widetilde{u}_\theta e^{c_1 B_1}(1 - \frac{\widetilde{u}_\theta}{u})\frac{u}{\widetilde{u}_\theta}\widetilde{u}_\theta(\lambda_1 - \varpi_{11}\widetilde{u}_\theta e^{c_1 B_1}\rho_1)]\,\mathrm{d}x \\
&+ \int_\Omega \xi \rho_2 e^{c_2 B_2} v(\lambda_2 - \varpi_{21} u e^{c_1 B_1}\rho_1 - \varpi_{22} v e^{c_2 B_2}\rho_2)\,\mathrm{d}x \\
\le &- \int_\Omega \mu_1 e^{c_1 B_1} u^2 |\nabla \frac{\widetilde{u}_\theta}{u}|^2\,\mathrm{d}x \\
&- \int_\Omega \rho_1 \widetilde{u}_\theta e^{c_1 B_1}(u - \widetilde{u}_\theta)(-\varpi_{11} e^{c_1 B_1}\rho_1(u - \widetilde{u}_\theta) - \varpi_{12} v e^{c_2 B_2}\rho_2)\,\mathrm{d}x \\
&+ \int_\Omega \xi \rho_2 e^{c_2 B_2} v[(\lambda_2 - \varpi_{21}\widetilde{u}_\theta e^{c_1 B_1}\rho_1) - \varpi_{21} e^{c_1 B_1}\rho_1(u - \widetilde{u}_\theta) - \varpi_{22} v e^{c_2 B_2}\rho_2]\,\mathrm{d}x \\
\le &- \int_\Omega \rho_1^2 \widetilde{u}_\theta e^{2c_2 B_2}\varpi_{11}(u - \widetilde{u}_\theta)^2\,\mathrm{d}x - \int_\Omega \xi \rho_2^2 e^{2c_2 B_2}\varpi_{22} v^2\,\mathrm{d}x \\
&- \int_\Omega \rho_1 \rho_2 e^{c_1 B_1 + c_2 B_2}(\varpi_{12}\widetilde{u}_\theta + \xi \varpi_{21})(u - \widetilde{u}_\theta)v\,\mathrm{d}x.
\end{aligned}$$

The following discussion will refer to the part (i), then we will not repeat it.

(iii) Clearly, (2) has a semi-trivial steady-state $(0, \dfrac{\lambda_2(x)}{\rho_2(x)\varpi_{22}(x)e^{c_2 B_2(x)}})$. Let us define a function $\Phi : [0, \infty) \to R$,

$$\Phi(t) = \int_\Omega \rho_1 e^{c_1 B_1} u\,\mathrm{d}x + \int_\Omega \xi \rho_2 e^{c_2 B_2}(v - \widetilde{v}_\theta - \widetilde{v}_\theta \ln \frac{v}{\widetilde{v}_\theta})\,\mathrm{d}x,$$

where $\xi(x) = \frac{\varpi_{12}(x)}{\varpi_{21}(x)} > 0$ and $\widetilde{v}_\theta(x) = \dfrac{\lambda_2(x)}{\rho_2(x)\varpi_{22}(x)e^{c_2 B_2(x)}}$. From (22), we have

$$\begin{aligned}
\Phi'(t) = &\int_\Omega \rho_1 e^{c_1 B_1} u(\lambda_1 - \varpi_{11} u e^{c_1 B_1}\rho_1 - \varpi_{12} v e^{c_2 B_2}\rho_2)\,\mathrm{d}x \\
&+ \int_\Omega \xi \rho_2 e^{c_2 B_2} v(\lambda_2 - \varpi_{21} u e^{c_1 B_1}\rho_1 - \varpi_{22} v e^{c_2 B_2}\rho_2)\,\mathrm{d}x \\
&- \int_\Omega \xi \rho_2 e^{c_2 B_2}\frac{v}{\widetilde{v}_\theta}\widetilde{v}_\theta(\lambda_2 - \varpi_{22}\widetilde{v}_\theta e^{c_2 B_2}\rho_2)]\,\mathrm{d}x \\
= &\int_\Omega \rho_1 e^{c_1 B_1} u[(\lambda_1 - \varpi_{12}\widetilde{v}_\theta e^{c_2 B_2}\rho_2) - \varpi_{11} u e^{c_1 B_1}\rho_1 - \varpi_{12} e^{c_2 B_2}\rho_2(v - \widetilde{v}_\theta)]\,\mathrm{d}x \\
&- \int_\Omega \xi \rho_2 e^{2c_2 B_2}(v - \widetilde{v}_\theta)[-\varpi_{21} u e^{c_1 B_1}\rho_1 - \varpi_{22} e^{c_2 B_2}\rho_2(v - \widetilde{v}_\theta)]\,\mathrm{d}x \\
\le &- \int_\Omega \rho_1^2 e^{2c_2 B_2}\varpi_{11} u^2\,\mathrm{d}x - \int_\Omega \xi \rho_2^2 e^{2c_2 B_2}\varpi_{22}(v - \widetilde{v}_\theta)^2\,\mathrm{d}x \\
&- \int_\Omega \rho_1 \rho_2 e^{c_1 B_1 + c_2 B_2}(\varpi_{12} + \xi \varpi_{21})(v - \widetilde{v}_\theta)u\,\mathrm{d}x.
\end{aligned}$$

The following discussion is similar to the part (i), so we omit it. □

4. Example

See the following parabolic problem:

$$\begin{cases} u_t = \dfrac{e^{-cB(x)}}{\rho(x)}\nabla[\mu_1(x)e^{cB(x)}\nabla u] + u[\bar{\lambda}_1\varphi(x)e^{cB(x)}\rho(x) + \varepsilon_1 g_1(x) & \\ \quad -\varpi_{11}\varphi(x)ue^{cB(x)}\rho(x) - \varpi_{12}\varphi(x)ve^{cB(x)}\rho(x)], & in\ \Omega\times\mathbb{R}^+, \\ v_t = \dfrac{e^{-cB(x)}}{\rho(x)}\nabla[\mu_2(x)e^{cB(x)}\nabla v] + v[\bar{\lambda}_2\varphi(x)e^{cB(x)}\rho(x) + \varepsilon_2 g_2(x) & \\ \quad -\varpi_{21}\varphi(x)ue^{cB(x)}\rho(x) - \varpi_{22}\varphi(x)ve^{cB(x)}\rho(x)], & in\ \Omega\times\mathbb{R}^+, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\ u(x,0) = e^{-cB(x)}\frac{U_0(x)}{\rho(x)} \geq, \not\equiv 0, v(x,0) = e^{-cB(x)}\frac{V_0(x)}{\rho(x)} \geq, \not\equiv 0, & in\ \Omega, \end{cases} \tag{25}$$

where $\bar{\lambda}_i, \varpi_{ij}, \varepsilon_i$ are all positive constants, $B, \rho \in C^2(\overline{\Omega}), \mu_i \in C^{1+\varrho}(\overline{\Omega}), \varphi, g_i \in C^{\varrho}(\overline{\Omega})$ and $\varphi(x), \mu_i(x) > 0$ on $\overline{\Omega}$.

Proposition 1. *If $0 \leq \varepsilon_i \ll 1$ and $\frac{\varpi_{21}}{\varpi_{11}} < \frac{\bar{\lambda}_2}{\bar{\lambda}_1} < \frac{\varpi_{22}}{\varpi_{12}}, \dfrac{\varpi_{11}\varpi_{22}}{\varpi_{12}\varpi_{21}} > 1$, then there exists $\eta_1 > 0, \eta_2 > 0$ such that*

$$\frac{\varpi_{11}\varpi_{22}}{\varpi_{12}\varpi_{21}} > \sqrt{\frac{\eta_2}{\eta_1}} \tag{26}$$

and the system (25) admits a positive non-homogeneous steady-state $(\widetilde{u_\theta}(x), \widetilde{v_\theta}(x))$, which satisfies $\eta_1 \leq \frac{\widetilde{u_\theta}(x)}{\widetilde{v_\theta}(x)} \leq \eta_2$.

Proof. The steady-state of (25) satisfies the following elliptic problem

$$\begin{cases} \dfrac{e^{-cB(x)}}{\rho(x)}\nabla[\mu_1(x)e^{cB(x)}\nabla u] + u[\bar{\lambda}_1\varphi(x)e^{cB(x)}\rho(x) + \varepsilon_1 g_1(x) & \\ -\varpi_{11}\varphi(x)ue^{cB(x)}\rho(x) - \varpi_{12}\varphi(x)ve^{cB(x)}\rho(x)] = 0, & in\ \Omega, \\ \dfrac{e^{-cB(x)}}{\rho(x)}\nabla[\mu_2(x)e^{cB(x)}\nabla v] + v[\bar{\lambda}_2\varphi(x)e^{cB(x)}\rho(x) + \varepsilon_2 g_2(x) & \\ -\varpi_{21}\varphi(x)ue^{cB(x)}\rho(x) - \varpi_{22}\varphi(x)ve^{cB(x)}\rho(x)] = 0, & in\ \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & on\ \partial\Omega. \end{cases} \tag{27}$$

Set $\bar{k}_i = \max\limits_{\overline{\Omega}} \dfrac{g_i(x)}{\varphi(x)e^{cB(x)}\rho(x)}, \underline{k}_i = \min\limits_{\overline{\Omega}} \dfrac{g_i(x)}{\varphi(x)e^{cB(x)}\rho(x)}$ for $i = 1,2$. Applying $0 < \varepsilon_i \ll 1$ and $\frac{\varpi_{21}}{\varpi_{11}} < \frac{\bar{\lambda}_2}{\bar{\lambda}_1} < \frac{\varpi_{22}}{\varpi_{12}}$, we have the linear system

$$\begin{cases} \bar{\lambda}_1 + \varepsilon_1\underline{k}_1 - \varpi_{11}\underline{u} - \varpi_{12}\bar{v} = 0, \\ \bar{\lambda}_2 + \varepsilon_2\bar{k}_2 - \varpi_{21}\underline{u} - \varpi_{22}\bar{v} = 0, \\ \bar{\lambda}_1 + \varepsilon_1\bar{k}_1 - \varpi_{11}\bar{u} - \varpi_{12}\underline{v} = 0, \\ \bar{\lambda}_2 + \varepsilon_2\underline{k}_2 - \varpi_{21}\bar{u} - \varpi_{22}\underline{v} = 0. \end{cases}$$

Then

$$\bar{u} = \frac{\varpi_{22}(\bar{\lambda}_1 + \varepsilon_1\bar{k}_1) - \varpi_{12}(\bar{\lambda}_2 + \varepsilon_2\underline{k}_2)}{\varpi_{11}\varpi_{22} - \varpi_{12}\varpi_{21}}, \underline{u} = \frac{\varpi_{22}(\bar{\lambda}_1 + \varepsilon_1\underline{k}_1) - \varpi_{12}(\bar{\lambda}_2 + \varepsilon_2\bar{k}_2)}{\varpi_{11}\varpi_{22} - \varpi_{12}\varpi_{21}},$$

$$\bar{v} = \frac{\varpi_{11}(\bar{\lambda}_2 + \varepsilon_2\bar{k}_2) - \varpi_{21}(\bar{\lambda}_1 + \varepsilon_1\underline{k}_1)}{\varpi_{11}\varpi_{22} - \varpi_{12}\varpi_{21}}, \underline{v} = \frac{\varpi_{11}(\bar{\lambda}_2 + \varepsilon_2\underline{k}_2) - \varpi_{21}(\bar{\lambda}_1 + \varepsilon_1\bar{k}_1)}{\varpi_{11}\varpi_{22} - \varpi_{12}\varpi_{21}}.$$

Hence, the system (25) has a positive non-homogeneous steady-state $(\widetilde{u_\theta}(x), \widetilde{v_\theta}(x))$ and $0 < \underline{u} < \widetilde{u_\theta}(x) < \bar{u}$ and $0 < \underline{v} < \widetilde{v_\theta}(x) < \bar{v}$. Let

$$\eta_1 = \frac{\underline{u}}{\bar{v}}, \eta_2 = \frac{\bar{u}}{\underline{v}}. \tag{28}$$

we have $\eta_1 \leq \frac{\widetilde{u_\theta}(x)}{\widetilde{v_\theta}(x)} \leq \eta_2$. Applying (28), we get $\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\eta_2}{\eta_1} = 1$. Hence, for $0 < \varepsilon_i \ll 1$,

$$\min_{\overline{\Omega}} \frac{\bar{\varpi}_{11}\varphi(x)\bar{\varpi}_{22}\varphi(x)}{\bar{\varpi}_{12}\varphi(x)\bar{\varpi}_{21}\varphi(x)} = \frac{\bar{\varpi}_{11}\bar{\varpi}_{22}}{\bar{\varpi}_{12}\bar{\varpi}_{21}} > \sqrt{\frac{\eta_2}{\eta_1}}.$$

The proof is completed. □

Example 1. *In the above (25), let $c = 2, B(x) = x, \rho(x) = e^{-x}, \mu_1(x) = \mu_2(x) = e^{-x}, R_1(x) = R_2(x) = \frac{1}{2}e^{-x}, \varphi(x) = e^{-x}, g_1(x) = g_2(x) = 1 + \cos(\frac{\pi}{2}x), \bar{\lambda}_1 = 1, \bar{\lambda}_2 = 2, \bar{\varpi}_{11} = \bar{\varpi}_{12} = \bar{\varpi}_{21} = 1, \bar{\varpi}_{22} = 3$, and $\varepsilon_1 = \varepsilon_2 = \frac{1}{3}, x \in \Omega = [0, 10]$. Then the problem (25) becomes the following model*

$$\begin{cases} u_t = e^{-x}\nabla[e^x\nabla u] + u[1 + \frac{1}{3}(1 + \cos(\frac{\pi}{2}x)) - u - v], & in\ \Omega \times \mathbb{R}^+, \\ v_t = e^{-x}\nabla[e^x\nabla v] + v[2 + \frac{1}{3}(1 + \cos(\frac{\pi}{2}x)) - u - 3v], & in\ \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & on\ \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = e^{-x}(2 + \cos(\pi x)) \geq, \not\equiv 0, v(x, 0) = e^{-x}(2 + \cos(\pi x)) \geq, \not\equiv 0, & in\ \Omega, \end{cases} \tag{29}$$

where $u_0(x), v_0(x) \geq, \not\equiv 0$. It is not difficult to verify that (H_1) and (H_2) hold. We can find $\eta_1 = 1 > 0, \eta_2 = \frac{7}{3} > 0$, such that $\eta_1 \leq \frac{\widetilde{u_\theta}(x)}{\widetilde{v_\theta}(x)} \leq \eta_2$ and $\sqrt{\frac{\eta_2}{\eta_1}} < \min_{\overline{\Omega}} \frac{\varpi_{11}\varpi_{22}}{\varpi_{12}\varpi_{21}}$. According to Theorem 2, the model (29) admits a solution $(u(x,t), v(x,t))$ that satisfies

$$\lim_{t\to\infty} u(x,t) = \widetilde{u_\theta}(x), \lim_{t\to\infty} v(x,t) = \widetilde{v_\theta}(x) \ in\ C^2(\overline{\Omega}).$$

Indeed, the steady-state of (29) satisfies the following elliptic problem

$$\begin{cases} e^{-x}\nabla[e^x\nabla u] + u[1 + \frac{1}{3}(1 + \cos(\frac{\pi}{2}x)) - u - v] = 0, & in\ \Omega, \\ e^{-x}\nabla[e^x\nabla v] + v[2 + \frac{1}{3}(1 + \cos(\frac{\pi}{2}x)) - u - 3v] = 0, & in\ \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & on\ \partial\Omega. \end{cases} \tag{30}$$

It is not difficult to see that $\bar{k}_1 = \bar{k}_2 = 2, \underline{k}_1 = \underline{k}_2 = 0$. By calculation, we can obtain

$$\begin{cases} 1 - \underline{u} - \bar{v} = 0, \\ 2 + 2\varepsilon_2 - \underline{u} - 3\bar{v} = 0, \\ 1 + 2\varepsilon_1 - \bar{u} - \underline{v} = 0, \\ 2 - \bar{u} - 3\underline{v} = 0. \end{cases}$$

Then

$$\bar{u} = \frac{1 + 6\varepsilon_1}{2} = \frac{3}{2} > 0,\ \underline{u} = \frac{1 - 2\varepsilon_2}{2} = \frac{1}{6} > 0,$$
$$\bar{v} = \frac{1 + 2\varepsilon_2}{2} = \frac{5}{6} > 0,\ \underline{v} = \frac{1 - 2\varepsilon_1}{2} = \frac{1}{6} > 0.$$

Hence, $0 < \underline{u} < \widetilde{u_\theta}(x) < \bar{u}$ and $0 < \underline{v} < \widetilde{v_\theta}(x) < \bar{v}$, which yield that there exists a positive non-homogeneous steady-state $(\widetilde{u_\theta}(x), \widetilde{v_\theta}(x))$ of (29).

5. Discussion

In this paper, by using the Lyapunov functional method, we mainly analyzed the global stability of non-homogeneous steady-state for the Lotka–Volterra competition–diffusion–advection system between two competing biological organisms in heterogeneous environments, where two biological organisms are competing for different fundamental resources, their advection and diffusion strategies follow different positive diffusion distributions, and the functions of specific competition ability are variable. Moreover, we also obtained the global stability result when one of the two organisms has no diffusion ability and is not affected by advection.

At the end of this section, we propose an interesting research problem. To the best of our knowledge, for the Lotka–Volterra competition–diffusion–advection system between two competing biological organisms in heterogeneous environments, we did not obtain any results under the condition of cross-diffusion, such as the existence and stability of nontrivial positive steady state. We leave this challenge to future investigations.

Author Contributions: All authors contributed equally and significantly in writing this article. Conceptualization, L.C.; Formal analysis, Y.Z.; Funding acquisition, L.C.; Writing—original draft, S.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Shandong Provincial Natural Science Foundation under grant ZR2020MA006 and the Introduction and Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to express our thanks to the anonymous referees and the editor for their constructive comments and suggestions, which greatly improved this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cantrell, R.; Cosner, C. *Spatial Ecology via Reaction-Diffusion Equations*; Wiley: New York, NY, USA, 2003.
2. He, X.; Ni, W. Global dynamics of the Lotka-Volterra competition-diffusion system: Diffusion and spatial heterogeneity I. *Commun. Pure Appl. Math.* **2016**, *69*, 981–1014. [CrossRef]
3. Korobenko, L.; Braverman, E. On evolutionary stability of carrying capacity driven dispersal in competition with regularly diffusing populations. *J. Math. Biol.* **2014**, *69*, 1181–1206. [CrossRef] [PubMed]
4. Hsu, S.; Waltman, P. On a system of reaction-diffusion equations arising from competition in an unstirred chemostat. *SIAM J. Appl. Math.* **1993**, *53*, 1026–1044. [CrossRef]
5. Lam, K.; Ni, W. Uniqueness and complete dynamics in heterogeneous competition–diffusion systems. *SIAM J. Appl. Math.* **2012**, *72*, 1695–1712. [CrossRef]
6. Ni, W.; Shi, J.; Wang, M. Global stability of nonhomogeneous equilibrium solution for the diffusive Lotka-Volterra competition model. *Calc. Var. Partial Differ. Equ.* **2020**, *59*, 267–270. [CrossRef]
7. Cosner, C. Reaction-diffusion-advection models for the effects and evolution of dispersal. *Discret. Contin. Dyn. Syst.* **2014**, *34*, 1701–1745. [CrossRef]
8. Lou, Y.; Zhao, X.; Zhou, P. Global dynamics of a Lotka-Volterra competition-diffusion-advection system in heterogeneous environments. *J. Math. Pures Appl.* **2019**, *121*, 47–82. [CrossRef]
9. Zhou, P.; Xiao, D. Global dynamics of a classical Lotka-Volterra competition-diffusion-advection system. *J. Funct. Anal.* **2018**, *275*, 356–380. [CrossRef]
10. Wang, Q. On a Lotka-Volterra competition-diffusion-advection model in general heterogeneous environments. *J. Math. Anal. Appl.* **2020**, *489*, 124127. [CrossRef]
11. Li, Z.; Dai, B. Stability and Hopf bifurcation analysis in a Lotka–Volterra competition–diffusion–advection model with time delay effect. *Nonlinearity* **2021**, *34*, 3271–3313. [CrossRef]
12. Li, Z.; Dai, B.; Dong, X. Global stability of nonhomogeneous steady-state solution in a Lotka-Volterra competition-diffusion-advection model. *Appl. Math. Lett.* **2020**, *107*, 106480. [CrossRef]
13. Ma, L.; Guo, S. Bifurcation and stability of a two-species reaction-diffusion-advection competition model. *Nonlinear Anal-Real* **2021**, *59*, 103241. [CrossRef]
14. Zhou, P.; Tang, D.; Xiao, D. On Lotka-Volterra competitive parabolic systems: Exclusion, coexistence and bistability. *J. Differ. Equ.* **2021**, *282*, 596–625. [CrossRef]
15. Wang, M. A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment. *J. Funct. Anal.* **2016**, *270*, 483–508. [CrossRef]
16. Wang, M.; Zhang, Y. Dynamics for a diffusive prey-predator model with different free boundaries. *J. Differ. Equ.* **2018**, *264*, 3527–3558. [CrossRef]
17. Wang, M. Note on the Lyapunov functional method. *Appl. Math. Lett.* **2018**, *75*, 102–107. [CrossRef]
18. Pao, C. *Nonlinear Parabolic and Elliptic Equations*; Plenum Press: New York, NY, USA, 1992.