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Positive Solvability for Conjugate Fractional Differential Inclusion of $(k,\ n-k)$ Type without Continuity and Compactness

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Abstract: The monotonicity of multi-valued operators serves as a guideline to prove the existence of the results in this article. This theory focuses on the existence of solutions without continuity and compactness conditions. We study these results for the $(k,\ n-k)$ conjugate fractional differential inclusion type with $\lambda > 0$, $1 \le k \le n-1$.

Keywords: $(n-k,k)$ conjugate operator; existence and uniqueness; monotone operator; spectral radius

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1. Introduction

We suggest the $(k,\ n-k)$-type conjugate boundary value problem for the nonlinear fractional differential inclusion

$$(-1)^{n-k}\ {}_{CH}D^{\rho}w(\tau) \in \lambda\Theta(\tau, w(\tau)), \quad \tau \in [b_0,\ b], \tag{1}$$

$$w^{(r)}(b_0) = 0, \qquad r = 1, \ldots\ldots, n-1, \tag{2}$$

$$w(b) = 0, \tag{3}$$

where $0 < b_0 < b \le +\infty$, $n \ge 3$, $\rho \in (n-1, n]$, $k \in \{1, \ldots\ldots, n-1\}$, and $\lambda > 0$. ${}_{CH}D^{\rho}$ denotes Caputo-Hadamard fractional differential derivative and $\Theta : [b_0, b] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a monotone multi-valued mapping.

Fractional calculus is a very strong way to generalize the models of ordinary differential problems. It plays important roles in the dynamics description for many complex systems. It is worth noting the practical progress in the field of fractional calculus and its theory (for example, in the analysis of fractional–order numerical schemes, viscoelasticity, transport processes, elastodynamics, the behavior of multifacted media, random flow processes, ..., etc). To show its importance, we refer to [1–8] and the references therein.

The $(k,\ n-k)$ conjugate differential equation of second order $(n=2)$ has been studied in [9]. After that, the main results with a high order were presented in [10–13] and the references given therein. The development, by adding the fractional derivative, is important and necessary to prove the strong extent of nonlinearity theory and its applications. The research into the conjugate fractional type of problem began with the case $(n-1,1)$ in [14,15]. As far as we know, there are no papers exploring the existence of solutions for the $(k,\ n-k)$ conjugate differential inclusion of fractional order.

Oscillation theory began with Sturm's work in 1836, and was further developed for the fifty years before 1996. At present, it is a full, self-contained, discipline, turning more towards nonlinear and functional differential equations. On one hand, oscillation theory has two research fields; one with linear operators and the other with nonlinear functional operators. On the other hand, it has two different fields under the conjugate and disconjugate operators topics. See [16] for a good overview. This theory strongly influences investigatations of strong solution results for $(k,\ n-k)$ conjugate differential boundary value problems.

The aim of this paper is to take one more step with oscillation theory, to develop the previous results from another aspect, which is to study $(n-k, k)$ fractional conjugate problems with multi-valued mappings instead of single–valued mappings. These results are devoted to the sufficient conditions for the existence of a positive solution to the problem

$$(-1)^{n-k}\ {}_{CH}D^{\rho}w(\tau) \in \lambda\Theta(\tau, w(\tau)),$$

where Θ is a monotone multi-valued map.

There are several contributions that generalize differential equations and inclusions and study their solvability. They depend on investigations into the properties of the solutions (existence, uniqueness, stability, controllability, . . . , etc.), see [17,18] and the references given therein.

It is worth mentioning that the literature on the existence and uniqueness of solutions to fractional differential equations is expanding at present, and this problem has drawn the attention of many contributors [19–27].

In the next section, we provide some basic definitions, properties, lemmas and theorems used to investigate the main upshots. The main theorems and results are included in Section 3. Consequently, Section 4 comes with some applications. Finally, Section 5 is formed by a brief overview of current and future works.

2. Preliminaries

2.1. Fractional Calculus

In this subsection, we recall the definitions and some fundamental facts of Caputo-Hadamard fractional integral and the corresponding derivative [28,29].

Definition 1 (Caputo-Hadamard Fractional Integral). *Let $\rho \geq 0,\ n = [\rho]+1, w(\tau) \in L^p[b_0, b],\ 0 < b_0 < b \leq \infty$, and $1 \leq p < \infty$. Then, the Caputo-Hadamard integral ${}_{CH}I^{\rho}_{b_0}$ of fractional order ρ is written by*

$${}_{CH}I^{\rho}_{b_0}w(\tau) = \frac{1}{\Gamma(\rho)}\int_{b_0}^{\tau}\left(\log\frac{\tau}{s}\right)^{\rho-1} w(s)\frac{ds}{s}.$$

Definition 2 (Caputo-Hadamard Fractional Derivative). *Let $\rho \geq 0,\ n = [\rho]+1,\ w(\tau) \in AC^n_{\delta}[b_0, b]$ and $0 < b_0 < b < \infty$. Then, the Caputo-Hadamard fractional derivative ${}_{CH}D^{\rho}_{b_0}$ exists everywhere on $[b_0, b]$ and*

(a) *if $\rho \notin \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,*

$${}_{CH}D^{\rho}_{b_0}w(\tau) = \frac{1}{\Gamma(n-\rho)}\int_{b_0}^{\tau}\left(\log\frac{\tau}{s}\right)^{n-\rho-1}\delta^n w(s)\frac{ds}{s} = {}_{CH}I^{n-\rho}_{b_0}\delta^n w(\tau),$$

(b) *If $\rho \in \mathbb{N}$,*

$${}_{CH}D^{\rho}_{b_0}w(\tau) = \delta^n w(\tau).$$

(c) *If $\rho = 0$,*

$${}_{CH}D^{\rho}_{b_0}w(\tau) = w(\tau),$$

where $\delta := \tau\frac{d}{d\tau}$ and

$$AC^n_{\delta}[b_0, b] = \left\{w : [b_0, b] \to \mathbb{R} : \delta^{n-1}w(\tau) \in AC[b_0, b]\right\}.$$

Lemma 1. *Let $\rho \geq 0$, $n = [\rho] + 1$ and $w(\tau) \in C[b_0, b]$. Then*

(a) *if $\rho \neq 0$, or $\rho \in \mathbb{N}$, then*
$$ {}_{CH}D^{\rho}_{b_0}\left({}_{CH}I^{\rho}_{b_0} w\right)(\tau) = w(\tau). $$

(b) *If $w(\tau) \in AC^n_\delta[b_0, b]$, or $C^n_\delta[b_0, b]$, then*
$$ {}_{CH}I^{\rho}_{b_0}\left({}_{CH}D^{\rho}_{b_0} w\right)(\tau) = w(\tau) - \sum_{r=0}^{n-1} c_r \left(\log \frac{\tau}{b_0}\right)^r, $$
where
$$ C^n_\delta[b_0, b] = \{w : [b_0, b] \to \mathbb{R} : \delta^n w(\tau) \in C[b_0, b]\}. $$

(c) *If $w(\tau) \in C^n_\delta[b_0, b]$ and $\beta > \rho \geq 0$, then*
$$ {}_{CH}D^{\rho}_{b_0}\left({}_{CH}I^{\beta}_{b_0} w\right)(\tau) = {}_{CH}I^{\beta-\rho}_{b_0} w(\tau). $$

2.2. *Monotone Multi-Valued Operators and Corresponding Fixed Point Theorems*

We recall the following definitions and results from [30–32].

Definition 3. *A multi-valued map $A : [b_0, b] \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$(the nonempty compact convex subsets of $\mathbb{R}$) is known as a Caratheodory map if:*

(1) *For all $w \in \mathbb{R}$; $\tau \to A(\tau, w)$ is measurable.*
(2) *For a.e $\tau \in [b_0, b]$; $w \to A(\tau, w)$ is upper semi–continuous.*

In addition to assumptions (1) and (2), the map A is a L^1–Caratheodory map if for each $k > 0$, $\exists\, \phi_k \in L^1[b_0, b]$ satisfying $\sup_{\tau \geq 0}|\phi_k(\tau)| < +\infty$ and $\phi_k > 0$, and a nondecreasing map Ł for which:
$$ \|A(\tau, w)\| = \sup\{|a| : a(\tau) \in A(\tau, w)\} \leq \phi_k(\tau)Ł(\|w\|), $$
for all $\|w\| < k$, $\tau \in [b_0, b]$.

A has a closed graph if, whenever $v_n \to v_*$, $z_n \to z_*$ and $z_n \in A(v_n)$, it holds $z_* \in A(v_*)$.

Let $(\mathbb{E}, \|.\|)$ be a real Banach space and P a normal cone of $\mathbb{E}$. A partial ordering "$\preceq$" is induced by the cone P, namely, for any $w, z \in \mathbb{E}$, $w \preceq z$ if and only if $z - w \in P$.

Let $\mathbb{X}$ and $\mathbb{Y}$ be subsets of $\mathbb{E}$. If, for all $w \in \mathbb{X}$, there exists $z \in \mathbb{Y}$ such that $w \preceq z$, then we write $\mathbb{X} \preceq \mathbb{Y}$. For a nonempty subset D of $\mathbb{X}$ and $A : D \to 2^{\mathbb{X}}/\emptyset$, we say that A is increasing (decreasing) upward if and only if, for all $u, v \in D$ with $u \preceq v$, it is true that, for any $w \in A(u)$, there exists $z \in A(v)$ such that $w \preceq z$ $(w \succeq z)$. A is increasing (decreasing) downward if and only if, for all $u, v \in D$ with $u \preceq v$ it is true that, for any $z \in A(v)$, there exists $w \in A(u)$ such that $w \preceq z$ $(w \succeq z)$. If A is increasing (decreasing) upward and downward, we say that A is increasing (decreasing).

Lemma 2. *Let Σ be a Banach space, $J = [b_0, b]$, $\Psi : J \times \Sigma \to P_{cp,cv}(\Sigma)$ be a L^1-Caratheodory multi-valued map and $P : L^1(J, \Sigma) \to C(J, \Sigma)$ be a continuous and linear map. Define the operator S_Ψ by*
$$ S_\Psi : z \in L^1(J, \Sigma) \to S_\Psi(z) = \Big\{\psi(\tau, z(\tau)) : \ \psi(\tau, z(\tau)) \in \Psi(\tau, z(\tau)) \cap L^1(J, \Sigma)\Big\}. $$
Then, the operator
$$ P \circ S_\Psi : C(J, \Sigma) \to P_{cp,cv}(C(J, \Sigma)) $$
defined as $(P \circ S_\Psi)(z) = P(S_\Psi(z))$ is an operator with a closed graph.

Theorem 1. *Let $\mathbb{X}$ be a real Banach space and P a normal cone of $\mathbb{X}$. Suppose that $T : \mathbb{X} \to 2^{\mathbb{X}}/\{\emptyset\}$ is an increasing multi-valued operator satisfying:*

(1) *For any $w \in \mathbb{X}$, $T(w)$ is a nonempty and closed subset of $\mathbb{X}$.*
(2) *There exists a linear operator $L : \mathbb{X} \to \mathbb{X}$ with a spectral radius $r(L) < 1$ and $L(P) \subset P$ such that, for any $w, z \in \mathbb{X}$ with $w \preceq z$,*
   (i) *for any $u \in T(w)$ there exists $v \in T(z)$ satisfying*
$$0 \preceq v - u \preceq L(z - w).$$
   (ii) *For any $v \in T(z)$ there exists $u \in T(w)$ satisfying*
$$0 \preceq v - u \preceq L(z - w).$$
Then T has a fixed point in $\mathbb{X}$.

Theorem 2. *Let $\mathbb{X}$ be a real Banach space and P a normal cone of $\mathbb{X}$. Suppose that $T : \mathbb{X} \to 2^{\mathbb{X}}/\{\emptyset\}$ is a decreasing multi-valued operator satisfying;*
(1) *For any $w \in \mathbb{X}$, $T(w)$ a nonempty and a closed subset of $\mathbb{X}$.*
(2) *There exists a constant $c \in (0, 1)$ such that, for any $w, z \in \mathbb{X}$ with $w \preceq z$,*
   (i) *for any $u \in T(w)$ there exists $v \in T(z)$ satisfying*
$$-c(z - w) \preceq v - u \preceq 0.$$
   (ii) *For any $v \in T(z)$ there exists $u \in T(w)$ satisfying*
$$-c(z - w) \preceq v - u \preceq 0.$$
Then T has a fixed point in $\mathbb{X}$.

3. Main Results

To show the main results, we need to explain some basic facts. Let $\Sigma = \mathbb{R}$, $J = [b_0, b]$ and define the set–valued map $S_\Theta(w)$ by

$$S_\Theta(w) = \Big\{\theta(\tau, w(\tau)) |\, \theta(\tau, w(\tau)) \in \Theta(\tau, w(\tau)) \cap L^1([b_0, b], \mathbb{R})\Big\}. \tag{4}$$

Then we have below Lemmas:

3.1. Some Auxiliary Results

Lemma 3. *Let $\eta(\tau) \in L^1([b_0, b], \mathbb{R})$ and consider the following problem*

$$_{CH}D^\rho w(\tau) = \eta(\tau), \quad \tau \in [b_0, b], \tag{5}$$
$$w^{(r)}(b_0) = 0, \qquad r = 1, \ldots.., n-1, \tag{6}$$
$$w(b) = 0, \tag{7}$$

then, the unique solution is given by

$$w(\tau) = \int_{b_0}^{b} K(\tau, s)\eta(s)ds, \tag{8}$$

where

$$K(\tau, s) = \frac{1}{\Gamma(\rho)} \begin{cases} \dfrac{(\log \frac{\tau}{s})^{\rho-1}}{s} - \dfrac{\left(\log \frac{b}{s}\right)^{\rho-1}}{s}, & b_0 \le s < \tau \le b, \\ -\dfrac{\left(\log \frac{b}{s}\right)^{\rho-1}}{s}, & b_0 \le \tau < s \le b \end{cases} \tag{9}$$

Proof. To get (8) and (9) we apply the integral operator ${}_{CH}I_{b_0}^{\rho}$ to the both sides of (5). We obtain

$$w(\tau) = {}_{CH}I_{b_0}^{\rho}\eta(\tau) + \sum_{r=0}^{n-1} c_r \left(\log\frac{\tau}{b_0}\right)^r.$$

Using the conditions in (6), we find $c_r = 0,\ \forall r = 1,\dots,n-1$ and then

$$w(\tau) = {}_{CH}I_{b_0}^{\rho}\eta(\tau) + c_0.$$

Under the effect of the condition (7) we have

$$c_0 = -\,{}_{CH}I_{b_0}^{\rho}\eta(b),$$

which completes the proof. □

Define the normal cone $P \subset \mathbb{E}$ by the set of all non–negative functions $P = \{w(\tau) | w(\tau) \geq 0\}$. Consider the multi-valued map

$$H_k^{\lambda}(\tau, w(\tau)) = \lambda(-1)^{n-k}\Theta(\tau, w(\tau)).$$

Let $\eta_\lambda^k(\tau) \in H_k^{\lambda}(\tau, w(\tau))$ and consider the problem

$$ {}_{CH}D^{\rho}w(\tau) = \eta_\lambda^k(\tau), \quad \tau \in [b_0, b], \tag{10}$$
$$ w^{(r)}(b_0) = 0, \qquad r = 1, \dots\dots, n-1, \tag{11}$$
$$ w(b) = 0, \tag{12}$$

which has a solution

$$w(\tau) = \int_{b_0}^{b} K(\tau,s)\eta_\lambda^k(s)ds = \lambda(-1)^{n-k}\int_{b_0}^{b} K(\tau,s)\theta(s)ds. \tag{13}$$

We fix the set K_+ with values of (τ, s) such that

$$(-1)^{n-k}K(\tau,s) > 0. \tag{14}$$

Depending on (14), we define the green functions $G_{K_+}(\tau, s)$ by

$$G_{K_+}(\tau,s) = \frac{1}{\Gamma(\rho)}\begin{cases} \dfrac{\left(\log\frac{b}{s}\right)^{\rho-1}}{s} - \dfrac{(\log\frac{\tau}{s})^{\rho-1}}{s}, & b_0 \leq s < \tau \leq b, \\ \dfrac{\left(\log\frac{b}{s}\right)^{\rho-1}}{s}, & b_0 \leq \tau < s \leq b \end{cases} \tag{15}$$

Lemma 4. *Let $\theta(\tau) \in L^1([b_0, b], \mathbb{R})$ and $\eta_\lambda^k(\tau) = \lambda(-1)^{n-k}\theta(\tau)$, then the problem (10)–(12) admits a unique solution with respect to (14) given by*

$$w(\tau) = \lambda\int_{b_0}^{b} G_{K_+}(\tau,s)\eta(s)ds, \tag{16}$$

where $G_{K_+}(\tau, s)$ is defined by (15).

Proof. Using Lemma 3 and (14) we obtain the result. □

Lemma 5. *Consider $G_{K_+}(\tau, s)$ defined by (15). Then*

$$G_{K_+}(\tau,s) \leq (\rho-1)M(s), \qquad \forall s,\tau \in [b_0, b], \tag{17}$$

where $M(s) = \frac{1}{s\Gamma(\rho)}\left(\log\left(\frac{b}{s}\right)\right)^{\rho-1}$.

Proof. The proof is divided in two cases:

Case 1: If $b_0 \leq s < \tau \leq b$, then, by using the fact that $\log w < w$ and $\log w \leq \log z$ if $w \leq z$, we have

$$\begin{aligned}\Gamma(\rho)G_{K_+}(\tau,s) &= \frac{1}{s}\left[\left(\log\frac{b}{s}\right)^{\rho-1} - \left(\log\frac{\tau}{s}\right)^{\rho-1}\right] = \frac{\rho-1}{s}\int_\tau^b \frac{1}{r}\left(\log\frac{r}{s}\right)^{\rho-2}dr \\ &\leq \frac{(\rho-1)}{s}\left(\log\frac{b}{s}\right)^{\rho-2}\int_\tau^b \frac{s}{r}\frac{1}{s}dr \leq \frac{(\rho-1)}{s}\left(\log\frac{b}{s}\right)^{\rho-2}\left(\log\frac{b}{\tau}\right) \\ &\leq (\rho-1)\Gamma(\rho)M(s).\end{aligned}$$

Case 2: If $b_0 \leq \tau < s \leq b$, one has

$$\Gamma(\rho)G_{K_+}(\tau,s) = \frac{\left(\log\frac{b}{s}\right)^{\rho-1}}{s} \leq \frac{(\rho-1)}{s}\left(\log\frac{b}{s}\right)^{\rho-1} \leq (\rho-1)\Gamma(\rho)M(s).$$

From both cases, we get (17). □

Now, define the linear operator $\Delta^\lambda_{K_+} : [b_0, b] \times S_\Theta(w) \to 2^{\mathbb{R}}$ as follows

$$\Delta^\lambda_{K_+}(\theta)(\tau) = \lambda\int_{b_0}^b G_{K_+}(\tau,s)\theta(s,w(s))ds. \tag{18}$$

Define the set Λ by

$$\Lambda(\theta) = \left\{\lambda > 0 |\ \Delta^\lambda_{K_+}(\theta)(\tau) \geq 0,\ \forall\tau \in [b_0,b],\ \forall w \in P,\ \forall\theta \in S_\Theta(w)\right\} \tag{19}$$

Consequently, define the multi-valued operator $N^\Lambda_{K_+}(w)(\tau)$ by the relation

$$N^\Lambda_{K_+}(w)(\tau) = \left\{\eta^\lambda_k(\tau) |\ \eta^\lambda_k(\tau) = \Delta^\lambda_k(\theta)(\tau),\ \theta(\tau) \in \overline{S_\Theta(w)},\ \lambda \in \Lambda\right\}. \tag{20}$$

3.2. Main Results

Consider $\mathbb{X} = L^1([b_0,b],\mathbb{R})$ and $P = \{w \in \mathbb{X} |\ w \geq 0\}$ as a normal cone in $\mathbb{X}$. Then we can study two different cases.

Theorem 3 (Increasing Map). *Suppose that $\Theta : [b_0,b] \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is a L^1–Caratheodory multi-valued map subject to the following conditions:*

($\mathcal{N}_1$) *S_Θ is an increasing multi-valued map.*

($\mathcal{N}_2$) *There exists a nondecreasing function $\psi_R \in L^\infty([0,R],\mathbb{R}_+)$ with*

$$\|\Theta(\tau,w)\| \leq \psi_R(\|w\|), \quad \forall\ \|w\| \leq R.$$

($\mathcal{N}_3$) *There exists a nondecreasing function $\beta(\tau) \in L^\infty([b_0,b],\mathbb{R}_+)$ such that, for any $\tau \in [b_0,b]$, $\theta_w \in S_\Theta(w)$, $\theta_z \in S_\Theta(z)$ and $w,z \in P$ with $w \preceq z$, it holds*

$$\int_{b_0}^\tau (\theta_z - \theta_w)(s)ds \leq \beta(\tau)(z-w)(\tau).$$

Then, the problem (1)–(3) has at least one positive solution.

Proof. Here, bearing in mind Theorem 1, the proof is shown in the following steps:

Step1: We claim that $N^{\Lambda}_{K_+}$ has a closed graph. Indeed, let us consider $u_n(\tau) \in N^{\Lambda}_{K_+}(w_n)$, where $u_n \to u^*$ and $w_n \to w^*$. It follows that there exists $\theta_{w_n} \in \overline{S_\Theta(w_n)}$ such that $u_n(\tau) = \Delta^{\lambda}_{K_+}\theta_{w_n}(\tau)$. Since the operator $\Delta^{\lambda}_{K_+}$ is a closed linear operator (Lemma 2) and $u_n \to u^*$; then, there exists $\theta_{w^*} \in \overline{S_\Theta(w^*)}$ such that

$$u_n(\tau) = \Delta^{\lambda}_{K_+}\theta_{w_n}(\tau) \to \Delta^{\lambda}_{K_+}\theta_{w^*}(\tau).$$

Take $u^*(\tau) = \Delta^{\lambda}_{K_+}\theta_{w^*}(\tau)$: then $u^*(\tau) \in N^{\Lambda}_{K_+}(w^*)$ concludes the proof of the claim.

Step2: Define the linear operator $Lw = \lambda(\rho-1)M(b_0)\int_{b_0}^{b}\beta(s)w(s)ds,\ w \in \mathbb{X}$. Then, $L(P) \subseteq P$ and

$$\begin{aligned} r(L) &= \lim_{n\to\infty}(\|L^n\|)^{\frac{1}{n}} \\ &\leq \lambda(\rho-1)M(b_0)\beta \lim_{n\to\infty}\sup_{\|w\|=1}\left|\frac{1}{\Gamma(n)}\int_{b_0}^{b}(b-s)^{n-1}w(s)ds\right|^{\frac{1}{n}} \\ &\leq \lambda(\rho-1)M(b_0)\beta \lim_{n\to\infty}\left|\frac{1}{\Gamma(n+1)}(b-b_0)^n\right|^{\frac{1}{n}} \\ &\leq \lambda(\rho-1)M(b_0)\beta(b-b_0)\lim_{n\to\infty}\left|\frac{1}{\Gamma(n+1)}\right|^{\frac{1}{n}} = 0 \end{aligned}.$$

Then

(1) Let $w, z \in P$ with $w \preceq z$. If $\eta_w(\tau) \in N^{\Lambda}_{K_+}(w)$, there exists $\theta_w \in \overline{S_\Theta(w)}$ such that

$$\eta_w(\tau) = \lambda\int_{b_0}^{b} G_{K_+}(\tau,s)\theta_w(s)ds.$$

Since S_Θ is increasing upward, then there exists $\theta_z \in \overline{S_\Theta(z)}$ with $\theta_w \preceq \theta_z$. Consequently,

$$\eta_w(\tau) \leq \lambda\int_{b_0}^{b} G_{K_+}(\tau,s)\theta_z(s)ds.$$

Defining

$$\eta_z(\tau) = \lambda\int_{b_0}^{b} G_{K_+}(\tau,s)\theta_z(s)ds,$$

it holds $\eta_z(\tau) \in N^{\Lambda}_{K_+}(z)$ and $0 \leq \eta_z(\tau) - \eta_w(\tau)$.

(2) Similarly to (1), we can prove that if $\eta_z(\tau) \in N^{\Lambda}_{K_+}(z)$ there exists $\eta_w(\tau) \in N^{\Lambda}_{K_+}(w)$, for which $0 \leq \eta_z(\tau) - \eta_w(\tau)$.

(3) Using Lemma 5, one has

$$\begin{aligned} \eta_z(\tau) - \eta_w(\tau) &= \lambda\int_{b_0}^{b} G_{K_+}(\tau,s)[\theta_z - \theta_w](s)ds \\ &\leq \lambda(\rho-1)\int_{b_0}^{b} M(s)[\theta_z - \theta_w](s)ds \\ &\leq \lambda(\rho-1)\int_{b_0}^{b} M(s)\int_{b_0}^{s}[\theta_z - \theta_w](z)dzds \\ &\leq \lambda(\rho-1)M(b_0)\int_{b_0}^{b}\beta(s)[z-w](s)ds \\ &\leq \lambda(\rho-1)M(b_0)\beta\int_{b_0}^{b}[z-w](s)ds \\ &\leq L(z-w). \end{aligned}$$

By Theorem 1, the previous steps imply that the problem (1)–(3) admits at least a solution in P, i.e, a positive solution. □

Theorem 4. **(Decreasing Map)** *Suppose that Θ is a $L^1 - -$Caratheodory multi-valued map subject to the following conditions:*

($\mathcal{N}_4$) *S_Θ is a decreasing multi-valued map.*

($\mathcal{N}_5$) *There exists a nondecreasing function $\Upsilon_b(\tau) \in L^\infty([b_0, b], \mathbb{R}_+)$ such that for any $\tau \in [b_0, b]$, $\theta_w \in S_\Theta(w)$, $\theta_z \in S_\Theta(z)$ and $w, z \in P$ with $w \preceq z$ it holds*

$$\int_{b_0}^{b} (\theta_z - \theta_w)(s)ds \geq -\Upsilon_b(\tau)(z - w)(\tau).$$

($\mathcal{N}_6$) *For the function $M(s)$ defined as in Lemma 5 we have the condition*

$$C = \lambda(\rho - 1)M(b_0)\Upsilon_b^* < 1, \ \lambda \in \Lambda,$$

where Λ is defined in (19) and $\Upsilon_b^* = \|\Upsilon_b\|_\infty$.

Then, the problem (1)–(3) has at least one positive solution.

Proof. **Step1:** Similarly to Step1 in the proof of Theorem 3.

Step2 (1) Let $w, z \in P$ with $w \preceq z$. If $\eta_w(\tau) \in N_{K_+}^{\Lambda}(w)$, it follows that there exists $\theta_w \in \overline{S_\Theta(w)}$ such that

$$\eta_w(\tau) = \lambda \int_{b_0}^{b} G_{K_+}(\tau, s)\theta_w(s)ds.$$

Since S_Θ is decreasing upward, then there exists $\theta_z \in \overline{S_\Theta(z)}$ with $\theta_z \preceq \theta_w$. Consequently,

$$\eta_w(\tau) \geq \lambda \int_{b_0}^{b} G_{K_+}(\tau, s)\theta_z(s)ds.$$

Defining

$$\eta_z(\tau) = \lambda \int_{b_0}^{b} G_{K_+}(\tau, s)\theta_z(s)ds,$$

$\eta_z(\tau) \in N_{K_+}^{\Lambda}(z)$ and $\eta_z(\tau) - \eta_w(\tau) \leq 0$.

(2) Similarly to (1), we can prove that if $\eta_z(\tau) \in N_{K_+}^{\Lambda}(z)$, then there exists $\eta_w(\tau) \in N_{K_+}^{\Lambda}(w)$, for which $\eta_z(\tau) - \eta_w(\tau) \leq 0$.

(3) Using Lemma 5, one has

$$\begin{aligned}
\eta_z(\tau) - \eta_w(\tau) &= \lambda \int_{b_0}^{b} G_{K_+}(\tau, s)[\theta_z - \theta_w](s)ds \\
&\geq -\lambda(\rho - 1) \int_{b_0}^{b} M(s)[\theta_w - \theta_z](s)ds \\
&\geq \lambda(\rho - 1)M(b_0) \int_{b_0}^{b} [\theta_z - \theta_w](s)ds \\
&\geq -\lambda(\rho - 1)M(b_0)\Upsilon_b(\tau)[z - w](\tau) \\
&\geq -\lambda(\rho - 1)M(b_0)\Upsilon_b^*[z - w](\tau) \\
&\geq -C(z - w)(\tau),
\end{aligned}$$

where $C = \lambda(\rho - 1)M(b_0)\Upsilon_b^* < 1$.

By Theorem 2, the pervious steps show that the problem (1)–(3) is solvable in P, i.e, admits a positive solution. □

4. Applications

Here, we present some examples related to the main results. To obtain the desired conditions, we make use of the Poincaré inequality in $L^1(J,\mathbb{R})$.

Example 1. *Consider the problem*

$$(-1)^{4-k}\ {}_{CH}D^{\rho}w(\tau)\in\Theta(\tau,w(\tau)), J=[1,2], \rho=\frac{7}{2}, \lambda=1 \tag{21}$$

$$\Theta(\tau,w)=\left[[\rho]^n\sin\left(\frac{\pi w}{2^n(\pi+w)}\right)\right],\ n\leq N,\ n,N\in\mathbb{N}. \tag{22}$$

$$w^{(r)}(1)=0,\qquad r=1,2,3 \tag{23}$$

$$w(2)=0 \tag{24}$$

Then

(1) $\|\Theta\|_\infty=\sup\left|[\rho]^n\sin\left(\frac{\pi w}{2(\pi+w)}\right)\right|\leq[\rho]^N$, *which implies* $\psi_R=[\rho]^N$.

(2) *It is known that the function* $\sin w$ *is increasing in the compact interval* $[\frac{-\pi}{2},\frac{\pi}{2}]$. *So,* Θ *is increasing since* $0<\frac{\pi w}{2^n(\pi+w)}<\frac{\pi}{2^n}\leq\frac{\pi}{2}$.

(3) *For all* $0\leq w\preceq z$ *one has*

$$\begin{aligned}
&[\rho]^n\int_1^\tau\left[\sin\left(\frac{\pi z}{2^n(\pi+z)}\right)(s)-\sin\left(\frac{\pi w}{2^n(\pi+w)}\right)(s)\right]ds\\
&=[\rho]^n\int_1^\tau 2\left[\sin\left(\frac{\pi z(\pi+w)-\pi w(\pi+z)}{2^{n+1}(\pi+z)(\pi+w)}\right)(s)\cos\left(\frac{\pi z(\pi+w)+\pi w(\pi+z)}{2^{n+1}(\pi+z)(\pi+w)}\right)(s)\right]ds\\
&\leq 2[\rho]^N\int_1^\tau\left[\sin\left(\frac{\pi(z-w)(\pi+z)}{2^{n+1}(\pi+z)(\pi+w)}\right)(s)\right]ds\\
&=2[\rho]^N\int_1^\tau\left[\sin\left(\frac{\pi(z-w)}{2^{n+1}(\pi+w)}\right)(s)\right]ds\\
&\leq 2[\rho]^N\int_1^\tau\left[\sin\left(\frac{\pi(z-w)}{2^{n+1}(\pi+w)}\right)(s)\right]ds\\
&\leq 2[\rho]^N\int_1^\tau\left[\frac{1}{2^{n+1}}(z-w)(s)\right]ds\\
&\leq\frac{[\rho]^N}{2^n}C_J\int_1^\tau\nabla[z-w](s)ds\\
&=\frac{[\rho]^N}{2^n}C_J(z-w)(\tau)\left[1-\frac{(z-w)(1)}{(z-w)(\tau)}\right]\\
&\leq\frac{[\rho]^N}{2}C_J(z-w)(\tau)\left[1-\frac{(z-w)(1)}{\|z-w\|}\right]\\
&\leq\frac{[\rho]^N}{2}C_J(z-w)(\tau),
\end{aligned}$$

where we used the fact that $u(\tau)\leq\|u\|_\infty,\ \forall\tau\in J$. *Hence,* $\beta(\tau)=\frac{[\rho]^N}{2}C_J$
Comparing (1)–(3) with Theorem 3 we find the solvability of the problem (21)–(24) in the cone P.

Example 2. *If we replace (22) by the multi-valued map*

$$\Theta(\tau,w)=\left[\frac{1}{[\rho]^n}\cos\left(\frac{\pi w}{2^n(\pi+w)}\right)\right],\ n\in\mathbb{N}. \tag{25}$$

Then we have the followings

(1) $\|\Theta\|_\infty=\sup\left|\frac{1}{[\rho]^n}\cos\left(\frac{\pi w}{2^n(\pi+w)}\right)\right|\leq\frac{1}{3}$, *which implies* $\psi_R=\frac{1}{3}$.

(2) *It is known that the function* $\cos w$ *is decreasing in the compact interval* $[0, \pi]$. *Therefore,* Θ *is decreasing since* $0 < \frac{\pi w}{2^n(\pi+w)} < \frac{\pi}{2}$.

(3) *For all* $0 \leq w \preceq z$ *we have*

$$
\begin{aligned}
&\frac{1}{[\rho]^n}\int_1^2\left[\cos\left(\frac{\pi z}{2^n(\pi+z)}\right)(s)-\cos\left(\frac{\pi w}{2^n(\pi+w)}\right)(s)\right]ds\\
&=\frac{1}{[\rho]^n}\int_1^2 -2\left[\sin\left(\frac{\pi z(\pi+w)-\pi w(\pi+z)}{2^{n+1}(\pi+z)(\pi+w)}\right)(s)\sin\left(\frac{\pi z(\pi+w)+\pi w(\pi+z)}{2^{n+1}(\pi+z)(\pi+w)}\right)(s)\right]ds\\
&\geq \frac{-2}{[\rho]^n}\int_1^2\left[\sin\left(\frac{\pi(z-w)(\pi+w)}{2^{n+1}(\pi+z)(\pi+w)}\right)(s)\right]ds\\
&=\frac{-2}{[\rho]^n}\int_1^2\left[\sin\left(\frac{\pi(z-w)}{2^{n+1}(\pi+z)}\right)(s)\right]ds\\
&\geq \frac{-2}{[\rho]^n}\int_1^2\left(\frac{\pi(z-w)}{2^{n+1}(\pi+z)}\right)(s)ds\\
&\geq \frac{-\pi}{2^n[\rho]^n}\int_1^2\left(\frac{(z-w)}{(\pi+z)}\right)(s)ds\\
&\geq \frac{-1}{2[\rho]^n}C_J\int_1^2 \nabla(z-w)(s)ds\\
&=\frac{-1}{2[\rho]^n}C_J[(z-w)(2)-(z-w)(1)]\\
&=\frac{1}{2[\rho]^n}C_J(z-w)(1)\\
&\geq 0 \geq \frac{-1}{[\rho]}(z-w)(\tau),
\end{aligned}
$$

which tends to take $\Upsilon_b(\tau) = \frac{1}{[\rho]}$.

(4) *We have the following values*

$$
\begin{aligned}
&\Gamma(\frac{7}{2}) = \frac{15\sqrt{\pi}}{8} \approx 3.323350970445,\\
&M(1) \approx 0.1496057282,\\
&\Upsilon_b^* = \frac{1}{3}\\
&\lambda = 1
\end{aligned}
.
$$

$C = \lambda(\rho-1)M(b_0)\Upsilon_b^* \approx 0.1246714402 < 1$

By linking results for positive solutions of the problem (1)–(3) with Theorem 4, we can see that the problem (21), (23)–(25) has a positive solution.

5. Conclusions

For monotone-type multi-valued operator, we investigate the existence results and provide some applications for them. Our analysis relies on nonlinear monotone fixed point theorems and is connected with oscillation theory in the sense of $(k,\ n-k)$ conjugate-type differential operator. It is worth generalizing the results on fractional differential equation by multi-valued maps in order to get new extents for phenomena modeling.

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References

1. Seadawy, A.R.; Lu, D.; Iqbal, M. Application of mathematical methods on the system of dynamical equations for the ion sound and Langmuir waves. *Pramana J. Phys.* **2019**, *93*, 10. [CrossRef]
2. Kilbas, A.A.; Sirvastava, H.M.; Trujilo, J.J. *Theory and Applications of Fractional Differential Equation*; Elsvier: Amsterdam, The Netherlands, 2006.
3. Barbour, A.D.; Pugliese, A. Asymptotic behavior of a metapopulation model. *Adv. Appl. Prabob.* **2005**, *15*, 1306–1338. [CrossRef]
4. Patnaiky, S.; Sidhardh, S.; Semperlottiy, F. A Ritz-based finite element method for a fractional-order boundary value problem of nonlocal elasticity. *Int. J. Solids Struct.* **2020**, *202*, 398–417. [CrossRef]
5. Patnaik, S.; Semperlotti, F. A generalized fractional-order elastodynamic theory for non-local attenuating media. *Proc. R. Soc. A* **2020**, *476*, 20200200. [CrossRef] [PubMed]
6. Sandev, T.; Tomovski, Ž. *Fractional Equations and Models*; Springer: Cham, Switzerland, 2019.
7. Sandev, T.; Metzler, R.; Tomovski, Ž. Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative. *J. Phys. Math. Theor.* **2011**, *44*, 255203. [CrossRef]
8. Sumelka, W. Fractional Viscoplasticity. *Mech. Res. Commun.* **2014**, *56*, 31–36. [CrossRef]
9. Agarwal, R.P.; O'Regan, D. Singular boundary value problems for superlinear second order ordinary and delay differential equations. *J. Differ. Equ.* **1996**, *130*, 333–355. [CrossRef]
10. Agarwal, R.P.; Bohner, M.; Wong, P.J.Y. Positive solutions and eigenvalues of conjugate boundary value problems. *Proc. Edinb. Math. Soc.* **1999**, *42*, 349–374. [CrossRef]
11. Kong, L.; Lu, T. Positive solutions of singular $(k, n-k)$ conjugate boundary value problem. *J. Appl. Math. Bioinform.* **2015**, *5*, 13–24.
12. CuiB, Y.; Zou, Y. Monotone iterative technique for $(k, n-k)$ conjugate boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *69*, 1–12.
13. Sun, Q.; Cui, Y. Existence results for $(k, n-k)$ conjugate boundary value problems with integral boundary conditions at resonance with dim ker L = 2. *Bound. Value Probl.* **2017**, *27*, 1–14.
14. Yuan, C. Multiple positive solutions for $(n-1, 1)$-type semipositone conjugate boundary value problems of nonlinear fractional differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *36*, 1–12. [CrossRef]
15. Yuan, C. Multiple positive solutions for $(n-1, 1)$-type semipositone conjugate boundary value problems for coupled systems of nonlinear fractional differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *13*, 1–12. [CrossRef]
16. Elias, U. *Oscillation Theory of Two-Term Differential Equations*; Mathematics and Its Applications; Springer Science & Business Media: Berlin/Heidelberg, Germany, 1997.
17. Salem, A.; Al-dosari, A. A Countable System of Fractional Inclusions with Periodic. Almost, and Antiperiodic Boundary Conditions. *Complexity* **2021**, *2021*, 6653106. [CrossRef]
18. Salem, A.; Al-Dosari, A. Existence results of solution for fractional Sturm-Liouville inclusion involving composition with multi-maps. *J. Taibah Univ. Sci.* **2020**, *14*, 721–733. [CrossRef]
19. Baghani, H.; Nieto, J.J. On fractional Langevin equation involving two fractional orders in different intervals. *Nonlinear Anal. Model. Control* **2019**, *24*, 884–897. [CrossRef]
20. Agarwal, R.P.; Alsaedi, A.; Alghamdi, N.; Ntouyas, S.K.; Ahmad, B. Existence results for multi-term fractional differential equations with nonlocal multi-point and multi-strip boundary conditions. *Adv. Differ. Equ.* **2018**, *342*. [CrossRef]
21. Salem, A.; Mshary, N. On the Existence and Uniqueness of Solution to Fractional-Order Langevin Equation. *Adv. Math. Phys.* **2020**, *2020*, 8890575. [CrossRef]
22. Salem, A.; Alghamdi, B. Multi-Strip and Multi-Point Boundary Conditions for Fractional Langevin Equation. *Fractal Fract.* **2020**, *4*, 18. [CrossRef]
23. Salem, A.; Alnegga, M. Measure of Noncompactness for Hybrid Langevin Fractional Differential Equations. *Axioms* **2020**, *9*, 59. [CrossRef]
24. Salem, A. Existence results of solutions for anti-periodic fractional Langevin equation. *J. Appl. Anal. Comput.* **2020**, *10*, 2557–2574.
25. Alqahtani, B.; Abbas, S.; Benchohra, M.; Alzaid, S.S. Fractional q-difference inclusions in Banach spaces. *Mathematics* **2020**, *8*, 91. [CrossRef]
26. Ren, J.; Zhai, C. Characteristic of unique positive solution for a fractional q-difference equation with multistrip boundary conditions. *Math. Commun.* **2019**, *24*, 181–192.
27. Lachouri, A.; Abdo, M.S.; Ardjouni, A.; Abdalla, B.; Abdeljawad, T. Hilfer fractional differential inclusions with Erdélyi–Kober fractional integral boundary condition. *Adv. Differ. Equ.* **2021**, *2021*, 244. [CrossRef]

28. Jarad, F.; Abdeljawad, T.; Baleanu, D. Caputo-type modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2012**, 142. [CrossRef]
29. Gambo, Y.; Jarad, F.; Baleanu, D.; Abdeljawad, T. On Caputo modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2014**, 10. [CrossRef]
30. Aubin, J.P.; Frankowska, H. *Set-Valued Analysis*; Springer: Bosten, MA, USA, 1990.
31. Bhat, S.P.; Bernstein, D.S. Finit-time stability of continuous autonomous systems. *SIAM J. Control Optim.* **2000**, *38*, 751–757. [CrossRef]
32. Feng, Y.; Wang, Y. Fixed points of multi-valued monotone operators and the solvability of a fractional integral inclusion. *Fixed Point Theory Appl.* **2016**, 64. [CrossRef]