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Some New Fractional Estimates of Inequalities for LR- p -Convex Interval-Valued Functions by Means of Pseudo Order Relation

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Abstract: It is a familiar fact that interval analysis provides tools to deal with data uncertainty. In general, interval analysis is typically used to deal with the models whose data are composed of inaccuracies that may occur from certain kinds of measurements. In interval analysis, both the inclusion relation (\subseteq) and pseudo order relation (\leq_p) are two different concepts. In this article, by using pseudo order relation, we introduce the new class of nonconvex functions known as LR- p -convex interval-valued functions (LR- p -convex-IVFs). With the help of this relation, we establish a strong relationship between LR- p -convex-IVFs and Hermite-Hadamard type inequalities (*HH*-type inequalities) via Katugampola fractional integral operator. Moreover, we have shown that our results include a wide class of new and known inequalities for LR- p -convex-IVFs and their variant forms as special cases. Useful examples that demonstrate the applicability of the theory proposed in this study are given. The concepts and techniques of this paper may be a starting point for further research in this area.

Keywords: LR- p -convex interval-valued function; Katugampola fractional integral operator; Hermite-Hadamard type inequality; Hermite-Hadamard-Fejér inequality

1. Introduction

Hermite [1] and Hadamard [2] derived the familiar inequality known as Hermite-Hadamard inequality (*HH* inequality). This inequality establishes a strong relationship with a convex function such that:

Let $f : I \rightarrow \mathbb{R}$ be a convex function defined on an interval $I \subseteq \mathbb{R}$ and $u, v \in I$ such that $v > u$. Then

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x)dx \leq \frac{f(u)+f(v)}{2} \quad (1)$$

If f is a concave function, then both inequalities are reversed. We note that *HH*-inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. In the last few decades, *HH*-inequality has attracted many authors to devote themselves to this field. Therefore, many authors have proposed different varieties of convexities to introduce *HH*-type inequalities such as harmonic convexity [3], quasi convexity [4], Schur convexity [5,6], strong convexity [7,8], h -convexity [9], p -convexity [10], fuzzy

convexity [11,12], fuzzy pre-invexity [13] and generalized convexity [14], P -convexity [15], etc. Fejér [16] considered the major generalization of HH -inequality which is known as HH -Fejér inequality. It can be expressed as follows:

Let $f : [u, v] \rightarrow \mathbb{R}$ be a convex function on an interval $[u, v]$ with $u \leq v$, and let $\mathcal{W} : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ with $\mathcal{W} \geq 0$ be an integrable and symmetric function with respect to $\frac{u+v}{2}$. Then, we have the following inequality:

$$f\left(\frac{u+v}{2}\right) \int_u^v \mathcal{W}(x) dx \leq \int_u^v f(x) \mathcal{W}(x) dx \leq \frac{f(u)+f(v)}{2} \int_u^v \mathcal{W}(x) dx \quad (2)$$

If f is concave, then the double inequality (2) is reversed. If $\mathcal{W}(x) = 1$, then we obtain (1) from (2). With the assistance of inequality (2), several classical inequalities can be obtained through special convex functions. In addition, these inequalities have a very significant role for convex functions in both pure and applied mathematics. We urge the readers for a further analysis of the literature on the applications and properties of generalized convex functions and HH -integral inequalities, see [17–19] and the references therein.

On the other hand, it is a well-known fact that the interval-valued analysis was introduced as an attempt to overcome interval uncertainty, which occurs in the computer or mathematical models of some deterministic real-world phenomena. A classic example of an interval closure is Archimedes' technique, which is associated with the computation of the circumference of a circle. In 1966, Moore [20] gave the concept of interval analysis in his book and discussed its applications in computational Mathematics.

After that, several authors have developed a strong relationship between inequalities and IVFs by means of inclusion relation via different integral operators, as one can see by Costa [21], Costa and Roman-Flores [22], Roman-Flores et al. [23,24], and Chalco-Cano et al. [25,26], but also to more general set-valued maps by Nikodem et al. [27], and Matkowski and Nikodem [28]. In particular, Zhang et al. [29] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation and proved that these Jensen's inequalities generalized a form of Costa Jensen's inequalities [21].

In the last two decades, in the development of pure and applied mathematics, fractional calculus has played a key role. Yet, it attains magnificent deliberation in the ongoing research work, which is due to its application in various directions such as image processing, signal processing, physics, biology, control theory, computer networking, and fluid dynamics [30–33].

As a further extension, several authors have introduced the refinements of classical inequalities through fractional integrals and discussed their applications, such as Budak et al. [34], who established a strong relationship between fractional interval HH -inequality and convex-IVF.

Through Katugampola fractional integral [35], Toplu et al. [36] established the following HH -inequality for p -convex functions:

Let f be a real-valued Lebesgue integrable function and $p, \alpha > 0$. If $f \in SX([u, v], \mathbb{R}^+, p)$, then

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p^\alpha \Gamma(\alpha + 1)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u) \right] \leq \frac{f(u) + f(v)}{2}. \quad (3)$$

Due to the vast applications of convexity and fractional HH -inequality in mathematical analysis and optimization, many authors have discussed the applications, refinements, generalizations, and extensions, see [37–56] and the references therein.

Inspired by the ongoing research work, we generalize the class of p -convex function known as LR- p -convex-IVF, and establish the relationship between HH -type inequalities and LR- p -convex-IVF via Katugampola fractional integral.

2. Preliminaries

Let \mathbb{R} be the set of real numbers and \mathbb{R}_I be the collection of all closed and bounded intervals of \mathbb{R} that is $\mathbb{R}_I = \{ [\underline{\zeta}, \bar{\zeta}] : \underline{\zeta}, \bar{\zeta} \in \mathbb{R} \text{ and } \underline{\zeta} \leq \bar{\zeta} \}$. If $\underline{\zeta} \geq 0$, then $[\underline{\zeta}, \bar{\zeta}]$ is called positive interval. The set of all positive intervals is denoted by \mathbb{R}_I^+ and defined as

$$\mathbb{R}_I^+ = \{ [\underline{\zeta}, \bar{\zeta}] : [\underline{\zeta}, \bar{\zeta}] \in \mathbb{R}_I \text{ and } \underline{\zeta} \geq 0 \}.$$

Let $\rho \in \mathbb{R}$ and $\rho\zeta$ be defined as

$$\rho\zeta = \begin{cases} [\rho\underline{\zeta}, \rho\bar{\zeta}], & \rho > 0, \\ \{0\}, & \rho = 0, \\ [\rho\bar{\zeta}, \rho\underline{\zeta}], & \rho < 0. \end{cases} \tag{4}$$

Then, the addition $\zeta_1 + \zeta_2$ and Minkowski difference $\zeta_1 - \zeta_2$ for $\zeta_1, \zeta_2 \in \mathbb{R}_I$ are defined by

$$\zeta_1 + \zeta_2 = [\underline{\zeta}_1, \bar{\zeta}_1] + [\underline{\zeta}_2, \bar{\zeta}_2] = [\underline{\zeta}_1 + \underline{\zeta}_2, \bar{\zeta}_1 + \bar{\zeta}_2] \tag{5}$$

and

$$\zeta_1 - \zeta_2 = [\underline{\zeta}_1, \bar{\zeta}_1] - [\underline{\zeta}_2, \bar{\zeta}_2] = [\underline{\zeta}_1 - \bar{\zeta}_2, \bar{\zeta}_1 - \underline{\zeta}_2] \tag{6}$$

respectively.

The inclusion relation “ \supseteq ” means that

$$\zeta_2 \supseteq \zeta_1 \Leftrightarrow [\underline{\zeta}_2, \bar{\zeta}_2] \supseteq [\underline{\zeta}_1, \bar{\zeta}_1] \Leftrightarrow [\underline{\zeta}_1 \geq \underline{\zeta}_2, \bar{\zeta}_2 \geq \bar{\zeta}_1] \tag{7}$$

Remark 1. ([29]). (i) The relation “ \leq_p ” defined on \mathbb{R}_I by

$$[\underline{\zeta}, \bar{\zeta}] \leq_p [\underline{\zeta}, \bar{\zeta}] \text{ if and only if } \underline{\zeta} \leq \underline{\zeta}, \bar{\zeta} \leq \bar{\zeta}, \tag{8}$$

for all $[\underline{\zeta}, \bar{\zeta}], [\underline{\zeta}, \bar{\zeta}] \in \mathbb{R}_I$ is a pseudo order relation. In the interval analysis case, both the pseudo order relation (\leq_p) and partial order relation (\leq) behave alike, thus the relation $[\underline{\zeta}, \bar{\zeta}] \leq_p [\underline{\zeta}, \bar{\zeta}]$ is coincident to $[\underline{\zeta}, \bar{\zeta}] \leq [\underline{\zeta}, \bar{\zeta}]$ on \mathbb{R}_I , for more details see, [21,29].

(ii) It can be easily seen that “ \leq_p ” looks similar to “left and right” on the real line \mathbb{R} , so we call “ \leq_p ” is “left and right” (or “LR” order, in short).

The concept of Riemann integral for IVF first introduced by Moore [20] is defined as follows:

Theorem 1. ([20]). Let $f : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}_I$ is an IVF such that $f(x) = [\underline{f}(x), \bar{f}(x)]$. Then, f is Riemann integrable over $[u, v]$ if and only if, \underline{f} and \bar{f} both are Riemann integrable over $[u, v]$ such that

$$(IR) \int_u^v f(x)dx = \left[(R) \int_u^v \underline{f}(x)dx, (R) \int_u^v \bar{f}(x)dx \right] \tag{9}$$

Now, we discuss the concept of Katugampola fractional integral operator for IVF.

Let $q \geq 1$, $c \in \mathbb{R}$ and $\mathcal{X}_c^q(u, v)$ be the set of all complex-valued Lebesgue integrable IVFs f on $[u, v]$ for which the norm $\|f\|_{\mathcal{X}_c^q}$ is defined by

$$\|f\|_{\mathcal{X}_c^q} = \left(\int_u^v |q^c f(x)|^q \frac{dq}{q} \right)^{\frac{1}{q}} < \infty$$

For $1 \leq q < \infty$ and

$$\|f\|_{\mathcal{X}_c^\infty} = \text{ess sup}_{u \leq \varrho \leq v} \varrho^c |f(\varrho)|$$

Katugampola [35] presented a new fractional integral to generalize the Riemann Liouville and Hadamard fractional integrals under certain conditions.

Let $p, \alpha > 0$ and $f \in \mathfrak{S}\mathcal{L}_{[u,v]}$ be the collection of all complex-valued Lebesgue integrable IVFs on $[u, v]$. Then, the interval left and right Katugampola fractional integrals of $f \in \mathfrak{S}\mathcal{L}_{[u,v]}$ with order are defined by

$$\mathcal{I}_{u^+}^{p,\alpha} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_u^x (x^p - \zeta^p)^{\alpha-1} \zeta^{p-1} f(\zeta) d(\zeta) \quad (x > u), \tag{10}$$

and

$$\mathcal{I}_{v^-}^{p,\alpha} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^v (\zeta^p - x^p)^{\alpha-1} \zeta^{p-1} f(\zeta) d(\zeta) \quad (x < v) \tag{11}$$

respectively, where $\Gamma(x) = \int_0^\infty \zeta^{x-1} u^{-\zeta} d(\zeta)$ is the Euler gamma function.

The concept of p -convex functions were established by Zhang and Wang [10], and a number of properties of the functions were introduced.

Definition 1. ([54]). Let $p \in \mathbb{R}$ with $p \neq 0$. Then, the interval I is said to be p -convex if

$$[\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}} \in I, \tag{12}$$

for all $x, y \in I, \varrho \in [0, 1]$, where $p = 2n + 1$ and $n \in \mathbb{N}$ or p is an odd number.

Definition 2. ([10]). Let $p \in \mathbb{R}$ with $p \neq 0$ and $I = [u, v] \subseteq \mathbb{R}$. Then, the function $f : [u, v] \rightarrow \mathbb{R}^+$ is said to be p -convex function if

$$f\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \varrho f(x) + (1 - \varrho)f(y), \tag{13}$$

for all $x, y \in [u, v], \varrho \in [0, 1]$. If the inequality (13) is reversed, then f is called p -concave function. The set of all p -convex (LR- p -concave, LR- p -affine) functions is denoted by

$$SX([u, v], \mathbb{R}^+, p) \quad (SV([u, v], \mathbb{R}^+, p),).$$

Firstly, we introduce the new class of LR- p -convex-IVF.

3. LR- p -Convex Interval-Valued Functions

Now, we introduce LR- p -convex interval-valued functions.

Definition 3. The IVF $f : [u, v] \rightarrow \mathbb{R}_I^+$ is said to be LR- p -convex-IVF if for all $x, y \in [u, v]$ and $\varrho \in [0, 1]$ we have

$$f\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq_p \varrho f(x) + (1 - \varrho)f(y). \tag{14}$$

If inequality (14) is reversed, then f is said to be LR- p -concave on $[u, v]$. The set of all LR- p -convex (LR- p -concave) IVFs is denoted by

$$LRSX([u, v], \mathbb{R}_I^+, p) \quad (LRSV([u, v], \mathbb{R}_I^+, p),).$$

Remark 2. If $p = 1$, then LR- p -convex-IVF reduces to LR-convex-IVF, see [24].

If $p = -1$, then we obtain the class of harmonically convex functions, which is also a new one.

The next Theorem 2 establishes the relationship between Definition 3 and end point functions of IVFs.

Theorem 2. Let $f : [u, v] \rightarrow \mathbb{R}_I^+$ be an IVF defined by $f(x) = [\underline{f}(x), \bar{f}(x)]$, for all $x \in [u, v]$. Then, $f \in LRSX([u, v], \mathbb{R}_I^+, p)$ if and only if, $\underline{f}, \bar{f} \in SX([u, v], \mathbb{R}^+, p)$.

Proof. Assume that $\underline{f}, \bar{f} \in SX([u, v], \mathbb{R}^+, p)$. Then, for all $x, y \in [u, v]$, $\varrho \in [0, 1]$, we have

$$\underline{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \varrho \underline{f}(x) + (1 - \varrho)\underline{f}(y)$$

and

$$\bar{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \varrho \bar{f}(x) + (1 - \varrho)\bar{f}(y)$$

From Definition 3 and order relation \leq_p , we have

$$\begin{aligned} & \left[\underline{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right), \bar{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \right] \\ & \leq_p \left[\varrho \underline{f}(x) + (1 - \varrho)\underline{f}(y), \varrho \bar{f}(x) + (1 - \varrho)\bar{f}(y) \right] \\ & = \varrho \left[\underline{f}(x), \bar{f}(x) \right] + (1 - \varrho) \left[\underline{f}(y), \bar{f}(y) \right] \end{aligned}$$

That is

$$f\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq_p \varrho f(x) + (1 - \varrho)f(y), \forall x, y \in [u, v], \varrho \in [0, 1].$$

Hence, $f \in LRSX([u, v], \mathbb{R}_I^+, p)$.

Conversely, let $f \in LRSX([u, v], \mathbb{R}_I^+, p)$. Then, for all $x, y \in [u, v]$ and $\varrho \in [0, 1]$, we have

$$f\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq_p \varrho f(x) + (1 - \varrho)f(y).$$

That is

$$\begin{aligned} & \left[\underline{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right), \bar{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \right] \leq_p \varrho \left[\underline{f}(x), \bar{f}(x) \right] + (1 - \varrho) \left[\underline{f}(y), \bar{f}(y) \right] \\ & = \left[\varrho \underline{f}(x) + (1 - \varrho)\underline{f}(y), \varrho \bar{f}(x) + (1 - \varrho)\bar{f}(y) \right] \end{aligned}$$

It follows that

$$\underline{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \varrho \underline{f}(x) + (1 - \varrho)\underline{f}(y),$$

and

$$\bar{f}\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \varrho \bar{f}(x) + (1 - \varrho)\bar{f}(y),$$

Hence, the result follows. \square

Remark 3. If $\underline{f}(x) = \bar{f}(x)$, then p -convex-IVF reduces to the classical p -convex function, see [10].

If $\underline{f}(x) = \bar{f}(x)$ with $\gamma = 1$ and $p = 1$, then p -convex-IVF reduces to the classical convex function.

Example 1. Let p be an odd number, $\alpha = \frac{1}{2}$, $x \in [2, 3]$ and $f(x) = \left[-x^{\frac{p}{2}}, 2 - x^{\frac{p}{2}}\right]$. Then, we clearly see that both end point functions $\underline{f}(x) = -x^{\frac{p}{2}}$ and $\bar{f}(x) = 2 - x^{\frac{p}{2}}$ are p -convex functions. Hence, $f \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$.

Fractional Hermite-Hadamard Type Inequalities

In this section, we will prove some new Hermite-Hadamard type inequalities for LR- p -convex-IVFs by means of the pseudo order relation via Katugampola fractional integral operator.

Theorem 3. Let $p, \alpha > 0, u, v \in I$ such that $v > u$, $f \in \mathfrak{L}_{([u,v])}$. If $f \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$, then

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p^\alpha \Gamma(\alpha + 1)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)\right] \leq_p \frac{f(u) + f(v)}{2}. \tag{15}$$

If $f \in \text{LRSV}([u, v], \mathbb{R}_I^+, p)$ ($[u, v], \mathbb{R}_I^+, p$), then

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \geq_p \frac{p^\alpha \Gamma(\alpha + 1)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)\right] \geq_p \frac{f(u) + f(v)}{2}. \tag{16}$$

Proof. Let $f \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$. Then, by hypothesis, we have

$$2f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p f\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right) + f\left([(1 - q)u^p + qv^p]^{\frac{1}{p}}\right) \tag{17}$$

Multiplying both sides (17) by $q^{\alpha-1}$ and integrating the obtained result with respect to q over $(0, 1)$, we have

$$\begin{aligned} & 2 \int_0^1 q^{\alpha-1} f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) dq \\ & \leq_p \int_0^1 q^{\alpha-1} \left[f\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right) + f\left([(1 - q)u^p + qv^p]^{\frac{1}{p}}\right) \right] dq \end{aligned} \tag{18}$$

From (18), we get

$$\begin{aligned} 2 \int_0^1 q^{\alpha-1} f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) dq &= 2 \left[\int_0^1 q^{\alpha-1} \underline{f}\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) dq, \int_0^1 q^{\alpha-1} \bar{f}\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) dq \right] \\ &= 2 \left[\frac{1}{\alpha} \underline{f}\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right), \frac{1}{\alpha} \bar{f}\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \right] \\ &= 2 \frac{1}{\alpha} f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right). \end{aligned} \tag{19}$$

and

$$\begin{aligned} & \int_0^1 q^{\alpha-1} \left[f\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right) + f\left([(1 - q)u^p + qv^p]^{\frac{1}{p}}\right) \right] dq \\ &= \int_0^1 q^{\alpha-1} \left[\underline{f}\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right), \underline{f}\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right) \right] dq \\ &+ \int_0^1 q^{\alpha-1} \left[\bar{f}\left([(1 - q)u^p + qv^p]^{\frac{1}{p}}\right), \bar{f}\left([(1 - q)u^p + qv^p]^{\frac{1}{p}}\right) \right] dq \end{aligned}$$

Let $q \in [0, 1]$, $x^p = qu^p + (1 - q)v^p$ and $y^p = (1 - q)u^p + qv^p$. Then, we have

$$\begin{aligned} &= \frac{p}{(v^p - u^p)^\alpha} \left[\int_u^v (v^p - y^p)^{\alpha-1} \frac{f(y)}{y^{1-p}} dy, \int_u^v (v^p - y^p)^{\alpha-1} \frac{\bar{f}(y)}{y^{1-p}} dy \right] \\ &+ \frac{p}{(v^p - u^p)^\alpha} \left[\int_u^v (x^p - u^p)^{\alpha-1} \frac{f(x)}{x^{1-p}} dx, \int_u^v (x^p - u^p)^{\alpha-1} \frac{\bar{f}(x)}{x^{1-p}} dx \right], \tag{20} \\ &= \frac{p}{(v^p - u^p)^\alpha} \left[\int_u^v (v^p - u^p)^{\alpha-1} \frac{f(y)}{y^{1-p}} dy, \int_u^v (x^p - u^p)^{\alpha-1} \frac{f(x)}{x^{1-p}} dx \right], \\ &\leq_p \frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u) \right]. \end{aligned}$$

Since $f \in LRSX([u, v], \mathbb{R}_I^+, p)$, we obtain

$$f\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right) \leq_p qf(u) + (1 - q)f(v) \tag{21}$$

and

$$f\left([qv^p + (1 - q)u^p]^{\frac{1}{p}}\right) \leq_p qf(v) + (1 - q)f(u) \tag{22}$$

Adding (21) and (22), we get

$$f\left([qu^p + (1 - q)v^p]^{\frac{1}{p}}\right) + f\left([qv^p + (1 - q)u^p]^{\frac{1}{p}}\right) \leq_p f(u) + f(v) \tag{23}$$

Multiplying both sides (23) by $q^{\alpha-1}$ and integrating both sides of the obtained result with respect to q over $(0, 1)$, we get

$$\frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u) \right] \leq_p \frac{f(u) + f(v)}{\alpha} \tag{24}$$

From (20) and (24), (19) becomes

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p^\alpha \Gamma(\alpha + 1)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u) \right] \leq_p \frac{f(u) + f(v)}{2},$$

and the theorem has been proved. \square

Remark 4. Let $p = 1$. Then, Theorem 3 reduces to the result for LR-convex-IVF, which is also a new one:

$$f\left(\frac{u + v}{2}\right) \leq_p \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} \left[\mathcal{I}_{u^+}^\alpha f(v) + \mathcal{I}_{v^-}^\alpha f(u) \right] \leq_p \frac{f(u) + f(v)}{2}.$$

If $\alpha = 1$, then Theorem 3 reduces to the result for LR- p -convex-IVF, which is also a new one:

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p - u^p} (IR) \int_u^v x^{p-1} f(x) dx \leq_p \frac{f(u) + f(v)}{2}$$

Let $p = \alpha = 1$. Then, Theorem 3 reduces to the result for LR- p -convex-IVF, which is also a new one:

$$f\left(\frac{u + v}{2}\right) \leq_p \frac{1}{v - u} (IR) \int_u^v f(x) dx \leq_p \frac{f(u) + f(v)}{2}$$

If $\underline{f} = \bar{f}$, then we get inequality (13) from Theorem 3.

If $p = 1$ and $\underline{f} = \bar{f}$, then from Theorem 3, we obtain fractional *HH*-inequality for convex function, see [41]:

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [\mathcal{I}_{u^+}^\alpha f(v) + \mathcal{I}_{v^-}^\alpha f(u)] \leq \frac{f(u)+f(v)}{2}.$$

If $\alpha = 1$, and $\underline{f} = \bar{f}$, then Theorem 3 reduces to the result for LR- p -convex-IVF, see [10]:

$$f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p} \int_u^v x^{p-1} f(x) dx \leq \frac{f(u)+f(v)}{2}$$

If $\alpha = p = 1$ and $\underline{f} = \bar{f}$, then we obtain the classical inequality (1) from Theorem 3.

Example 2. Let p be an odd number, $\alpha = \frac{1}{2}$, $x \in [2, 3]$ and $f(x) = \left[2 - x^{\frac{p}{2}}, 2\left(2 - x^{\frac{p}{2}}\right)\right]$. Then, we clearly see that $f \in \mathfrak{S}\mathcal{L}_{([u,v])}$ and $f \in LRSX([u, v], \mathbb{R}_I^+, p)$. Since $\underline{f}(x) = 2 - x^{\frac{p}{2}}$ and $\bar{f}(x) = 2\left(2 - x^{\frac{p}{2}}\right)$. Now, we compute the following:

$$\underline{f}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = \underline{f}\left(\frac{5}{2}\right) = \frac{4-\sqrt{10}}{2}$$

$$\bar{f}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = \bar{f}\left(\frac{5}{2}\right) = 4 - \sqrt{10},$$

$$\frac{f(u)+f(v)}{2} = 2 - \frac{\sqrt{2}-\sqrt{3}}{2},$$

$$\frac{\bar{f}(u)+\bar{f}(v)}{2} = 4 - \sqrt{2} - \sqrt{3}.$$

Note that

$$\begin{aligned} \frac{p^\alpha \Gamma(\alpha+1)}{2(v^p-u^p)^\alpha} [\mathcal{I}_{u^+}^{p,\alpha} \underline{f}(v) + \mathcal{I}_{v^-}^{p,\alpha} \underline{f}(u)] &= \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_2^3 (3^p - x^p)^{-\frac{1}{2}} x^{p-1} [2 - x^{\frac{p}{2}}, 2(2 - x^{\frac{p}{2}})] dx \\ &+ \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_2^3 (x^p - 2^p)^{-\frac{1}{2}} x^{p-1} [2 - x^{\frac{p}{2}}, 2(2 - x^{\frac{p}{2}})] dx \\ &= \frac{1}{4} \left[\frac{7393}{5000} + \frac{9501}{5000} \right] = \frac{8447}{10,000} \end{aligned}$$

and

$$\begin{aligned} \frac{p^\alpha \Gamma(\alpha)}{(v^p-u^p)^\alpha} [\mathcal{I}_{u^+}^{p,\alpha} \bar{f}(v) + \mathcal{I}_{v^-}^{p,\alpha} \bar{f}(u)] &= \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_2^3 (3^p - x^p)^{-\frac{1}{2}} x^{p-1} (2 - x^{\frac{p}{2}}) dx \\ &+ \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_2^3 (x^p - 2^p)^{-\frac{1}{2}} x^{p-1} (2 - x^{\frac{p}{2}}) dx \\ &= \frac{1}{4} \left[\frac{7393}{10,000} + \frac{9501}{10,000} \right] = \frac{8447}{20,000}. \end{aligned}$$

Therefore, we have

$$\frac{4-\sqrt{10}}{2} \leq \frac{8447}{20,000} \leq 2 - \frac{\sqrt{2}+\sqrt{3}}{2}$$

$$4 - \sqrt{10} \leq \frac{8447}{10,000} \leq 4 - \sqrt{2} - \sqrt{3}$$

and Theorem 3 is verified.

The next Theorem 4 gives the *HH*-Fejér type inequality for LR- p -convex-IVFs.

Theorem 4. Let $p, \alpha > 0, u, v \in I$ with $v > u, f \in \mathfrak{S}\mathcal{L}_{([u,v])}$ and $\mathcal{W}(x) = \mathcal{W}\left([u^p + v^p - x^p]^{\frac{1}{p}}\right) \geq 0$ for $x \in I$. If $f \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$, then we have the HH-Fejér type inequality as follows:

$$\begin{aligned} & f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u)\right] \\ & \leq_p \left[\mathcal{I}_{u^+}^{p,\alpha} f\mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} f\mathcal{W}(u)\right] \leq_p \frac{f(u)+f(v)}{2} \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u)\right]. \end{aligned} \tag{25}$$

If $f \in \text{LRSV}([u, v], \mathbb{R}_I^+, p)$, then

$$\begin{aligned} & f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u)\right] \geq_p \left[\mathcal{I}_{u^+}^{p,\alpha} f\mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} f\mathcal{W}(u)\right] \\ & \geq_p \frac{f(u)+f(v)}{2} \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u)\right]. \end{aligned} \tag{26}$$

Proof. Since $f \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$, then for $\varrho \in [0, 1]$, we have

$$f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{1}{2} \left(f\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + f\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\right). \tag{27}$$

Since $\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) = \mathcal{W}\left([\varrho v^p + (1-\varrho)u^p]^{\frac{1}{p}}\right)$, then multiplying both sides of (27) by $\varrho^{\alpha-1} \mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)$, and integrating it with respect to ϱ over $[0, 1]$, we have

$$\begin{aligned} & 2 \int_0^1 \varrho^{\alpha-1} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho \\ & \leq_p \int_0^1 \varrho^{\alpha-1} f\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho \\ & + \int_0^1 \varrho^{\alpha-1} f\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) \mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho \\ & = \int_0^1 \varrho^{\alpha-1} \left[\underline{f}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right), \bar{f}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) \right] \times \mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho + \\ & \int_0^1 \varrho^{\alpha-1} \left[\underline{f}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right), \bar{f}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) \right] \times \mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho. \end{aligned}$$

Let $x^p = \varrho v^p + (1 - \varrho)u^p$. Then, we have

$$\begin{aligned} & \frac{2p}{(v^p - u^p)^\alpha} f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \int_u^v (x^p - u^p)^{\alpha-1} \mathcal{W}(x) dx \\ & \leq_p \frac{p}{(v^p - u^p)^\alpha} \int_u^v (x^p - u^p)^{\alpha-1} \left[\underline{f}\left(\left[u^p - v^p - x^p\right]^{\frac{1}{p}}\right), \bar{f}\left(\left[u^p + v^p - x^p\right]^{\frac{1}{p}}\right) \right] \mathcal{W}(x) x^{p-1} dx \\ & + \int_u^v (x^p - u^p)^{\alpha-1} \left[\underline{f}(x), \bar{f}(x) \right] \mathcal{W}(x) x^{p-1} dx, \\ & = \frac{p}{(v^p - u^p)^\alpha} \int_u^v (x^p - u^p)^{\alpha-1} \left[\underline{f}(x), \bar{f}(x) \right] \mathcal{W}\left(\left[u^p - v^p - x^p\right]^{\frac{1}{p}}\right) x^{p-1} dx \\ & + \int_u^v (x^p - u^p)^{\alpha-1} \left[\underline{f}(x), \bar{f}(x) \right] \mathcal{W}(x) x^{p-1} dx, \\ & = \frac{p}{(v^p - u^p)^\alpha} \int_u^v (x^p - u^p)^{\alpha-1} \left[\underline{f}(x), \bar{f}(x) \right] \mathcal{W}(x) x^{p-1} dx \\ & + \int_u^v (x^p - u^p)^{\alpha-1} \left[\underline{f}(x), \bar{f}(x) \right] \mathcal{W}(x) x^{p-1} dx, \\ & = \frac{p}{(v^p - u^p)^\alpha} \left[\int_u^v (v^p - x^p)^{\alpha-1} f(x) \mathcal{W}(x) x^{p-1} dx + \int_u^v (x^p - u^p)^{\alpha-1} f(x) \mathcal{W}(x) x^{p-1} dx \right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u) \right] \\ & \leq_p \frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} f \mathcal{W}(u) \right]. \end{aligned} \tag{28}$$

Now taking the multiplication of (23) by $q^{\alpha-1} \mathcal{W}\left(\left[\varrho v^p + (1 - \varrho)u^p\right]^{\frac{1}{p}}\right)$, and integrating it with respect to q over $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 q^{\alpha-1} \mathcal{W}\left(\left[\varrho v^p + (1 - \varrho)u^p\right]^{\frac{1}{p}}\right) f\left(\left[\varrho u^p + (1 - \varrho)v^p\right]^{\frac{1}{p}}\right) d\varrho \\ & + \int_0^1 q^{\alpha-1} \mathcal{W}\left(\left[\varrho v^p + (1 - \varrho)u^p\right]^{\frac{1}{p}}\right) f\left(\left[\varrho v^p + (1 - \varrho)u^p\right]^{\frac{1}{p}}\right) d\varrho \\ & \leq_p [f(u) + f(v)] \int_0^1 q^{\alpha-1} \mathcal{W}\left(\left[\varrho v^p + (1 - \varrho)u^p\right]^{\frac{1}{p}}\right) d\varrho. \end{aligned}$$

Therefore, we have

$$\frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} f \mathcal{W}(u) \right] \leq_p \frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} \cdot \frac{\mathcal{F}(u) + \mathcal{F}(v)}{2} \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u) \right]. \tag{29}$$

Combining (20) and (21), we get

$$\begin{aligned} & f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u) \right] \\ & \leq_p \left[\mathcal{I}_{u^+}^{p,\alpha} f \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} f \mathcal{W}(u) \right] \leq_p \frac{f(u) + f(v)}{2} \left[\mathcal{I}_{u^+}^{p,\alpha} \mathcal{W}(v) + \mathcal{I}_{v^-}^{p,\alpha} \mathcal{W}(u) \right] \end{aligned}$$

and the theorem has been proved. \square

Remark 5. Let $p = 1$. Then, Theorem 4 reduces to the result for LR-convex-IVF, which is also a new one:

$$f\left(\frac{u + v}{2}\right) \left[\mathcal{I}_{u^+}^\alpha \mathcal{W}(v) + \mathcal{I}_{v^-}^\alpha \mathcal{W}(u) \right] \leq_p \left[\mathcal{I}_{u^+}^\alpha f \mathcal{W}(v) + \mathcal{I}_{v^-}^\alpha f \mathcal{W}(u) \right] \leq_p \frac{f(u) + f(v)}{2} \left[\mathcal{I}_{u^+}^\alpha \mathcal{W}(v) + \mathcal{I}_{v^-}^\alpha \mathcal{W}(u) \right].$$

Let $\alpha = 1$. Then, Theorem 4 reduces to the result for LR- p -convex-IVF, which is also a new one:

$$f\left(\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{1}{\int_u^v x^{p-1} \mathcal{W}(x) dx} \int_u^v x^{p-1} f(x) \mathcal{W}(x) dx \leq_p \frac{f(u) + f(v)}{2}$$

Let $p = \alpha = 1$. Then, Theorem 4 reduces to the result for LR-convex-IVF, which is also a new one:

$$f\left(\frac{u + v}{2}\right) \leq_p \frac{1}{\int_u^v \mathcal{W}(x) dx} \int_u^v f(x) \mathcal{W}(x) dx \leq_p \frac{f(u) + f(v)}{2}$$

If $f = \bar{f}$ and $\alpha = 1$, then from Theorem 4, we get Theorem 5 of [39].

If $f = \bar{f}$ and $\alpha = 1$, then from Theorem 4, we obtain the classical HH-Fejér type inequality (2).

If $f = \bar{f}$ and $\mathcal{W}(x) = p = \alpha = 1$, then from Theorem 4, we get the classical HH-inequality (1).

If $\mathcal{W}(x) = 1$, then from Theorem 4, we get Theorem 3.

Theorem 5. Let $p, \alpha > 0, u, v \in I$ with $v > u$ and $f, g \in \mathfrak{S}\mathcal{L}_{([u, v])}$. If $f, g \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$, then we have

$$\frac{p^\alpha \Gamma(\alpha)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p, \alpha} f(v)g(v) + \mathcal{I}_{v^-}^{p, \alpha} f(u)f(u) \right] \leq_p \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) M(u, v) + \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) N(u, v). \quad (30)$$

If $f, g \in \text{LRSV}([u, v], \mathbb{R}_I^+, p)$, then

$$\frac{p^\alpha \Gamma(\alpha)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p, \alpha} f(v)f(v) + \mathcal{I}_{v^-}^{p, \alpha} f(u)f(u) \right] \geq_p \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) M(u, v) + \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) N(u, v) \quad (31)$$

where

$$M(u, v) = [f(u)g(u) + f(v)g(v)]$$

and

$$N(u, v) = [f(u)g(v) + f(v)g(u)].$$

Proof. Since $f, g \in \text{LRSX}([u, v], \mathbb{R}_I^+, p)$, then for $\varrho \in [0, 1]$ we have

$$f\left(\left[\varrho u^p + (1 - \varrho)v^p\right]^{\frac{1}{p}}\right) \leq_p \varrho f(u) + (1 - \varrho)f(v),$$

and

$$g\left(\left[\varrho u^p + (1 - \varrho)v^p\right]^{\frac{1}{p}}\right) \leq_p \varrho g(u) + (1 - \varrho)g(v).$$

From the definition of p -convex-IVFs, it follows that $0 \leq_p f(x)$ and $0 \leq_p g(x)$, then we have

$$\begin{aligned} & f\left(\left[\varrho u^p + (1 - \varrho)v^p\right]^{\frac{1}{p}}\right) g\left(\left[\varrho u^p + (1 - \varrho)v^p\right]^{\frac{1}{p}}\right) \\ & \leq_p \varrho^2 f(u)g(u) + (1 - \varrho)^2 f(v)g(v) + \varrho(1 - \varrho)[f(v)g(u) + f(u)g(v)] \end{aligned} \quad (32)$$

Similarly, we have

$$\begin{aligned} & f\left(\left[(1 - \varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right) g\left(\left[(1 - \varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right) \\ & \leq_p (1 - \varrho)^2 g(u)f(u) + \varrho^2 f(v)g(v) + \varrho(1 - \varrho)[g(v)f(u) + g(u)f(v)] \end{aligned} \quad (33)$$

Adding (32) and (33), we get

$$\begin{aligned}
 & f\left([qu^p + (1 - \varrho)v^p]^{\frac{1}{p}}\right)g\left([qu^p + (1 - \varrho)v^p]^{\frac{1}{p}}\right) \\
 & + f\left([(1 - \varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)g\left([(1 - \varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) \\
 & \leq_p \left[\varrho^2 + (1 - \varrho)^2\right][f(u)g(u) + f(v)g(v)] + 2\varrho(1 - \varrho)[f(v)g(u) + f(u)g(v)]
 \end{aligned} \tag{34}$$

Multiplying both sides of (34) by $\varrho^{\alpha-1}$ and integrating the obtained result with respect to ϱ over (0,1), we have

$$\begin{aligned}
 & \int_0^1 \varrho^{\alpha-1} f\left([qu^p + (1 - \varrho)v^p]^{\frac{1}{p}}\right)g\left([qu^p + (1 - \varrho)v^p]^{\frac{1}{p}}\right) d\varrho \\
 & + \int_0^1 \varrho^{\alpha-1} f\left([(1 - \varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)g\left([(1 - \varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho \\
 & \leq_p M(u, v) \int_0^1 \varrho^{\alpha-1} \left[\varrho^2 + (1 - \varrho)^2\right] + 2N(u, v) \int_0^1 \varrho^{\alpha-1} \varrho(1 - \varrho) d\varrho.
 \end{aligned} \tag{35}$$

Form (35), we have

$$\begin{aligned}
 & \int_0^1 \varrho^{\alpha-1} f\left([qu^p + (1 - \varrho)v^p]^{\frac{1}{p}}\right)g\left([qu^p + (1 - \varrho)v^p]^{\frac{1}{p}}\right) d\varrho \\
 & + \int_0^1 \varrho^{\alpha-1} f\left([(1 - \varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)g\left([(1 - \varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho \\
 & = \frac{p^\alpha \Gamma(\alpha)}{(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v)g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)g(u) \right].
 \end{aligned} \tag{36}$$

and

$$\begin{aligned}
 & M(u, v) \int_0^1 \varrho^{\alpha-1} \left[\varrho^2 + (1 - \varrho)^2\right] + 2N(u, v) \int_0^1 \varrho^{\alpha-1} \varrho(1 - \varrho) d\varrho \\
 & = \frac{2}{\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) M(u, v) + \frac{2}{\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right) N(u, v).
 \end{aligned} \tag{37}$$

From (36) and (37), we have

$$\frac{p^\alpha \Gamma(\alpha)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v)g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)g(u) \right] \leq_p \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) M(u, v) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right) N(u, v)$$

and the required result has been obtained. \square

Example 3. Let p be an odd number, $[u, v] = [0, 2]$, $\alpha = \frac{1}{2}$, $f(x) = [e^{x^p} - 4, 2x^p]$, and $g(x) = [x^p - 3, 2x^p]$. Then, $fg \in \mathfrak{S}\mathcal{L}_{([u,v])}$ and

$$\begin{aligned}
 & \frac{p^\alpha \Gamma(1+\alpha)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v)g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)g(u) \right] \\
 & = \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (2^p - x^p)^{\frac{-1}{2}} x^{p-1} \left[(4 - e^{x^p})(3 - x^p), 4x^{2p} \right] dx + \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (x^p)^{\frac{-1}{2}} x^{p-1} \left[(4 - e^{x^p})(3 - x^p), 4x^{2p} \right] dx \\
 & \approx [2.6446, 5.8664].
 \end{aligned}$$

Note that

$$\begin{aligned}
 M(u, v) &= [f(u)g(u) + f(v)g(v)] = [13 - e^2, 16] \\
 N(u, v) &= [f(u)g(v) + f(v)g(u)] = [15 - 3e^2, 0].
 \end{aligned}$$

Therefore, we have

$$\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(u, v) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(u, v) = \frac{11}{15} [13 - e^2, 16] + \frac{2}{15} [15 - 3e^2, 0] \approx [3.1591, 11.7333].$$

It follows that

$$[2.6446, 5.8664] \leq_p [3.1591, 11.7333],$$

and Theorem 5 has been illustrated.

Theorem 6. Let $p, \alpha > 0, u, v \in I$ with $v > u$ and $f, g \in \mathfrak{S}\mathcal{L}_{([u, v])}$. If $f, g \in LRSX([u, v], \mathbb{R}_I^+, p)$, then we have

$$2f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p^\alpha\Gamma(\alpha+1)}{2(v^p-u^p)^\alpha} [\mathcal{I}_{u^+}^{p,\alpha} f(v)g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)g(u)] + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(u, v) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(u, v). \tag{38}$$

If $f, g \in LRSV([u, v], \mathbb{R}_I^+, p)$, then

$$f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \geq_p \frac{p^\alpha\Gamma(\alpha+1)}{4(v^p-u^p)^\alpha} [\mathcal{I}_{u^+}^{p,\alpha} f(v)g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u)g(u)] + \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(u, v) + \frac{1}{2}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(u, v) \tag{39}$$

where $M(u, v)$ and $N(u, v)$ are given in Theorem 5.

Proof. Since $f, g \in LRSX([u, v], \mathbb{R}_I^+, p)$, then by hypothesis, for $\varrho \in [0, 1]$ we have

$$\begin{aligned} & f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \\ &= f\left[\frac{[(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}}{2} + \frac{[\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}}{2}\right] \times g\left[\frac{[(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}}{2} + \frac{[\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}}{2}\right] \\ &\leq_p \frac{1}{4} \left[f\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) + f\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right) \right] \\ &\quad \times \left[g\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) + g\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right) \right] \\ &= \frac{1}{4} \left[f\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right)g\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) \right. \\ &\quad \left. + \left[f\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right)g\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right) \right] \right. \\ &\quad \left. + \left[g\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right)f\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) \right] \right. \\ &\quad \left. + \left[f\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right)g\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) \right] \right] \\ &\leq_p \frac{1}{4} \left[f\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) + g\left(\left[\varrho u^p + (1-\varrho)v^p\right]^{\frac{1}{p}}\right) \right] \\ &\quad + f\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right)g\left(\left[(1-\varrho)u^p + \varrho v^p\right]^{\frac{1}{p}}\right) \\ &\quad + \frac{1}{4}(2\varrho^2 - 2\varrho + 1)N(u, v) + \frac{1}{2}\varrho(1-\varrho)M(u, v). \end{aligned} \tag{40}$$

Taking both multiplications of (40) with $q^{\alpha-1}$ and integrating the result with respect to over $(0,1)$, we have

$$\begin{aligned} & \int_0^1 q^{\alpha-1} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) dq \\ & \leq_p \frac{1}{4} \left[\int_0^1 q^{\alpha-1} f\left(\left[qu^p + (1-q)v^p\right]^{\frac{1}{p}}\right) g\left(\left[qu^p + (1-q)v^p\right]^{\frac{1}{p}}\right) dq \right. \\ & \quad \left. + \int_0^1 q^{\alpha-1} f\left(\left[(1-q)u^p + qv^p\right]^{\frac{1}{p}}\right) g\left(\left[(1-q)u^p + qv^p\right]^{\frac{1}{p}}\right) dq \right] \\ & \quad + \frac{1}{4} \int_0^1 q^{\alpha-1} (2q^2 - 2q + 1) N(u, v) + \frac{1}{2} \int_0^1 q^{\alpha-1} q(1-q) M(u, v) dq. \end{aligned} \tag{41}$$

From (41), we get

$$\begin{aligned} & \int_0^1 q^{\alpha-1} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) dq \\ & = \left[\int_0^1 q^{\alpha-1} \underline{f}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \underline{g}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) dq, \int_0^1 q^{\alpha-1} \bar{f}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \bar{g}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) dq \right] \\ & = \left[\frac{1}{\alpha} \underline{f}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \underline{g}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right), \frac{1}{\alpha} \bar{f}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \bar{g}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \right] \\ & = \frac{1}{\alpha} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right). \end{aligned} \tag{42}$$

On the other hand, from (42) and taking $qx^p = qu^p + (1-q)v^p$ and $y^p = (1-q)u^p + qv^p$, we get

$$\begin{aligned} & \frac{1}{4} \left[\int_0^1 q^{\alpha-1} f\left(\left[qu^p + (1-q)v^p\right]^{\frac{1}{p}}\right) g\left(\left[qu^p + (1-q)v^p\right]^{\frac{1}{p}}\right) dq \right. \\ & \quad \left. + \int_0^1 q^{\alpha-1} f\left(\left[(1-q)u^p + qv^p\right]^{\frac{1}{p}}\right) g\left(\left[(1-q)u^p + qv^p\right]^{\frac{1}{p}}\right) dq \right] \\ & \quad + \frac{1}{4} \int_0^1 q^{\alpha-1} (2q^2 - 2q + 1) N(u, v) dq + \frac{1}{2} \int_0^1 q^{\alpha-1} q(1-q) M(u, v) dq \\ & = \frac{p}{4(v^p-u^p)^\alpha} \left[\int_u^v (v^p-x^p)^{\alpha-1} \underline{f}(x) \underline{g}(x) x^{p-1} dx + \int_u^v (y^p-u^p)^{\alpha-1} \underline{f}(y) \underline{g}(y) y^{p-1} dy, \right. \\ & \quad \left. \int_0^1 (v^p-x^p)^{\alpha-1} \bar{f}(x) \bar{g}(x) x^{p-1} dx + \int_u^v (y^p-u^p)^{\alpha-1} \bar{f}(y) \bar{g}(y) y^{p-1} dy \right] \\ & \quad + \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) N(u, v) + \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) M(u, v) \\ & = \frac{p^\alpha \Gamma(\alpha+1)}{4(v^p-u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u) g(u) \right] \\ & \quad + \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) N(u, v) + \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) M(u, v). \end{aligned} \tag{43}$$

From (42) and (43), (41) becomes

$$\begin{aligned} 2f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) & \leq_p \frac{p^\alpha \Gamma(\alpha+1)}{2(v^p-u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p,\alpha} f(v) g(v) + \mathcal{I}_{v^-}^{p,\alpha} f(u) g(u) \right] \\ & \quad + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) N(u, v) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) M(u, v) \end{aligned}$$

Hence, Theorem 6 has been proved. \square

Example 4. Let p be an odd number and $\alpha = 1$ for $q \in [0, 1]$, and the LR- p -convex $f : [u, \vartheta] = [2, 3] \rightarrow \mathbb{R}_I^+$ and LR- p -convex IVFs $g : [u, \vartheta] = [2, 3] \rightarrow \mathbb{R}_I^+$ are respectively defined by $f(x) =$

$\left[2 - x^{\frac{p}{2}}, 2\left(2 - x^{\frac{p}{2}}\right)\right]$ and $g(x) = [x^p, 2x^p]$. Since $f_*(x) = 2 - x^{\frac{p}{2}}$, $f^*(x) = 2\left(2 - x^{\frac{p}{2}}\right)$ and $g_*(x) = x^p$, $g^*(x) = 2x^p$, then we compute the following

$$\begin{aligned}
 2 f_* \left(\left[\frac{u^p + \vartheta^p}{2} \right]^{\frac{1}{p}} \right) \times g_* \left(\left[\frac{u^p + \vartheta^p}{2} \right]^{\frac{1}{p}} \right) &= \frac{20 - 5\sqrt{10}}{2} \\
 2 f^* \left(\left[\frac{u^p + \vartheta^p}{2} \right]^{\frac{1}{p}} \right) \times g^* \left(\left[\frac{u^p + \vartheta^p}{2} \right]^{\frac{1}{p}} \right) &= 40 - 10\sqrt{10}, \\
 \frac{p^\alpha \Gamma(\alpha + 1)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p, \alpha} f_*(v) \times g_*(v) + \mathcal{I}_{v^-}^{p, \alpha} f_*(u) \times g_*(u) \right] &= 1 \\
 \frac{p^\alpha \Gamma(\alpha + 1)}{2(v^p - u^p)^\alpha} \left[\mathcal{I}_{u^+}^{p, \alpha} f^*(v) \times g^*(v) + \mathcal{I}_{v^-}^{p, \alpha} f^*(u) \times g^*(u) \right] &= 4, \\
 \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \mathcal{M}_*(u, \vartheta) &= \frac{1}{6} (10 - 2\sqrt{2} - 3\sqrt{3}) \\
 \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \mathcal{M}^*(u, \vartheta) &= \frac{4}{6} (10 - 2\sqrt{2} - 3\sqrt{3}), \\
 \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \mathcal{N}_*(u, \vartheta) &= \frac{1}{3} (10 - 3\sqrt{2} - 2\sqrt{3}) \\
 \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \mathcal{N}^*(u, \vartheta) &= \frac{4}{3} (10 - 3\sqrt{2} - 2\sqrt{3}),
 \end{aligned}$$

that means

$$\begin{aligned}
 \frac{20 - 5\sqrt{10}}{2} &\leq \left(1 + \frac{30 - 8\sqrt{2} - 7\sqrt{3}}{6} \right), \\
 40 - 10\sqrt{10} &\leq \left(4 + \frac{60 - 16\sqrt{2} - 14\sqrt{3}}{3} \right),
 \end{aligned}$$

hence, Theorem 6 has been illustrated.

4. Conclusions

In this work, we introduced the new class of LR- p -convex interval-valued functions and established some new Hermite-Hadamard inequalities by means of the pseudo order relation via Katugampola fractional integral operator. Useful examples that verify the applicability of the theory developed in this study are presented. We intend to use various types of LR-convex interval-valued functions to construct interval inequalities of interval-valued functions. In the future, we will try to explore this concept for fuzzy-interval-valued functions by means of the fuzzy pseudo order relation.

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