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Characterization of Transitivity in L-Tolerance Spaces by Convergence and Closure

Gunther Jäger [1,*] and T. M. G. Ahsanullah [2]

[1] School of Mechanical Engineering, University of Applied Sciences Stralsund, 18435 Stralsund, Germany
[2] Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia; tmga1@ksu.edu.sa
* Correspondence: gunther.jaeger@hochschule-stralsund.de

Abstract: We show that the category of quantale-valued tolerance spaces is isomorphic to a category of quantale-valued convergence spaces. We define suitable quantale-valued closure functions and use them to characterize transitivity axioms. Furthermore, transitivity is characterized by convergence and diagonal axioms. Quantale-valued tolerance relations compatible with group structures are also characterized by convergence and it is shown that they are transitive.

Keywords: L-tolerance space; L-pretolerance convergence space; L-tolerance convergence space; L-closure function; L-equivalence relation; L-valued set; L-tolerance convergence group

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1. Introduction

The definition of a tolerance relation as a reflexive and symmetric relation $\tau \subseteq X \times X$ is due to Zeeman [1] but it traces back to the works of Poincaré and their concept of *physical continuum* [2]. The theory of tolerance spaces is developed in the thesis of Poston [3] and developed further, e.g., in [4]. Algebraic structures compatible with tolerance relations have also been studied, see, e.g., [5]. Applications of tolerance relations are, among others, in the fields of information systems and image analysis, see, e.g., [6].

Fuzzy generalizations of tolerance relations with the unit interval as lattice of truth values were studied since the early nineties of the last century, see, e.g., [7,8]. The idea here is to not only state that two elements x, y of the space are similar by $(x, y) \in \tau$, where τ denotes the tolerance relation, but to allow "grades of similarity". In this way, two points $x, y \in X$ get assigned a valued $\tau(x, y) \in [0, 1]$, indicating their grade of being similar. Hence, τ is considered as a 'fuzzy relation", $\tau : X \times X \longrightarrow [0, 1]$. Replacing the unit interval by a quantale, i.e., by a complete lattice $(L, \leq)$ with a suitable algebraic operation, leads to more general quantale-valued tolerance relations $\tau : X \times X \longrightarrow L$. These appear, e.g., in the work of Stout [9] who uses them in the study of a categorical logic suitable for fuzzy set theory. From the viewpoint of characterizing all tolerance classes for a given quantale-valued tolerance relation on a set, they were also studied in detail in [10].

This paper adds to the theory of quantale-valued tolerance spaces by providing a suitable theory of convergence, which allows the introduction of topological concepts.

Special examples of quantale-valued tolerance relations are quantale-valued (partial) metric spaces. The convergence theory that we develop in this paper parallels the theory that is available for quantale-valued metric spaces and was developed in terms of quantale-value convergence towers in [11]. However, for a meaningful theory we have to impose a left-continuity condition under which we can describe such a convergence tower equivalently by a quantale-valued convergence function. This is the viewpoint that we adopt here. In order to be self-contained, we provide all proofs for the basic theory in Section 3 although they can mostly be adapted from [11]. New is the use of the quantale-valued convergence function to define quantale-valued closure functions (Section 4) and the application of these to characterize the important property of transitivity. This is achieved both for quantale-valued equivalence relations (in Section 5) and for quantale-valued equalities as introduced by Höhle [12] (in Section 6). In Section 7, we characterize both transitivities by diagonal axioms. Lastly, we apply our convergence theory to quantale-valued tolerance groups in Section 8.

2. Preliminaries

Let $(L, \leq)$ be a complete lattice with distinct bottom and top elements $\perp \neq \top$. In a complete lattice $(L, \leq)$ we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j \in J} \beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is *completely distributive* if and only if we have $\alpha = \bigvee\{\beta \ : \ \beta \lhd \alpha\}$ for any $\alpha \in L$, [13]. For more results on lattices, we refer to [14].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called an *integral, cummutative quantale* [15] if $(L, *)$ is a commutative semigroup for which the top element acts as the unit, i.e., if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e.,

$$\beta * \left(\bigvee_{i \in J} \alpha_i \right) = \bigvee_{i \in J} (\beta * \alpha_i)$$

for all $\alpha_i, \beta \in L$, $i \in J$. In a quantale we define an *implication operator* by $\alpha \to \beta = \bigvee\{\gamma \in L \ : \ \alpha * \gamma \leq \beta\}$. Then $\delta \leq \alpha \to \beta$ if and only if $\delta * \alpha \leq \beta$.

We consider in this paper only integral, commutative quantales $\mathsf{L} = (L, \leq, *)$ with completely distributive lattices $(L, \leq)$ and simply speak of a quantale from now on.

Example 1.

(1) **Left-continuous t-norms:** *A* triangular norm *or* t-norm *is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $\mathsf{L} = ([0, 1], \leq, *)$ is a quantale if the t-norm is left-continuous. Examples for (left-continuous) t-norms are the minimum t-norm, $\alpha * \beta = \alpha \wedge \beta$, the product t-norm, $\alpha * \beta = \alpha \cdot \beta$, and the Lukasiewicz t-norm, $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.*

(2) **Lawevere's quantale:** *The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + \alpha = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale $\mathsf{L} = ([0, \infty], \geq, +)$, see, e.g., [16,17].*

(3) **Distance distribution functions:** *A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which satisfies $\varphi(x) = \sup_{z<x} \varphi(z)$ for all $x \in (0, \infty)$, is called a* distance distribution function *[18]. We note that such a function satisfies $\varphi(0) = 0$ and is non-decreasing. Furthermore, note that, in contrast to [18], we do not require the* finiteness condition $\varphi(\infty) = 1$. *The set of distance distribution functions is denoted by Δ^+ and is ordered pointwise. With this order Δ^+ becomes a complete lattice and it is shown in [16] that Δ^+ is completely distributive. A quantale operation, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ is called a* sup-continuous triangle function *in [18].*

Sometimes we need two further requirements on the quantale. First, we call an integral and commutative quantale $\mathsf{L} = (L, \leq, *)$ *divisible* [19] if for all $\alpha, \beta \in L$, whenever $\alpha \leq \beta$, there is $\gamma \in L$ such that $\alpha = \beta * \gamma$. This is equivalent to the requirement $\alpha * (\alpha \to \beta) = \alpha \wedge \beta$ for all $\alpha, \beta \in L$.

Second, we need the axiom

(DM2) $\alpha \to \bigvee_{j \in J} \beta_j = \bigvee_{j \in J} (\alpha \to \beta_j)$ for all $\alpha, \beta_j \in L$, $J \neq \emptyset$.

Lawvere's quantale satisfies (DM2), however (DM2) is not always satisfied in the probabilistic case $\mathsf{L} = (\Delta^+, \leq, *)$. We show this with the next example.

Example 2. *We consider* $\mathsf{L} = (\Delta^+, \leq, *)$ *with the pointwise multiplication as triangle function, i.e., we define* $\varphi * \psi$ *for* $\varphi, \psi \in \Delta^*$ *by* $\varphi * \psi(x) = \varphi(x) \cdot \psi(x)$ *for all* $x \in X$*. For a subset* $A \subseteq [0, \infty]$ *we denote the characteristic function by* 1_A*, defined by* $1_A(x) = 1$ *if* $x \in A$ *and* $1_A(x) = 0$ *for* $x \notin A$*. Let further* $\psi = 1_{(1,\infty]} \in \Delta^+$*. For* $n \in \mathbb{N}$*, we consider* $\varphi_n \in \Delta^+$ *defined by* $\varphi_n(x) = 0$ *for* $0 \leq x < 1$ *and* $\varphi_n(x) = n(x-1)$ *for* $1 \leq x \leq 1 + \frac{1}{n}$*. Then* $\bigvee_{n \in \mathbb{N}} \varphi_n = 1_{(1,\infty]}$ *and hence* $\psi \to \bigvee_{n \in \mathbb{N}} \varphi_n = 1_{(0,\infty]}$*, the top element in* Δ^+*. Furthermore, we have by the definition of the implication, for each* $n \in \mathbb{N}$*,* $\psi \to \varphi_n = \bigvee\{\eta \in \Delta^+ : \eta(z) \leq \frac{\varphi_n(z)}{\psi(z)} \wedge 1 \forall z \in [0, \infty]\}$*. If* $\eta \in \Delta^+$ *satisfies* $\eta(z) \leq \frac{\varphi_n(z)}{\psi(z)} \wedge 1$ *for all* $z \in [0, \infty]$*, then for* $z > 1$ *we have* $\psi(z) = 1$ *and hence* $\eta(z) \leq \varphi_n(z)$*. As* $\varphi_n(1) = 0$ *this also implies* $\eta(z) = 0$ *for* $0 \leq z \leq 1$*. Therefore, we obtain* $\psi \to \varphi_n = \varphi_n$ *and* $\bigvee_{n \in \mathbb{N}}(\psi \to \varphi_n) = \bigvee_{n \in \mathbb{N}} \varphi_n = 1_{(1,\infty]}$*. This shows that* $\psi \to \bigvee_{n \in \mathbb{N}} \varphi_n \neq \bigvee_{n \in \mathbb{N}}(\psi \to \varphi_n)$*.*

Although we do not use it in this paper, we give an interesting characterization of (DM2). We call the well-below relation *multiplicative* if $\alpha \lhd \beta$ and $\theta \neq \bot$ imply $\alpha * \theta \lhd \beta * \theta$, for all $\alpha, \beta \in L$.

Proposition 1. *We consider a quantale* $\mathsf{L} = (L, \leq, *)$ *with completely distributive underlying lattice* $(L, \leq)$*. Then (DM2) is satisfied if and only if the well-below relation is multiplicative.*

Proof. Let first the condition (DM2) be satisfies and let $\alpha \lhd \beta$ and let $\theta \neq \bot$. Let further $D \subseteq L$ such that $\theta * \beta \leq \bigvee D$. Then $\alpha \lhd \beta \leq \theta \to \bigvee D = \bigvee_{\delta \in D}(\theta \to \delta)$ by (DM2). Hence there is $\delta \in D$ such that $\alpha \leq \theta \to \delta$, i.e., $\theta * \alpha \leq \delta$, which shows $\alpha * \theta \lhd \beta * \theta$.

Let now the well-below relation be multiplicative and let $\epsilon \lhd \theta \to \bigvee D$ with $\theta \neq \bot$ and $D \subseteq L$. Then $\theta * \epsilon \lhd \theta * (\theta \to \bigvee D) \leq \bigvee D$. Then there is $\delta \in D$ such that $\theta * \epsilon \leq \delta$, i.e., $\epsilon \leq \theta \to \delta$. Hence $\epsilon \leq \bigvee_{\delta \in D}(\theta \to \delta)$ and we have, using the complete distributivity, $\theta \to \bigvee D \leq \bigvee_{\delta \in D}(\theta \to \delta)$. The converse inequality is always true and thus we have equality. □

For a set X, we denote its power set by $\mathsf{P}(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, \ldots$ on X by $\mathsf{F}(X)$. The set $\mathsf{F}(X)$ is ordered by set inclusion and maximal elements of $\mathsf{F}(X)$ in this order are called *ultrafilters*. The set of all ultrafilters on X is denoted by $\mathsf{U}(X)$. In particular, for each $x \in X$, the point filter $[x] = \{A \subseteq X : x \in A\} \in \mathsf{F}(X)$ is an ultrafilter. If $\mathbb{F} \in \mathsf{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathsf{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G \text{ for some } F \in \mathbb{F}\}$.

For notions from category theory, we refer to [20]. In particular, we denote for a category $\mathcal{C}$ the class of its objects by $|\mathcal{C}|$.

3. L-Tolerance Spaces as L-Convergence Spaces

For a quantale $\mathsf{L} = (L, \leq, *)$, an L-*tolerance space* [9] is a pair (X, τ) of a set X and an L-*tolerance relation* $\tau : X \times X \longrightarrow L$ such that

(LTOL1) $\tau(x, y) \leq \tau(x, x)$ for all $x, y \in X$ (*reflexivity*);

(LTOL2) $\tau(x, y) = \tau(y, x)$ for all $x, y \in X$ (*symmetry*).

A mapping between two L-tolerance spaces, $f : (X, \tau) \longrightarrow (X', \tau')$ is called *tolerance preserving* if $\tau(x_1, x_2) \leq \tau'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-tolerance spaces with tolerance preserving mappings by L-Tol.

In case $L = \{0, 1\}$, an L-tolerance space is a tolerance space [1,4]. In this case we identify the L-tolerance relation with the relation $\tau_1 = \{(x, y) \in X \times X : \tau(x, y) = 1\}$ and we say that x and y are *similar* if $(x, y) \in \tau_1$.

Sometimes, e.g., in [10], a stronger reflexivity axiom is required instead of (LTOL1):

(LTOL1s) $\tau(x, x) = \top$ for all $x \in X$.

For Lawvere's quantale, special instances of L-tolerance spaces are metric spaces and partial metric spaces [21].

Let X be a set. A function $\lambda : \mathsf{F}(X) \longrightarrow L^X$ is called an L-*pretolerance convergence function* if it satisfies the axioms

(LC1) $\lambda([x])(y) \leq \lambda([x])(x)$ for all $x, y \in X$;

(LC2) $\lambda(\mathbb{F}) \leq \lambda(\mathbb{G})$ whenever $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$ and $\mathbb{F} \leq \mathbb{G}$;

(LC3) $\lambda(\bigwedge_{j\in J} \mathbb{F}_j) = \bigwedge_{j\in J} \lambda(\mathbb{F}_j)$ for all families of filters $(\mathbb{F}_j)_{j\in J}$;

(LS) $\lambda([x])(y) = \lambda([y])(x)$ for all $x, y \in X$.

The pair (X, λ) is then called an L-*pretolerance convergence space*. A mapping $f : X \longrightarrow X'$ between the L-pretolerance convergence spaces (X, λ) and (X', λ'), is called *continuous* if, for all $x \in X$ and all $\mathbb{F} \in \mathsf{F}(X)$, $\lambda(\mathbb{F})(x) \leq \lambda'(f(\mathbb{F}))(f(x))$. The category of L-pretolerance convergence spaces with continuous mappings as morphisms is denoted by $\mathsf{L\text{-}PreTolConv}$.

Remark 1. *(1) The axiom (LS) is a* symmetry axiom*. If we do not want to impose this, we could reformulate the axiom (LC1) by $\lambda([x])(y) \leq \lambda([x])(x) \wedge \lambda([y])(y)$ for all $x \in X$.*

(2) Sometimes a stronger form of the axiom (LC1) is required: (LC1s) $\lambda([x])(x) = \top$ for all $x \in X$.

For a function $\tau : X \times X \longrightarrow L$, we define a function $\lambda^\tau : \mathsf{F}(X) \longrightarrow L^X$ by defining

$$\lambda^\tau(\mathbb{F})(x) = \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} \tau(x, y), \qquad \text{for } \mathbb{F} \in \mathsf{F}(X) \text{ and } x \in X.$$

Proposition 2. *Let $(X, \tau) \in |\mathsf{L\text{-}Tol}|$. Then $(X, \lambda^\tau) \in |\mathsf{L\text{-}PreTolConv}|$.*

Proof. (LC1) We first note that by definition $\lambda^\tau([x])(y) = \tau(y, x) = \tau(x, y)$. Then $\lambda^\tau([x])(y) = \tau(y, x) \leq \tau(x, x) = \lambda^\tau([x])(x)$ by (LTOL1). (LC2) is obvious.

(LC3) The one inequality $\leq$ is clear. To show the converse, let $\epsilon \lhd \bigwedge_{j\in J} \lambda^\tau(\mathbb{F}_j)$ and let $\epsilon \lhd \alpha$. Then for all $j \in J$ there is $F_j^\epsilon \in \mathbb{F}_j$ such that for all $y_j \in F_j^\epsilon$, $\tau(x, y_j) \geq \epsilon$. Then $F = \bigcup_{j\in J} F_j^\epsilon \in \bigwedge_{j\in J} \mathbb{F}_j$ and for all $y \in F$ we have $\tau(x, y) \geq \epsilon$. Hence $\lambda^\tau(\bigwedge_{j\in J} \mathbb{F}_j)(x) = \bigvee_{F\in\bigwedge_{j\in J}\mathbb{F}_j} \bigwedge_{y\in F} \tau(x, y) \geq \epsilon$. The complete distributivity yields $\lambda^\tau(\bigwedge_{j\in J} \mathbb{F}_j)(x) \geq \bigwedge_{j\in J} \lambda^\tau(\mathbb{F}_j)$.

(LS) is clear noting again $\lambda^\tau([x])(y) = \tau(y, x) = \tau(x, y) = \lambda^\tau([y])(x)$ and (LTOL2). □

Remark 2.

(1) For $L = \{0, 1\}$ we have $\lambda^\tau(\mathbb{F})(x) = 1$ iff $N^\tau(x) = \{y \in X : \tau(x, y) = 1\} \in \mathbb{F}$. If we consider a sequence and the filter generated by the endpieces of the sequence, then this means that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if there is an endpiece $\{x_{n_0}, x_{n_0+1}, ...\}$ such that $\tau(x_n, x) = 1$ for all $n \geq n_0$, i.e., such that all members of the endpiece are similar to x.

*(2) For Lawvere's quantale $\mathsf{L} = ([0, \infty], \geq, +)$ we have that $\lambda^\tau(\mathbb{F})(x) = \inf_{F\in\mathbb{F}} \sup_{y\in F} \tau(x, y) \leq \alpha$ iff for all $\beta > \alpha$ we have $N_\beta^\tau(x) = \{y \in X : \tau(x, y) < \beta\} \in \mathbb{F}$. If we define the α-*neighbourhood filter* of x, $\mathbb{U}_\alpha^\tau(x)$, as the filter generated by all $N_\beta^\tau(x)$ for $\beta > \alpha$, then this means that $\mathbb{F} \geq \mathbb{U}_\alpha^\tau(x)$. Again, in terms of sequences, this means that for each $\beta > \alpha$, there is an endpiece such that all members y of the endpiece are similar to x with $\tau(x, y) < \beta$.*

Proposition 3. *Let $f : (X, \tau) \longrightarrow (X', \tau')$ be tolerance preserving. Then $f : (X, \lambda^\tau) \longrightarrow (X', \lambda^{\tau'})$ is continuous.*

Proof. We have, for $\mathbb{F} \in \mathsf{F}(X)$ and $x \in X$,

$$\lambda^\tau(\mathbb{F})(x) = \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} \tau(x, y) \leq \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} \tau'(f(x), f(y))$$

$$= \bigvee_{F\in\mathbb{F}} \bigwedge_{y'\in f(F)} \tau'(f(x),y') \le \bigvee_{G\in f(\mathbb{F})} \bigwedge_{y'\in G} \tau'(f(x),y') = \lambda^{\tau'}(f(\mathbb{F}))(f(x)).$$

□

Hence, we have a functor from L-Tol into L-$\mathsf{PreTolConv}$. We note that if $\tau \neq \tau'$ then there are $x, y \in X$ such that $\lambda^{\tau}([x])(y) \neq \lambda^{\tau'}([x])(y)$. Therefore this functor is injective on objects.

We define now, for a function $\lambda : \mathsf{F}(X) \longrightarrow L^X$, a function $\tau^{\lambda} : X \times X \longrightarrow L$ by defining for $x, y \in X$, $\tau^{\lambda}(x,y) = \lambda([y])(x)$.

Proposition 4. *Let* $(X,\lambda) \in |\mathsf{L}\text{-}\mathsf{PreTolConv}|$. *Then* $(X,\tau^{\lambda}) \in |\mathsf{L}\text{-}\mathsf{Tol}|$.

Proof. (LTOL1) follows from (LC1) and (LTOL2) follows from the symmetry (LS). □

Proposition 5. *Let* $f : (X,\lambda) \longrightarrow (X',\lambda')$ *be continuous. Then* $f : (X,\tau^{\lambda}) \longrightarrow (X',\tau^{\lambda'})$ *is tolerance preserving.*

Proof. We have $\tau^{\lambda}(x,y) = \lambda([y])(x) \le \lambda'([f(y)])(f(x)) = \tau^{\lambda'}(f(x),f(y))$. □

Proposition 6. *Let* $(X,\tau) \in |\mathsf{L}\text{-}\mathsf{Tol}|$. *Then* $\tau^{(\lambda^{\tau})} = \tau$.

Proof. We have $\tau^{(\lambda^{\tau})}(x,y) = \lambda^{\tau}([y])(x) = \tau(x,y)$. □

Proposition 7. *Let* $(X,\lambda) \in |\mathsf{L}\text{-}\mathsf{PreTolConv}|$. *Then* $\lambda^{(\tau^{\lambda})}(\mathbb{F})(x) \le \lambda(\mathbb{F})(x)$.

Proof. Let $\epsilon \lhd \lambda^{(\tau^{\lambda})}(\mathbb{F})(x)$. Then there is $F_\epsilon \in \mathbb{F}$ such that for all $y \in F_\epsilon$ we have $\lambda([y])(x) = \tau^{\lambda}(x,y) \ge \epsilon$. From (LC3) we get, with $\bigwedge_{y\in F_\epsilon}[y] = [F_\epsilon] \le \mathbb{F}$, $\epsilon \le \bigwedge_{y\in F_\epsilon} \lambda([y])(x) = \lambda([F_\epsilon])(x) \le \lambda(\mathbb{F})(x)$. The complete distributivity of L yields the claim. □

Combining Propositions 2 to 7 we obtain the following result.

Theorem 1. *The category* L-Tol *can be coreflectively embedded into the category* L-$\mathsf{PreTolConv}$.

We introduce the following axiom for $(X,\lambda) \in |\mathsf{L}\text{-}\mathsf{PreTolConv}|$.

(LT) $\lambda(\mathbb{U})(x) = \bigwedge_{U\in\mathbb{U}} \bigvee_{y\in U} \lambda([y])(x)$ for all $\mathbb{U} \in \mathsf{U}(X)$ and all $x \in X$.

The following little result is proved for $L = [0,\infty]$ in [22], Proposition 1.8.29. It is proved in more generality as Lemma B in [11].

Lemma 1 ([12])**.** *Let* $(L,\le)$ *be completely distributive and let* $\mathbb{U} \in \mathsf{U}(X)$ *be an ultrafilter and let* $f : X \longrightarrow L$ *be a mapping. Then* $\bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} f(y) = \bigwedge_{U\in\mathbb{U}} \bigvee_{y\in U} f(y)$.

Proposition 8. *Let* $(X,\tau) \in |\mathsf{L}\text{-}\mathsf{Tol}|$. *Then* (X,λ^{τ}) *satisfies (LT).*

Proof. We consider, for a fixed $x \in X$, the function $f(y) = \tau(x,y) = \lambda^{\tau}([y])(x)$. Then the lemma above yields

$$\lambda^{\tau}(\mathbb{U})(x) = \bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} \lambda^{\tau}([y])(x) = \bigwedge_{U\in\mathbb{U}} \bigvee_{y\in U} \lambda^{\tau}([y])(x).$$

□

Proposition 9. *Let* $(X,\lambda) \in |\mathsf{L}\text{-}\mathsf{PreTolConv}|$ *satisfy the axiom (LT). Then* $\lambda^{(\tau^{\lambda})}(\mathbb{F})(x) = \lambda(\mathbb{F})(x)$ *for all* $\mathbb{F} \in \mathsf{F}(X)$ *and all* $x \in X$.

Proof. Let $\mathbb{U} \in \mathsf{U}(X)$ be an ultrafilter and let $\beta \lhd \lambda(\mathbb{U})(x)$. For $U \in \mathbb{U}$ we obtain from the axiom (LT) that there is $y_\beta \in U$ such that $\lambda([y_\beta])(x) \geq \beta$. Hence $y_\beta \in N_\beta^x = \{y \in X : \lambda([y])(x) \geq \beta\}$ and therefore $U \cap N_\beta^x \neq \varnothing$ and we conclude $N_\beta^x \in \mathbb{U}$. Hence

$$\lambda^{(\tau^\lambda)}(\mathbb{U})(x) = \bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} \lambda([y])(x) \geq \bigwedge_{y\in N_\beta^x} \lambda([y])(x) \geq \beta$$

and the complete distributivity yields $\lambda(\mathbb{U})(x) \leq \lambda^{(\tau^\lambda)}(\mathbb{U})(x)$. As both (X,λ) and $(X,\lambda^{(\tau^\lambda)})$ satisfy (LC3) we conclude $\lambda(\mathbb{F})(x) \leq \lambda^{(\tau^\lambda)}(\mathbb{F})(x)$ for all $\mathbb{F} \in \mathbb{F}(X)$. The converse inequality is always true and so we have the desired equality. □

If we denote the subcategory of L-$\mathsf{PreTolConv}$ with objects the L-pretolerance convergence spaces that satisfy the axiom (LT) by L-$\mathsf{TolConv}$, then we obtain the following main result.

Theorem 2. *The categories* L-$\mathsf{TolConv}$ *and* L-Tol *are isomorphic.*

Proof. We define the functors $\mathsf{G} : \mathsf{L}\text{-}\mathsf{Tol} \longrightarrow \mathsf{L}\text{-}\mathsf{TolConv}$ by $\mathsf{G}((X,\lambda)) = (X,\tau^\lambda)$ (and leaving morphism unchanged) and $\mathsf{H} : \mathsf{L}\text{-}\mathsf{TolConv} \longrightarrow \mathsf{L}\text{-}\mathsf{Tol}$ by $\mathsf{H}((X,\tau)) = (X,\lambda^\tau)$ (and again leaving morphisms unchanged). By Proposition 6 then $\mathsf{H} \circ \mathsf{G} = id_{\mathsf{L}\text{-}\mathsf{Tol}}$ and by Proposition 9 also $\mathsf{G} \circ \mathsf{H} = id_{\mathsf{L}\text{-}\mathsf{TolConv}}$. Hence, according to [20], Definition 3.24, G and H provide the required isomorphism. □

Therefore, if we define an L-*tolerance convergence space* (X,λ) by an L-*tolerance convergence function* $\lambda : \mathsf{F}(X) \longrightarrow L^X$ with the axioms (LC1), (LC2), (LC3), (LS) and (LT) then these spaces can be identified with L-tolerance spaces.

4. L-Tolerance Closures

The availability of a convergence notion allows us to introduce topological concepts. We shall discuss here a suitable concept of closure of sets. For a space $(X,\lambda) \in |\mathsf{L}\text{-}\mathsf{TolConv}|$ and a subset $A \subseteq X$, we define

$$c^\lambda(A)(x) = \bigvee_{\mathbb{U}\in\mathsf{U}(X), A\in\mathbb{U}} \lambda(\mathbb{U})(x).$$

We call $c^\lambda : \mathsf{P}(X) \longrightarrow L^X$ an L-*closure function*. Then $c^\lambda(A)$ generalizes the concept of closure of a set $A \subseteq X$ in a topological (or convergence) space in the sense that for $L = \{0,1\}$ it collapses to the "classical definition"

$$x \in \overline{A} \iff \exists \mathbb{U} \in \mathsf{U}(X), A \in \mathbb{U}, \mathbb{U} \longrightarrow x.$$

For an L-tolerance space (X,τ) we define $c^\tau(A)(x) := c^{(\lambda^\tau)}(A)(x)$.

Proposition 10. *Let* $(X,\tau) \in |\mathsf{L}\text{-}\mathsf{Tol}|$ *and let* $A \subseteq X, x \in X$. *Then* $c^\tau(A)(x) = \bigvee_{y\in A} \tau(x,y)$.

Proof. We have, using $\tau(x,y) = \lambda^\tau([y])(x)$,

$$\bigvee_{y\in A} \tau(x,y) = \bigvee_{y\in A} \lambda^\tau([y])(x) \leq \bigvee_{A\in\mathbb{U}\in\mathbb{U}(X)} \lambda^\tau(\mathbb{U})(x) = c^{(\lambda^\tau)}(A)(x).$$

Furthermore, using (LT) for (X,λ^τ), we obtain

$$c^{(\lambda^\tau)}(A)(x) = \bigvee_{A\in\mathbb{U}\in\mathbb{U}(X)} \bigwedge_{U\in\mathbb{U}} \bigvee_{y\in U} \lambda^\tau([y])(x) \leq \bigvee_{y\in A} \lambda^\tau([y])(x) = \bigvee_{y\in A} \tau(x,y).$$

□

We have $c^\tau(A)(x) \geq \alpha$ iff $\bigvee_{y \in A} \tau(x,y) \geq \alpha$ and this collapses in the case $L = \{0,1\}$ to

$$x \in \overline{A} \iff \exists y \in A \text{ s.t. } (x,y) \in \tau,$$

which is the image $\tau[A]$ of A under the relation τ. In [4], it is called the *widening of A* and [23] uses it as the definition of the closure of a set in a tolerance space. Proposition 10 justifies this name.

The following result collects the properties of the L-closure function c^τ. The observation that for all $x, y \in X$ we have $c^\tau(\{y\})(x) = \tau(x,y)$ makes the proof trivial.

Proposition 11. *Let* $(X, \tau) \in |\mathsf{L}\text{-}\mathsf{Tol}|$. *Then we have*

(LCl1) $c^\tau(\{x\})(y) \leq c^\tau(\{x\})(x)$ *for all* $x, y \in X$;

(LCl2) $c^\tau(\{y\})(x) = c^\tau(\{x\})(y)$ *for all* $x, y \in X$;

(LCl3) $c^\tau(A)(x) = \bigvee_{y \in A} c^\tau(\{y\})(x)$ *for all* $A \subseteq X$ *and all* $x, y \in X$.

On the other hand, if we have an L-closure function $c : \mathsf{P}(X) \longrightarrow L^X$ satisfying the properties (LCl1), (LCl2) and (LCl3), then $\tau^c(x,y) = c(\{y\})(x)$ defines an L-tolerance relation and we have, for $(X,\tau) \in |\mathsf{L}\text{-}\mathsf{Tol}|$ that $\tau^{(c^\tau)} = \tau$ and for an L-closure function with the properties (LCl1), (LCl2) and (LCl3), we have $c^{(\tau^c)} = c$. If, furthermore, we call a mapping between two L-tolerance closure spaces, $f : (X,c) \longrightarrow (X',c')$ *closure preserving* if $c(A)(x) \leq c'(f(A))(f(x))$ for all $A \subseteq X$ and all $x \in X$, then $f : (X, \tau^c) \longrightarrow (X', \tau^{c'})$ is tolerance preserving and, conversely, for a tolerance preserving mapping $f : (X,\tau) \longrightarrow (X',\tau')$, the mapping $f : (X, c^\tau) \longrightarrow (X', c^{\tau'})$ is closure preserving. Hence, the categories $\mathsf{L}\text{-}\mathsf{Tol}$ and $\mathsf{L}\text{-}\mathsf{TolCl}$ of L-tolerance closure spaces are isomorphic.

Remark 3. *An application of the* L*-closure function is given in [6]. They define in the case* $L = \{0,1\}$ *for a given tolerance relation* τ *on* X*, the so-called* Zeemann tolerance relation, $\overline{\tau}$ *on* $\mathsf{P}(X)$ *by* $(A,B) \in \overline{\tau} \iff A \subseteq c^\tau(B)$ *and* $B \subseteq c^\tau(A)$. *We can generalize this as follows. For* $A, B \subseteq X$ *we define*

$$\overline{\tau}(A,B) = \bigwedge_{a \in A} c^\tau(B)(a) \wedge \bigwedge_{b \in B} c^\tau(A)(b) = \bigwedge_{a \in A} \bigvee_{y \in B} \tau(a,b) \wedge \bigwedge_{b \in B} \bigvee_{a \in A} \tau(b,a).$$

For Lawvere's quantale $\mathsf{L} = ([0,\infty], \geq, +)$ *and a metric d as* L*-tolerance relation we recognize this as the Hausdorff distance between the subsets A and B in the metric space* (X,d). *We note that clearly the axiom (LTOL2) is satisfied for* $\overline{\tau}$. *Furthermore we have* $\bigvee_{b \in B} \tau(a,b) \leq \tau(a,a)$ *for all* $a \in A$ *and hence* $\overline{\tau}(A,B) \leq \bigwedge_{a \in A} \bigvee_{b \in B} \tau(a,b) \leq \bigwedge_{a \in A} \tau(a,a) \leq \bigwedge_{a \in A} \bigvee_{a' \in A} \tau(a,a') = \overline{\tau}(A,A)$ *and we have (LTOL1).*

5. Transitivity: L-Equivalence Relations

To date, we have used from the quantale $\mathsf{L} = (L, \leq, *)$ only the underlying lattice $(L, \leq)$ and made no reference to the quantale operation. This becomes different if we wish to consider the property of transitivity. A tolerance relation is transitive if $(x,y) \in \tau$ and $(y,z) \in \tau$ implies $(x,z) \in \tau$ for all $x,y,z \in X$. A transitive tolerance relation is an equivalence relation. Hence the logical connective "and" needs to be modelled in the quantale-valued case. This can be done in the following way.

An L-tolerance relation is an L*-equivalence relation on* X, see, e.g., [10], if $\tau : X \times X \longrightarrow L$ satisfies

(LTOL1s) $\tau(x,x) = \top$ for all $x \in X$ (strong reflexivity);

(LTOL2) $\tau(x,y) = \tau(y,x)$ for all $x, y \in X$ (symmetry);

(LTrans) $\tau(x,y) * \tau(y,z) \leq \tau(x,z)$ for all $x,y,z \in X$ (transitivity).

An L-pretolerance convergence space (X, λ) is called *transitive* if

(LCTrans) $\lambda([z])(y) * \lambda([y])(x) \le \lambda([z])(x)$ for all $x, y, z \in X$.

As $\lambda^\tau([y])(x) = \tau(x, y)$ for all $x, y \in X$ we immediately see that $(X, \tau) \in |\mathsf{L\text{-}Tol}|$ is transitive if and only if (X, λ^τ) is transitive. Similarly, as $\tau^\lambda(x, y) = \lambda([y])(x)$ for all $x, y \in X$, we see that $(X, \lambda) \in |\mathsf{L\text{-}PreTolConv}|$ is transitive if and only if (X, τ^λ) is transitive. As the axiom (LTOL1s) is satisfied for (X, τ) if and only if (X, λ^τ) satisfies (LC1s) and conversely, (X, λ) satisfies (LC1s) if and only if (X, τ^λ) satisfies (LTOL1s), we see that L-tolerance spaces with an L-equivalence relation can be identified with L-tolerance convergence spaces that satisfy (LC1s) and (LCTrans).

We are now going to characterize transitivity by the L-tolerance closure. First, we need the following result.

Lemma 2. *Let $(X, \lambda) \in |\mathsf{L\text{-}TolConv}|$ and let $A \subseteq X$ and $x \in X$. Then $c^\lambda(A)(x) = c^{(\tau^\lambda)}(A)(x)$.*

Proof. We have, using the axiom (LT), $c^\lambda(A)(x) = \bigvee_{A \in \mathbb{U} \in \mathsf{U}(X)} \bigwedge_{U \in \mathbb{U}} \bigvee_{y \in U} \lambda([y])(x)$. This is clearly $\le \bigvee_{y \in A} \lambda([y])(x)$ and, choosing for $y \in A$ the ultrafilter $[y]$, it is also $\ge \bigvee_{y \in A} \lambda([y])(x)$. Noticing $c^{(\tau^\lambda)}(A)(x) = \bigvee_{y \in A} \tau^\lambda(x, y) = \bigvee_{y \in A} \lambda([y])(x)$ then completes the proof. □

Theorem 3. *Let $(X, \lambda) \in |\mathsf{L\text{-}TolConv}|$. The following statements are equivalent.*

(1) (X, λ) is transitive.

(2) $c^\lambda(B)(x) * \bigwedge_{y \in B} c^\lambda(A)(y) \le c^\lambda(A)(x)$ for all $A, B \subseteq X$, $x \in X$.

Proof. Let first the axiom (LCTrans) be satisfied and let $A, B \subseteq X$ and let $x \in X$. Then for $b \in B$ we have, using the symmetry of τ^λ in the third step,

$$\tau^\lambda(x, b) * \left(\bigwedge_{y \in B} \bigvee_{a \in A} \tau^\lambda(a, b) \right) \le \tau^\lambda(x, b) * \bigvee_{a \in A} \tau^\lambda(a, b)$$

$$= \bigvee_{a \in A} (\tau^\lambda(x, b) * \tau^\lambda(b, a)) \le \bigvee_{a \in A} \tau^\lambda(x, a) = c^\lambda(A)(x).$$

Hence, using the distributivity of the quantale operation over joins, we obtain

$$c^\lambda(B)(x) * \bigwedge_{y \in B} c^\lambda(A)(y) = \bigvee_{b \in B} \left(\tau^\lambda(x, b) * \left(\bigwedge_{b \in B} \bigvee_{a \in A} \tau^\lambda(a, b) \right) \right) \le c^\lambda(A)(x).$$

For the converse, we choose $B = \{y\}$ and $A = \{z\}$ and conclude

$$\lambda([x])(y) * \lambda([y])(z) = c^\lambda(\{y\})(x) * \bigwedge_{u \in \{y\}} c^\lambda(\{z\})(u) \le c^\lambda(\{z\})(x) = \lambda([x])(z).$$

□

Corollary 1. *Let $(X, \tau) \in |\mathsf{L\text{-}Tol}|$. The following statements are equivalent.*

(1) (X, τ) is transitive.

(2) $c^\tau(B)(x) * \bigwedge_{y \in B} c^\tau(A)(y) \le c^\tau(A)(x)$ for all $A, B \subseteq X$, $x \in X$.

Remark 4. *(1) Property (2) of Theorem 3 was observed for Lawvere's quantale in the realm of approach spaces [22] and for quantale-valued topological spaces in [24]. In our setting with an integral, commutative quantale, the axioms of an L-valued topolocical space defined via an L-closure operator $c : \mathsf{P}(X) \longrightarrow L^X$ are $c(\{x\})(x) = \top$ for all $x \in X$, the transitivity as per Theorem 3, $c(\varnothing)(x) = \bot$ for all $x \in X$ and $c(A \cup B)(x) = c(A)(x) \vee c(B)(x)$ for all $A, B \subseteq X, x \in X$. The last two conditions are satisfied for c^λ using (LCl3), so that* *we conclude that a transitive* L*-tolerance convergence space* (X,λ) *satisfying (LC1s) is an* L*-valued topological space in the sense of [24].*

(2) *Property (2) in Theorem 3 is the* idempotency of the closure *in the following sense. For* $A \subseteq X$ *and* $\alpha \in L$ *we define* $\overline{A}^{\alpha} = \{x \in X : c^{\lambda}(A)(x) \geq \alpha\}$. *Then (2) if and only if* $\overline{\overline{A}^{\alpha}}^{\beta} \subseteq \overline{A}^{\alpha*\beta}$ *for all* $A \subseteq X$ *and all* $\alpha, \beta \in L$. *To see this, let first* $x \in \overline{\overline{A}^{\alpha}}^{\beta}$. *Then* $c^{\lambda}(\overline{A}^{\alpha})(x) \geq \beta$. *For* $y \in \overline{A}^{\alpha}$ *we know* $c^{\lambda}(A)(y) \geq \alpha$. *Hence* $\beta * \alpha \leq c^{\lambda}(\overline{A}^{\alpha})(x) * \bigwedge_{y \in \overline{A}^{\alpha}} c^{\lambda}(A)(y) \leq c^{\lambda}(A)(x)$, *which means* $x \in \overline{A}^{\alpha*\beta}$. *To show the converse, let* $\beta \leq c^{\lambda}(B)(x)$ *and* $\alpha \leq \bigwedge_{y\in B} c^{\lambda}(A)(y)$. *Then* $x \in \overline{B}^{\beta}$ *and* $B \subseteq \overline{A}^{\alpha}$. *From this it follows that* $x \in \overline{\overline{A}^{\alpha}}^{\beta} \subseteq \overline{A}^{\alpha*\beta}$, *which in turn means* $\alpha * \beta \leq c^{\lambda}(A)(x)$.

6. Transitivity: L-Valued Sets

In the absence of strong reflexivity (LT1s), a stronger form of transitivity can be formulated for an L-tolerance space as follows.

Definition 1 ([12])**.** *An* L-valued set *is a pair* (X,τ) *with an* L-valued equality $\tau : X \times X \longrightarrow L$ *such that*

(LT1) $\tau(x,y) \leq \tau(x,x)$ *for all* $x, y \in X$ *(reflexivity);*

(LT2) $\tau(x,y) = \tau(y,x)$ *for all* $x, y \in X$ *(symmetry);*

(LTrans*) $\tau(x,y) * (\tau(y,y) \to \tau(y,z)) \leq \tau(x,z)$ *for all* $x, y, z \in X$ *(strong transitivity).*

In the case of Lawvere's quantale, an L-valued set is a partial pseudometric space [21].

Clearly, in the presence of (LTOL1s), the axioms (LTrans) and (LTrans*) are equivalent. In general, as $\tau(y,z) \leq \tau(y,y) \to \tau(y,z)$, the axiom (LTrans*) implies the axiom (LTrans).

Example 3. *A simple example of an* L*-tolerance space which is transitive but not strongly transitive is* $\mathsf{L} = ([0,1], \leq, *)$ *with the Lukasiewicz t-norm,* $X = [0,1]$ *with* $\tau(\alpha,\beta) = \alpha * \beta = (\alpha + \beta - 1) \vee 0$. *Clearly,* $(\alpha * \beta) * (\beta * \gamma) \leq \alpha * \gamma$, *but, e.g., for* $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$ *and* $\gamma = \frac{1}{3}$ *we obtain* $\tau(\alpha,\beta) * (\tau(\beta,\beta) \to \tau(\beta,\gamma)) = \frac{5}{4} * (0 \to 0) = \frac{5}{4}$, *while* $\tau(\alpha,\gamma) = \frac{1}{12}$.

We can again define suitable convergence functions that characterize L-valued equalities, so that also here a convergence theory can be developed. In this respect, an L-pretolerance convergence space (X,λ) is called *strongly transitive* if the function λ satisfies the axiom

(LCTrans*) $(\lambda([y])(y) \to \lambda([z])(y)) * \lambda([y])(x) \leq \lambda([z])(x)$ for all $x, y, z \in X$.

Strongly transitive L-tolerance convergence spaces can be identified with L-valued sets.

Before we proceed and study suitable L-closure operators, we introduce an *auxiliary* L*-convergence function*. For $(X,\lambda) \in |\mathsf{L\text{-}TolConv}|$ we define $\overline{\lambda} : \mathsf{F}(X) \longrightarrow L^X$ by

$$\overline{\lambda}(\mathbb{F})(x) = \lambda([x])(x) \to \lambda(\mathbb{F})(x).$$

Clearly, if (X,λ) satisfies (LC1s), then $\overline{\lambda} = \lambda$. The L-convergence function $\overline{\lambda}$ then satisfies the axioms (LC1s), (LC2), (LC3) and [(LSw)] $\lambda([x])(x) = \lambda([y])(y)$ implies $\overline{\lambda}([x])(y) = \overline{\lambda}([y])(x)$. If the quantale satisfies (DM2), then also (LT) is satisfied. This all follows using elementary properties of the implication operator. We note that $(X,\overline{\lambda})$ is a natural example of a non-symmetric L-convergence space.

Similarly, for an L-tolerance space (X,τ), we define the *auxiliary* L*-relation* $\overline{\tau} : X \times X \longrightarrow L$ by $\overline{\tau}(x,y) = \tau(x,x) \to \tau(x,y)$ for all $x, y \in X$. The function $\overline{\tau}$ then satisfies the properties (LTOL1s) $\overline{\tau}(x,x) = \top$ for all $x \in X$ and (LTOL2w) $\overline{\tau}(x,y) = \overline{\tau}(y,x)$ whenever $\tau(x,x) = \tau(y,y)$.

Example 4. *We consider the* L*-tolerance on* L*, defined by* $\tau(\alpha, \beta) = \alpha \wedge \beta$*. Here we obtain* $\overline{\tau}(\alpha, \beta) = \alpha \to (\alpha \wedge \beta) = (\alpha \to \alpha) \wedge (\alpha \to \beta) = \top \wedge (\alpha \to \beta) = \alpha \to \beta$*.*

It is not difficult to show that $\tau^{\overline{\lambda}} = \overline{\tau^{\lambda}}$ and that with (DM2) we have $\lambda^{\overline{\tau}} = \overline{\lambda^{\tau}}$. For the compostions, we have $\overline{\tau^{\overline{\lambda^{\tau}}}} = \overline{\tau}$ and, again with (DM2), $\overline{\lambda^{\tau^{\overline{\lambda}}}} = \overline{\lambda}$. Hence, we have again one-to-one correspondences between $(X, \overline{\tau})$ and $(X, \lambda^{\overline{\tau}})$, and between $(X, \overline{\lambda})$ and $(X, \tau^{\overline{\lambda}})$.

It is of interest to study L-closure operators also in this context. The key to what follows is the following result.

Proposition 12. *Let the quantale* L *be divisible and let* $(X, \lambda) \in |\mathsf{L\text{-}TolConv}|$*. Then* λ *satisfies (LCTrans*) if and only if* $\overline{\lambda}$ *satisfies (LCTrans).*

Proof. We first note that from (LC1) and the divisibilty of L we obtain, for all $x, y \in X$,

$$\lambda([y])(x) = \lambda([y])(x) \wedge \lambda([x])(x) = \lambda([x])(x) * (\lambda([x])(x) \to \lambda([y])(x)).$$

With this, the adjunction $\delta \leq \alpha \to \beta \iff \delta * \alpha \leq \beta$ yields

$$\lambda([y])(x) * (\lambda([y])(y) \to \lambda([z])(y)) \ \leq \ \lambda([z])(x)$$

if and only if

$$\lambda([x])(x) * (\lambda([x])(x) \to \lambda([y])(x)) * (\lambda([y])(y) \to \lambda([z])(y)) \ \leq \ \lambda([z])(x),$$

from which the claimed equivalence immediately follows. □

We will therefore use an L-closure operator for $(X, \overline{\lambda})$ to characterize strong transitivity. We define, for an L-tolerance convergence space (X, λ) and $A \subseteq X$ and $x \in X$, $C^{\lambda} : \mathsf{P}(X) \longrightarrow L^X$ by $C^{\lambda}(A)(x) = \bigvee_{y \in A}(\lambda([x])(x) \to \lambda([y])(x))$. We point out that for this definition of L-closure operator we need the quantale operation (via the implication), whereas the L-closure operator c^{λ} does not make use of the quantale operation in L but only depends on the lattice $(L, \leq)$. In order that $C^{\lambda}(A)(x) = \lambda([x])(x) \to \bigvee_{y \in A} \lambda([y])(x) = \lambda([x])(x) \to c^{\lambda}(A)(x)$, so that C^{λ} is indeed the L-closure operator for $(X, \overline{\lambda})$, we need the axiom (DM2).

Furthermore, for $(X, \tau) \in |\mathsf{L\text{-}Tol}|$, we define $C^{\tau}(A)(x) = C^{(\lambda^{\tau})}(A)(x)$. This definition is motivated by Section 4 and taylored to give $C^{\tau}(A)(x) = \bigvee_{y \in A}(\tau(x, x) \to \tau(x, y))$. Moreover, we then also trivially have $C^{(\tau^{\lambda})} = C^{\lambda}$.

Theorem 4. *Let* $\mathsf{L} = (L, \leq, *)$ *be divisible and let* $(X, \lambda) \in |\mathsf{L\text{-}TolConv}|$*. The following statements are equivalent.*

(1) (X, λ) *is strongly transitive.*
(2) $C^{\lambda}(B)(x) * \bigwedge_{y \in B} C^{\lambda}(A)(y) \leq C^{\lambda}(A)(x)$ *for all* $A, B \subseteq X$, $x \in X$.

Proof. If we impose the axiom (DM2), then the proof follows from Theorem 3. We present here a proof without using (DM2), although it is very similar. Let (X, λ) be strongly transitive and let $A, B \subseteq X$ and let $x \in X$. We use $C^{(\tau^{\lambda})} = C^{\lambda}$ and conclude

$$
\begin{aligned}
& C^\lambda(B)(x) * \bigwedge_{y\in B} C^\lambda(A)(y) \\
= \; & \left(\bigvee_{b\in B} \left(\tau^\lambda(x,x) \to \tau^\lambda(x,b) \right) \right) * \bigwedge_{y\in B} \left(\bigvee_{a\in A} (\tau^\lambda(y,y) \to \tau^\lambda(y,a)) \right) \\
\leq \; & \bigvee_{b\in B} \left(\tau^\lambda(x,x) \to \tau^\lambda(x,b) \right) * \bigvee_{a\in A} \left(\tau^\lambda(b,b) \to \tau^\lambda(b,a) \right) \\
= \; & \bigvee_{b\in B} \bigvee_{a\in A} \left(\tau^\lambda(x,x) \to \tau^\lambda(x,b) \right) * \left(\tau^\lambda(b,b) \to \tau^\lambda(b,a) \right) \\
\leq \; & \bigvee_{b\in B} \bigvee_{a\in A} \left(\tau^\lambda(x,x) \to \left(\tau^\lambda(x,b) * \left(\tau^\lambda(b,b) \to \tau^\lambda(b,a) \right) \right) \right) \\
\leq \; & \bigvee_{b\in B} \bigvee_{a\in A} \left(\tau^\lambda(x,x) \to \tau^\lambda(x,a) \right) \; = \; C^\lambda(A)(x).
\end{aligned}
$$

The converse follows again with $B = \{y\}$ and $A = \{z\}$. Condition (2) then yields

$$\left(\tau^\lambda(x,x) \to \tau^\lambda(x,y) \right) * \left(\tau^\lambda(y,y) \to \tau^\lambda(y,z) \right) \leq \tau^\lambda(x,x) \to \tau^\lambda(x,z).$$

By divisibility and (LTOL1) we have $\tau^\lambda(x,x) * (\tau^\lambda(x,x) \to \tau^\lambda(x,y)) = \tau^\lambda(x,y) \wedge \tau^\lambda(x,x) = \tau^\lambda(x,y)$ and we conclude $\tau^\lambda(x,y) * (\tau^\lambda(y,y) \to \tau^\lambda(y,z)) \leq \tau^\lambda(x,z)$. Noting $\tau^\lambda(u,v) = \lambda([v])(u)$ for all $u, v \in X$ then completes the proof. $\square$

Corollary 2. *Let* $\mathsf{L} = (L, \leq, *)$ *be divisible and let* $(X, \tau) \in |\mathsf{L}\text{-}\mathsf{Tol}|$*. The following statements are equivalent.*

(1) (X, τ) *is strongly transitive.*
(2) $C^\tau(B)(x) * \bigwedge_{y\in B} C^\tau(A)(y) \leq C^\tau(A)(x)$ *for all* $A, B \subseteq X$*,* $x \in X$*.*

7. Characterization of Transitivity by Diagonal Axioms

Following Kowalsky [25] we define, for $\mathbb{V}, \mathbb{U}_y \in \mathsf{U}(X)$, $(y \in X)$, the *diagonal filter* $\kappa(\mathbb{V}, (\mathbb{U}_y)_{y\in X}) = \bigcup_{V\in\mathbb{V}} \bigwedge_{y\in V} \mathbb{U}_y$. According to [22], $\kappa(\mathbb{V}, (\mathbb{U}_y)_{y\in X}) \in \mathsf{U}(X)$. For an L-convergence function $\lambda : \mathbb{F}(X) \longrightarrow L^X$ we define the axiom

$$(LUK) \qquad \forall \mathbb{V}, \mathbb{U}_y \in \mathsf{U}(X), (y \in X) : \; \lambda(\mathbb{V})(x) * \bigwedge_{y\in X} \lambda(\mathbb{U}_y)(y) \leq \lambda(\kappa(\mathbb{V}, (\mathbb{U}_y)_{y\in X}))(x).$$

Diagonal axioms in the theory of convergence spaces have a long history, see, e.g., [25,26]. For quantale-valued convergence functions, they are appearing, e.g., for Lawvere's quantale in the theory of approach spaces [22,27,28] and in the theory of quantale-valued topological spaces in [24].

Proposition 13. *Let the* L*-convergence function* $\lambda : \mathbb{F}(X) \longrightarrow L^X$ *satisfy the axiom (LT). Then* λ *is transitive if and only if it satisfies (LUK).*

Proof. Let first λ be transitive and let $\mathbb{V}, \mathbb{U}_y \in \mathsf{U}(X), (y \in X)$. Let further $\alpha \lhd \lambda(\mathbb{V})(x)$ and $\beta \lhd \bigwedge_{y\in Y} \lambda(\mathbb{U}_y)(y)$. Using (LT) then there is $V_\alpha \in \mathbb{V}$ such that for all $v \in V_\alpha$ we have $\lambda([v])(x) \geq \alpha$ and for all $y \in X$ there is $U^y_\beta \in \mathbb{U}_y$ such that for all $u \in U^y_\beta$ we have $\lambda([u])(y) \geq \beta$. The set $H = \bigcup_{v\in V_\alpha} U^v_\beta \in \bigwedge_{v\in V_\alpha} \mathbb{U}_v \leq \kappa(\mathbb{V}, (\mathbb{U}_y)_{y\in Y})$ and for $z \in H$ we have $z \in U^x_\beta$ for some $v \in V_\alpha$. Hence, $\lambda([z])(v) \geq \beta$ and $\lambda([v])(x) \geq \alpha$ which implies by transitivity $\lambda([z])(x) \geq \alpha * \beta$. We conclude

$$\lambda(\kappa(\mathbb{V},(\mathbb{U}_y)_{y\in X}))(x) = \bigvee_{H\in\kappa(\mathbb{V},(\mathbb{U}_y)_{y\in Y})} \bigwedge_{z\in H} \lambda([z])(x) \geq \alpha * \beta.$$

The complete distributivity then yields the axiom (LUK).

For the converse, we choose $\mathbb{V} = [y]$ and $\mathbb{U}_y = [z]$ for all $y \in X$. Then $\kappa(\mathbb{V},(\mathbb{U}_y)_{y\in X}) = [z]$ and (LUK) reads $\lambda([y])(x) * \lambda([z])(y) \leq \lambda([z])(x)$, which is the transitivity of λ. □

Corollary 3. *Let $(X,\lambda) \in |\mathsf{L\text{-}TolConv}|$. The following statements are equivalent.*

(1) *(X,λ) is transitive.*
(2) *(X,λ) satisfies the axiom (LUK).*

Noting that the auxiliary L-convergence function for an L-tolerance convergence space (X,λ) satisfies (LT), we deduce with Proposition 12 the following characterization of strong transitivity.

Corollary 4. *Let $\mathsf{L} = (L,\leq,*)$ be divisible and satisfy (DM2) and let $(X,\lambda) \in |\mathsf{L\text{-}TolConv}|$. The following statements are equivalent.*

(1) *(X,λ) is strongly transitive.*
(2) $\forall \mathbb{V},\mathbb{U}_y \in \mathsf{U}(X), y \in X :\ \lambda(\mathbb{V})(x) * \bigwedge_{y\in X}(\lambda([y])(y) \to \lambda(\mathbb{U}_y)(y)) \leq \lambda(\kappa(\mathbb{V},(\mathbb{U}_y)_{y\in X}))(x).$

Proof. Using the definition of $\overline{\lambda}$ we only need to remark that for an ultrafilter $\mathbb{V}$ we have with (LT) and the symmetry of λ that

$$\lambda(\mathbb{V})(x) = \bigwedge_{V\in\mathbb{V}} \bigvee_{v\in V} \lambda([v])(x) = \bigwedge_{V\in\mathbb{V}} \bigvee_{v\in V} \lambda([x])(v) \leq \lambda([x])(x).$$

Hence, by the divisibility of L, we get $\lambda([x])(x) * (\lambda([x])(x) \to \lambda(\mathbb{V})(x)) = \lambda([x])(x) \wedge \lambda(\mathbb{V})(x) = \lambda(\mathbb{V})(x)$ and (2) is equivalent to the axiom (LUK) for $\overline{\lambda}$. □

8. Transitivity of L-Tolerance Groups and L-Pretolerance Convergence Groups

Let $(X,\cdot)$ be a group with neutral element e. For filters $\mathbb{F},\mathbb{G} \in \mathsf{F}(X)$, the filter $\mathbb{F}\odot\mathbb{G}$ is generated by the sets $F\odot G = \{xy\ :\ x\in F, y\in G\}$ for $F\in\mathbb{F}$ and $G\in\mathbb{G}$ and the filter $\mathbb{F}^{-1}$ is generated by the sets $F^{-1} = \{x^{-1}\ :\ x\in F\}$ for $F\in\mathbb{F}$.

Definition 2. *A triple $(X,\cdot,\lambda)$, where $(X,\cdot)$ is a group and (X,λ) is an* L*-pretolerance convergence space, is called an* L-pretolerance convergence group *if for all $x,y \in X$ and all $\mathbb{F},\mathbb{G} \in \mathsf{F}(X)$*

(LCGM) $(\lambda([x])(x) \to \lambda(\mathbb{F})(x)) * (\lambda([y])(y) \to \lambda(\mathbb{G})(y)) \leq \lambda([xy])(xy) \to \lambda(\mathbb{F}\odot\mathbb{G})(xy)$;
(LCGI) $\lambda([x])(x) \to \lambda(\mathbb{F})(x) \leq \lambda([x^{-1}])(x^{-1}) \to \lambda(\mathbb{F}^{-1})(x^{-1})$.

The category of L*-pretolerance convergence groups and continuous group homomorphisms is denoted by* L-PreTolConvGrp.

Using the auxiliary L-convergence function $\overline{\lambda}$ the properties (LCGM) and (LCGI) can be written more concisely as

(LCGM) $\overline{\lambda}(\mathbb{F})(x) * \overline{\lambda}(\mathbb{G})(y) \leq \overline{\lambda}(\mathbb{F}\odot\mathbb{G})(xy)$;
(LCGI) $\overline{\lambda}(\mathbb{F})(x) \leq \overline{\lambda}(\mathbb{F}^{-1})(x^{-1})$.

We note that in the presence of (LC1s), $\lambda([x])(x) = \top$ for all $x \in X$, the axioms (LCGM) and (LCGI) become much simpler as we have $\overline{\lambda} = \lambda$ then.

Definition 3. *A triple* $(X, \cdot, \tau)$ *where* $(X, \cdot) \in |\mathsf{Grp}|$*, and* $(X, \tau) \in |\mathsf{L\text{-}Tol}|$*, is called an* L-tolerance group*, if the following conditions are fulfilled:*

(LTGM) $(\tau(x,x) \to \tau(x,y)) * (\tau(x',x') \to \tau(x',y')) \le \tau(xx',xx') \to \tau(xx',yy')$*;*

(LTGI) $\tau(x,x) \to \tau(x,y) \le \tau(x^{-1},x^{-1}) \to \tau(x^{-1},y^{-1})$

In this case, we call τ *an* L-group tolerance*. The category of* L*-tolerance groups and* L*-tolerance preserving group homomorphisms is denoted by* $\mathsf{L\text{-}TolGrp}$*.*

Again, using the auxiliary L-relation $\overline{\tau}$, the axioms can be stated as follows.

(LTGM) $\overline{\tau}(x,y) * \overline{\tau}(x',y') \le \overline{\tau}(xx',yy')$;

(LTGI) $\overline{\tau}(x,y) \le \overline{\tau}(x^{-1},y^{-1})$.

If the strong reflexivity axiom (LTOL1s), $\tau(x,x) = \top$ for all $x \in X$, is satisfied, then the axioms become much simpler as then $\overline{\tau} = \tau$. In the case $L = \{0,1\}$, tolerance groups and algebraic structures compatible with a tolerance relation have been extensively studied, cf. e.g., [5,29–31]. For $\mathsf{L} = ([0,1], \le, \wedge)$ see also [32].

Sometimes it is sufficient to consider only the axiom (LTGM).

Lemma 3. *Let* $(X, \cdot)$ *be a group and let* $(X, \tau) \in |\mathsf{L\text{-}Tol}|$ *satisfy* $\tau(e,e) = \tau(x,x)$ *for all* $x \in X$*. Then (LTGM) implies (LTGI).*

Proof. Let $x, y \in X$. Then we have $\overline{\tau}(x,y) = \overline{\tau}(x,y) * \overline{\tau}(x^{-1},x^{-1}) \le \overline{\tau}(xx^{-1},yx^{-1}) = \overline{\tau}(e,yx^{-1}) = \overline{\tau}(y^{-1},y^{-1}) * \overline{\tau}(e,yx^{-1}) \le \overline{\tau}(y^{-1},y^{-1}yx^{-1}) = \overline{\tau}(y^{-1},x^{-1}) = \overline{\tau}(x^{-1},y^{-1})$, where we have used in the last step $\tau(x,x) = \tau(e,e) = \tau(y,y)$. □

We note that the condition $\tau(e,e) = \tau(x,x)$ for all $x \in X$ is implied by (LTOL1s).

Proposition 14. *Let* L *satsify (DM2). If* $(X, \cdot, \tau) \in |\mathsf{L\text{-}TolGrp}|$ *then* $(X, \cdot, \lambda^\tau) \in |\mathsf{L\text{-}PreTolConvGrp}|$*.*

Proof. Let $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$ and let $x, y \in X$. If $\alpha \lhd \lambda^\tau(\mathbb{F})(x)$ and $\beta \lhd \lambda^\tau(\mathbb{G})(y)$, then there are $F_\alpha \in \mathbb{F}$ and $G_\beta \in \mathbb{G}$ such that for all $u \in F_\alpha$ we have $\tau(x,u) \ge \alpha$ and for all $v \in G_\beta$ we have $\tau(y,v) \ge \beta$. Then $F_\alpha \odot G_\beta \in \mathbb{F} \odot \mathbb{G}$ and we conclude

$$\begin{aligned}
\tau(xy,xy) \to \lambda^\tau(\mathbb{F} \odot \mathbb{G})(xy) &\ge \tau(xy,xy) \to \bigwedge_{u \in F_\alpha, v \in G_\beta} \tau(xy,uv) \\
&= \bigwedge_{u \in F_\alpha, v \in G_\beta} \tau(xy,xy) \to \tau(xy,uv) \\
&\ge \bigwedge_{u \in F_\alpha, v \in G_\beta} (\tau(x,x) \to \tau(x,u)) * (\tau(y,y) \to \tau(y,v)) \\
&\ge (\tau(x,x) \to \alpha) * (\tau(y,y) \to \beta).
\end{aligned}$$

The complete distributivity L and the distributivity of the quantale operation over joins and the property (DM2) leads to

$$\begin{aligned}
\tau(xy,xy) \to \lambda^\tau(\mathbb{F} \odot \mathbb{G})(xy) &\ge \bigvee_{\alpha \lhd \lambda^\tau(\mathbb{F})(x)} (\tau(x,x) \to \alpha) * \bigvee_{\beta \lhd \lambda^\tau(\mathbb{G})(y)} (\tau(y,y) \to \beta) \\
&= (\tau(x,x) \to \lambda^\tau(\mathbb{F})(x)) * (\tau(y,y) \to \lambda^\tau(\mathbb{G})(y)).
\end{aligned}$$

Noting that $\lambda^\tau([u])(u) = \tau(u,u)$ then yields (LCGM).

(LCGI) We have for $\mathbb{F} \in \mathsf{F}(X)$ and $x \in X$, using the axiom (DM2),

$$\begin{aligned}
\lambda^\tau([x])(x) \to \lambda^\tau(\mathbb{F})(x) &= \tau(x,x) \to \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} \tau(x,y) \\
&= \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} (\tau(x,x) \to \tau(x,y)) \\
&\leq \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} (\tau(x^{-1},x^{-1}) \to \tau(x^{-1},y^{-1})) \\
&\leq \tau(x^{-1},x^{-1}) \to \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} \tau(x^{-1},y^{-1}) \\
&= \lambda^\tau([x^{-1}])(x^{-1}) \to \lambda^\tau(\mathbb{F}^{-1})(x^{-1}).
\end{aligned}$$

□

We note again, that with (LTOL1s) the proof becomes simpler and we do not need (DM2) then.

We call an L-pretolerance convergence group $(X,\cdot,\lambda)$ L-*tolerance induced* if there is a L-group tolerance τ on X such that $\lambda = \lambda^\tau$.

Theorem 5. *Let* L *satisfy (DM2). An* L*-pretolerance convergence group* $(X,\cdot,\lambda)$ *is* L*-tolerance induced if and only if* (X,λ) *satisfies the axiom (LT).*

Proof. If $(X,\cdot,\lambda)$ is L-tolerance induced, then $(X,\lambda) = (X,\lambda^\tau)$ and hence satisfies the axiom (LT). Let now (X,λ) satisfy (LT). We then define $\tau = \tau^\lambda$ and we have $\lambda^{(\tau^\lambda)} = \lambda$. Noting that $[u] \odot [v] = [uv]$ and $[u]^{-1} = [u^{-1}]$ immediately establishes (LTGM) and (LTGI) for $(X,\cdot,\tau^\lambda)$. □

If we call an L-pretolerance convergence group that satisfies (LT) an L-*tolerance convergence group* and denote the subcategory of these spaces by L-$\mathsf{TolConvGrp}$, then we conclude the following theorem.

Theorem 6. *Let* L *satisfy (DM2). Then the categories* L-TolGrp *and* L-$\mathsf{TolConvGrp}$ *are isomorphic.*

We now turn to transitivity. First we need the *homogeneity* of an L-pretolerance convergence group.

Proposition 15. *Let* $(X,\cdot,\lambda) \in |\mathsf{L}\text{-}\mathsf{PreTolConvGrp}|$. *If* $\mathbb{F} \in \mathsf{F}(X)$ *and* $x \in X$ *then*

$$\overline{\lambda}(\mathbb{F})(x) = \overline{\lambda}([x^{-1}] \odot \mathbb{F})(e).$$

Proof. We have

$$\begin{aligned}
\lambda([x])(x) \to \lambda(\mathbb{F})(x) &= (\lambda([x^{-1}])(x^{-1}) \to \lambda([x^{-1}])(x^{-1})) * (\lambda([x])(x) \to \lambda(\mathbb{F})(x)) \\
&\leq \lambda([x^{-1}x])(x^{-1}x) \to \lambda([x^{-1}] \odot \mathbb{F})(x^{-1}x) \\
&= \lambda([e])(e) \to \lambda([x^{-1}] \odot \mathbb{F})(e) \\
&= (\lambda([x])(x) \to \lambda([x])(x)) * (\lambda([e])(e) \to \lambda([x^{-1}] \odot \mathbb{F})(e)) \\
&\leq \lambda([xe])(xe) \to \lambda([x] \odot [x^{-1}] \odot \mathbb{F})(xe) \ = \ \lambda([x])(x) \to \lambda(\mathbb{F})(x).
\end{aligned}$$

□

Theorem 7. *Let* L *be divisible. Then* $(X,\cdot,\lambda) \in |\mathsf{L}\text{-}\mathsf{PreTolConvGrp}|$ *is strongly transitive.*

Proof. Using the homogeneity from Proposition 15, we conclude for $x, y, z \in X$

$$\begin{aligned}\overline{\lambda}([z])(y) * \overline{\lambda}([y])(x) &= \overline{\lambda}([y^{-1}] \odot [z])(e) * \overline{\lambda}([x^{-1}] \odot [y])(e) \\ &\leq \overline{\lambda}([x^{-1}] \odot [z])(e) \;=\; \overline{\lambda}([z])(x).\end{aligned}$$

Hence $\overline{\lambda}$ satisfies (LCTrans) and, by Proposition 12, λ satisfies (LCTrans*). □

Therefore, for a divisible quantale, an L-pretolerance convergence group is also transitive. With regard to L-equivalence relations we note the following corollary, where we do not need the divisibilty of L.

Corollary 5. *Let $(X, \cdot, \lambda) \in |\mathsf{L\text{-}PreTolConvGrp}|$ satisfy (LC1s). Then $(X, \cdot, \lambda)$ is transitive.*

Remark 5. *We consider a group $(X, \cdot)$ and an L-tolerance relation $\tau : X \times X \longrightarrow L$ that satisfies (LTOL1s) and (LTrans). Then (LTGM) is equivalent to the* invariance of τ*, i.e., to $\tau(x, y) = \tau(xz, yz)$ for all $x, y, z \in X$. In fact, using (LTOL1s) and (LTGM) we obtain $\tau(x, y) = \tau(x, y) * \tau(z, z) \leq \tau(xz, yz) = \tau(xz, yz) * \tau(z^{-1}, z^{-1}) \leq \tau(xzz^{-1}, yzz^{-1}) = \tau(x, y)$. On the other hand, invariance implies, using transitivity, $\tau(x, y) * \tau(x', y') = \tau(xx', yx') * \tau(yx', yy') \leq \tau(xx', yy')$.*

9. Conclusions

In this paper, we introduced the tool of convergence into the theory of quantale-valued tolerance spaces. We used it to characterize the important property of transitivity in two ways: one by using a closure operation for the subsets of the space (derived from the convergence notion in a natural way), and the other by using so-called diagonal axioms that are well-known in the theory of convergence spaces.

Transitivity for quantale-valued tolerance spaces comes in two forms. The one makes the quantale-valued tolerance relation to a quantale-valued equivalence relation. The other leads to so-called quantale-valued sets, which generalize, e.g., partial metric spaces. Transitivity, besides being an often required "natural property" of a similarity relation, is, e.g., useful when one tries to determine so-called tolerance classes. Without transitivity, this determination becomes rather involved, as can be seen, e.g., [10], whereas for equivalence relations, the fact that the equivalence classes form a partition of the space and we can therefore from the classes retrieve the equivalence relation, makes things usually much simpler.

The theory developped in this paper can also be extended to define, for a quantale-valued tolerance space, a grade of being transitive. This can, e.g., be achieved by generalizing Corollary 1 and using the implication operation and defining the "grade of transitivity" of a space by

$$trans((X, \tau)) = \bigwedge_{A, B \subseteq X, x \in X} \left(\left(c^{\tau}(B)(x) * \bigwedge_{y \in B} c^{\tau}(A)(y) \right) \to c^{\tau}(A)(x) \right).$$

The higher this grade, the more transitive a space is. A similar approach using the classes of a quantale-valued tolerance space is used in [10]. The possibility of "numerically evaluating the grade to which a property holds" is an advantage of considering quantale-valued tolerance spaces.

The use of "topological notions" that can be derived from convergence establishes also connections to topics seemingly unrelated to similarity like the Hausdorff metric. This connection becomes more transparent in the quantale-valued case, where a suitable choice of the quantale relates similarity to metrics in a natural way. We are therefore convinced that the study of quantale-valued generalizations of classical concepts like similarity or convergence are useful and find applications in other branches of mathematics.

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