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Basic Core Fuzzy Logics and Algebraic Routley–Meyer-Style Semantics

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Abstract: Recently, algebraic Routley–Meyer-style semantics was introduced for basic substructural logics. This paper extends it to fuzzy logics. First, we recall the basic substructural core fuzzy logic **MIAL** (Mianorm logic) and its axiomatic extensions, together with their algebraic semantics. Next, we introduce two kinds of ternary relational semantics, called here linear *Urquhart-style* and *Fine-style* Routley–Meyer semantics, for them as *algebraic* Routley–Meyer-style semantics.

Keywords: operational semantics; Routley–Meyer-style semantics; algebraic semantics; (core) fuzzy logics; implicational tonoid fuzzy logics

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1. Introduction

The author [1] recently introduced algebraic Routley–Meyer-style (ARM for simplicity) semantics for basic substructural logics. Here, the term *ARM semantics* means semantics with operations interpreting ternary relations, the frames of which have the same structures as algebraic semantics. This paper extends it to *fuzzy* logics. To this end, we first recall some historical facts associated with Routley–Meyer semantics.

Using binary accessibility relations, Kripke [2–4] first established relational semantics, the so-called *Kripke Semantics*, for modal and intuitionistic logics. Since then, many semantics have been introduced as its generalizations. In particular, Urquhart provided operational semantics, called Urquhart semantics in [1]; for relevant implication see [5–7]. From an operational semantic point of view, this semantics is interesting since instead of binary relations for accessibility it has groupoid operations. More precisely, it provides the valuation of implication using the binary operation $\circ$ such that

$(\rightarrow_{\circ_U})$ $a \Vdash A \rightarrow B$ if and only if (iff) for any $b \in X$, $b \Vdash A$ implies $a \circ b \Vdash B$,

instead of using the binary relation R such that

$(\rightarrow_{R_K})$ $a \Vdash A \rightarrow B$ iff for any $b \in X$, aRb and $b \Vdash A$ imply $b \Vdash B$.

Urquhart semantics has the following additional valuations for extensional conjunction and disjunction: For sentences A, B,

$(\wedge)$ $a \Vdash A \wedge B$ iff $a \Vdash A$ and $a \Vdash B$; and

$(\vee)$ $a \Vdash A \vee B$ iff $a \Vdash A$ or $a \Vdash B$.

As is well known, these three valuation conditions do not work together for substructural logics in general. As Urquhart himself mentioned in [7,8], while sentences such as (a) $((A \rightarrow (B \vee C)) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$ are valid in their semantics, the distributive substructural logic **R** of relevance does not prove such sentences. Because of this negative fact, Routley–Meyer [9–11] instead introduced the so-called Routley–Meyer semantics for implication as a ternary relational semantics (see [12]).

Please note that Urquhart [7] provided the binary operational valuation for implication $(\rightarrow_{\circ_U})$, whereas Fine [13] did the following ternary relational valuation for implication.

($\to_{\circ_F}$) $a \Vdash A \to B$ iff for all $b, c \in X$, $a \circ b \leq c$ and $b \Vdash A$ imply $c \Vdash B$.

Although these two valuations are not free from the above negative fact, they have been extensively used in substructural logics: Using ($\to_{\circ_U}$), many logicians such as Došen and Ono have introduced similar semantics for modal and substructural logics [14–16]; with the title "Kripke-style semantics", Montagna–Ono [17], Montagna–Sacchetti [18], and Yang [19,20] introduced similar semantics for substructural fuzzy logics. Using ($\to_{\circ_F}$), logicians such as Ono–Komori [21], Ishihara [22], and Kamide [23] have introduced analogous semantics for some (modal) substructural logics (For more detailed introduction of these semantics, see [1]).

The starting point for the current work is the observation that, as the author [1] mentioned, using ternary relation $Rabc$, the valuations ($\to_{\circ_U}$) and ($\to_{\circ_F}$) can be rephrased as:

($\to_{R_U}$) $a \Vdash A \to B$ iff for all $b, c \in X$, if $Rabc$ (:= $a \circ b = c$ (df_U)) and $b \Vdash A$, then $c \Vdash B$, (We take c in place of $a * b$ in $a * b \Vdash B$ because $a \circ b = c$. This was introduced by Dunn in [24]) and

($\to_{R_F}$) $a \Vdash A \to B$ iff for all $b, c \in X$, if $Rabc$ (:= $a \circ b \leq c$ (df_F)) and $b \Vdash A$, then $c \Vdash B$, (As Došen [15] and Dunn [25,26] already mentioned, Fine [13] interpreted $Rabc$ as $a \circ b \leq c$. Although Urquhart [7] did not consider to reinterpret ($\to_{\circ_U}$) using ternary relation, Bimbó and Dunn [27] and Restall [12] introduced such reformulation.), respectively.

In particular, using ($\to_{R_U}$) and ($\to_{R_F}$), the author first introduced ARM semantics for basic substructural logics in general. Then, since fuzzy logics are also substructural logics and further prove sentences such as (a), one can ask the following.

Q: Could one establish ARM semantics, i.e., operational and ternary relational semantics equivalent to algebraic semantics, for basic substructural fuzzy logics, using the clauses ($\wedge$), ($\vee$), and either ($\to_{R_U}$) or ($\to_{R_F}$) together?

As a positive answer to this question, we introduce such semantics with the conditions ($\wedge$), ($\vee$) and the corresponding implication conditions for basic (core) fuzzy logics. (A logic L is called *fuzzy* if it is complete on linearly ordered models, and *core* fuzzy if it is fuzzy on $[0, 1]$ (see [28,29])). This will verify that the clauses ($\wedge$), ($\vee$), and either ($\to_{R_U}$) or ($\to_{R_F}$) work together for basic substructural fuzzy logics.

The more detailed other reasons to study this are as follows: The first and most important reason is that while algebraic Kripke-style (briefly AK) semantics (The term *AK semantics* means semantics with operations in place of binary accessibility relations, the frames of which have the same structures as algebraic semantics.) for substructural fuzzy logics have been introduced extensively (see, e.g., [17–20,30–32]), ARM semantics for such logics have not. Only, the author [33,34] introduced such semantics for **MTL** (Monoidal t-norm logic) and its involutive extension **IMTL**. In particular, the author [1] introduced ARM semantics for substructural logics in general, whereas he did not for substructural fuzzy logics. This is the direct specific reason to consider ARM semantics for *fuzzy logics* in general.

The following are more reasons related to ARM semantics itself, some of which are mentioned in [1]. First, "the definitions (df_U) and (df_F) provide more intuitive ways to understand or interpret the ternary relation R. Please note that using the ternary relation R in $Rxyz$ itself we cannot say how to understand or interpret R, whereas we can say it using $x * y = z$ and $x * y \leq z$". Second, this semantics provides a *direct* way to understand *equivalence relations* between algebraic and relational semantics. "An n-ary operation is an n+1-ary relation, but not always conversely. If one shows its converse, one can state an equivalence between the operation and the relation". Associated with this, most well-known method to consider this equivalence is to use 'canonical extensions' investigated with the titles such as 'representation' and 'duality' (see [35–41]). However, the way to use (df_U) and (df_F) is different from it and more direct in the sense that the way defines ternary relations by virtue of binary operations and (in)equations. The third is the fact that ARM semantics uses forcing relations. It means that this semantics is a study still in the tradition of relational semantic research. The last but not least one is that ARM semantics is a common area between algebraic semantics and ternary relational semantics. Since algebraic semantics and ARM semantics are both based on the same algebraic structures, this last semantics gives a chance to study similarities and differences between algebraic semantics and relational semantics.

We organize the paper as follows. In Section 2, we first recall some basic (core) fuzzy logics, together with their algebraic semantics. In Section 3, we introduce ARM semantics for them. More precisely, we introduce ARM semantics with ($\to_{R_U}$) in Section 3.1 and that with ($\to_{R_F}$) in Sectio 3.2. In Section 4, we consider advantages and limitations of these two semantics as ARM semantics.

We finally note that, as in [1], our ARM semantics in Sections 3.1 and 3.2 provides frames as some reducts of their corresponding algebras and defines ternary relations using binary operations and (in)equations. However, unlike the semantics in [1], this semantics is provided based on linear theories. More precisely, it is an ARM semantics with linearly ordered models. In this sense, this semantics is a *novel* one to connect n-nary operations and n+1-nary relations. By *ARM$^\ell$ semantics*, we henceforth mean this kind of ARM semantics.

2. Algebraic Semantics for Basic Core Fuzzy Logics

Here we recall the most basic substructural core fuzzy logic **MIAL** and its axiomatic extensions (extensions for short) and their algebraic semantics (See [42] for more detailed introduction of these logics and semantics). The language for these logics is provided over a countable propositional language with Fm (a set of formulas) built from VAR (a set of propositional variables), propositional constants t, f, $\mathbf{F}$, $\mathbf{T}$, and connectives $\to$, $\rightsquigarrow$, $\wedge$, $\vee$, &. We further define $A \leftrightarrow B$ and A_t as $(A \to B) \wedge (B \to A)$ and $A \wedge t$, respectively.

The variables are denoted by lowercase Latin letters $p, q, r, \ldots$ and the formulas by uppercase ones $A, B, C \ldots$. Theories as sets of formulas are denoted by uppercase Greek letters $\Gamma, \Delta, \ldots$. Please note that variables are also formulas. We provide a consequence relation, denoted by $\vdash$, on axiom systems.

Definition 1 ([43,44])**.** ***MIAL*** *consists of the axioms and rules below:*

$(A \wedge B) \to A$, $(A \wedge B) \to B$ *($\wedge$-elimination, $\wedge$-E);*
$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ *($\wedge$-introduction, $\wedge$-I);*
$A \to (A \vee B)$, $B \to (A \vee B)$ *($\vee$-introduction, $\vee$-I);*
$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ *($\vee$-elimination, $\vee$-E);*
$\boldsymbol{F} \to A$ *(ex falsum quodlibet, EF);*
$(t \to A) \leftrightarrow A$ *(push and pop, PP);*
$A \to (B \to (B\&A))$ *(&-adjunction, &-Adj);*
$A \to (B \rightsquigarrow (A\&B))$ *(&-adjunction, &-Adj$_{\rightsquigarrow}$);*
$(A_t\&B_t) \to (A \wedge B)$ *(&$\wedge$);*
$(B\&(A\&(A \to (B \to C)))) \to C$ *(residuation, Res$'$);*
$((A\&(A \rightsquigarrow (B \to C)))\&B) \to C$ *(residuation, Res$'_{\rightsquigarrow}$);*
$((A \to (A\&(A \to B)))\&(B \to C)) \to (A \to C)$ *(transitivity, T$'$);*
$((A \rightsquigarrow ((A \rightsquigarrow B)\&A))\&(B \to C)) \to (A \rightsquigarrow C)$ *(transitivity, T$'_{\rightsquigarrow}$);*
$(A \to B)_t \vee ((C\&D) \to (C\&(D\&(B \to A)_t)))$ *(prelinearity, PL$_{\alpha_{C,D}}$);*
$(A \to B)_t \vee ((C\&D) \to ((C\&(B \to A)_t)\&D))$ *(prelinearity, PL$_{\alpha'_{C,D}}$);*
$(A \to B)_t \vee ((C \to (D \to ((D\&C)\&(B \to A)_t)))$ *(prelinearity, PL$_{\beta_{C,D}}$);*
$(A \to B)_t \vee ((C \to (D \rightsquigarrow ((C\&D)\&(B \to A)_t)))$ *(prelinearity, PL$_{\beta'_{C,D}}$);*
$A \to B, A \vdash B$ *(modus ponens, mp);*
$A \vdash A_t$ *(adj$_U$);*
$A \vdash (C\&D) \to (C\&(D\&A))$ *(α);*
$A \vdash (C\&D) \to ((C\&A)\&D)$ *(α');*

$A \vdash C \to (D \to ((D\&C)\&A))$ *(β);*
$A \vdash C \to (D \rightsquigarrow ((C\&D)\&A))$ *(β′).*

A logic is called an extension of a logic **L** if it is obtained from **L** by adding further axioms.

Definition 2. *The following are basic structural axioms:*
(exchange, e) $A\&B \to B\&A$;
(expansion, p) $(A\&A) \to A$;
(contraction, c) $A \to (A\&A)$;
(left weakening, i) $A \to (B \to A)$;
(right weakening, o) $f \to A$;
(associativity, a) $A\&(B\&C) \leftrightarrow (A\&B)\&C$.
$\boldsymbol{MIAL}_S$, $S \subseteq \{e, p, c, i, o, a\}$, *is a substructural (core) fuzzy logic extending* ***MIAL****.*

Example 1. *The following well-known core fuzzy logics are extensions of* ***MIAL****.*

(1) *Micanorm logic* ***MICAL*** *is* $\boldsymbol{MIAL}_e$.
(2) *Uninorm logic* ***UL*** *is* $\boldsymbol{MIAL}_{ea}$.
(3) *Monoidal t-norm logic* ***MTL*** *is* $\boldsymbol{MIAL}_{eai}$.

By L^ℓs, we denote the set of substructural fuzzy logics introduced in Definition 2, i.e., L^ℓs = $\{\mathbf{MIAL}_S : S \subseteq \{e, p, c, i, o, a\}\}$.

A theory of $L^\ell \in L^\ell$s (L^ℓ-theory for short) is a set Γ of formulas such that $\Gamma \vdash_{L^\ell} A$ entails $A \in \Gamma$. Since $\emptyset \subseteq \Gamma$, the set of theorems of L^ℓ is a subset of all L^ℓ-theories. We define a *proof* in an L^ℓ-theory Γ as a sequence of formulas, the elements of which are either axioms of L^ℓ, members of Γ, or derived from its precedent elements using rules of L^ℓ. For each pair of formulas A, B and a theory Γ, if $\Gamma \vdash A \to B$ or $\Gamma \vdash B \to A$, we call Γ a *linear* theory.

Definition 3. *A* bounded, pointed residuated lattice-ordered groupoid with unit (*bprlu*-groupoid for simplicity) *is an algebra* $\mathcal{A} = (A, \bot, \top, 0, 1, \backslash, /, \wedge, \vee, \circ)$ *such that:* $(A, \circ, 1)$ *is a unital groupoid;* $(A, \bot, \top, \wedge, \vee)$ *is a bounded lattice;* 0 *is an arbitrary element in* A*; for all* $a, b, c \in A$, $a \circ b \leq c$ *iff* $a \leq c/b$ *iff* $b \leq a\backslash c$ *(residuation).*

Please note that the notations '$\wedge$' and '$\vee$' are used both as propositional connectives and as algebraic operators.

Definition 4 (L^ℓ-algebras)**.** *Let* a_1 *be* $a \wedge 1$*. A bprlu-groupoid is a* **MIAL**-algebra *if it satisfies the following prelinearity properties: for all* $a, b, c, d \in A$,
($PL^{\mathcal{A}}_{\alpha_{C,D}}$) $1 \leq (a\backslash b)_1 \vee ((c \circ d)\backslash(c \circ (d \circ (b\backslash a)_1)))$;
($PL^{\mathcal{A}}_{\alpha'_{C,D}}$) $1 \leq (a\backslash b)_1 \vee ((c \circ d)\backslash((c \circ (b\backslash a)_1) \circ d))$;
($PL^{\mathcal{A}}_{\beta_{C,D}}$) $1 \leq (a\backslash b)_1 \vee ((c\backslash(d\backslash((d \circ c) \circ (b\backslash a)_1)))$;
($PL^{\mathcal{A}}_{\beta'_{C,D}}$) $1 \leq (a\backslash b)_1 \vee ((c\backslash(((d \circ c) \circ (b\backslash a)_1)/d))$.
The following are the (in)equations corresponding to the structural axioms above: for all $a, b, c \in A$,
($e^{\mathcal{A}}$) $a \circ b \leq b \circ a$;
($p^{\mathcal{A}}$) $a \circ a \leq a$;
($c^{\mathcal{A}}$) $a \leq a \circ a$;
($i^{\mathcal{A}}$) $a \leq 1$;
($o^{\mathcal{A}}$) $0 \leq a$;
($a^{\mathcal{A}}$) $a \circ (b \circ c) = (a \circ b) \circ c$.
Thus, for any $S \subseteq \{e^{\mathcal{A}}, p^{\mathcal{A}}, c^{\mathcal{A}}, i^{\mathcal{A}}, o^{\mathcal{A}}, a^{\mathcal{A}}\}$, $\mathbf{MIAL}_S$-algebras *are defined with* S*. We call all these algebras* L^ℓ-algebras *and linearly ordered* L^ℓ*-algebras* L^ℓ*-chains.*

Given an L^ℓ-algebra $\mathcal{A}$, an $\mathcal{A}$-*valuation* is defined as a map $v : Fm \to \mathcal{A}$ such that $v(\#(A_1,\ldots,A_n)) = \#^{\mathcal{A}}(v(A_1,\ldots,A_n))$, where $\# \in \{\mathbf{F}, \mathbf{T}, f, t, \to, \rightsquigarrow, \wedge, \vee, \&\}$ and $\#^{\mathcal{A}} \in \{\bot, \top, 0, 1, \backslash, /, \wedge, \vee, \circ\}$. A formula A is said to be an $\mathcal{A}$-*tautology* if for each $\mathcal{A}$-valuation v, $1 \le v(A)$. An $\mathcal{A}$-valuation v is said to be an $\mathcal{A}$-*model* of an L^ℓ-theory Γ if $1 \le v(A)$ for all $A \in \Gamma$. By *Mod*(Γ, $\mathcal{A}$), we denote the class of all $\mathcal{A}$-models of Γ. Over a class $\mathcal{L}^\ell$ of L^ℓ-algebras, a formula A is called a *semantic consequence* of Γ, denoted by $\Gamma \models_{\mathcal{L}^\ell} A$, if $Mod(\Gamma \cup \{A\}, \mathcal{A}) = Mod(\Gamma, \mathcal{A})$ for all $\mathcal{A} \in \mathcal{L}^\ell$. If $\mathcal{A}$ is a semantic consequence of Γ with respect to regarding $\{\mathcal{A}\}$ whenever A is provable in Γ on L^ℓ, $\mathcal{A}$ is called an $\boldsymbol{L^\ell}$-*algebra*. By *MOD*(L^ℓ) and *MOD*$^\ell$(L^ℓ), we denote the set of such algebras and the set of linearly ordered ones, respectively, and write $\Gamma \models_{L^\ell} A$ and $\Gamma \models^\ell_{L^\ell} A$ instead of $\Gamma \models_{MOD(L^\ell)} A$ and $\Gamma \models_{MOD^\ell(L^\ell)} A$, respectively.

Theorem 1 (*Completeness*)**.** *For a theory Γ over $L^\ell \in L^\ell$s and a formula A, $\Gamma \vdash_{L^\ell} A$ iff $\Gamma \models_{L^\ell} A$ iff $\Gamma \models^\ell_{L^\ell} A$.*

Proof. As a corollary of Theorem 3.1.8 in [45], we obtain the claim. □

An L^ℓ-algebra is said to be *standard* if it has the real interval $[0, 1]$ as its carrier set.

Theorem 2 ([42])**.** *Let ε be a unit element in* $[0, 1]$.

(*i*) *For $L^\ell \in L^\ell{}_T = \{\boldsymbol{MIAL}_T : T \subseteq \{e, p, c, i, o\}\}$, $\Gamma \vdash_{L^\ell} A$ iff for each standard L^ℓ-algebra and for each valuation v, $\varepsilon \le v(B)$ for all $B \in \Gamma$ implies $\varepsilon \le v(A)$.*

(*ii*) *For $L^\ell \in L^\ell{}_U$ such that $\{i, a\} \subseteq U \subseteq \{e, p, c, i, o, a\}$, $\Gamma \vdash_{L^\ell} A$ iff for each standard L^ℓ-algebra and for each valuation v, $\varepsilon \le v(B)$ for all $B \in \Gamma$ implies $\varepsilon \le v(A)$.*

Example 2. *For $L^\ell \in \{\boldsymbol{MIAL}_a, \boldsymbol{MIAL}_{ac}, \boldsymbol{MIAL}_{ap}, \boldsymbol{MIAL}_{ao}, \boldsymbol{MIAL}_{aco}, \boldsymbol{MIAL}_{cpo}, \boldsymbol{MIAL}_{acpo}\}$, L^ℓ is not standard complete since (A) $1 \le (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash a)))$ or (B) $x \circ y \le 1$ iff $y \circ x \le 1$ holds in standard L^ℓ-algebras but not in general in linearly ordered L^ℓ-algebras (see [42,46]).*

3. ARM$^\ell$ Semantics

In this section, we deal with the ARM semantics for fuzzy extensions of the basic substructural logics introduced in [1]. As in [1], we introduce two kinds of ARM$^\ell$ semantics: one is the semantics with the definition (df$_U$) and linearly ordered models, called here *linear Urquhart-style* Routley–Meyer semantics (briefly U-RM$^\ell$ semantics), and the other is the semantics with the definition (df$_F{}'$) below and linearly ordered models, called here *linear Fine-style* Routley–Meyer semantics (briefly F-RM$^\ell$ semantics). Please note that unlike the semantics in [1], these two semantics are provided using *linear theories* in place of closed theories. However, these semantics still have the same structures as algebraic semantics and so are ARM semantics.

3.1. U-RM$^\ell$ Semantics

Here we consider U-RM$^\ell$ semantics for $L^\ell_{eq} = \{\mathbf{MIAL}_{eq} : eq \subseteq \{e, o, a\}\}$. We first define several Routley–Meyer (RM for short) frames.

Definition 5.

(*i*) (*RM frames [1]) An* RM frame *is a structure $\boldsymbol{F} = (F, 1, R)$ such that 1 is a special element in F and $R \subseteq F^3$. We call the elements of $\boldsymbol{F}$* nodes.

(*ii*) (*Linear RM frames*) *A* linear RM (briefly, RM$^\ell$) frame *is an RM frame $\boldsymbol{F} = (F, 1, \le, R)$, where $(F, \le)$ is a linearly order set.*

(*iii*) (*(Residuated) Urquhart operational RM$^\ell$ frames*) *An* Urquhart operational *RM$^\ell$ (briefly, U-RM$^\ell$) frame is an RM$^\ell$ frame $\boldsymbol{F} = (F, 1, \circ, \le, R)$, where $\circ$ is a groupoid operation satisfying (df$_U$) $a \circ b = c := Rabc$. A U-RM$^\ell$ frame is called* residuated *if for any $a, b \in F$, the sets $\{c : a \circ c \le b\}$ and $\{c : c \circ a \le b\}$ have suprema.*

(iv) *(Bounded, pointed U-RM$^\ell$ frames) A* U-RM$^\ell$ frame *is said to be* pointed *if it further has an arbitrary element* 0*, and* bounded *if it further has top and bottom elements* $\top$ *and* $\bot$*.*

(v) *(U-RM$^\ell$ MIAL frames) A* U-RM$^\ell$ MIAL frame *is a bounded pointed residuated U-RM$^\ell$ frame, where* $\circ$ *is left-continuous and conjunctive and* R *satisfies the postulates below: for any* $a \in F$,

- $p1.$ $R1aa$
- $p2.$ $Ra1a.$

(vi) *(U-RM$^\ell$ L^ℓ_{eq} frames) Consider the definitions and postulates below: for any* $a, b, c, d \in F$,

- *df1.* $R^2a(bc)d := (\exists x)(Raxd \wedge Rbcx)$
- *df2.* $R^2abcd := (\exists x)(Rabx \wedge Rxcd)$
- $p_e.$ $Rabc$ *implies* $Rbac$*.*
- $p_o.$ $R10a$ *iff* $R1\bot a$*, where* $\bot$ *is the bottom element in* F*.*
- $p_a.$ R^2abcd *iff* $R^2a(bc)d$*.*

For any $eq \subseteq \{p_e, p_o, p_a\}$, U-RM$^\ell$ MIAL$_{eq}$ frames *are defined, along with their corresponding postulates. We call all these frames* U-RM$^\ell$ L^ℓ_{eq} frames *(briefly U-L^ℓ_{eq} frames).*

Remark 1. *Definition 1 has some interesting facts to note.*

(1) *([1]) The definition of an RM frame in* (i) *is the same as that of a frame structure for* $\mathbf{R}^+$ *(the positive* **R***), which eliminates all the definitions and postulates for the ternary relation R introduced in [8].*

(2) *The definitions in* (ii) *and* (iii) *are a fuzzy specification of partially ordered RM frames and (residuated) Urquhart operational RM frames introduced in [1]. (Please note that if* $\leq$ *is a partial ordering in place of a linear ordering in* (ii) *and* (iii)*, the definitions in* (ii) *and* (iii) *form partially ordered RM frames and (residuated) Urquhart operational RM frames introduced in [1].)*

(3) *([1]) The postulates* $p1$ *and* $p2$ *in* (v) *are for a unit element since we have that* $1 \circ a = a = a \circ 1$ *using (df$_U$).*

(4) *([1]) The indices of the postulates in* (vi) *denote their corresponding axioms. For example, the postulate* p_e *is for the exchange axiom e. In particular, (df$_U$) assures that the postulates satisfy the equational forms of their corresponding algebraic properties. For instance, using the postulate* p_o *and (df$_U$), we have that* $0 = 1 \circ 0 = 1 \circ \bot = \bot$*, i.e.,* $0 = \bot$*.*

A *valuation* on a bounded pointed residuated U-RM$^\ell$ frame is a forcing relation $\Vdash$ between the nodes and the propositional variables, propositional constants, and formulas satisfying the below conditions. For each propositional variable p,

(AHC) $b \leq a$ and $a \Vdash p$ imply $b \Vdash p$;

(min) $\bot \Vdash p$,

for the propositional constants f, t, and **F**,

(0) $a \Vdash f$ iff $a \leq 0$;

(1) $a \Vdash t$ iff $a \leq 1$;

$(\bot)$ $a \Vdash \mathbf{F}$ iff $a = \bot$, and

for formulas A, B,

$(\to)$ $a \Vdash A \to B$ iff for all $b, c \in F$, $Rbac$ and $b \Vdash A$ imply $c \Vdash B$;

$(\rightsquigarrow)$ $a \Vdash A \rightsquigarrow B$ iff for all $b, c \in F$, $Rabc$ and $b \Vdash A$ imply $c \Vdash B$;

$(\wedge)$ $a \Vdash A \wedge B$ iff $a \Vdash A$ and $a \Vdash B$;

(∨) $a \Vdash A \vee B$ iff $a \Vdash A$ or $a \Vdash B$;

(&) $a \Vdash A\&B$ iff there exist $b, c \in F$ such that $b \Vdash A$, $c \Vdash B$, and $a \leq b \circ c$.

A valuation on a U-L^{ℓ}_{eq} frame is a valuation further satisfying that (max) for every propositional variable p, $\{a : a \Vdash p\}$ has a maximum.

Definition 6 (U-L^{ℓ}_{eq} model)**.** *An* U-L^{ℓ}_{eq} model *is a pair* $(\boldsymbol{F}, \Vdash)$, *where* $\boldsymbol{F}$ *is a U-*L^{ℓ}_{eq} *frame and* $\Vdash$ *is a valuation on* $\boldsymbol{F}$. *This model is said to be* complete *if* $\boldsymbol{F}$ *is a complete frame and* $\Vdash$ *is a valuation on* $\boldsymbol{F}$.

Definition 7. *For a U-*L^{ℓ}_{eq} *model* $(\boldsymbol{F}, \Vdash)$, *a node* a *of* $\boldsymbol{F}$ *and a formula* A, a *is said to* force A *if* $a \Vdash A$. A *is said to be* true *in* $(\boldsymbol{F}, \Vdash)$ *if* $1 \Vdash A$, *and* valid *in the frame* $\boldsymbol{F}$ *if* A *is true in* $(\boldsymbol{F}, \Vdash)$ *for any valuation* $\Vdash$ *on* $\boldsymbol{F}$. *For a class* $\mathcal{UL}^{\ell}$ *of U-*L^{ℓ}_{eq} *frames and for a theory* Γ, *by* $\Gamma \models_{\mathcal{UL}^{\ell}} A$, *we mean that* A *is valid in* $\boldsymbol{F} \in \mathcal{UL}^{\ell}$ *whenever* B *is valid in it for all* $B \in \Gamma$. *This* A *is called a* semantic consequence *of* Γ *on* $\mathcal{UL}^{\ell}$.

Now we consider soundness and completeness of L^{ℓ}_{eq}.

Lemma 1 (Hereditary Lemma)**.**

(i) *Let* $\boldsymbol{F}$ *be a residuated U-RM*$^{\ell}$ *frame. For any formula* A *and for any nodes* $a, b \in \boldsymbol{F}$, *if* $a \Vdash A$ *and* $b \leq a$, *then* $b \Vdash A$.

(ii) *Let* $\Vdash$ *be a forcing relation on a U-*L^{ℓ}_{eq} *frame and* A *be a formula. Then the set* $\{a \in F : a \Vdash A\}$ *has a maximum.*

Proof. It is easy to prove (i). For the proof of (ii), see Proposition 3.3 in [30] and Lemma 2.11 in [18]. □

Lemma 2. $1 \Vdash A \to B$ *iff for any* $a \in F$, $a \Vdash A$ *implies* $a \Vdash B$.

Proof. (⇒) See Lemma 3 in [1]. (⇐) Suppose $Ra1b$ and $a \Vdash A$ and that $a \Vdash A$ implies $a \Vdash B$. We prove $b \Vdash B$. Using the suppositions and (df$_U$), we obtain that $a \Vdash B$ and $a = a \circ 1 = b$; therefore, $b \Vdash B$. □

Theorem 3 (*Soundness*)**.** *For a linear theory* Γ *over* $L^{\ell} \in L^{\ell}_{eq}$, *a formula* A, *and a class* $\mathcal{UL}^{\ell}$ *of all U-*L^{ℓ}_{eq} *frames,* $\Gamma \vdash_{L^{\ell}} A$ *only if* $\Gamma \models_{\mathcal{UL}^{\ell}} A$.

Proof. For the system **MIAL**, we consider the axiom (EF) as an example. For (EF), by Lemma 2, we assume that $a \Vdash \mathbf{F}$ and show that $a \Vdash A$. This result directly follows from the supposition and the condition (⊥). The other axioms and rules for **MIAL** can be proved similarly.

For the other systems, we need to consider the other structural axioms, i.e., $S \in \{e, o, a\}$.

(e): Suppose that $a \Vdash A\&B$. We have to prove that $a \Vdash B\&A$. By the supposition and the condition (&), there are $b, c \in F$ such that $b \Vdash A$, $a \leq b \circ c$, and $c \Vdash B$. Then, p_e and (df$_U$) ensure that $b \circ c = c \circ b$ and so $a \leq c \circ b$. Hence, we obtain $a \Vdash B\&A$ by (&).

(o): Suppose that $a \Vdash f$. We have to prove $a \Vdash A$. Please note that **MIAL**$_o$ proves $\mathbf{F} \leftrightarrow f$. Then, since p_o and (df$_U$) assure that $0 = \bot$, we can obtain that $a \Vdash A$ using (⊥) and (EF).

(a): Suppose $a \Vdash A\&(B\&C)$. We have to prove $a \Vdash (A\&B)\&C$. By the supposition, the condition (&), and (df$_U$), there are $b, c \in F$ so that $b \Vdash A$, $a \leq b \circ c$, and $c \Vdash B\&C$; therefore, for some d, e, we have that $d \Vdash B$, $c \leq d \circ e$, and $e \Vdash C$. Then, $b \circ c \leq b \circ (d \circ e)$ and so $b \circ c \leq (b \circ d) \circ e$ since p_a, df1, df2, and (df$_U$) assure that $b \circ (d \circ e) = (b \circ d) \circ e$. Since $b \circ d \leq b \circ d$ and $a \leq (b \circ d) \circ e$, we may take some x so that $a \leq x \circ e$ and $x \leq b \circ d$. Hence, by the condition (&), we obtain $x \Vdash A\&B$; therefore, $a \Vdash (A\&B)\&C$. The proof for the other direction is analogous. □

The following shows a connection between postulates for U-L^{ℓ}_{eq} frames and algebraic (in)equations for the structural axioms of $L^{\ell} \in L^{\ell}_{eq}$.

Proposition 1. *The postulates for U-L^{ℓ}_{eq} frames introduced in Definition 1 as p_e, p_o, and p_a are reducible to algebraic (in)equations $e^{\mathcal{A}}$, $o^{\mathcal{A}}$, and $a^{\mathcal{A}}$, respectively.*

Proof. We show that $p_{eq}, eq \in \{e, o, a\}$, is reducible to $eq^{\mathcal{A}}$.

(e): Using p_e and (df$_U$), we obtain that for arbitrary $a, b, c \in F$, $c = a \circ b$ implies $c = b \circ a$; therefore, $a \circ b \leq b \circ a$, i.e., $e^{\mathcal{A}}$, since $a \circ b = b \circ a$.

(o): Using p_o and (df$_U$), we obtain that $a = 1 \circ 0 = 0$ iff $a = 1 \circ \bot = \bot$ for any $a \in F$ and so $0 \leq a$, i.e., $0^{\mathcal{A}}$, since $0 = \bot$.

(a): Using p_a, df1, df2, and (df$_U$), we obtain that for arbitrary $a, b, c, d \in F$, there is x such that $x = a \circ b$ and $d = x \circ c$ and thus $d = (a \circ b) \circ c$ iff $x' = b \circ c$ and $d = a \circ x'$ for some x' and so $d = a \circ (b \circ c)$; therefore, $(a \circ b) \circ c = a \circ (b \circ c)$, i.e., $a^{\mathcal{A}}$. □

Corollary 1. *Every U-L^{ℓ}_{eq}, $eq \in \{o, e\}$, frame is embeddable into a complete U-L^{ℓ}_{eq} frame.*

Proof. This corollary directly follows from Theorem 2 and Proposition 1. □

The next proposition connects algebraic semantics and U-RM$^{\ell}$ semantics for $L^{\ell} \in L^{\ell}_{eq}$.

Proposition 2.

(i) *The $\{\top, \bot, 1, 0, \leq, \circ\}$ reduct of an L^{ℓ}-chain $\mathcal{A}$ is a U-L^{ℓ}_{eq} frame, which is complete iff $\mathcal{A}$ is complete.*

(ii) *Let $\mathbf{F} = (F, \top, \bot, 1, 0, \leq, \circ)$ be a U-L^{ℓ}_{eq} frame. Then, the structure $\mathcal{A} = (F, \top, \bot, 1, 0, min, max, \backslash, /, \circ)$ is an L^{ℓ}-algebra (where min and max are meant on $\leq$).*

(iii) *If $\mathbf{F}$ is the $\{\top, \bot, 1, 0, \leq, \circ\}$ reduct of an L^{ℓ}-chain $\mathcal{A}$ and v is a valuation in $\mathcal{A}$, then $(\mathbf{F}, \Vdash)$ is a U-L^{ℓ}_{eq} model and for any formula A and for any $a \in \mathcal{A}$, it holds that $a \Vdash A$ iff $a \leq v(A)$.*

(iv) *Let $(\mathbf{F}, \Vdash)$ be a U-L^{ℓ}_{eq} model and $\mathcal{A}$ be the L^{ℓ}-algebra defined as in (ii). Define for every propositional variable p, $v(p) = max\{a \in F : a \Vdash p\}$. Then, for every formula A, $v(A) = max\{a \in F : a \Vdash A\}$.*

Proof. Here we consider (iii) because the proof for (i) and (ii) is easy and (iv) follows almost directly from (iii) and Lemma 1 (ii). We consider the induction steps, where $A = B \to C$ and $A = B \rightsquigarrow C$. For the induction step of $A = B \& C$, see Proposition 3.9 in [33]. The proof for the other cases is easy.

Suppose $A = B \to C$. By the condition ($\to$), $a \Vdash B \to C$ iff for any $b, c \in F$, $Rbac$ and $b \Vdash B$ entail $c \Vdash C$, hence by the induction hypothesis, iff for any $b, c \in F$, $b \circ a = c$ and $b \leq v(B)$ entail $c \leq v(C)$ and so iff $v(B) \circ a \leq v(C)$; therefore, iff $a \leq v(B) \to v(C) = v(B \to C)$ by residuation. The proof for the case $A = B \rightsquigarrow C$ is analogous. □

Theorem 4 (*Completeness*)**.** *Let Γ be a linear theory on $L^{\ell} \in L^{\ell}_{eq}$, A a formula, and $\mathcal{UL}^{\ell}$ a class of all U-L^{ℓ}_{eq} frames.*

(i) *$\Gamma \vdash_{L^{\ell}} A$ iff $\Gamma \models_{\mathcal{UL}^{\ell}} A$.*

(ii) *Let $L^{\ell} \in L^{\ell}_{eq'}$, $eq' = \{e, o\}$, and $\mathcal{UL}^{\ell}_c$ a class of all complete U-$L^{\ell}_{eq'}$ frames. Then, $\Gamma \vdash_{L^{\ell}} A$ iff $\Gamma \models_{\mathcal{UL}^{\ell}_c} A$.*

Proof. (i) follows from Proposition 2 and Theorems 1 and 3 and (ii) from Proposition 2 and Theorems 2 (i) and 3. □

Example 3. *Among the examples introduced in Example 1, the systems **MICAL** and **UL** have U-RM$^\ell$ semantics but **MTL** does not since the postulates e, a are equationally definable by (df_U) but the postulate i is not. Therefore, we can say the following.*

*(1) Micanorm logic **MICAL** has a U-RM$^\ell$ semantics.*

*(2) Uninorm logic **UL** has a U-RM$^\ell$ semantics.*

*(3) Monoidal t-norm logic **MTL** does not have a U-RM$^\ell$ semantics.*

Please note that one is capable of defining the ternary relation R using (df_U) and the forcing relation $\Vdash$ using $\leq$. This means that for $L^\ell \in L^\ell_{eq}$, U-RM$^\ell$ semantics can be considered in the context of algebraic semantics and vice versa.

As in [1], using the definition (df_U) and the valuation conditions ($\rightarrow$) and ($\rightsquigarrow$), we can show the following derived conditions of a valuation.

Proposition 3 ([1]).

(i) *For any $b \in F$, $b \Vdash A$ implies $b \circ a \Vdash B$ iff for any $b, c \in F$, Rbac and $b \Vdash A$ imply $c \Vdash B$.*

(ii) *For any $b \in F$, $b \Vdash A$ implies $a \circ b \Vdash B$ iff for any $b, c \in F$, Rabc and $b \Vdash A$ imply $c \Vdash B$.*

Proposition 3 ensures that, as far as we accept (df_U), the conditions of a valuation are reducible to those of AK semantics for L^ℓ_{eq}. Thus, U-RM$^\ell$ semantics for L^ℓ_{eq} can be reduced to the AK semantics for L^ℓ_{eq} with the definition (df_U). Therefore, this semantics can be called ARM$^\ell$ semantics reducible to AK semantics.

3.2. F-RM$^\ell$ Semantics

Here we consider F-RM$^\ell$ semantics for L^ℓs. We first define some further RM frames.

Definition 8.

(i) *(Operational RM frame [1]) An* operational RM frame *is a structure $\mathbf{F} = (F, 1, \leq, \circ, R)$, where $(F, 1, R)$ is an RM frame, $(F, 1, \circ)$ is a groupoid with unit, and R satisfies the postulates below: for all $a, b, c \in F$,*

p_s. *R1ab and R1ba imply $a = b$;*

p_t. *R1ab and R1bc imply R1ac;*

$p_{\leq}$. *$a \leq b$ iff R1ba.*

(ii) *((Residuated) Fine operational RM$^\ell$ frame) Linear RM frames are defined as in Definition 5 (ii). A* Fine operational *RM$^\ell$ frame (F-RM$^\ell$ frame for short) is an operational RM frame, where $\circ$ satisfies ($df_F{}'$) $a \circ b \geq c :=$ Rabc (Notice that $\leq$ in ($df_F{}'$) is considered order reversely. Please compare it with $\leq$ in (df_F).) and R satisfies the postulate below: for all $a, b \in F$,*

$p0$. *R1ab or R1ba.*

Residuated *F-RM$^\ell$ frames are defined as in Definition 5 (iii).*

(iii) *(Bounded, pointed F-RM$^\ell$ MIAL frames) Bounded, pointed F-RM$^\ell$ frames are defined as in Definition 5 (iv). An* F-RM$^\ell$ MIAL frame *is a bounded pointed residuated F-RM$^\ell$ frame, where $\circ$ is conjunctive and left-continuous.*

(iv) *(F-L$^\ell$ frames) Consider the definitions and postulates df1, df2, p_e, p_o, p_a and the below additional postulates: for all $a, b, c, d \in F$,*

p_p. *Raab implies R1ab.*

p_c. *Raaa*

p_i. *Rabc implies R1bc.*

For any $S \subseteq \{p_e, p_p, p_c, p_i, p_o, p_a\}$*,* F-RM$^\ell$ MIAL$_S$ frames *are defined, along with their corresponding postulates. We call all these frames* F-L^ℓ frames.

Remark 2.

(1) *One is capable of showing that* $(F, \leq)$ *is a linearly ordered set, using* $(df_F{}')$*, identity* $a \leq a$*,* p_s*,* p_t*, and* $p0$ *in F-RM*$^\ell$ *frames and so these frames are linearly ordered.*

(2) *([1]) The indices of the postulates in* (iv) *denote their corresponding axioms. For example, the postulate* p_p *is for the expansion axiom* p*. Moreover,* $(df_F{}')$ *assures that those postulates satisfy their corresponding algebraic properties.*

The conditions for a valuation on an F-L^ℓ frame are the same as in Section 3.1 except for the following:

$(\&_R)$ $a \Vdash A\&B$ iff there exist $b, c \in F$ so that $Rbca$, $b \Vdash A$ and $c \Vdash B$.

We can prove Proposition 3 and the condition $(\&)$ using $(\mathrm{df}_F{}')$. Moreover, we can further show the following additional derived condition.

Proposition 4. $a \leq v(A) \circ v(B)$ *iff there exist* $b, c \in F$ *such that* $Rcba$*,* $b \leq v(A)$ *and* $c \leq v(B)$*.*

Proof. $(\Rightarrow)$ Assume that $a \leq v(A) \circ v(B)$. We take b, c satisfying that $b = v(A)$ and $c = v(B)$. Then, using $(\mathrm{df}_F{}')$, we can obtain that $Rcba$, $b \leq v(A)$, and $c \leq v(B)$. $(\Leftarrow)$ Suppose that $Rcba$, $b \leq v(A)$ and $c \leq v(B)$. Then, using $(\mathrm{df}_F{}')$, we obtain $a \leq b \circ c$, $b \leq v(A)$, and $c \leq v(B)$; therefore, $a \leq v(A) \circ v(B)$. $\square$

Notice that Lemmas 1 and 2 also hold for F-L^ℓ frames and models.

Theorem 5 (*Soundness*)**.** *For a linear theory* Γ *over* $L^\ell \in L^\ell$*s, a formula* A*, and a class* $\mathcal{FL}^\ell$ *of all F-*L^ℓ *frames,* $\Gamma \vdash_{L^\ell} A$ *only if* $\Gamma \models_{\mathcal{FL}^\ell} A$*.*

Proof. We need to consider the structural axioms c, p, i. For c, we assume that $a \Vdash A$ and show that $a \Vdash A\&A$. By the supposition, p_c, and $(\&)$, we obtain that $a \Vdash A\&A$. The proof for the other ones p, i is analogous. $\square$

Now, we recall a connection between postulates for F-L^ℓ frames and algebraic (in)equations for the structural axioms of $L^\ell \in L^\ell$s.

Proposition 5 ([1])**.** *The postulates for F-*L^ℓ *frames introduced in Definition 1 are reducible to algebraic (in)equations for the structural axioms of* L^ℓ *introduced in Definition 4.*

The next proposition connects F-RM$^\ell$ semantics and algebraic semantics for L^ℓs.

Proposition 6.

(i) *The* $\{\top, \bot, 1, 0, \leq, \circ\}$ *reduct of an* L^ℓ*-chain* $\mathcal{A}$ *is an F-*L^ℓ *frame, which is complete iff* $\mathcal{A}$ *is complete.*

(ii) *Let* $\boldsymbol{F} = (F, \top, \bot, 1, 0, \leq, \circ)$ *be an F-*L^ℓ *frame. Then, the structure* $\mathcal{A} = (F, \top, \bot, 1, 0, max, min, \circ, \backslash, /)$ *is an* L^ℓ*-algebra.*

(iii) *If* $\boldsymbol{F}$ *is the* $\{\top, \bot, 1, 0, \leq, \circ\}$ *reduct of an* L^ℓ*-chain* $\mathcal{A}$ *and* v *is a valuation in* $\mathcal{A}$*, then* $(\boldsymbol{F}, \Vdash)$ *is an F-*L^ℓ *model and for all formulas* A *and for all* $a \in \mathcal{A}$*, we obtain that* $a \Vdash A$ *iff* $a \leq v(A)$*.*

(iv) *Let* $(\boldsymbol{F}, \Vdash)$ *be an F-*L^ℓ *model and* $\mathcal{A}$ *be the* L^ℓ*-algebra defined as in* (ii)*. Define for every propositional variable* p*,* $v(p) = max\{a \in F : a \Vdash p\}$*. Then, for every formula* A*,* $v(A) = max\{a \in F : a \Vdash A\}$*.*

Proof. As above, we prove (iii). We consider the induction steps, where $A = B\&C$, $A = B \to C$ and $A = B \rightsquigarrow C$, since the other cases can be easily proved.

Suppose $A = B\&C$. The condition $(\&_R)$ assures that $a \Vdash B\&C$ iff there are $b, c \in F$ so that $b \Vdash B, c \Vdash C$, and $Rbca$, hence by the induction hypothesis and $(\mathrm{df}_F{}')$, iff there are $b, c \in F$ such that $b \leq v(B), c \leq v(C)$, and $a \leq b \circ c$. It then holds true that $a \leq b \circ c \leq v(B) \circ v(C) = v(B\&C)$. Conversely, suppose $a \leq v(B) \circ v(C) = v(B\&C)$ and take $b = v(B)$ and $c = v(C)$. Then we can obtain $b \Vdash B$, $c \Vdash C$ and $a \leq b \circ c$; hence, $a \Vdash B\&C$ by $(\&_R)$ and $(\mathrm{df}_F{}')$.

Suppose $A = B \to C$. The condition $(\to)$ assures that $a \Vdash B \to C$ iff for all $b, c \in F$, $Rbac$ and $b \Vdash B$ entail $c \Vdash C$, hence by the induction hypothesis and $(\mathrm{df}_F{}')$, iff for all $b, c \in F$, $c \leq b \circ a$ and $b \leq v(B)$ entail $c \leq v(C)$. Then, we obtain $a \leq v(B)\backslash v(C) = v(B \to C)$ since $c \leq b \circ a \leq v(B) \circ a \leq v(C)$. Suppose conversely that $a \leq v(B)\backslash v(C) = v(B \to C)$. Then we have that $v(B) \circ a \leq v(C)$ and so $c \leq v(C)$ for $c \leq b \circ a$ and $b \leq v(B)$. This ensures that $Rbac$ and $b \Vdash B$ entail $c \Vdash C$; hence, $a \Vdash B \to C$ by $(\to)$.

The proof of the case $A = B \rightsquigarrow C$ is analogous to the case $A = B \to C$. □

Theorem 6 (*Completeness*)**.** *Let Γ be a linear theory over $L^\ell \in L^\ell$s, A a formula, and $\mathcal{FL}^\ell$ a class of all F-L^ℓ frames.*

(i) $\Gamma \vdash_{L^\ell} A$ *iff* $\Gamma \models_{\mathcal{FL}^\ell} A$.

(ii) *Let L^ℓ be a member of $L^\ell{}_{S'} = \{\boldsymbol{MIAL}_{S'} : S' \subseteq \{e, p, c, i, o\}\}$ or $L^\ell_{S''}$ such that $\{i, a\} \subseteq S'' \subseteq \{e, p, c, i, o, a\}$, and $\mathcal{FL}^\ell_c$ a class of all complete F-L^ℓ frames. Then, $\Gamma \vdash_{L^\ell} A$ iff $\Gamma \models_{\mathcal{FL}^\ell_c} A$.*

Proof. (i) follows from Proposition 6 and Theorems 1 and 5 and (ii) from Proposition 6 and Theorems 2 and 5. □

Example 4. *All the systems introduced in Example 1 have F-RM$^\ell$ semantics since the postulates e, a, i are inequationally definable by $(\mathrm{df}_F{}')$. However, the non-fuzzy system $\boldsymbol{R}^+$ (the positive $\boldsymbol{R}$) does not have such semantics since, while F-RM$^\ell$ semantics validates sentences such as (a) in Section 1, $\boldsymbol{R}^+$ does not proves such sentences. Therefore, we can say the following.*

*(1) Micanorm logic **MICAL** has an F-RM$^\ell$ semantics.*

*(2) Uninorm logic **UL** has an F-RM$^\ell$ semantics.*

*(3) Monoidal t-norm logic **MTL** has an F-RM$^\ell$ semantics.*

(4) The positive relevance logic $\boldsymbol{R}^+$ does not have an F-RM$^\ell$ semantics.

As above, one is capable of defining the ternary relation R using $(\mathrm{df}_F{}')$ and the forcing relation $\Vdash$ using $\leq$. This means that for $L^\ell \in L^\ell$s, F-RM$^\ell$ semantics can be considered in the context of algebraic semantics and vice versa.

4. Advantages and Limitations of ARM$^\ell$ Semantics

4.1. Advantages and Limitations: General

Here we consider the advantages and limitations of U-RM$^\ell$ and F-RM$^\ell$ semantics as ARM$^\ell$ semantics in general. The most important advantage of these semantics is that the clauses $(\wedge)$, $(\vee)$, and either $(\to_{R_U})$ or $(\to_{R_F})$ can be used together. Please note that these clauses are not working together on distributive substructural logic systems in general, whereas they are still working on linearly ordered related substructural systems (see Examples 3 and 4). This means that these semantics use the standard clauses $(\wedge)$, $(\vee)$, and so are more powerful than such semantics introduced in [1] in a *pragmatic* sense. Because most people working for semantics of a formal system would be familiar with these standard clauses and thus U-RM$^\ell$ and F-RM$^\ell$ semantics would be easier to understand to them. (Please note that in fuzzy logic prime theories are interchangeable with linear theories (see, e.g., [29]) and so the clause $(\vee)$ can be used in ARM$^\ell$ semantics. Note also that the ARM semantics in [1] uses closed theories in place of linear theories.)

ARM$^\ell$ semantics is also very powerful in a *philosophical* sense that one may understand concretely the connections between nodes to force a formula. For instance, the inequation $(e^{\mathcal{A}})$ shows that any pair of nodes a, b must be *exchangeable* in order to force the formula $(A\&B) \to (B\&A)$ in the logic $\mathbf{MIAL}_e$ (= **MICAL**). Hence, one can say that the basic feature of the postulate p_e is the commutativity property. Please note that it is not easy for us to understand that the postulate p_e has the commutativity feature if we do not interpret it using the definition (df_U) (or ($\mathrm{df}_F{}'$)). The definition (df_U) says that the ternary relation R for accessibility is replaced by the operator $\circ$ and equality $=$ and similarly for the definition ($\mathrm{df}_F{}'$). Using these definitions, one can achieve an intuitive understanding of the meaning of p_e.

As is well known, Routley–Meyer semantics for relevance logics need not require structures equivalent to algebraic semantics for those logics. For example, the Routley–Meyer semantics for **R** introduced by Dunn [8] has the ternary relation R indefinable by (df_U) and ($\mathrm{df}_F{}'$) and the postulates for this semantics cannot be interpreted by those definitions (see [1,8]). Thus, related to accessibility, we may introduce two different approaches to Routley–Meyer-style semantics: One is Routley–Meyer-style semantics based on accessibility relations themselves and the other is that instead based on accessibility operations. The Routley–Meyer semantics for **R** above is a representative example of the former sort and the ARM$^\ell$ semantics for the substructural fuzzy logics here is a representative example of the latter sort. Therefore, ARM$^\ell$ semantics as a representative of the latter sort is *very useful* to deal with substructural fuzzy logics.

However, on the other side of the same coin, ARM$^\ell$ semantics has clear limitations. First, it does not have its own semantics distinguished from algebraic semantics in that frames for the ARM$^\ell$ semantics have the same structures as algebraic semantics. In this sense, ARM$^\ell$ semantics is not very interesting in terms of the novelty of semantics. Second, it is not applicable to Routley–Meyer frames not having such algebraic structures. Thirdly, it is not applicable to non-fuzzy substructural logics in general. These three are the limitations of U-RM$^\ell$ and F-RM$^\ell$ semantics as ARM$^\ell$ semantics in general.

4.2. Advantages and Limitations: Specific

Here we deal with comparative advantages and limitations between U-RM$^\ell$ and F-RM$^\ell$ semantics. We first introduce advantages of each semantics. First, consider U-RM$^\ell$ semantics. As mentioned in Section 3.1, Position 3 implies that this semantics can be regarded as ARM$^\ell$ semantics reducible to AK semantics. This shows that U-RM$^\ell$ semantics is related to both Kripke-style semantics and algebraic semantics. More precisely, U-RM$^\ell$ semantics belongs to a common area of algebraic, Kripke-style, and Routley–Meyer-style semantics. This is the most important *logical* advantage of U-RM$^\ell$ semantics in that it provides a chance to investigate similarities and differences between the three sorts of semantics. U-RM$^\ell$ frames have the postulates $p1, p2$, whereas F-RM$^\ell$ frames do not. This means that U-RM$^\ell$ frames have relational conditions for a unit element but F-RM$^\ell$ frames do not. This is another advantage of U-RM$^\ell$ semantics in a *technical* point of view for relational semantics when it is compared to F-RM$^\ell$ semantics.

Next, consider F-RM$^\ell$ semantics. Proposition 6 assures that if one accepts ($\mathrm{df}_F{}'$), one is capable of providing ARM$^\ell$ semantics being equivalent to algebraic semantics for L^ℓs. This implies that F-RM$^\ell$ semantics can cover all the systems introduced in Definition 2. This is the most important *logical* advantage of F-RM$^\ell$ semantics in that this semantics as one sort of ARM$^\ell$ semantics is as powerful as algebraic semantics. Please note that, as is shown in Section 3.2, substructural (core) fuzzy logics being algebraically complete are also complete regarding this sort of semantics and vice versa. F-RM$^\ell$ frames have the postulates $p_s, p_t, p_\leq, p0$, whereas U-RM$^\ell$ frames do not. This means that F-RM$^\ell$ frames have relational conditions for the *linear ordering* of $\leq$ but U-RM$^\ell$ frames do not. Moreover, it has the relational clause ($\&_R$) defined using the ternary relation R for a valuation of the intensional conjunction &. These provide a *technical* advantage of F-RM$^\ell$ semantics for relational semantics when it is compared to U-RM$^\ell$ semantics.

We next introduce limitations of each semantics. Like the face and back of a coin, the advantages of U-RM$^\ell$ provide limits of F-RM$^\ell$ semantics, and the advantages of F-RM$^\ell$ give limitations of U-RM$^\ell$ semantics. We state this. For the limitations of U-RM$^\ell$ semantics, first consider the following example.

Example 5. *As in [1], (df_U) does not work for the postulates p_p, p_c, and p_i introduced in Section 3.2. To verify this, apply (df_U) to p_c, p_p, and p_i. Then, we obtain the following:*

p'_c. $a \circ a = a$.

p'_p. $b = a \circ a$ *implies* $b = 1 \circ a$.

p'_i. $c = a \circ b$ *implies* $c = 1 \circ b$.

*Since p'_c implies p'_p and vice versa, the postulates p_c and p_p are the same; therefore, the MIAL$_c$-RM (MIAL$_p$-RM resp) frame validates the axiom p (c resp), which is not provable in **MIAL**$_c$ (**MIAL**$_p$ resp). Similarly, the postulate p_i implies that $b = a * b$ and thus the MIAL$_i$-RM frame validates formulas such as $\varphi \to (\psi \& \varphi)$, which is not provable in **MIAL**$_i$.*

This example shows that as semantics for basic (core) fuzzy logics U-RM$^\ell$ semantics is *less powerful* than F-RM$^\ell$ semantics in a *logical* point of view. Moreover, using the ternary relation R, one cannot provide the postulates for the above linear ordering and the clause ($\&_R$). These two are the limitations of U-RM$^\ell$ semantics compared to F-RM$^\ell$ semantics.

For the limitations of F-RM$^\ell$ semantics, first consider the following example, which can be easily verified.

Example 6. *Using the definition ($df_F{}'$) and the valuation conditions ($\to$) and ($\rightsquigarrow$), one can prove the following.*

(i) *For any $b \in F$, $b \Vdash A$ implies $b \circ a \Vdash B$ only if for any $b, c \in F$, Rbac and $b \Vdash A$ imply $c \Vdash B$.*

(ii) *For any $b \in F$, $b \Vdash A$ implies $a \circ b \Vdash B$ only if for any $b, c \in F$, Rabc and $b \Vdash A$ imply $c \Vdash B$.*

However, one cannot prove the reverse direction of each (i) and (ii) and so cannot do Proposition 3.

This example shows that F-RM$^\ell$ semantics is not reducible to AK semantics. Therefore, such semantics can be called ARM$^\ell$ semantics irreducible to AK semantics for L^ℓs. It implies that F-RM$^\ell$ semantics is not helpful to study a common area between Kripke-style semantics and Routley–Meyer-style semantics. Hence, F-RM$^\ell$ semantics is *less powerful* than U-RM$^\ell$ semantics in a logical research of common areas between different sorts of semantics. Moreover, using the ternary relation R, one cannot provide the postulates for the unit element. These two are the limitations of F-RM$^\ell$ semantics compared to U-RM$^\ell$ semantics.

As a summary we note the following facts. First, U-RM$^\ell$ semantics covers all the (core) fuzzy systems with equationally definable substructural axioms, whereas F-RM$^\ell$ semantics those systems with inequationally definable substructural axioms. Second, in these two sorts of ARM$^\ell$ semantics, (df$_U$) and (df$_F{}'$) provide ways to interpret the ternary relation R using binary operation $\circ$ and (in)equation. Third, these two semantics are not applicable to Routley–Meyer-style semantics irreducible to algebraic one. For instance, while the RM semantics for **R** introduced in [8] requires the postulate *Raaa* for the rule (mp), U-RM$^\ell$ and F-RM$^\ell$ semantics cannot have $a = a \circ a$ (by (df$_U$)) and $a \leq a \circ a$ (by (df$_F{}'$)), respectively, as semantic postulates for **MIAL**.

5. Discussion and Conclusions

We investigated ARM semantics for substructural (core) fuzzy logics based on mianorms. More precisely, we provided U-RM$^\ell$ and F-RM$^\ell$ semantics as two sorts of ARM semantics for them. We in particular deal with advantages and limitations of these semantics.

We note that U-RM$^{\ell}$ and F-RM$^{\ell}$ semantics provide frames as some reducts of their corresponding algebras and so are ARM$^{\ell}$ semantics for fuzzy extensions of substructural logics. Especially, those semantics define ternary relations using binary operations and (in)equations like the semantics for substructural logics in [1]. However, unlike these semantics, they are provided based on linear theories and conditions for linear ordering, and so work for linearly ordered models. As mentioned in Section 1, ARM$^{\ell}$ semantics as a relational semantics for substructural fuzzy logics in general is a *novel* one to connect n-nary operations and n+1-nary relations.

The author [47] introduced implicational tonoid *fuzzy* logics as fuzzy extensions of implicational tonoid logics, the class of logics satisfying transitivity, reflexivity, tonicity, and modus ponens introduced by the author and Dunn [48]. Please note that all the logic systems introduced in Definition 2 can be regarded as such fuzzy logics because they also satisfy the conditions for an implicational tonoid logic and are complete over linearly ordered models. Hence, this investigation can be thought of as an introduction of ARM semantics for *concrete* implicational tonoid fuzzy logics.

However, any more exact connection between semantics for implicational tonoid fuzzy logics and those for substructural (core) fuzzy logics is not studied here. For instance, while the former logics do not introduce any concrete connectives, the substructural (core) fuzzy logics do. For these logics, the connectives $\vee$ and $\wedge$ need to be interpreted by *join* and *meet* as lattice operators and need their corresponding relational consideration. Thus, in the context of implicational tonoid fuzzy logics, these things have to be dealt with. Furthermore, while non-operational RM semantics can be established for substructural logics, e.g., the system **R** (see [8]), such semantics for the fuzzy logics is not considered either. The author has a plan to study these two in the future, i.e., leave these for another day. By these two works, we can fill gaps between abstract logic (implicational tonoid fuzzy logics) and concrete logic (substructural (core) fuzzy logics) and between algebraic and non-algebraic Routley–Meyer-style semantics.

It is well known that lattices can be defined as ordered sets and as algebraic structures. To show the equivalence between the first relational definition of a lattice and its second algebraic definition, one has to have some definitions such as (b) $x \leq y$ iff $join(x, y) = y$ iff $meet(x, y) = x$. Similarly, we can consider the algebraic and ARM semantics and the definitions (df_U) and ($\mathrm{df}_F{}'$) regarding substructural (core) fuzzy logics. Associated with this, the author [49] studied basic logico-algebraic properties of micanorms characterizing the logic **MICAL** such as (left-)continuity, residuated implications, conjunctive and disjunctive micanorms, idempotent, nilpotent, and divisor micanorms, and so on. This implies that such theoretic applications of micanorms can be considered in the context of U-RM$^{\ell}$ and F-RM$^{\ell}$ frames. Namely, one can treat such properties as applications of U-RM$^{\ell}$ and F-RM$^{\ell}$ frames. More exact treatment of such applications is an another problem to solve in the future.

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