axioms

MDPI

Article

On q-Horn Hypergeometric Functions $\mathbf{H}_6$ and $\mathbf{H}_7$

Ayman Shehata [1,2]

[1] Department of Mathematics, College of Science and Arts, Qassim University, Unaizah 56264, Qassim, Saudi Arabia; A.Ahmed@qu.edu.sa
[2] Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt; aymanshehata@science.aun.edu.eg

Abstract: This work aims to construct various properties for basic Horn functions $\mathbf{H}_6$ and $\mathbf{H}_7$ under conditions on the numerator and denominator parameters, such as several q-contiguous function relations, q-differential relations, and q-differential equations. Special cases of our main results are also demonstrated.

Keywords: q-calculus; q-Horn hypergeometric series; q-derivatives; q-contiguous function relations

MSC: 33D15; 33D70; 05A30

check for updates

Citation: Shehata, A. On q-Horn hYpergeometric Functions $\mathbf{H}_6$ and $\mathbf{H}_7$. *Axioms* **2021**, *10*, 336. https://doi.org/10.3390/axioms10040336

Academic Editors: Serkan Araci, Martin Bohner, Roberto B. Corcino and Sunil Dutt Purohit

Received: 15 October 2021
Accepted: 25 November 2021
Published: 8 December 2021

1. Introduction

The theory of quantum calculus or q-calculus has a wide range of applications in several fields of mathematics, engineering, physics, partition theory, number theory, Lie theory, combinatorial analysis, integral transforms, fractional calculus, and quantum theory, etc. Several authors have contributed works on this subject: (see for example, [1–15]). Sahai and Verma [16,17], Guo and Schlosser [18], Verma and Sahai [19], Verma and Yadav [20], Wei and Gong [21] studied and investigated some properties for various families of the q-hypergeometric, q-Appell and q-Lauricella series by applying operators of quantum calculus. In [22], Ernst obtained the q-analogues of Srivastava's triple hypergeometric functions. Araci et al. [23–25] studied some properties of q-Bernoulli, q-Euler, and q-Frobenius–Euler polynomials based on q-exponential functions. Duran et al. [26,27] investigated q-Bernoulli, q-Genocchi, and q-Euler polynomials and introduced the q-analogues of familiar earlier formulas. In [28], Pathan et al. derived the certain new formulas for the classical Horn's hypergeometric functions $H_1, H_2, \ldots H_{11}$. In [29,30], the author introduced the (p,q)-Humbert, (p,q)-Bessel functions. In [31], Shehata has earlier investigated the results for basic Horn hypergeometric functions H_3 and H_4. The reason of interest for this family of basic Horn's hypergeometric functions is due to their intrinsic mathematical physics importance.

Throughout this work, we assume that the expression $0<|q|<1, q\in\mathbb{C}$, we use the following abbreviated notations: let $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}=\{0,1,2,\ldots\}$ be the sets of complex, natural and non-negative numbers.

The q-shifted factorial (q-Pochhammer symbol) $(\eta;q)_m$ is defined by (see [32]):

$$(\eta;q)_m=\begin{cases}(1-\eta)(1-\eta q)\ldots(1-\eta q^{m-1}), & m\in\mathbb{N},\eta\in\mathbb{C}\setminus\{1,q^{-1},q^{-2},\ldots,q^{1-m}\};\\ 1, & m=0,\eta\in\mathbb{C},\end{cases}\tag{1}$$

for negative subscripts,

$$(\eta;q)_{-m}=\frac{1}{(\eta q^{-m};q)_m}=\frac{1}{(1-\eta q^{-1})(1-\eta q^{-2})\ldots(1-\eta q^{-m})}=\frac{(-\eta^{-1}q)^m q^{\binom{m}{2}}}{(\eta^{-1}q;q)_m},\tag{2}$$
$$\eta\neq q^{\pm1},q^{\pm2},q^{\pm3},q^{\pm4},\ldots,q^{\pm m}, m=1,2,3,\ldots.$$

For $\eta \in \mathbb{C}$, the q-number or q-bracket $[\eta]_q$ is defined as (see [32])

$$[\eta]_q = \begin{cases} \frac{1-q^\eta}{1-q}, & \eta \in \mathbb{C}; \\ 1, & \eta = 0. \end{cases} . \tag{3}$$

Let m be a non-negative integer number, the q-number $[m]_q$ and q-factorial $[m]_q!$ are defined by (see [13,27])

$$[m]_q = \begin{cases} \frac{1-q^m}{1-q}, & m \in \mathbb{N}; \\ 1, & m = 0 \end{cases} \tag{4}$$

and

$$[m]_q! = \begin{cases} \prod_{r=1}^{m} [r]_q = [m]_q[m-1]_q \dots [2]_q[1]_q = \frac{(q;q)_m}{(1-q)^m}, & m \in \mathbb{N}; \\ 1, & m = 0. \end{cases} \tag{5}$$

We recall the notations for $m, k \in \mathbb{N}$, $\eta \in \mathbb{C}$, which are used in the sequel (see [32])

$$\begin{aligned} (\eta;q)_m &= (1-\eta)(\eta q;q)_{m-1} \\ &= (1-\eta q^{m-1})(\eta;q)_{m-1}, \end{aligned} \tag{6}$$

$$\begin{aligned} (\eta q;q)_m &= \frac{1-\eta q^m}{1-\eta}(\eta;q)_m \\ &= (1-\eta q^m)(\eta q;q)_{m-1}, \end{aligned} \tag{7}$$

$$\begin{aligned} (\eta q^{-1};q)_m &= \frac{1-\eta q^{-1}}{1-\eta q^{m-1}}(\eta;q)_m \\ &= (1-\eta q^{-1})(\eta;q)_{m-1} \end{aligned} \tag{8}$$

and

$$\begin{aligned} (\eta;q)_{m+k} &= (\eta;q)_m(\eta q^m;q)_k \\ &= (\eta;q)_k(\eta q^k;q)_m. \end{aligned} \tag{9}$$

The q-difference operator $D_{z,q}$ of a function f at $z \neq 0 \in \mathbb{C}$ is defined as (see [33]).

$$D_{z,q}f(z) = \frac{f(z) - f(qz)}{(1-q)z}, z \neq 0, \tag{10}$$

and $D_{z,q}f(0) = \frac{df(z)}{dz}|_{z=0} = f'(0)$, provided that f is differentiable at $z = 0$, and defined differential operator $\theta_{z,q} = zD_{z,q}$.

For $0 < |q| < 1$, $q \in \mathbb{C}$ and (1), we give the definition of the basic Horn functions $\mathbf{H}_6$, $\mathbf{H}_7$, $\mathbb{H}_6$ and $\mathbb{H}_7$ as follows

$$\mathbf{H}_6(\alpha;\beta;q,x,y) = \sum_{r,s=0}^{\infty} \frac{(\alpha;q)_{2r+s}}{(\beta;q)_{r+s}(q;q)_r(q;q)_s} x^r y^s, \beta \neq 1, q^{-1}, q^{-2}, \dots, \tag{11}$$

$$\mathbf{H}_7(\alpha;\beta,\gamma;q,x,y) = \sum_{r,s=0}^{\infty} \frac{(\alpha;q)_{2r+s}}{(\beta;q)_r(\gamma;q)_s(q;q)_r(q;q)_s} x^r y^s, \beta, \gamma \neq 1, q^{-1}, q^{-2}, \dots, \tag{12}$$

$$\mathbb{H}_6(q^\alpha;q^\beta;q,x,y) = \sum_{r,s=0}^{\infty} \frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_{r+s}(q;q)_r(q;q)_s} x^r y^s, q^\beta \neq 1, q^{-1}, q^{-2}, \dots \tag{13}$$

and

$$\mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, x, y) = \sum_{r,s=0}^{\infty} \frac{(q^\alpha; q)_{2r+s}}{(q^\beta; q)_r (q^\gamma; q)_s (q; q)_r (q; q)_s} x^r y^s, q^\beta, q^\gamma \neq 1, q^{-1}, q^{-2}, \ldots. \quad (14)$$

Note that for $q \longrightarrow 1^-$, the basic Horn functions $\mathbf{H}_6(\alpha; \beta; q, x, y)$ and $\mathbf{H}_7(\alpha; \beta, \gamma; q, x, y)$ reduces to Horn functions $\mathbf{H}_6$ and $\mathbf{H}_7$ [34].

So as to simplify the following notations, we are writing $\mathbf{H}_6$ for the function $\mathbf{H}_6(\alpha; \beta; q, x, y)$, $\mathbf{H}_6(\alpha q^{\pm 1})$ for the function $\mathbf{H}_6(\alpha q^{\pm 1}; \beta; q, x, y)$, $\mathbf{H}_6(\beta q^{\pm 1})$ for the function $\mathbf{H}_6(\alpha; \beta q^{\pm 1}; q, x, y)$, $\mathbf{H}_7$ for the function $\mathbf{H}_7(\alpha; \beta, \gamma; q, x, y)$, $\mathbf{H}_7(\gamma q^{\pm 1})$ stands for the function $\mathbf{H}_7(\alpha; \beta, \gamma q^{\pm 1}; q, x, y)$, $\mathbb{H}_6$ for the function $\mathbb{H}_6(q^\alpha; q^\beta; q, x, y)$, $\mathbb{H}_6(q^{\alpha \pm 1})$ for the function $\mathbb{H}_6(q^{\alpha \pm 1}; q^\beta; q, x, y)$, $\mathbb{H}_6(q^{\beta \pm 1})$ for the function $\mathbb{H}_6(q^\alpha; q^{\beta \pm 1}; q, x, y)$, and $\mathbb{H}_7$ for the function $\mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, x, y)$, ... etc.

Our present study is primarily motivated by the former works in quantum calculus. We express a family of extended forms for the functions $\mathbf{H}_6$ and $\mathbf{H}_7$. In Section 2, the q-contiguous relations, q-differential relations, and q-differential equations for the functions $\mathbf{H}_6$, $\mathbb{H}_6$, $\mathbf{H}_7$ and $\mathbb{H}_7$ under conditions on the numerator and denominator parameters are derived. Finally, some concluding remarks for the functions $\mathbf{H}_6$, $\mathbb{H}_6$, $\mathbf{H}_7$ and $\mathbb{H}_7$ are determined in Section 3.

2. Main Result

Here we establish various properties as well as the q-contiguous function relations and q-differential equations for the functions $\mathbf{H}_6$ with $\beta \neq 1, q^{-1}, q^{-2}, \ldots$, $\mathbf{H}_7$ with α, $\beta \neq 1, q^{-1}, q^{-2}, \ldots$, $\mathbb{H}_6$ with $q^\beta \neq 1, q^{-1}, q^{-2}, \ldots$, and $\mathbb{H}_7$ with $q^\beta, q^\gamma \neq 1, q^{-1}, q^{-2}, \ldots$ which will be useful in the sequel.

Theorem 1. *The relations of the functions $\mathbf{H}_6$ and $\mathbf{H}_7$ with the numerator parameter α*

$$\begin{aligned} \mathbf{H}_6(\alpha q) &= \mathbf{H}_6 + \frac{\alpha x(1-\alpha q)}{1-\beta} \mathbf{H}_6(\alpha q^2; \beta q; q, x, y) \\ &+ \frac{\alpha x q(1-\alpha q)}{1-\beta} \mathbf{H}_6(\alpha q^2; \beta q; q, xq, y) + \frac{\alpha y}{1-\beta} \mathbf{H}_6(\alpha q; \beta q; xq^2, y), \beta \neq 1, \end{aligned} \quad (15)$$

$$\begin{aligned} \mathbf{H}_6(\alpha q) &= \mathbf{H}_6 + \frac{\alpha y}{1-\beta} \mathbf{H}_6(\alpha q; \beta q; q, x, y) + \frac{\alpha x q(1-\alpha q)}{1-\beta} \mathbf{H}_6(\alpha q^2; \beta q; q, xq, yq) \\ &+ \frac{\alpha x(1-\alpha q)}{1-\beta} \mathbf{H}_6(\alpha q^2; \beta q; q, x, yq), \beta \neq 1, \end{aligned} \quad (16)$$

$$\begin{aligned} \mathbf{H}_7(\alpha q) &= \mathbf{H}_7 + \frac{\alpha x(1-\alpha q)}{1-\beta} \mathbf{H}_7(\alpha q^2; \beta q, \gamma; q, x, y) + \frac{\alpha y}{1-\gamma} \mathbf{H}_7(\alpha q; \beta, \gamma q; q, xq^2, y) \\ &+ \frac{\alpha x q(1-\alpha q)}{1-\beta} \mathbf{H}_7(\alpha q^2; \beta q, \gamma; q, xq, y), \beta, \gamma \neq 1 \end{aligned} \quad (17)$$

and

$$\begin{aligned} \mathbf{H}_7(\alpha q) &= \mathbf{H}_7 + \frac{\alpha y}{1-\gamma} \mathbf{H}_7(\alpha q; b, \gamma q; q, x, y) + \frac{\alpha x q(1-\alpha q)}{1-\beta} \mathbf{H}_7(\alpha q^2; \beta q, \gamma; q, xq, yq) \\ &+ \frac{\alpha x(1-\alpha q)}{1-\beta} \mathbf{H}_7(\alpha q^2; \beta q, \gamma; q, x, yq), \beta, \gamma \neq 1. \end{aligned} \quad (18)$$

Proof. To prove (15). Replacing α by αq in (11) and using (6)–(7), we get

$$\begin{aligned}
\mathbf{H}_6(\alpha q) - \mathbf{H}_6 &= \sum_{r,s=0}^{\infty} \frac{1}{(\beta;q)_{r+s}(q;q)_r(q;q)_s} \Big[(\alpha q;q)_{2r+s} - (\alpha;q)_{2r+s} \Big] x^r y^s \\
&= \sum_{r,s=0}^{\infty} \frac{\alpha(1-q^{2r+s})(\alpha q;q)_{2r+s-1}}{(\beta;q)_{r+s}(q;q)_r(q;q)_s} x^r y^s \\
&= \sum_{r=1,s=0}^{\infty} \frac{\alpha(\alpha q;q)_{2r+s-1}}{(\beta;q)_{r+s}(q;q)_{r-1}(q;q)_s} x^r y^s + \sum_{r=1,s=0}^{\infty} \frac{\alpha q^r(\alpha q;q)_{2r+s-1}}{(\beta;q)_{r+s}(q;q)_{r-1}(q;q)_s} x^r y^s \\
&+ \sum_{r=0,s=1}^{\infty} \frac{\alpha q^{2r}(\alpha q;q)_{2r+s-1}}{(\beta;q)_{r+s}(q;q)_r(q;q)_{s-1}} x^r y^s \\
&= \frac{\alpha x(1-\alpha q)}{1-\beta} \mathbf{H}_6(\alpha q^2;\beta q;q,x,y) + \frac{\alpha x q(1-\alpha q)}{1-\beta} \mathbf{H}_6(\alpha q^2;\beta q;q,xq,y) \\
&+ \frac{\alpha y}{1-\beta} \mathbf{H}_6(\alpha q;\beta q;q,xq^2,y), \beta \neq 1.
\end{aligned}$$

Similarly, by using the relation

$$1 - q^{2r+s} = 1 - q^s + q^s(1-q^r) + q^{r+s}(1-q^r),$$

we get (16). Equations (17) and (18) can be proven on as same lines as the Equation (15). □

Theorem 2. *The functions* $\mathbf{H}_7$ *and* $\mathbf{H}_6$ *satisfy the q-derivative equations*

$$D^r_{x,q}\mathbf{H}_7 = \frac{(\alpha;q)_{2r}}{(\beta;q)_r(1-q)^r} \mathbf{H}_7(\alpha q^{2r};\beta q^r,\gamma;q,x,y), D_{x,q} = \frac{\partial}{\partial x}, \tag{19}$$

$$D^r_{y,q}\mathbf{H}_7(\alpha;\beta,\gamma;q,x,y) = \frac{(\alpha;q)_r}{(\gamma;q)_r(1-q)^r} \mathbf{H}_7(\alpha q^r;\beta,\gamma q^r;q,x,y), D_{y,q} = \frac{\partial}{\partial y}, \tag{20}$$

$$D^r_{x,q}\mathbf{H}_6 = \frac{(\alpha;q)_{2r}}{(\beta;q)_r(1-q)^r} \mathbf{H}_6(\alpha q^{2r};\beta q^r;q,x,y) \tag{21}$$

and

$$D^r_{y,q}\mathbf{H}_6 = \frac{(\alpha;q)_r}{(\beta;q)_r(1-q)^r} \mathbf{H}_6(\alpha q^r;\beta q^r;q,x,y). \tag{22}$$

Proof. From (12) and (10), we get q-derivatives of $\mathbf{H}_7$ with respect to x and y as follows

$$\begin{aligned}
D_{x,q}\mathbf{H}_7 &= \sum_{r,s=0}^{\infty} \left[\frac{1-q^r}{1-q}\right] \frac{(\alpha;q)_{2r+s}}{(\beta;q)_r(\gamma;q)_s(q;q)_{r-1}(q;q)_s} x^{r-1} y^s \\
&= \sum_{r=1,s=0}^{\infty} \left[\frac{1}{1-q}\right] \frac{(\alpha;q)_{2r+s}}{(\beta;q)_r(\gamma;q)_s(q;q)_{r-1}(q;q)_s} x^{r-1} y^s \\
&= \sum_{r,s=0}^{\infty} \left[\frac{1}{1-q}\right] \frac{(\alpha;q)_{2m+n+2}}{(\beta;q)_{m+1}(\gamma;q)_n(q;q)_r(q;q)_s} x^r y^s \\
&= \frac{(1-\alpha)(1-\alpha q)}{(1-\beta)(1-q)} \mathbf{H}_7(\alpha q^2;\beta q,\gamma;q,x,y)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
D_{y,q}\mathbf{H}_7 &= \sum_{r,s=0}^{\infty}\left[\frac{1-q^s}{1-q}\right]\frac{(\alpha;q)_{2r+s}}{(\beta;q)_r(\gamma;q)_s(q;q)_r(q;q)_s}x^r y^{s-1} \\
&= \sum_{r=0,s=1}^{\infty}\left[\frac{1}{1-q}\right]\frac{(\alpha;q)_{2r+s}}{(\beta;q)_r(\gamma;q)_s(q;q)_r(q;q)_{s-1}}x^r y^{s-1} \\
&= \sum_{r,s=0}^{\infty}\left[\frac{1}{1-q}\right]\frac{(\alpha;q)_{2r+s+1}}{(\beta;q)_m(\gamma;q)_{n+1}(q;q)_r(q;q)_s}x^r y^{s} \\
&= \frac{(1-\alpha)}{(1-\gamma)(1-q)}\mathbf{H}_7(\alpha q;\beta,\gamma q;q,x,y).
\end{aligned} \tag{24}$$

Iterating this technique r times on $\mathbf{H}_7$, we obtain (19) and (20). In the same way, the Equations (21) and (22) can be proven. □

Theorem 3. *The functions $\mathbf{H}_6$ and $\mathbf{H}_7$ satisfy the q-derivative formulas:*

$$\left[\alpha\theta_{x,q}+\frac{1-\alpha}{1-q}\right]\mathbf{H}_6+\alpha\theta_{x,q}\mathbf{H}_6(xq)+\alpha\theta_{y,q}\mathbf{H}_6(xq^2)=\frac{1-\alpha}{1-q}\mathbf{H}_6(\alpha q), \tag{25}$$

where $\theta_{x,q}=xD_{x,q}$ and $\theta_{y,q}=yD_{y,q}$ are differential operators,

$$\left[\alpha\theta_{y,q}+\frac{1-\alpha}{1-q}\right]\mathbf{H}_6+\alpha\theta_{x,q}\mathbf{H}_6(yq)+\alpha\theta_{x,q}\mathbf{H}_6(xq,yq)=\frac{1-\alpha}{1-q}\mathbf{H}_6(\alpha q), \tag{26}$$

$$\begin{aligned}
&\left[\alpha q^{-1}\theta_{x,q}+\frac{1-\alpha q^{-1}}{1-q}\right]\mathbf{H}_6(\alpha q^{-1})+\alpha q^{-1}\theta_{y,q}\mathbf{H}_6(\alpha q^{-1},xq,yq) \\
&+\alpha q^{-1}\theta_{y,q}\mathbf{H}_6(\alpha q^{-1},xq)=\frac{1-\alpha q^{-1}}{1-q}\mathbf{H}_6,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\left[\alpha q^{-1}\theta_{y,q}+\frac{1-\alpha q^{-1}}{1-q}\right]\mathbf{H}_6(\alpha q^{-1})+\alpha q^{-1}\theta_{x,q}\mathbf{H}_6(\alpha q^{-1},xq,yq) \\
&+\alpha q^{-1}\theta_{x,q}\mathbf{H}_6(\alpha q^{-1},yq)=\frac{1-\alpha q^{-1}}{1-q}\mathbf{H}_6,
\end{aligned} \tag{28}$$

$$\left[\alpha\theta_{x,q}+\frac{1-\alpha}{1-q}\right]\mathbf{H}_7+\alpha\theta_{x,q}\mathbf{H}_7(xq)+\alpha\theta_{y,q}\mathbf{H}_7(xq^2)=\frac{1-\alpha}{1-q}\mathbf{H}_7(\alpha q), \tag{29}$$

$$\left[\alpha\theta_{y,q}+\frac{1-\alpha}{1-q}\right]\mathbf{H}_7+\alpha\theta_{x,q}\mathbf{H}_7(yq)+\alpha\theta_{x,q}\mathbf{H}_7(xq,yq)=\frac{1-\alpha}{1-q}\mathbf{H}_7(\alpha q), \tag{30}$$

$$\begin{aligned}
&\left[\alpha q^{-1}\theta_{x,q}+\frac{1-\alpha q^{-1}}{1-q}\right]\mathbf{H}_7(\alpha q^{-1})+\alpha q^{-1}\theta_{y,q}\mathbf{H}_7(\alpha q^{-1},xq) \\
&+\alpha q^{-1}\theta_{y,q}\mathbf{H}_7(\alpha q^{-1},xq,yq)=\frac{1-\alpha q^{-1}}{1-q}\mathbf{H}_7
\end{aligned} \tag{31}$$

and

$$\begin{aligned}&\left[\alpha q^{-1}\theta_{y,q}+\frac{1-\alpha q^{-1}}{1-q}\right]\mathbf{H}_7(\alpha q^{-1})+\alpha q^{-1}\theta_{x,q}\mathbf{H}_7(\alpha q^{-1},yq)\\&+\alpha q^{-1}\theta_{x,q}\mathbf{H}_7(\alpha q^{-1},xq,yq)=\frac{1-\alpha q^{-1}}{1-q}\mathbf{H}_7.\end{aligned}\tag{32}$$

Proof. Using (23) and (24) for $\mathbf{H}_6$, we get the results (25)–(28). Equations (29)–(32) would run a parallel to Equations (25)–(28). □

Theorem 4. *The relations of $\mathbf{H}_7$ and $\mathbf{H}_6$ with denominator parameters β and γ hold true*

$$\mathbf{H}_7(\beta q^{-1})=\mathbf{H}_7+\frac{\beta x(1-\alpha)(1-\alpha q)}{(q-\beta)(1-\beta)}\mathbf{H}_7(\alpha q^2;\beta q,\gamma;q,x,y);\beta,\beta q^{-1}\neq 1,\tag{33}$$

$$\mathbf{H}_7(\gamma q^{-1})=\mathbf{H}_7+\frac{\gamma y(1-\alpha)}{(q-\gamma)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta,\gamma q;q,x,y);\gamma,\gamma q^{-1}\neq 1,\tag{34}$$

$$\begin{aligned}\mathbf{H}_6(\beta q^{-1})=&\,\mathbf{H}_6+\frac{\beta x(1-\alpha)(1-\alpha q)}{(q-\beta)(1-\beta)}\mathbf{H}_6(\alpha q^2;\beta q;q,x,y)\\&+\frac{\beta(1-\alpha)y}{(q-\beta)(1-\beta)}\mathbf{H}_6(\alpha q;\beta q;q,xq,y),\beta,\beta q^{-1}\neq 1\end{aligned}\tag{35}$$

and

$$\begin{aligned}\mathbf{H}_6(\beta q^{-1})=&\,\mathbf{H}_6+\frac{\beta y(1-\alpha)}{(q-\beta)(1-\beta)}\mathbf{H}_6(\alpha q;\beta q;q,x,y)\\&+\frac{\beta x(1-\alpha)(1-\alpha q)}{(q-\beta)(1-\beta)}\mathbf{H}_6(\alpha q^2;\beta q;q,x,yq),\beta,\beta q^{-1}\neq 1,\end{aligned}\tag{36}$$

Proof. Using the identities (6), (8) and replacing β by βq^{-1} in (12), we get

$$\begin{aligned}\mathbf{H}_7(\beta q^{-1})-\mathbf{H}_7&=\sum_{r,s=0}^{\infty}\frac{(\alpha;q)_{2r+s}}{(\gamma;q)_n(q;q)_r(q;q)_s}\frac{(\beta;q)_r-(\beta q^{-1};q)_r}{(\beta q^{-1};q)_r(\beta;q)_r}x^ry^s\\&=\sum_{r,s=0}^{\infty}\frac{\beta(1-q^r)}{q-\beta}\frac{(\alpha;q)_{2r+s}}{(\beta;q)_r(\gamma;q)_s(q;q)_r(q;q)_s}x^ry^s\\&=\frac{\beta x(1-\alpha)(1-\alpha q)}{(q-\beta)(1-\beta)}\mathbf{H}_7(\alpha q^2;\beta q;q,x,y),\beta,\beta q^{-1}\neq 1.\end{aligned}$$

The proof (34) is a very similar to those of Equation (33). Similarly, by using the relation $\frac{1-q^{r+s}}{q-\gamma}=\frac{1-q^r}{q-\gamma}+q^r\frac{1-q^s}{q-\gamma}$ and $\frac{1-q^{r+s}}{q-\gamma}=\frac{1-q^s}{q-\gamma}+q^s\frac{1-q^r}{q-\gamma}$, we get (35) and (36). □

Theorem 5. *The q-contiguous relations hold true for the denominator parameters β and γ of the functions $\mathbf{H}_6$ and $\mathbf{H}_7$*

$$\mathbf{H}_6(\beta q^{-1})=\tfrac{\beta}{\beta-q}\mathbf{H}_6(\alpha;\beta;q,xq,yq)-\tfrac{q}{\beta-q}\mathbf{H}_6(\alpha;\beta;q,x,y),\beta\neq p,\tag{37}$$

$$\mathbf{H}_7(\beta q^{-1})=\frac{\beta}{\beta-q}\mathbf{H}_7(\alpha;\beta,\gamma;q,xq,y)-\frac{q}{\beta-q}\mathbf{H}_7(\alpha;\beta,\gamma;q,x,y),\beta\neq p\tag{38}$$

and

$$\mathbf{H}_7(\gamma q^{-1}) = \frac{\gamma}{\gamma - q}\mathbf{H}_7(\alpha;\beta,\gamma;q,x,yq) - \frac{q}{\gamma - q}\mathbf{H}_7(\alpha;\beta,\gamma;q,x,y), \gamma q^{-1} \neq 1, \tag{39}$$

Proof. Using the definition of $\mathbf{H}_6$ in (11) with the relation $\frac{1}{(\beta q^{-1};q)_{r+s}} = \frac{1}{(\beta;q)_{r+s}}\left[\frac{\beta}{\beta-q}q^{r+s} - \frac{q}{\beta-q}\right]$, we have

$$\begin{aligned}\mathbf{H}_6(\beta q^{-1}) &= \sum_{r,s=0}^{\infty}\left[\frac{\beta}{\beta-q}q^{r+s} - \frac{q}{\beta-q}\right]\frac{(\alpha;q)_{2r+s}}{(\beta;q)_{r+s}(q;q)_r(q;q)_s}x^r y^s \\ &= \frac{\beta}{\beta-q}\mathbf{H}_6(\alpha;\beta;q,xq,yq) - \frac{q}{\beta-q}\mathbf{H}_6(\alpha;\beta;q,x,y); \beta \neq q.\end{aligned}$$

The proof of the Equations (38) and (39) are on the same lines as of Equation (37). □

Theorem 6. *The q-derivative formulas for $\mathbf{H}_6$ and $\mathbf{H}_7$ are satisfied:*

$$\left[\beta q^{-1}\theta_{x,q} + \frac{1-\beta q^{-1}}{1-q}\right]\mathbf{H}_6 + \beta q^{-1}\theta_{y,q}\mathbf{H}_6(xq) = \frac{1-\beta q^{-1}}{1-q}\mathbf{H}_6(\beta q^{-1}), \tag{40}$$

$$\left[\beta q^{-1}\theta_{y,q} + \frac{1-\beta q^{-1}}{1-q}\right]\mathbf{H}_6 + \beta q^{-1}\theta_{x,q}\mathbf{H}_6(yq) = \frac{1-\beta q^{-1}}{1-q}\mathbf{H}_6(\beta q^{-1}), \tag{41}$$

$$\left[\beta q^{-1}\theta_{x,q} + \frac{1-\beta q^{-1}}{1-q}\right]\mathbf{H}_7 = \frac{1-\beta q^{-1}}{1-q}\mathbf{H}_7(\beta q^{-1}) \tag{42}$$

and

$$\left[\gamma q^{-1}\theta_{y,q} + \frac{1-\gamma q^{-1}}{1-q}\right]\mathbf{H}_7 = \frac{1-\gamma q^{-1}}{1-q}\mathbf{H}_7(\gamma q^{-1}). \tag{43}$$

Proof. By using Equations (23)–(24), we obtain Equations (40)–(43). □

Theorem 7. *For $\beta, \beta q \neq 1$, the relations for $\mathbf{H}_6$ hold true*

$$\begin{aligned}\mathbf{H}_6(\alpha q;\beta q;q,x,y) &= \mathbf{H}_6 + \frac{(\alpha-\beta)x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,y) \\ &+ \frac{(\alpha-\beta)y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,xq,y) + \frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,xq,yq),\end{aligned} \tag{44}$$

$$\begin{aligned}\mathbf{H}_6(\alpha q;\beta q;q,x,y) &= \mathbf{H}_6 + \frac{\alpha y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,x,y) \\ &+ \frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,xq,yq) + \frac{\alpha x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,yq) \\ &- \frac{\beta x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,y) - \frac{\beta y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,xq,y),\end{aligned} \tag{45}$$

$$\mathbf{H}_6(\alpha q;\beta q;q,x,y)=\mathbf{H}_6+\frac{\alpha x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,y)$$
$$+\frac{\alpha y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,xq,y)+\frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,xq,yq) \quad (46)$$
$$-\frac{\beta y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,x,y)-\frac{\beta x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,yq)$$

and

$$\mathbf{H}_6(\alpha q;\beta q;q,x,y)=\mathbf{H}_6+\frac{(\alpha-\beta)y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,x,y)$$
$$+\frac{(\alpha-\beta)x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,yq)+\frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,xq,yq). \quad (47)$$

Proof. To prove (44). In (11), replacing α by αq and β by βq, and by using relation (7), we have

$$\mathbf{H}_6(\alpha q;\beta q;q,x,y)-\mathbf{H}_6=\sum_{r,s=0}^{\infty}\frac{1}{(q;q)_r(q;q)_s}\left[\frac{(\alpha q;q)_{2r+s}}{(\beta q;q)_{r+s}}-\frac{(\alpha;q)_{2r+s}}{(\beta;q)_{r+s}}\right]x^ry^s$$
$$=\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}}{(q;q)_r(q;q)_s}\left[\frac{\alpha(1-q^{2r+s})-\beta(1-q^{r+s})-\alpha\beta q^{r+s}(1-q^r)}{(1-\beta)(\beta q;q)_{r+s}}\right]x^ry^s$$
$$=\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}}{(q;q)_r(q;q)_s}\left[\frac{(\alpha-\beta)(1-q^r)+(\alpha-\beta)q^r(1-q^s)+\alpha(1-\beta)q^{r+s}(1-q^r)}{(1-\beta)(\beta q;q)_{r+s}}\right]x^ry^s$$
$$=\frac{(\alpha-\beta)}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s+1}}{(q;q)_r(q;q)_s(\beta q;q)_{r+s+1}}x^{r+1}y^s+\frac{(\alpha-\beta)}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{q^r(\alpha q;q)_{2r+s}}{(q;q)_r(q;q)_s(\beta q;q)_{r+s+1}}x^ry^{s+1}$$
$$+\alpha\sum_{r,s=0}^{\infty}\frac{q^{r+s+1}(\alpha q;q)_{2r+s+1}}{(q;q)_r(q;q)_s(\beta q;q)_{r+s+1}}x^{r+1}y^s$$
$$=\frac{(\alpha-\beta)x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,x,y)+\frac{(\alpha-\beta)y}{(1-\beta)(1-\beta q)}\mathbf{H}_6(\alpha q;\beta q^2;q,xq,y)$$
$$+\frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_6(\alpha q^2;\beta q^2;q,xq,yq).$$

Using the relations

$$\alpha(1-q^s)+\alpha q^s(1-q^r)+\alpha(1-\beta)q^{r+s}(1-q^r)-\beta(1-q^r)-\beta q^r(1-q^s),$$

$$\alpha(1-q^r)+\alpha q^r(1-q^s)+\alpha(1-\beta)q^{r+s}(1-q^r)-\beta(1-q^s)-\beta q^s(1-q^r)$$

and

$$(\alpha-\beta)(1-q^s)+(\alpha-\beta)q^s(1-q^r)+\alpha(1-\beta)q^{r+s}(1-q^r),$$

we prove in a similar way that of Equations (45)–(47) would run a parallel to Equation (44). □

Theorem 8. *The following results of* $\mathbf{H}_7$ *are valid:*

$$\mathbf{H}_7(\alpha q;\beta q,\gamma;q,x,y)=\mathbf{H}_7+\frac{(\alpha-\beta)x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,y)$$
$$+\frac{\alpha y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq,y)+\frac{\alpha xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,yq) \quad (48)$$
$$-\frac{\alpha\beta xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,y)-\frac{\alpha\beta}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq^2,y),\beta,\beta q,\gamma\neq 1,$$

$$\begin{aligned}\mathbf{H}_7(\alpha q;\beta q,\gamma;q,x,y) &= \mathbf{H}_7 + \frac{(\alpha-\beta)x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,y)\\ &+ \frac{\alpha y}{(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq,y) + \frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,yq), \beta,\beta q,\gamma\neq 1,\end{aligned} \tag{49}$$

$$\begin{aligned}\mathbf{H}_7(\alpha q;\beta q,\gamma;q,x,y) &= \mathbf{H}_7 + \frac{\alpha y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,x,y)\\ &+ \frac{\alpha x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,yq) + \frac{\alpha xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,yq)\\ &- \frac{\beta x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,y) - \frac{\alpha\beta xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,y)\\ &- \frac{\alpha\beta y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq^2,y), \beta,\beta q,\gamma\neq 1,\end{aligned} \tag{50}$$

$$\begin{aligned}\mathbf{H}_7(\alpha q;\beta q,\gamma;q,x,y) &= \mathbf{H}_7 + \frac{\alpha y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,x,y)\\ &+ \frac{\alpha x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,yq) + \frac{\alpha xq(1-\alpha q)}{(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,yq)\\ &- \frac{\beta x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,y) - \frac{\alpha\beta y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq,y), \beta,\beta q,\gamma\neq 1,\end{aligned} \tag{51}$$

$$\begin{aligned}\mathbf{H}_7(\alpha q;\beta q,\gamma;q,x,y) &= \mathbf{H}_7 + \frac{\alpha y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,x,y)\\ &+ \frac{\alpha x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,yq) + \frac{\alpha xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,yq)\\ &- \frac{\beta x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,y) - \frac{\alpha\beta xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,y)\\ &- \frac{\alpha\beta y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq^2,y), \beta,\beta q,\gamma\neq 1,\end{aligned} \tag{52}$$

$$\begin{aligned}\mathbf{H}_7(\alpha q;\beta,\gamma q;q,x,y) &= \mathbf{H}_7 + \frac{\alpha x(1-\alpha q)}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q^2;\beta q,\gamma q;q,x,y)\\ &+ \frac{\alpha y}{(1-\gamma)(1-\gamma q)}\mathbf{H}_7(\alpha q;\beta,\gamma q^2;q,xq,y) + \frac{\alpha xq(1-\alpha q)}{(1-\beta)}\mathbf{H}_7(\alpha q^2;\beta q,\gamma q;q,xq,yq)\\ &- \frac{\beta y}{(1-\gamma)(1-\gamma q)}\mathbf{H}_7(\alpha q;\beta,\gamma q^2;q,x,y) - \frac{\alpha\gamma x(1-\alpha q)}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q^2;\beta q,\gamma q;q,x,yq), \beta,\gamma,\gamma q\neq 1\end{aligned} \tag{53}$$

and

$$\begin{aligned}\mathbf{H}_7(\alpha q;\beta,\gamma q;q,x,y) &= \mathbf{H}_7 + \frac{(\alpha-\gamma)y}{(1-\gamma)(1-\gamma q)}\mathbf{H}_7(\alpha q;\beta,\gamma q^2;q,x,y)\\ &+ \frac{\alpha x(1-\alpha q)}{(1-\beta)}\mathbf{H}_7(\alpha q^2;\beta q,\gamma q;q,x,yq) + \frac{\alpha xq(1-\alpha q)}{(1-\beta)}\mathbf{H}_7(\alpha q^2;\beta q,\gamma q;q,xq,yq), \beta,\gamma,\gamma q\neq 1.\end{aligned} \tag{54}$$

Proof. To prove (48). In Equation (12), replacing α and β by αq and βq, respectively, we have

$$
\begin{aligned}
&\mathbf{H}_7(\alpha q;\beta q,\gamma;q,x,y)-\mathbf{H}_7\\
&=\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}(\beta q;q)_{r-1}}{(q;q)_r(q;q)_s(\gamma;q)_s}\left[\frac{(1-\alpha q^{2r+s})(1-\beta)-(1-\alpha)(1-\beta q^r)}{(1-\beta q^r)(\beta q;q)_{r-1}(\beta;q)_r}\right]x^ry^s\\
&=\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}}{(q;q)_r(q;q)_s(\gamma;q)_s}\left[\frac{\alpha(1-q^{2r+s})-\beta(1-q^r)-\alpha\beta q^r(1-q^{r+s})}{(1-\beta q^r)(\beta;q)_r}\right]x^ry^s\\
&=\frac{(\alpha-\beta)}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}(1-q^r)}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_s}x^ry^s+\frac{\alpha}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}q^r(1-q^s)}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_s}x^ry^s\\
&+\frac{\alpha}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}q^{r+s}(1-q^r)}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_s}x^ry^s-\frac{\alpha\beta}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}q^r(1-q^r)}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_s}x^ry^s\\
&-\frac{\alpha\beta}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s-1}q^{2s}(1-q^s)}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_s}x^ry^s\\
&=\frac{(\alpha-\beta)}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s+1}q^{r+s+1}}{(q;q)_r(q;q)_s(\beta q;q)_{r+1}(\gamma;q)_s}x^{r+1}y^s+\frac{\alpha}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s+1}}{(q;q)_r(q;q)_s(\beta q;q)_{r+1}(\gamma;q)_s}x^{r+1}y^s\\
&+\frac{\alpha}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s}q^r}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_{s+1}}x^ry^{s+1}-\frac{\alpha\beta}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s+1}q^{r+1}}{(q;q)_r(q;q)_s(\beta q;q)_{r+1}(\gamma;q)_s}x^{r+1}y^s\\
&-\frac{\alpha\beta}{(1-\beta)}\sum_{r,s=0}^{\infty}\frac{(\alpha q;q)_{2r+s}q^{2s}}{(q;q)_r(q;q)_s(\beta q;q)_r(\gamma;q)_{s+1}}x^ry^{s+1}\\
&=\frac{(\alpha-\beta)x(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,x,y)+\frac{\alpha y}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq,y)\\
&+\frac{\alpha xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,yq)-\frac{\alpha\beta xq(1-\alpha q)}{(1-\beta)(1-\beta q)}\mathbf{H}_7(\alpha q^2;\beta q^2,\gamma;q,xq,y)\\
&-\frac{\alpha\beta}{(1-\beta)(1-\gamma)}\mathbf{H}_7(\alpha q;\beta q,\gamma q;q,xq^2,y).
\end{aligned}
$$

Using the relations

$$(\alpha-\beta)(1-q^r)+\alpha(1-\beta)q^r(1-q^s)+\alpha(1-\beta)q^{r+s}(1-q^r),$$

$$\alpha(1-q^s)+\alpha q^s(1-q^r)+\alpha q^{r+s}(1-q^r)-\beta(1-q^r)-\alpha\beta q^r(1-q^r)-\alpha\beta q^{2s}(1-q^s),$$

$$\alpha(1-q^s)+\alpha q^s(1-q^r)+\alpha(1-\beta)q^{r+s}(1-q^r)-\beta(1-q^r)-\alpha\beta q^r(1-q^s),$$

$$\alpha(1-q^r)+\alpha q^r(1-q^s)+\alpha(1-\gamma)q^{r+s}(1-q^r)-\gamma(1-q^s)-\alpha\gamma q^s(1-q^r)$$

and

$$(\alpha-\gamma)(1-q^s)+\alpha(1-\gamma)q^s(1-q^r)+\alpha(1-\gamma)q^{r+s}(1-q^r)$$

and simplifying, we obtain (49)–(54). □

Theorem 9. *The q-contiguous relations for* $\mathbf{H}_6$ *and* $\mathbf{H}_7$ *hold true*

$$\mathbf{H}_6(\beta q)=\mathbf{H}_6+\frac{\beta}{1-\beta}\mathbf{H}_6(\alpha;\beta q;q,xq,yq)-\frac{\beta}{1-\beta}\mathbf{H}_6(\alpha;\beta q;q,x,y),\beta\neq 1 \tag{55}$$

$$\mathbf{H}_7(\beta q)=\mathbf{H}_7+\frac{\beta}{1-\beta}\mathbf{H}_7(\alpha;\beta q,\gamma;q,xq,y)-\frac{\beta}{1-\beta}\mathbf{H}_7(\alpha;\beta q,\gamma;q,x,y),\beta\neq 1 \tag{56}$$

and

$$\mathbf{H}_7(\gamma q) = \mathbf{H}_7 + \frac{\gamma}{1-\gamma}\mathbf{H}_7(\alpha;\beta,\gamma q;q,x,qy) - \frac{\gamma}{1-\gamma}\mathbf{H}_7(\alpha;\beta,\gamma q;q,x,y), \gamma \neq 1. \tag{57}$$

Proof. By using the definition of $\mathbf{H}_6$, we get

$$\mathbf{H}_6(\beta q) - \mathbf{H}_6 = \sum_{r,s=0}^{\infty} \frac{(\alpha;q)_{2r+s}(\beta q;q)_{r+s-1}}{(q;q)_r(q;q)_s}\left[\frac{(1-\beta)-(1-\beta q^{r+s})}{(1-\beta q^{r+s})(\beta q;q)_{r+s-1}(\beta;q)_{r+s}}\right]x^r y^s$$
$$= \frac{\beta}{1-\beta}\mathbf{H}_6(\alpha;\beta q;q,xq,yq) - \frac{\beta}{1-\beta}\mathbf{H}_6(\alpha;\beta q;q,x,y).$$

We prove in a similar way of (56) and (57). □

Theorem 10. *For $\alpha \neq 1$, the basic Horn hypergeometric functions $\mathbf{H}_6$ and $\mathbf{H}_7$ with respect to parameters satisfy the difference equations*

$$D_{\alpha,q}\mathbf{H}_6 = -\frac{1}{1-\alpha}\left[\theta_{x,q}\mathbf{H}_6 + q\theta_{x,q}\mathbf{H}_6(qx) + \theta_{y,q}\mathbf{H}_6(q^2x)\right], \tag{58}$$

$$D_{\alpha,q}\mathbf{H}_6 = -\frac{1}{1-\alpha}\left[\theta_{y,q}\mathbf{H}_6 + \theta_{x,q}\mathbf{H}_6(qy) + q\theta_{x,q}\mathbf{H}_6(qx,qy)\right], \tag{59}$$

$$D_{\alpha,q}\mathbf{H}_7 = -\frac{1}{1-\alpha}\left[\theta_{x,q}\mathbf{H}_7 + q\theta_{x,q}\mathbf{H}_7(qx) + \theta_{y,q}\mathbf{H}_7(q^2x)\right] \tag{60}$$

and

$$D_{\alpha,q}\mathbf{H}_7 = -\frac{1}{1-\alpha}\left[\theta_{y,q}\mathbf{H}_7 + \theta_{x,q}\mathbf{H}_7(qy) + q\theta_{x,q}\mathbf{H}_6(qx,qy)\right]. \tag{61}$$

Proof. Applying the q-difference operator $D_{\alpha,q}$ and using (7), (23) and (24), we have

$$\begin{aligned}
D_{\alpha,q}\mathbf{H}_6 &= \sum_{r,s=0}^{\infty} \frac{(\alpha;q)_{2r+s} - (\alpha q;q)_{2r+s}}{(1-q)\alpha(\beta;q)_{r+s}(q;q)_r(q;q)_s}x^r y^s \\
&= \sum_{r,s=0}^{\infty}\left[1 - \frac{1-\alpha q^{2r+s}}{1-\alpha}\right]\frac{(\alpha;q)_{2r+s}}{(1-q)\alpha(\beta;q)_{r+s}(q;q)_r(q;q)_s}x^r y^s \\
&= -\frac{1}{1-\alpha}\sum_{r,s=0}^{\infty}\left[\frac{1-q^r}{1-q} + q^r\frac{1-q^r}{1-q} + q^{2r}\frac{1-q^s}{1-q}\right]\frac{(\alpha;q)_{2r+s}}{(\beta;q)_{r+s}(q;q)_r(q;q)_s}x^r y^s \\
&= -\frac{1}{1-\alpha}\left[\theta_{x,q}\mathbf{H}_6 + q\theta_{x,q}\mathbf{H}_6(qx) + \theta_{y,q}\mathbf{H}_6(q^2x)\right].
\end{aligned}$$

By using the relation $1-q^{2r+s} = 1-q^s + q^s(1-q^r) + q^{n+m}(1-q^r)$ and after simplification of the resulting equation, we arrive at the Equation (59). Similarly, by the same technique we obtain (60) and (61). □

Theorem 11. *The functions $\mathbf{H}_6$ and $\mathbf{H}_7$ with respect to parameters satisfy the difference equations*

$$D_{\beta,q}\mathbf{H}_6 = \frac{1}{1-\beta}\left[\theta_{x,q}\mathbf{H}_6(\beta q) + \theta_{y,q}\mathbf{H}_6(\beta q,qx)\right], \beta \neq 1, \tag{62}$$

$$D_{\beta,q}\mathbf{H}_6 = \frac{1}{1-\beta}\left[\theta_{y,q}\mathbf{H}_6(\beta q) + \theta_{x,q}\mathbf{H}_6(\beta q,qy)\right], \beta \neq 1 \tag{63}$$

$$D_{\beta,q}\mathbf{H}_7 = \frac{1}{1-\beta}\theta_{x,q}\mathbf{H}_7(\beta q), \beta \neq 1 \tag{64}$$

and

$$D_{\gamma,q}\mathbf{H}_7 = \frac{1}{1-\gamma}\theta_{x,q}\mathbf{H}_7(\gamma q), \gamma \neq 1. \tag{65}$$

Proof. From the q-difference operator (10), we have

$$\begin{aligned}
D_{\beta,q}\mathbf{H}_6 &= \sum_{r,s=0}^{\infty}\left[1 - \frac{1-\beta}{1-\beta q^{r+s}}\right]\frac{(\alpha;q)_{2r+s}}{(1-q)\beta(\beta;q)_{r+s}(q;q)_r(q;q)_s}x^r y^s \\
&= \sum_{r,s=0}^{\infty}\left[\frac{1-q^{r+s}}{1-\beta q^{r+s}}\right]\frac{(\alpha;q)_{2r+s}}{(1-q)(\beta;q)_{r+s}(q;q)_r(q;q)_s}x^r y^s \\
&= \frac{1}{1-\beta}\left[\theta_{x,q}\mathbf{H}_6(\beta q) + \theta_{y,q}\mathbf{H}_6(\beta q, qx)\right].
\end{aligned}$$

By use of the relation $1 - q^{r+s} = 1 - q^s + q^s(1-q^r)$ and after simplifying and rearranging the terms, we obtain (63). The proofs (64) and (65) are similar technique for parameters β and γ to the proof (62). □

Theorem 12. *For the functions $\mathbb{H}_6$ and $\mathbb{H}_7$, we have the relations*

$$[\theta_{x,q}]_q\mathbb{H}_6 = \frac{(1-q^\alpha)(1-q^{\alpha+1})}{(1-q^\beta)(1-q)}x\mathbb{H}_6(q^{\alpha+2};q^{\beta+1};q,x,y), \tag{66}$$

$$[\theta_{y,q}]_q\mathbb{H}_6 = \frac{(1-q^\alpha)}{(1-q^\beta)(1-q)}y\mathbb{H}_6(q^{\alpha+1};q^{\beta+1};q,x,y), \tag{67}$$

$$[\theta_{x,q}]_q\mathbb{H}_7 = \frac{(1-q^\alpha)(1-q^{\alpha+1})}{(1-q^\beta)(1-q)}x\mathbb{H}_7(q^{\alpha+2};q^{\beta+1},q^\gamma;q,x,y) \tag{68}$$

and

$$[\theta_{y,q}]_q\mathbb{H}_7 = \frac{(1-q^\alpha)}{(1-q^\gamma)(1-q)}y\mathbb{H}_7(q^{\alpha+1};q^\beta,q^{\gamma+1};q,x,y). \tag{69}$$

Proof. Applying the operator $\theta_{x,q}$ to both sides of (13) with respect to x, we have

$$\begin{aligned}
[\theta_{x,q}]_q\mathbb{H}_6 &= \sum_{r,s=0}^{\infty}\left[\frac{1-q^r}{1-q}\right]\frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_{r+s}(q;q)_{r-1}(q;q)_s}x^r y^s \\
&= \sum_{r=1,s=0}^{\infty}\left[\frac{1}{1-q}\right]\frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_{r+s}(q;q)_{r-1}(q;q)_s}x^r y^s \\
&= \frac{(1-q^\alpha)(1-q^{\alpha+1})}{(1-q^\beta)(1-q)}x\mathbb{H}_6(q^{\alpha+2};q^{\beta+1};q,x,y).
\end{aligned}$$

By the same way, the proof of Equations (67)–(69) are similar lines to the proof of Equation (66). □

Theorem 13. *The functions $\mathbb{H}_6(q^\alpha;q^\beta;q,x,y)$ and $\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,y)$ satisfies the q-differential relations*

$$[2\theta_{x,q} + \theta_{y,q} + \alpha]_q\mathbb{H}_6 = [\alpha]_q\mathbb{H}_6(q^{\alpha+1}), \tag{70}$$

$$[\theta_{x,q}+\theta_{y,q}+\beta-1]_q\mathbb{H}_6=[\beta-1]_q\mathbb{H}_6(q^{\beta-1}), \tag{71}$$

$$[2\theta_{x,q}+\theta_{y,q}+\alpha]_q\mathbb{H}_7=[\alpha]_q\mathbb{H}_7(q^{\alpha+1}), \tag{72}$$

$$[\theta_{x,q}+\beta-1]_q\mathbb{H}_7=[\beta-1]_q\mathbb{H}_7(q^{\beta-1}) \tag{73}$$

and

$$[\theta_{y,q}+\gamma-1]_q\mathbb{H}_7=[\gamma-1]_q\mathbb{H}_7(q^{\gamma-1}). \tag{74}$$

Proof. For proving the theorem, we start from the definitions of (13) and (14), using the relation

$$(q^{\alpha+1};q)_{2r+s}=\frac{1-q^{\alpha+2r+s}}{1-q^{\alpha}}(q^{\alpha};q)_{2r+s}=\frac{[\alpha+2r+s]_q}{[\alpha]_q}(q^{\alpha};q)_{2r+s}$$

and applying to the q-derivatives operators (10) and (4) to get

$$\begin{aligned}[2\theta_{x,q}+\theta_{y,q}+\alpha]_q\mathbb{H}_6=&\sum_{r,s=0}^{\infty}\frac{[a+2r+s]_q(q^{\alpha};q)_{2r+s}}{(q^{\beta};q)_{r+s}(q;q)_r(q;q)_s}x^ry^s\\ &=[\alpha]_q\sum_{r,s=0}^{\infty}\frac{(q^{\alpha+1};q)_{2r+s}}{(q^{\beta};q)_{r+s}(q;q)_r(q;q)_s}x^ry^s=[\alpha]_q\mathbb{H}_6(q^{\alpha+1}).\end{aligned}$$

Using the relation

$$(q^{\beta};q)_{r+s}=\frac{1-q^{\beta+r+s-1}}{1-q^{\beta-1}}(q^{\beta-1};q)_{r+s}=\frac{[\beta+r+s-1]_q}{[\beta-1]_q}(q^{\beta-1};q)_{r+s},$$

we obtain

$$\begin{aligned}&[\theta_{x,q}+\theta_{y,q}+\beta-1]_q\mathbb{H}_6=\sum_{r,s=0}^{\infty}\frac{[\beta+r+s-1]_q(q^{\alpha};q)_{2r+s}}{(q^{\beta};q)_{r+s}(q;q)_r(q;q)_s}x^ry^s\\ &=[\beta-1]_q\sum_{r,s=0}^{\infty}\frac{[\beta+r+s-1]_q(q^{\alpha};q)_{2r+s}}{[\beta+r+s-1]_q(q^{\beta-1};q)_{r+s}(q;q)_r(q;q)_s}x^ry^s=[\beta-1]_q\mathbb{H}_6(q^{\beta-1}).\end{aligned}$$

This is the proof of Equation (70), and using the same procedure leads to the results (71)–(74). We omit the details. □

Theorem 14. *For the functions $\mathbb{H}_6$ and $\mathbb{H}_7$, we have*

$$\begin{aligned}\mathbb{H}_7(q^{\alpha+1};q^{\beta},q^{\gamma};q,x,y)=&\mathbb{H}_7+\frac{1-q}{1-q^{\alpha}}\Big[q^{\alpha}[\theta_{y,q}]_q\mathbb{H}_7\\ &+q^{\alpha}[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,x,qy)+q^{\alpha}[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,qy)\Big],\end{aligned} \tag{75}$$

$$\begin{aligned}\mathbb{H}_7(q^{\alpha+1};q^{\beta},q^{\gamma};q,x,y)=&\mathbb{H}_7+\frac{1-q}{1-q^{\alpha}}\Big[q^{\alpha}[\theta_{x,q}]_q\mathbb{H}_7\\ &+q^{\alpha}[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,y)+q^{\alpha}[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^2x,y)\Big],\end{aligned} \tag{76}$$

$$\mathbb{H}_6(q^{\alpha+1};q^\beta;q,x,y) = \mathbb{H}_6(q^\alpha;q^\beta;q,x,y) + \frac{1-q}{1-q^\alpha}\Big[q^\alpha[\theta_{y,q}]_q\mathbb{H}_6(q^\alpha;q^\beta;q,x,y) + q^\alpha[\theta_{x,q}]_q\mathbb{H}_6(q^\alpha;q^\beta;q,x,qy) + q^\alpha[\theta_{x,q}]_q\mathbb{H}_6(q^\alpha;q^\beta;q,qx,qy)\Big] \tag{77}$$

and

$$\mathbb{H}_6(q^{\alpha+1};q^\beta;q,x,y) = \mathbb{H}_6(q^\alpha;q^\beta;q,x,y) + \frac{1-q}{1-q^\alpha}\Big[q^\alpha[\theta_{x,q}]_q\mathbb{H}_6(q^\alpha;q^\beta;q,x,y) + q^\alpha[\theta_{x,q}]_q\mathbb{H}_6(q^\alpha;q^\beta;q,qx,y) + q^\alpha[\theta_{y,q}]_q\mathbb{H}_6(q^\alpha;q^\beta;q,q^2x,y)\Big]. \tag{78}$$

Proof. Using (14) and the relation

$$1-q^{\alpha+2r+s} = 1-q^\alpha + q^\alpha(1-q^s) + q^{\alpha+s}(1-q^r) + q^{\alpha+s+r}(1-q^r),$$

we have

$$\begin{aligned}
&\mathbb{H}_7(q^{\alpha+1};q^\beta,q^\gamma;q,x,y) = \sum_{r,s=0}^{\infty}\left[\frac{1-q^{\alpha+2r+s}}{1-q^\alpha}\right]\frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_r(q^\gamma;q)_s(q;q)_r(q;q)_s}x^ry^s \\
&= \sum_{r,s=0}^{\infty}\left[1+q^\alpha\frac{1-q^s}{1-q^\alpha} + q^{\alpha+s}\frac{1-q^r}{1-q^\alpha} + q^{\alpha+s+r}\frac{1-q^r}{1-q^\alpha}\right]\frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_r(q^\gamma;q)_s(q;q)_r(q;q)_s}x^ry^s \\
&= \mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,y) + \frac{1-q}{1-q^\alpha}\Big[q^\alpha[\theta_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,y) \\
&+ q^\alpha[\theta_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,qy) + q^\alpha[\theta_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,qx,qy)\Big].
\end{aligned}$$

Using the relation

$$1-q^{\alpha+2r+s} = 1-q^\alpha + q^\alpha(1-q^r) + q^{\alpha+r}(1-q^r) + q^{\alpha+2r}(1-q^s).$$

The proof of Equations (76)–(78) are similar lines to the proof of Equation (75). □

Theorem 15. *The following identity holds true for the functions $\mathbb{H}_6$ and $\mathbb{H}_7$*

$$\begin{aligned}
&[\alpha+1]_q\mathbb{H}_7(q^{\alpha+2}) = \mathbb{H}_7(q^{\alpha+1}) + qx[\alpha]_q[\alpha]_q\mathbb{H}_7 + 2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q\mathbb{H}_7 \\
&+ 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,qy) + 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,qx,qy) \\
&+ 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,qy) + 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,qx,qy) \\
&+ 2q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,qx,q^2y) + q^{2\alpha+1}x[\theta^2_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,y) \\
&+ q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,q^2y) + q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,q^2x,q^2y),
\end{aligned} \tag{79}$$

$$\begin{aligned}
&[\alpha+1]_q\mathbb{H}_7(q^{\alpha+2}) = \mathbb{H}_7(q^{\alpha+1}) + qx[\alpha]_q[\alpha]_q\mathbb{H}_7 + 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7 \\
&+ 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,qx,y) + 2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,q^2x,y) \\
&+ 2q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,qx,y) + 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,q^2x,y) \\
&+ 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,q^3x,y) + q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,x,y) \\
&+ q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,q^2x,y) + q^{2\alpha+1}x[\theta^2_{y,q}]_q\mathbb{H}_7(q^\alpha;q^\beta,q^\gamma;q,q^4x,y),
\end{aligned} \tag{80}$$

$$
\begin{aligned}
&[\alpha+1]_q \mathbb{H}_6(q^{\alpha+2}) = \mathbb{H}_6(q^{\alpha+1}) + qx[\alpha]_q[\alpha]_q \mathbb{H}_6 + 2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q \mathbb{H}_6 \\
&+ 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, x, qy) + 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, qx, qy) \\
&+ 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, x, qy) + 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, qx, qy) \\
&+ 2q^{2\alpha+1}x[\theta^2_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, qx, q^2y) + q^{2\alpha+1}x[\theta^2_{y,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, x, y) \\
&+ q^{2\alpha+1}x[\theta^2_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, x, q^2y) + q^{2\alpha+1}x[\theta^2_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, q^2x, q^2y)
\end{aligned} \tag{81}
$$

and

$$
\begin{aligned}
&[\alpha+1]_q \mathbb{H}_6(q^{\alpha+2}) = \mathbb{H}_6(q^{\alpha+1}) + qx[\alpha]_q[\alpha]_q \mathbb{H}_6 + 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q \mathbb{H}_6 \\
&+ 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, qx, y) + 2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, q^2x, y) \\
&+ 2q^{2\alpha+1}x[\theta^2_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, qx, y) + 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, q^2x, y) \\
&+ 2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q \mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, q^3x, y) + q^{2\alpha+1}x[\theta^2_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, x, y) \\
&+ q^{2\alpha+1}x[\theta^2_{x,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, q^2x, y) + q^{2\alpha+1}x[\theta^2_{y,q}]_q \mathbb{H}_6(q^\alpha; q^\beta; q, q^4x, y).
\end{aligned} \tag{82}
$$

Proof. From (14), we have

$$
\begin{aligned}
&[\alpha+1]_q \mathbb{H}_7(q^{\alpha+2}) - \mathbb{H}_7(q^{\alpha+1}) = \sum_{r,s=0}^{\infty} \frac{1-q^{\alpha+2r+s}}{1-q^\alpha}\left[\frac{1-q^{\alpha+2r+s+1}}{1-q} - 1\right] \frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_r(q^\gamma;q)_s(q;q)_r(q;q)_s} x^r y^s \\
&= \frac{q}{1-q^\alpha}\sum_{r,s=0}^{\infty}\left[\frac{(1-q^{\alpha+2r+s})(1-q^{\alpha+2r+s})}{1-q}\right]\frac{(q^\alpha;q)_{2r+s}}{(q^\beta;q)_r(q^\gamma;q)_s(q;q)_r(q;q)_s} x^r y^s
\end{aligned}
$$

and using

$$
\begin{aligned}
&\frac{(1-q^{\alpha+2r+s})(1-q^{\alpha+2r+s})}{1-q} = [\alpha]_q\Big[1-q^\alpha + 2q^\alpha(1-q^s) + 2q^{\alpha+s}(1-q^r) + 2q^{\alpha+s+r}(1-q^r)\Big] \\
&+ \frac{1}{1-q}\Big[2q^{2\alpha+s}(1-q^r)(1-q^s) + 2q^{2\alpha+s+r}(1-q^r)(1-q^s) + 2q^{2\alpha+2s+r}(1-q^r)^2 \\
&+ q^{2\alpha}(1-q^s)^2 + q^{2\alpha+2s}(1-q^r)^2 + q^{2\alpha+2s+2r}(1-q^r)^2\Big],
\end{aligned}
$$

after simplification we obtain (79). The proof of Equations (80)–(82) are similar to the proof of Equation (79). □

Theorem 16. *The functions $\mathbb{H}_6$ and $\mathbb{H}_7$ satisfies the partial q-differential equations*

$$
\begin{aligned}
&\left(q^{\gamma-1}[\theta_{y,q}]_q[\theta_{y,q}]_q - q^{\gamma-1}[\theta_{y,q}]_q + [\gamma]_q[\theta_{y,q}]_q - \frac{q^{\alpha+1}y}{1-q}[\theta_{y,q}]_q - yq^\alpha[2\theta_{x,q}+\theta_{y,q}]_q - \frac{y[\alpha]_q}{1-q}\right)\mathbb{H}_7 \\
&= \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q \mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, x, qy) + \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q \mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, qx, qy),
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
&\left(q^{\gamma-1}[\theta_{y,q}]_q[\theta_{y,q}]_q - q^{\gamma-1}[\theta_{y,q}]_q + [c]_q[\theta_{y,q}]_q - \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q - yq^\alpha[2\theta_{x,q}+\theta_{y,q}]_q - \frac{y[\alpha]_q}{1-q}\right)\mathbb{H}_7 \\
&= \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q \mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, qx, y) + \frac{q^{\alpha+1}y}{1-q}[\theta_{y,q}]_q \mathbb{H}_7(q^\alpha; q^\beta, q^\gamma; q, q^2x, y),
\end{aligned} \tag{84}
$$

$$\begin{aligned}&\Big(q^{\beta-1}[\theta_{x,q}]_q[\theta_{x,q}]_q - q^{\beta-1}[\theta_{x,q}]_q + [\beta]_q[\theta_{y,q}]_q - q^{2\alpha+1}x[\theta^2_{y,q}]_q - 2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q\\&- xq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q - qx[\alpha]_q[\alpha]_q - x[\alpha]_q\Big)\mathbb{H}_7 = 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,x,qy)\\&+2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,qy)+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,x,qy)\\&+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,qy)+2q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,q^2y)\\&+q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,x,q^2y)+q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^2x,q^2y),\end{aligned} \tag{85}$$

$$\begin{aligned}&\Big(q^{\beta-1}[\theta_{x,q}]_q[\theta_{x,q}]_q - q^{\beta-1}[\theta_{x,q}]_q + [\beta]_q[\theta_{y,q}]_q - q^{2\alpha+1}x[\theta^2_{x,q}]_q - 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q - xq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q\\&- x[\alpha]_q - qx[\alpha]_q[\alpha]_q\Big)\mathbb{H}_7 = 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,y)\\&+2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^2x,y)+2q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,y)\\&+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^2x,y)+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^3x,y)\\&+q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^2x,y)+q^{2\alpha+1}x[\theta^2_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^4x,y),\end{aligned} \tag{86}$$

$$\begin{aligned}&\Big(q^{\beta-1}[\theta_{y,q}]_q[\theta_{x,q}+\theta_{y,q}]_q - q^{\beta-1}[\theta_{y,q}]_q + [\beta]_q[\theta_{y,q}]_q - \frac{q^{\alpha+1}y}{1-q}[\theta_{y,q}]_q - yq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q - \frac{y[\alpha]_q}{1-q}\Big)\mathbb{H}_6\\&=\frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,x,qy)+\frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,qy),\end{aligned} \tag{87}$$

$$\begin{aligned}&\Big(q^{\beta-1}[\theta_{y,q}]_q[\theta_{x,q}+\theta_{y,q}]_q - q^{\beta-1}[\theta_{y,q}]_q - \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q + [\beta]_q[\theta_{y,q}]_q - yq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q - \frac{y[\alpha]_q}{1-q}\Big)\mathbb{H}_6\\&=\frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,y)+\frac{q^{\alpha+1}y}{1-q}[\theta_{y,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,q^2x,y)\Big],\end{aligned} \tag{88}$$

$$\begin{aligned}&\Big(q^{\beta-1}[\theta_{x,q}]_q[\theta_{x,q}+\theta_{y,q}]_q - 2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q - q^{\beta-1}[\theta_{x,q}] + [\beta]_q[\theta_{y,q}]_q - q^{2\alpha+1}x[\theta^2_{y,q}]_q - xq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q\\&- x[\alpha]_q - qx[\alpha]_q[\alpha]_q\Big)\mathbb{H}_6 = 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,x,qy)\\&+2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,qy)+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,x,qy)\\&+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,qy)+2q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,q^2y)\\&+q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,x,q^2y)+q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,q^2x,q^2y)\end{aligned} \tag{89}$$

and

$$\begin{aligned}&\Big(q^{\beta-1}[\theta_{x,q}]_q[\theta_{x,q}+\theta_{y,q}]_q - q^{\beta-1}[\theta_{x,q}]_q + [\beta]_q[\theta_{y,q}]_q - 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q - xq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q - q^{2\alpha+1}x[\theta^2_{x,q}]\\&- x[\alpha]_q - qx[\alpha]_q[\alpha]_q\Big)\mathbb{H}_6 = 2q^{\alpha+1}x[\alpha]_q[\theta_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,y)\\&+2q^{\alpha+1}x[\alpha]_q[\theta_{y,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,q^2x,y)+2q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,qx,y)\\&+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,q^2x,y)+2q^{2\alpha+1}x[\theta_{x,q}]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,q^3x,y)\\&+q^{2\alpha+1}x[\theta^2_{x,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,q^2x,y)+q^{2\alpha+1}x[\theta^2_{y,q}]_q\mathbb{H}_6(q^{\alpha};q^{\beta};q,q^4x,y).\end{aligned} \tag{90}$$

Proof. From (68) and (70), we obtain

$$\begin{aligned}&\Big([\theta_{y,q}]_q[\theta_{y,q}+\gamma-1]_q - y[2\theta_{x,q}+\theta_{y,q}+\alpha]_q\Big)\mathbb{H}_7 = [\gamma-1]_q[\theta_{y,q}]_q\mathbb{H}_7(q^{\gamma-1}) - y[\alpha]_q\mathbb{H}_7(q^{\alpha+1})\\ &= [\gamma-1]_q\frac{(1-q^{\alpha})y}{(1-q^{\gamma-1})(1-q)}\mathbb{H}_7(q^{\alpha+1};q^{\beta},q^{\gamma};q,x,y) - y[\alpha]_q\mathbb{H}_7(q^{\alpha+1})\\ &= \left([\gamma-1]_q\frac{(1-q^{\alpha})y}{(1-q^{\gamma-1})(1-q)} - y[\alpha]_q\right)\mathbb{H}_7(q^{\alpha+1};q^{\beta},q^{\gamma};q,x,y)\\ &= \left(\frac{y}{1-q}[\alpha]_q - y[\alpha]_q\right)\mathbb{H}_7(q^{\alpha+1};q^{\beta},q^{\gamma};q,x,y) = \left(\frac{qy}{1-q}\right)[\alpha]_q\mathbb{H}_7(q^{\alpha+1};q^{\beta},q^{\gamma};q,x,y).\end{aligned} \tag{91}$$

By using the relation

$$\begin{aligned}[n-k]_q &= q^{-k}\Big([n]_q - [k]_q\Big),\\ [n+k]_q &= [n]_q + q^s[k]_q\end{aligned}$$

and we write the q-differential operator

$$\begin{aligned}&\Big([\theta_{y,q}]_q[\theta_{y,q}+\gamma-1]_q - y[2\theta_{x,q}+\theta_{y,q}+\alpha]_q\Big)\\ &= \Big([\theta_{y,q}]_q\Big(q^{\gamma}[\theta_{y,q}-1]_q+[\gamma]_q\Big) - y\Big(q^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q+[\alpha]_q\Big)\Big)\\ &= \Big([\theta_{y,q}]_q\Big(q^{\gamma-1}[\theta_{y,q}]_q - q^{\gamma-1}[1]_q+[\gamma]_q\Big) - y\Big(q^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q+[\alpha]_q\Big)\Big)\\ &= \Big(q^{\gamma-1}[\theta_{y,q}]_q[\theta_{y,q}]_q - q^{\gamma-1}[\theta_{y,q}]_q + [\gamma]_q[\theta_{y,q}]_q - yq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q - y[\alpha]_q\Big).\end{aligned}$$

From the above relation and using (91), (71), we get

$$\begin{aligned}&\left(q^{\gamma-1}[\theta_{y,q}]_q[\theta_{y,q}]_q - q^{\gamma-1}[\theta_{y,q}]_q + [\gamma]_q[\theta_{y,q}]_q - \frac{q^{\alpha+1}y}{1-q}[\theta_{y,q}]_q - yq^{\alpha}[2\theta_{x,q}+\theta_{y,q}]_q - \frac{y[\alpha]_q}{1-q}\right)\mathbb{H}_7\\ &= \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,x,qy) + \frac{q^{\alpha+1}y}{1-q}[\theta_{x,q}]_q\mathbb{H}_7(q^{\alpha};q^{\beta},q^{\gamma};q,qx,qy).\end{aligned} \tag{92}$$

The proof Equations (84)–(90) are on the same lines as of Equation (83). We omit the proof of the given theorem. □

3. Concluding Remarks

This study is a continuation of the recent paper [31], we have investigated the q-analogues of hypergeometric Horn functions $\mathbf{H}_3$ and $\mathbf{H}_4$ and their various properties. In our present study, we have established several results, such as q-contiguous relations, q-differential relations and q-differential equations of the basic Horn functions $\mathbf{H}_6$ and $\mathbf{H}_7$ under conditions on the numerator and denominator parameters. In addition, we have deeply discussed new properties of these extended basic Horn functions $\mathbb{H}_6$ and $\mathbb{H}_7$ such as the iq-contiguous relations, q-differential relations, and q-differential equations. Note that, by setting $q \longrightarrow 1^-$, we obtain various known or unknown results for the Horn hypergeometric functions $\mathbf{H}_6$ and $\mathbf{H}_7$ established earlier in [28]. Therefore, other special types of these extensions are recommended for a parallel work of this study. More investigations will be carried out in the coming future results in other different fields of interest for quantum calculus on time scales and applications in the mathematical and physical sciences.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data that support the findings of this paper are available, as they are requested.

Acknowledgments: The researcher would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project. The author is very grateful to the referees, for their careful reading of the manuscript and insightful comments, which help to improve the quality of the paper. The Author expresses his sincere appreciation to Mohammed Eltayeb Elffaki Elasmaa (Department of English language, College of Science and Arts, Unaizah 56264, Qassim University, Qassim, Saudi Arabia) for his kind interest, encouragement, help, and correcting language errors in this paper.

Conflicts of Interest: The author of this paper declare that they have no conflict of interest.

References

1. Agarwal, R.P. Some basic hypergeometric identities. *Ann. Soc. Sci. Brux. Ser. I* **1953**, *67*, 186–202.
2. Andrews, G.E. Summations and transformations for basic Appell series. *J. Lond. Math. Soc.* **1972**, *4*, 618–622. [CrossRef]
3. Harsh, H.V.; Kim, Y.S.; Rakha, M.A.; Rathie, A.K. A study of q-contiguous function relations. *Commun. Korean Math. Soc.* **2016**, *31*, 65–94. [CrossRef]
4. Ismail, M.E.H.; Libis, C.A. Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. *J. Math. Anal. Appl.* **1989**, *141*, 349–372. [CrossRef]
5. Jain, V.K. Some expansions involving basic hypergeometric functions of two variables. *Pac. J. Math.* **1980**, *91*, 349–361. [CrossRef]
6. Jain, V.K. Certain transformations of basic hypergeometric series and their applications. *Pac. J. Math.* **1982**, *101*, 333–349. [CrossRef]
7. Jain, V.K.; Vertna, A. Transformations between basic hypergeometric series on different bases and identities of Rogers-Ramanujan type. *J. Math. Anal. Appl.* **1980**, *76*, 230–269.
8. Jain, V.K.; Vorma, A. Some transformations of basic hypergeometric functions, Part I. *SIAM J. Math. Anal.* **1981**, *12*, 943–956. [CrossRef]
9. Kim, Y.S.; Rathie, A.K.; Choi, J. Three term contiguous functional relations for basic hypergeometric series 2Φ1. *Commun. Korean Math. Soc.* **2005**, *20*, 395–403. [CrossRef]
10. Kim, Y.S.; Rathie, A.K.; Lee, C.H. On q-analog of Kummer's theorem and its contiguous results. *Commun. Korean Math. Soc.* **2003**, *18*, 151–157. [CrossRef]
11. Mishra, B.P. On certain transformation formulae involving basic hypergeometric functions. *J. Ramanujan Soc. Math. Math. Sci.* **2014**, *2*, 9–16.
12. Srivastava, H.M.; Jain, V.K. q-Series identities and reducibility of basic double hypergeometric functions. *Can. J. Math. (CJM)* **1986**, *38*, 215–231. [CrossRef]
13. Srivastava, H.M.; Shehata, A. A family of new q-Extensions of the Humbert functions. *Eur. J. Math. Sci.* **2018**, *4*, 13–26.
14. Srivastava, H.M.; Khan, S.; Araci, S.; Acikgoz, M.; Mumtaz Riyasat, M. A general class of the three-variable unified Apostol-Type q-polynomials and multiple Power q-Sums. *Bull. Iran. Math. Soc.* **2020**, *46*, 519–542. [CrossRef]
15. Swarttouw, R.F. The contiguous function relations for basic hypergeometric function. *J. Math. Anal. Appl.* **1990**, *149*, 151–159. [CrossRef]
16. Sahai, V.; Verma, A. Nth-order q-derivatives of Srivastava's General Triple q-hypergeometric Series with Respect to Parameters. *Kyungpook Math. J.* **2016**, *56*, 911–925. [CrossRef]
17. Sahai, V.; Verma, A. Recursion formulas for q-hypergeometric and q-Appell series. *Commun. Korean Math. Soc.* **2018**, *33*, 207–236.
18. Guo, V.J.W.; Schlosser, M.J. Some q-Supercongruences from Transformation Formulas for Basic Hypergeometric Series. *Constr. Approx.* **2021**, *53*, 155–200. [CrossRef]
19. Verma, A.; Sahai, V. Some recursion formulas for q-Lauricella series. *Afr. Mat.* **2020**, *31*, 643–686. [CrossRef]
20. Verma, A.; Yadav, S. Recursion formulas for Srivastava's general triple q-hypergeometric series. *Afr. Mat.* **2020**, *31*, 869–885. [CrossRef]
21. Wei, C.; Gong, D. q-Extensions of Gauss fifteen contiguous relation for ${}_2F_1$ series. *Commun. Comput. Inf. Sci.* **2011**, *105*, 85–92.
22. Ernst, T. On the q-Analogues of Srivastava's Triple Hypergeometric Functions. *Axioms* **2013**, *2*, 85–99 [CrossRef]
23. Araci, S.; Duran, U.; Acikgoz, M. Theorems on q-Frobenius-Euler polynomials under Sym (4). *Util. Math.* **2016**, *101*, 129–137.
24. Araci, S.; Acikgoz, M.; Sen, E. On the extended Kim's q-adic q-deformed fermionic integrals in the q-adic integer ring. *J. Number Theory* **2013**, *133*, 3348–3361. [CrossRef]
25. Araci, S.; Ağyuz, E.; Acikgoz, M. On a q-analog of some numbers and polynomials. *J. Inequalities Appl.* **2015**, *19*, 1–9. [CrossRef]

26. Duran, U.; Acikgoz, M.; Araci, S. Symmetric identities involving weighted q-Genocchi polynomials under S_4. *Proc. Jangjeon Math. Soc.* **2015**, *18*, 455–465.
27. Duran, U.; Acikgoz, M.; Araci, S. Research on some new results arising from multiple q-calculus. *Filomat* **2018**, *32*, 1–9. [CrossRef]
28. Pathan, M.A.; Shehata, A.; Moustafa, S.I. Certain new formulas for the Horn's hypergeometric functions H_1, H_{11}. *Acta Univ. Apulensis* **2020**, *64*, 137–170.
29. Shehata, A. On the (p, q)-Bessel functions from the view point of the generating functions method. *J. Interdiscip. Math.* **2020**, *23*, 1435–1448. [CrossRef]
30. Shehata, A. On the (p, q)-Humbert functions from the view point of the generating functions method. *J. Funct. Spaces* **2020**, *2020*, 4794571. [CrossRef]
31. Shehata, A. On basic Horn hypergeometric functions H_3 and H_4. *Adv. Differ. Equ.* **2020**, *2020*, 595. [CrossRef]
32. Gasper, G.; Rahman, M. *Basic Hypergeometric Series. Encyclopedia of Mathematics and Its Applications*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2004; Volume 96.
33. Jackson, F.H. Basic double hypergeometric functions (II). *Q. J. Math.* **1944**, *15*, 49–61. [CrossRef]
34. Erdélyi, A.; Mangus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA; Toronto, ON, Canada; London, UK, 1953; Volume I.