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Coefficient Inequalities for Multivalent Janowski Type q-Starlike Functions Involving Certain Conic Domains

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Abstract: In the current work, by using the familiar q-calculus, first, we study certain generalized conic-type regions. We then introduce and study a subclass of the multivalent q-starlike functions that map the open unit disk into the generalized conic domain. Next, we study potentially effective outcomes such as sufficient restrictions and the Fekete–Szegö type inequalities. We attain lower bounds for the ratio of a good few functions related to this lately established class and sequences of the partial sums. Furthermore, we acquire a number of attributes of the corresponding class of q-starlike functions having negative Taylor–Maclaurin coefficients, including distortion theorems. Moreover, various important corollaries are carried out. The new explorations appear to be in line with a good few prior commissions and the current area of our recent investigation.

Keywords: analytic functions; multivalent (p-valent) functions; q-derivative (q-difference) operator; differential subordination

MSC: Primary 30C45; 30C50; 30C80; Secondary 11B65; 47B38

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1. Introduction and Preliminaries

Let $\mathcal{A}(p)$ denote the multivalent (p-valent) class of functions j in the open unit disk $\mathbb{U}$ on the complex plane $\mathbb{C}$,

$$\mathbb{U} = \{\zeta : \zeta \in \mathbb{C} \quad \text{and} \quad |\zeta| < 1\},$$

with the series expansion

$$j(\zeta) = \zeta^p + \sum_{n=1}^{\infty} a_{n+p}\zeta^{n+p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1}$$

It is clear that

$$\mathcal{A}(1) = \mathcal{A},$$

where $\mathcal{A}$ is the class of normalized analytic functions j with

$$j(0) = 0 \quad \text{and} \quad j'(0) = 1. \tag{2}$$

The multivalent (p-valent) subclass of functions is signified by $\mathcal{S}^*(p)$, which comprises functions $j \in \mathcal{A}(p)$ satisfying

$$\Re\left(\frac{\zeta j'(\zeta)}{j(\zeta)}\right) > 0 \quad (\zeta \in \mathbb{U}).$$

It can be observed that

$$\mathcal{S}^*(1) = \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the prominent class of starlike functions.

For two analytic functions j and g in $\mathbb{U}$, it is affirmed in [1] that j is subordinate to g, expressed as

$$j \prec g, \text{ or } j(\zeta) \prec g(\zeta),$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(\zeta)| < 1$$

such that

$$j(\zeta) = g(w(\zeta)).$$

Moreover, if j is subordinate to the analytic univalent function g in $\mathbb{U}$, then

$$j(0) = g(0) \text{ and } j(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{P}$ present the familiar Carathéodory class of analytic functions Φ in $\mathbb{U}$, normalized by (see, for example, [2])

$$\Phi(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n, \tag{3}$$

such that

$$\Re\{\Phi(\zeta)\} > 0 \quad (\zeta \in \mathbb{U}).$$

Definition 1. *A function h with $h(0) = 1$ is said to be in class $\mathcal{P}[N, L]$ if*

$$h(\zeta) \prec \frac{1 + N\zeta}{1 + L\zeta} \quad (-1 \leqq L < N \leqq 1).$$

In particular, the class $\mathcal{P}[N, L]$ of analytic functions was launched by Janowski [3], who established that $h(\zeta) \in \mathcal{P}[N, L]$ if and only if there exists a function $\Phi \in \mathcal{P}$ so that

$$h(\zeta) = \frac{(N+1)\Phi(\zeta) - (N-1)}{(L+1)\Phi(\zeta) - (L-1)} \quad (-1 \leqq L < N \leqq 1).$$

Definition 2. *A function $j \in \mathcal{A}$ is said to be in class $\mathcal{S}^*[N, L]$ if*

$$\frac{\zeta j'(\zeta)}{j(\zeta)} = \frac{(N+1)\Phi(\zeta) - (N-1)}{(L+1)\Phi(\zeta) - (L-1)} \quad (-1 \leqq L < N \leqq 1). \tag{4}$$

Kanas et al. [4–6] were the first who illustrated the conic domain Ω_k ($k \geqq 0$) as follows

$$\Omega_k = \left\{\zeta = u + iv \in \mathbb{C} : u > k\sqrt{(u-1)^2 + v^2}\right\} \tag{5}$$

and, subjected to this domain, they also initiated and examined the corresponding class k-$\mathcal{ST}$ of k-starlike functions (see Definition 3 below). In particular,

(a) If $k = 0$, Ω_0 acts on the conic region bounded sequentially by the imaginary axis;
(b) If $k = 1$, Ω_1 is a parabola;
(c) If $0 < k < 1$, Ω_k is the right-hand branch of hyperbola;

(d) If $k > 1$, Ω_k represents an ellipse.

For these conic regions, the following functions act as extremal functions:

$$\varphi_k(\zeta) = \begin{cases} \frac{1+\zeta}{1-\zeta} = 1 + 2\zeta + 2\zeta^2 + \cdots & (k = 0) \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2 & (k = 1) \\ 1 + \frac{2}{1-k^2} \sinh^2\{\left(\frac{2}{\pi} \arccos k\right) \operatorname{arctanh}\left(\sqrt{\zeta}\right)\} & (0 < k < 1) \\ 1 + \dfrac{1}{k^2-1} \sin\left(\dfrac{\pi}{2K(\kappa)} \displaystyle\int_0^{\frac{u(\zeta)}{\sqrt{\kappa}}} \dfrac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \dfrac{1}{k^2-1} & (k > 1) \end{cases} \tag{6}$$

where

$$u(\zeta) = \frac{\zeta - \sqrt{\kappa}}{1 - \sqrt{\kappa}\zeta} \quad (\zeta \in \mathbb{U})$$

and $\kappa \in (0,1)$ is selected so that $k = \cosh\left(\pi K'(\kappa)/(4K(\kappa))\right)$. Here, $K(\kappa)$ is Legendre's complete elliptic integral of the first type and $K'(\kappa) = K(\sqrt{1-\kappa^2})$, i.e., $K'(\kappa)$ is the complementary integral of $K(\kappa)$.

It was proven in [7] that if we assume that

$$\varphi_k(\zeta) = 1 + P_1\zeta + P_2\zeta^2 + \ldots \quad (\zeta \in \mathbb{U}),$$

then we have

$$P_1 = \begin{cases} \frac{2N^2}{1-k^2} & (0 \leqq k < 1) \\ \frac{8}{\pi^2}, & (k = 1) \\ \frac{\pi^2}{4k^2(\kappa)^2(1+\kappa)\sqrt{\kappa}} & (k > 1) \end{cases} \tag{7}$$

and

$$P_2 = D(k)P_1, \tag{8}$$

where

$$D(k) = \begin{cases} \frac{N^2+2}{3} & (0 \leqq k < 1) \\ \frac{8}{\pi^2} & (k = 1) \\ \frac{[4K(\kappa)]^2\left(\kappa^2+6\kappa+1\right)-\pi^2}{24[K(\kappa)]^2(1+\kappa)\sqrt{\kappa}} & (k > 1) \end{cases} \tag{9}$$

with

$$N = \frac{2}{\pi} \arccos k.$$

These conic regions were formulated and generalized by many authors; for instance see [8–10].

The class k-$\mathcal{ST}$ is defined as below.

Definition 3. *A function $j \in \mathcal{A}$ is said to be considered to be in the class $k - \mathcal{ST}$, if*

$$\frac{\zeta j'(\zeta)}{j(\zeta)} \prec \varphi_k(\zeta) \quad (\zeta \in \mathbb{U};\ k \geqq 0),$$

where $\varphi_k(\zeta)$ is given by (6).

Noor et al. [11] amalgamated the idea of the Janowski functions and the conic regions, and introduced the following definition.

Definition 4. *A function $h \in \mathcal{P}$ is said to be in class k-$\mathcal{P}[N,L]$ if*

$$h(\zeta) \prec \frac{(N+1)\varphi_k(\zeta) - (N-1)}{(L+1))\varphi_k(\zeta) - (L-1)} \qquad (-1 \leqq L < N \leqq 1;\ k \geqq 0) \tag{10}$$

where $\varphi_k(\zeta)$ is defined by (6).

Geometrically, each function $h \in k$-$\mathcal{P}[N,L]$ takes all points in the domain $\Omega_k[N,L]$ $(-1 \leqq L < N \leqq 1;\ k \geqq 0)$, which is defined as follows

$$\Omega_k[N,L] = \left\{ w \in \mathbb{C} : \Re\left(\frac{(L-1)w - (N-1)}{(L+1))w - (N+1)} \right) > k \left| \frac{(L-1)w - (N-1)}{(L+1))w - (N+1)} - 1 \right| \right\}.$$

Thus, $\Omega_k[N,L]$ is a set of complex numbers $w = u + iv$ such that

$$\begin{aligned} &\left[\left(L^2 - 1\right)\left(u^2 + v^2\right) - 2(NL-1)u + \left(N^2 - 1\right) \right]^2 \\ &> k\left[\left(-2(L+1))\left(u^2+v^2\right) + 2(N+L+2)u - 2(N+1)\right)^2 + 4(N-L)^2 v^2 \right]. \end{aligned}$$

Domain $\Omega_k[N,L]$ depicts the conic-type regions in detail (see [11]).

Definition 5. *A function $j \in \mathcal{A}$ is said to be in class k-$\mathcal{ST}[N,L]$ if*

$$\frac{\zeta j'(\zeta)}{j(\zeta)} \in k\text{-}\mathcal{P}[N,L] \quad (\zeta \in \mathbb{U};\quad k \geqq 0).$$

Now, we recollect some fundamental definitions and concepts of the q-calculus, which will be used in this paper. Unless we mention otherwise, we assume that $0 < q < 1$ and $k \in \mathbb{N} \cup \{0\}$.

Definition 6. *Let q be within $(0,1)$. The q-number $[\lambda]_q$ is defined by*

$$[\lambda]_q = \frac{1 - q^\lambda}{1 - q} \quad (\lambda \in \mathbb{C}).$$

The q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

In particular, if $\lambda = n \in \mathbb{N}$, then

$$[n]_q = \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1}.$$

Definition 7 ([12,13])**.** *The q-derivative (or q-difference) operator D_q of a function j in a subset of $\mathbb{C}$ is defined by*

$$(D_q j)(\zeta) = \begin{cases} \frac{j(\zeta) - j(q\zeta)}{(1-q)\zeta} & (\zeta \neq 0) \\ j'(0) & (\zeta = 0), \end{cases} \tag{11}$$

if $j'(0)$ exists.

For a differentiable function j in a subset of $\mathbb{C}$, we can see from Definition 7 that

$$\lim_{q\to 1-}(D_q j)(\zeta) = \lim_{q\to 1-}\frac{j(\zeta)-j(q\zeta)}{(1-q)\zeta} = j'(\zeta).$$

Also, the Equations (1) and (11) give

$$(D_q j)(\zeta) = 1 + \sum_{n=2}^{\infty}[n]_q a_n \zeta^{n-1}. \tag{12}$$

Furthermore, one can see from (1) and (11) that

$$\left(D_q^{(1)} j\right)(\zeta) = [p]_q \zeta^{p-1} + \sum_{n=1}^{\infty}[n+p]_q a_{n+p}\zeta^{n+p-1}, \tag{13}$$

$$\left(D_q^{(2)} j\right)(\zeta) = [p]_q[p-1]_q \zeta^{p-2} + \sum_{n=1}^{\infty}[n+p]_q[n+p-1]_q a_{n+p}\zeta^{n+p-2}. \tag{14}$$

Differentiating p times, we obtain

$$\left(D_q^{(p)} j\right)(\zeta) = [p]_q! + \sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}a_{n+p}\zeta^{n}, \tag{15}$$

where $\left(D_q^{(p)} j\right)(\zeta)$ is the p-th time q-deravative of $j(\zeta)$.

In geometric function theory, many subclasses of class $\mathcal{A}$ of normalized analytic functions have already been discussed within a contrasting frame of reference. The q-calculus provides a key instrument to explore the subclasses of the normalized analytic functions in $\mathcal{A}$. Historically, the class of q-starlike functions was first studied by Ismail et al. [14]. However, Srivastava ([15] p. 347 *et seq.*—also see [1,16,17])—was the first who used the q-calculus to develop some relations between hypergeometric functions and geometric function theory. Some interesting recent developments in this area can be found in [18]. Later, inspired by the prior work, extensive explorations played a key role in the development. For example, the q-analogue of Ruscheweyh's derivative operator was introduced in [8]. For some recent investigations on this subject, the readers may see [19–24].

The following notation $\mathcal{S}_q^*$ was first used by Sahoo and Sherma [25].

Definition 8 ([14,15])**.** *A function $j \in \mathcal{A}$ is said to be class $\mathcal{S}_q^*$ if*

$$\left|\frac{\zeta}{j(\zeta)}(D_q j)(\zeta) - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}. \tag{16}$$

Equivalently ([26]),

$$\frac{\zeta}{j(\zeta)}(D_q j)(\zeta) \prec \widehat{\varphi}(\zeta) \quad \left(\widehat{\varphi}(\zeta) = \frac{1+\zeta}{1-q\zeta}\right).$$

Now, making use of the principle of subordination between analytic functions and the above-mentioned q-calculus, we define the class k-$\mathcal{P}_q$.

Definition 9. *A function φ is said to be in class k-$\mathcal{P}_q$ if*

$$\varphi(\zeta) \prec \widehat{\varphi}_k(\zeta) \tag{17}$$

where $\varphi_k(\zeta)$ is defined by (6) and $\widehat{\varphi}_k(\zeta)$ is by

$$\widehat{\varphi}_k(\zeta) = \frac{2\varphi_k(\zeta)}{(1+q)+(1-q)\varphi_k(\zeta)}. \tag{18}$$

Geometrically, the function $\varphi \in k\text{-}\mathcal{P}_q$ takes all points in the domain $\Omega_{k,q}$, which is defined as follows ([21,23,27]):

$$\Omega_{k,q} = \left\{ w : \Re\left(\frac{(1+q)w}{(q-1)w+2}\right) > k\left|\frac{(1+q)w}{(q-1)w+2} - 1\right| \right\}.$$

Domain $\Omega_{k,q}$ represents a generalized conic region.

One way to generalize the class $k\text{-}\mathcal{P}[N,L]$ in Definition 5 is to replace the function $\varphi_k(\zeta)$ in (10) by the function $\widehat{\varphi}_k(\zeta)$, which is involved in (17). The appropriate definition of the corresponding q-extension of class $k\text{-}\mathcal{P}[N,L]$ is given below.

Definition 10. *A function $h \in \mathcal{P}$ is said to be class $\mathcal{P}(q,k,N,L)$ if*

$$h(\zeta) \prec \frac{(N+1)\widehat{\varphi}_k(\zeta) - (N-1)}{(L+1))\widehat{\varphi}_k(\zeta) - (L-1)} \qquad (-1 \leqq L < N \leqq 1;\ k \geqq 0),$$

where $\widehat{\varphi}_k(\zeta)$ is given by (18) and $\varphi_k(\zeta)$ is by (6).

Geometrically, the function $h \in \mathcal{P}(q,k,N,L)$ takes on all points in the domain $\Omega_k[q,N,L]$, which is defined as follows, and represents a generalized conic-type region:

$$\begin{aligned}\Omega_k[q,N,L] = \Bigg\{ w : \Re\left(\frac{(1+q)\{(L-1)w-(N-1)\}}{\{(L+3)+q(L-1)\}w - \{(N+3)+q(N-1)\}}\right) \\ > k\left|\frac{(1+q)\{(L-1)w-(N-1)\}}{\{(L+3)+q(L-1)\}w - \{(N+3)+q(N-1)\}} - 1\right| \Bigg\}.\end{aligned}$$

As an application of Definition 10, we introduce and study the corresponding q-extension of class $k\text{-}\mathcal{S}^*[N,L]$, which involves higher-order q-derivatives, below.

Definition 11. *A function $j \in \mathcal{A}(\ p)$ is said to be class $\mathcal{S}^*(\ q,k,N,L,p)$ if*

$$\Re(\mathcal{F}(q,k,N,L,p)) > k|\mathcal{F}(q,k,N,L,p) - 1|,$$

where

$$\mathcal{F}(q,k,N,L,p) = \frac{(1+q)\left\{(L-1)\left(\frac{\zeta\left(D_q^{(p)}j\right)(\zeta)}{\left(D_q^{(p-1)}j\right)(\zeta)}\right) - (N-1)\right\}}{\{(L+3)+q(L-1)\}\left(\frac{\zeta\left(D_q^{(p)}j\right)(\zeta)}{\left(D_q^{(p-1)}j\right)(\zeta)}\right) - \{(N+3)+q(N-1)\}}. \tag{19}$$

Equivalently,

$$\frac{\zeta(D_q j)(\zeta)}{j(\zeta)} \in \mathcal{P}(q,k,N,L). \tag{20}$$

It is worthwhile to note the following special subclasses of class $\mathcal{S}^*(q,k,N,L)$.

(a) For $k=0,\ N=1-2\alpha \ \ (0 \leq \alpha < 1), L=-1=-p$, and $q \to 1-$: it gives class $\mathcal{S}^*(\alpha)$ (see [28]).

(b) For $N=1=p, L=-1$, and $q \to 1-$: it gives class $k\text{-}\mathcal{ST}$ (see [5]).

(c) For $N=1-2\alpha \ \ (0 \leq \alpha < 1)$ Ł $=-1=-p$, and $q \to 1-$: it gives class $\mathcal{SD}(k,\alpha)$ (see [10]).

(d) For $k=0,\ N=1=p\ L=-1$, and $q \to 1-$: it gives class $\mathcal{S}_q^*$ (see [14]).

(e) For $p=1$ and $q \to 1-$: it gives class $k\text{-}\mathcal{S}^*[N,L]$ (see [11]).

(f) For $k=0=p-1$: it gives class $\mathcal{S}_q^*[N,L]$ (see [22]).

Lemma 1 ([29,30]). *Let*
$$\Phi(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \dots$$
be in class $\mathcal{P}$; then, for any complex number v,
$$\left|c_2 - vc_1^2\right| \leqq 2\max\{1, |1-2v|\}.$$
If v is a real parameter, by Lemma then
$$\left|c_2 - vc_1^2\right| \leqq \begin{cases} -4v+2 & (v \leqq 0) \\ 2 & (0 \leqq v \leqq 1) \\ 4v-2 & (v \geqq 1). \end{cases} \tag{21}$$

2. A Set of Main Results

Theorem 1. *A function $j \in \mathcal{A}(p)$ of the form given by (1) is in class $\mathcal{S}^*(q,k,N,L,p)$ if it satisfies the following condition*
$$\sum_{n=1}^{\infty} \Lambda(n,k,N,L,q,p)\left|a_{n+p}\right| < |L-N|(1+q), \tag{22}$$
where
$$\Lambda(n,k,N,L,q,p) = 4(k+1)q[n]_q + |\mathcal{L}(n,k,N,L,q,p)| \tag{23}$$
and
$$\mathcal{L}(n,k,N,L,q,p) = \{L(1+q)+3-q\}\frac{[n+p]_q![n+1]_q}{[n+1]_q!} - \{N(q+1)+3-q\}. \tag{24}$$

Proof. Assuming that (22) holds, by Definition 11, it suffices to show that
$$k|\mathcal{F}(q,k,N,L,p)-1| - \Re(\mathcal{F}(q,k,N,L,p)-1) < 1,$$
where $\mathcal{F}(q,k,N,L)$ is given by (19).

Now, we have
$$\begin{aligned}
&(k+1)|\mathcal{F}(q,k,N,L,p)-1| \\
&\leq (k+1)\left|\frac{(1+q)\left\{(L-1)\zeta\left(D_q^{(p)}j\right)(\zeta)-(N-1)\left(D_q^{(p-1)}j\right)(\zeta)\right\}}{\mathcal{M}(N,L,k,q,p)} - 1\right| \\
&= 4(k+1)\left|\frac{\left(D_q^{(p-1)}j\right)(\zeta)-\zeta\left(D_q^{(p)}j\right)(\zeta)}{\mathcal{M}(N,L,k,q,p)}\right| \\
&= 4(k+1)\left|\frac{\sum_{n=1}^{\infty}\frac{[n+p]q!}{[n+1]q!}\left(1-[n+1]_q\right)a_{n+p}\zeta^{n+1}}{(L-N)(1+q)[p]_q!\zeta+\sum_{n=1}^{\infty}\mathcal{L}(n,k,N,L,q,p)a_{n+p}\zeta^{n+1}}\right| \\
&\leq \frac{4(k+1)\sum_{n=1}^{\infty}\frac{[n+p]q!}{[n+1]q!}\left(1-[1+n]_q\right)\left|a_{n+p}\right|}{|(L-N)(1+q)|[p]_q!-\sum_{n=2}^{\infty}\mathcal{L}(n,k,N,L,q,p)\left|a_{n+p}\right|},
\end{aligned} \tag{25}$$
where
$$\mathcal{M}(N,L,k,q,p) = \{(L+3)+q(L-1)\}\zeta\left(D_q^{(p)}j\right)(\zeta) - \{(N+3)+q(N-1)\}D_q^{(p-1)}j(\zeta)$$
and $\mathcal{L}(n,k,N,L,q,p)$ is given by (24).

If (22) holds,

$$\sum_{n=1}^{\infty} \Lambda(n,k,N,L,q,p)\big|a_{n+p}\big| < |L-N|(1+q).$$

Using the identities (23) and (24), the last expression in (25) is bounded above by 1, and hence the proof is now completed. □

Upon letting $q \to 1-$, and $p = 1$, Theorem 1 yields the following known result.

Corollary 1 ([11]). *A function $j \in \mathcal{A}$ of the form given by (1) is in class k-$\mathcal{S}^*[N,L]$ if it satisfies the following condition*

$$\sum_{n=2}^{\infty} \{2n(k+1) + |(n+1)(L+1) - (N+1)|\}|a_n| < |L-N|.$$

In Theorem 1, if we set

$$k = 0, \;\; N = 1-2\alpha \quad (0 \le \alpha < 1) \quad \text{and} \quad L = -1 = -p,$$

and let $q \to 1-$, we are led to the following known result.

Corollary 2 ([28]). *A function $j \in \mathcal{A}$ of the form given by (1) is in class $\mathcal{S}^*(\alpha)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} (n+\alpha)|a_n| < 1-\alpha \qquad (0 \le \alpha < 1).$$

Furthermore, if we put

$$p = N = 1 \quad \text{and} \quad L = -1,$$

and let $q \to 1-$, then Theorem 1 implies the following corollary.

Corollary 3 ([5]). *A function $j \in \mathcal{A}$ of the form given by (1) is in class $k - \mathcal{ST}$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n(k+1)+1\}|a_n| < 1.$$

Moreover, if we put

$$N = 1-2\alpha \qquad (0 \le \alpha < 1) \quad \text{and} \quad L = -1 = -p,$$

and let $q \to 1-$, then Theorem 1 implies the following known result.

Corollary 4 ([10]). *A function $j \in \mathcal{A}$ of the form given by (1) is in class $\mathcal{SD}(k,\alpha)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n(k+1)+\alpha\}|a_n| < (1-\alpha).$$

Theorem 2. *Let $j(\zeta)$ be a function with (1) in class $\mathcal{S}^*(q,k,N,L,p)$, where $0 \le k \le 1$. (a) If μ is a complex number, then*

$$\left|a_{p+2}-\mu a_{p+1}^{2}\right| \leq \left(\frac{(1-q^{2})(1-q^{3})}{(1-q^{p+1})(1-q^{p+2})}\right)\left(\frac{N-L}{8q}\right)P_{1} \cdot \max\left\{1,\left|\frac{P_{2}}{P_{1}}+\frac{\Upsilon(q)}{4q}P_{1}-\frac{\mu(N-L)(1+q)^{2}}{4q}P_{1}\right|\right\}. \tag{26}$$

(b) If μ is a real parameter, then

$$\left|a_{p+2}-\mu a_{p+1}^{2}\right| \leq \begin{cases} \left(\frac{(1-q^{2})(1-q^{3})}{(1-q^{p+1})(1-q^{p+2})}\right)\left(\frac{N-L}{4q}\right)\left(P_{2}+\frac{\Upsilon(q)}{4q}P_{1}^{2}-\frac{\mu(1+q)^{2}}{4q}P_{1}^{2}\right) & (\mu<\sigma_{1}) \\ \left(\frac{(1-q^{2})(1-q^{3})}{(1-q^{p+1})(1-q^{p+2})}\right)\left(\frac{N-L}{4q}\right)P_{1} & (\sigma_{1}\leq\mu\leq\sigma_{2}) \\ \left(\frac{(1-q^{2})(1-q^{3})}{(1-q^{p+1})(1-q^{p+2})}\right)\left(\frac{L-N}{4q}\right)\left(P_{2}+\frac{\Upsilon(q)}{4q}P_{1}^{2}-\frac{\mu(1+q)^{2}}{4q}P_{1}^{2}\right) & (\mu>\sigma_{2}), \end{cases} \tag{27}$$

where

$$\Upsilon(q)=\left[(N-L)+(N-5L-3)q+(1-L)q^{2}\right], \tag{28}$$

$$\sigma_{1}=\frac{4q[p+1]_{q}[3]_{q}}{(N-L)(1+q)^{2}[2]_{q}[p+2]_{q}P_{1}^{2}}\left\{\frac{\Upsilon(q)}{4q}P_{1}^{2}-P_{1}+P_{2}\right\}$$

$$\sigma_{2}=\frac{4q[p+1]_{q}[3]_{q}}{(N-L)(1+q)^{2}[2]_{q}[p+2]_{q}P_{1}^{2}}\left\{P_{1}+P_{2}+\frac{\Upsilon(q)}{4q}P_{1}^{2}\right\}$$

where P_1, P_2 are defined by (7), (8), respectively. The result is sharp for a real parameter μ.

Proof. We begin by showing that the inequalities (26) and (27) are true for $j \in \mathcal{S}^*(q,k,N,L,p)$. Let us consider a function $\Psi(\zeta)$ given by

$$\Psi(\zeta)=\frac{\zeta\left(D_{q}^{(p)}j\right)(\zeta)}{D^{(p-1)}j(\zeta)} \qquad (\forall\zeta\in\mathbb{U}).$$

Since $j \in \mathcal{S}^*(q,k,N,L,p)$, we have the following subordination:

$$\Psi(\zeta)\prec\phi(\zeta), \tag{29}$$

where

$$\phi(\zeta)=\frac{(1+q)(N+1)(p_{k}(\zeta)-1)+2(p_{k}(\zeta)+1-q(p_{k}(\zeta)-1))}{(1+q)(L+1)(p_{k}(\zeta)-1)+2(p_{k}(\zeta)+1-q(p_{k}(\zeta)-1))}.$$

Suppose that

$$p_{k}(\zeta)=1+P_{1}\zeta+P_{2}\zeta^{2}+...,$$

then we can find, after some simplification, that

$$\phi(\zeta)=1+\frac{1}{4}(N-L)(q+1)P_{1}\zeta+\frac{1}{16}(N-L)(q+1) \cdot\left\{4P_{2}-(3-q+(q+1)L)P_{1}^{2}\right\}\zeta^{2}+...$$

Using the subordination relation (29), we see that the function $h(\zeta)$ given by

$$h(\zeta)=\frac{1+\phi^{-1}(\Psi(\zeta))}{1-\phi^{-1}(\Psi(\zeta))}=1+c_{1}\zeta+c_{2}\zeta^{2}+... \quad (\forall\zeta\in\mathbb{U})$$

belongs to the class $\mathcal{P}$. We also have

$$\Psi(\zeta) = \phi\left(\frac{h(\zeta)-1}{h(\zeta)+1}\right). \tag{30}$$

where

$$\begin{aligned}\Psi(\zeta) \quad &= \quad \frac{\zeta\left(D_q^{(p)}j\right)(\zeta)}{\left(D_q^{(p-1)}j\right)(\zeta)} = 1 + \frac{q}{1+q}\left(\frac{1-q^{p+1}}{1-q}\right)a_{p+1}\zeta \\ &+\left\{\frac{q\left(1-q^{p+1}\right)\left(1-q^{p+2}\right)}{(1-q)(1-q^3)}a_{p+2} - \frac{q\left(1-q^{p+1}\right)^2}{(1-q)^2(1+q)^2}a_{p+1}^2\right\}\zeta^2 \\ &+\ldots\end{aligned} \tag{31}$$

and

$$\begin{aligned}\phi\left(\frac{\Phi(\zeta)-1}{\Phi(\zeta)+1}\right) \quad &= 1 + \frac{1}{8}(N-L)(q+1)P_1c_1\zeta + \frac{1}{8}(N-L)(q+1) \\ &\cdot\left\{P_1c_2 + \left(\frac{P_2}{2} - \frac{(3-q+(q+1)L)}{8}P_1^2 - \frac{P_2}{2}\right)c_1^2\right\}\zeta^2 + \ldots.\end{aligned} \tag{32}$$

From Equations (31) and (32) ,

$$a_{p+1} = \frac{(N-L)(q+1)^2}{8q[p+1]_q}P_1c_1 \tag{33}$$

and

$$a_{p+2} = \frac{(N-L)[2]_q[3]_q}{8q[p+1]_q[p+2]_q}\left[P_1c_2 + \left(\frac{P_2}{2} - \frac{P_1}{2} + \frac{\Upsilon(q)P_1^2}{8q}\right)c_1^2\right], \tag{34}$$

where $\Upsilon(q)$ is given by (28).

Thus,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \left(\frac{(N-L)[2]_q[3]_q}{8q[p+1]_q[p+2]_q}\right)P_1\left|c_2 - \zeta c_1^2\right|, \tag{35}$$

where

$$\zeta = \frac{1}{2}\left(1 - \frac{P_2}{P_1} - \frac{\Upsilon(q)P_1}{4q} + \frac{\mu(N-L)(1+q)^2[2]_q[p+2]_qP_1}{4q[p+1]_q[3]_q}\right).$$

Now applying for Lemma 1 with (35), we obtain the results stated in (27). $\square$

3. Partial Sums for the Function Class $\mathcal{S}^*(q,k,N,L,p)$

We assume that j is of the form (1) unless otherwise stated. Thus, its sequence of partial sums is given by

$$j_m(\zeta) = \zeta^p + \sum_{n=1}^{m} a_{n+p}\zeta^{n+p}.$$

we then examine the ratio of a function j to its sequence of partial sums.

Also, we obtain the sharp lower bounds for

$$\Re\left(\frac{j(\zeta)}{j_m(\zeta)}\right), \left(\frac{j_m(\zeta)}{j(\zeta)}\right), \Re\left(\frac{D_q^{(p)}j(\zeta)}{D_q^{(p)}j_m(\zeta)}\right) \text{ and } \Re\left(\frac{\left(D_q^{(p)}j_m\right)(\zeta)}{\left(D_q^{(p)}j\right)(\zeta)}\right).$$

Theorem 3. *If j of the form (1) satisfies condition (22), then*

$$\Re\left(\frac{j(\zeta)}{j_m(\zeta)}\right) \geq 1 - \frac{1}{\rho_{j+1}} \qquad (\zeta \in \mathbb{U}) \tag{36}$$

and
$$\Re\left(\frac{j_m(\zeta)}{j(\zeta)}\right) \geq \frac{\rho_{j+1}}{1+\rho_{j+1}} \quad (\zeta \in \mathbb{U}), \tag{37}$$
where
$$\rho_m = \frac{\Lambda(j,k,N,L,q,p)}{(1+q)|N-L|} \tag{38}$$
and $\Lambda(j,k,N,L,q,p)$ *is given by* (23).

Proof. In order to prove the inequality (36), we set
$$\rho_{j+1}\left[\frac{j(\zeta)}{j_m(\zeta)} - \left(1 - \frac{1}{\rho_{j+1}}\right)\right] = \frac{1 + \sum\limits_{n=1}^{m} a_{n+p}\zeta^n + \rho_{j+1} \sum\limits_{n=j+1}^{\infty} a_{n+p}\zeta^n}{1 + \sum\limits_{n=2}^{m} a_{n+p}\zeta^n}$$
$$= \frac{1+h_1(\zeta)}{1+h_2(\zeta)}.$$
If we set,
$$\frac{1+h_1(\zeta)}{1+h_2(\zeta)} = \frac{1+w(\zeta)}{1-w(\zeta)},$$
then we have,
$$w(\zeta) = \frac{h_1(\zeta)-h_2(\zeta)}{2+h_1(\zeta)+h_2(\zeta)}.$$
It follows that
$$w(\zeta) = \frac{\rho_{j+1} \sum\limits_{n=j+1}^{\infty} a_{n+p}\zeta^n}{2 + 2\sum\limits_{n=1}^{m} a_{n+p}\zeta^n + \rho_{j+1} \sum\limits_{n=j+1}^{\infty} a_{n+p}\zeta^n}$$
and
$$|w(\zeta)| \leq \frac{\rho_{j+1} \sum\limits_{n=j+1}^{\infty} |a_{n+p}|}{2 - 2\sum\limits_{n=1}^{m} |a_{n+p}| - \rho_{j+1} \sum\limits_{n=j+1}^{\infty} |a_{n+p}|}.$$
It is not difficult to see that
$$|w(\zeta)| \leq 1$$
is equivalent to
$$2\rho_{j+1} \sum_{n=j+1}^{\infty} |a_{n+p}| \leq 2 - 2\sum_{n=1}^{m} |a_{n+p}|,$$
which implies that
$$\sum_{n=1}^{m} |a_{n+p}| + \rho_{j+1} \sum_{n=j+1}^{\infty} |a_{n+p}| \leq 1. \tag{39}$$
To prove our assertion (36), it is enough to show that (39) is bounded above by $\sum\limits_{n=2}^{\infty} \rho_n |a_n|$, which can be written as
$$\sum_{n=1}^{m} (1-\rho_n)|a_{n+p}| + \sum_{n=j+1}^{\infty} (\rho_{j+1}-\rho_n)|a_{n+p}| \geq 0. \tag{40}$$
Thus, the proof of inequality in (36) is completed now.

Next, in order to prove the inequality (37), we set

$$(1+\rho_m)\left(\frac{j_m(\zeta)}{j(\zeta)}-\frac{\rho_m}{1+\rho_m}\right)=\frac{1+\sum\limits_{n=1}^{m}a_{n+p}\zeta^n-\rho_{j+1}\sum\limits_{n=j+1}^{\infty}a_{n+p}\zeta^n}{1+\sum\limits_{n=1}^{\infty}a_{n+p}\zeta^n}=\frac{1+w(\zeta)}{1-w(\zeta)},$$

where

$$|w(\zeta)|\leq\frac{(1+\rho_{j+1})\sum\limits_{n=j+1}^{\infty}|a_{n+p}|}{2-2\sum\limits_{n=1}^{m}|a_{n+p}|-(\rho_{j+1}-1)\sum\limits_{n=j+1}^{\infty}|a_{n+p}|}\leq 1. \tag{41}$$

The right-hand side inequality in (42) is equivalent to

$$\sum_{n=1}^{m}|a_{n+p}|+\rho_{j+1}\sum_{n=j+1}^{\infty}|a_{n+p}|\leq 1. \tag{42}$$

Also, the left-hand side inequality in (42) is bounded above by $\sum\limits_{n=2}^{\infty}\rho_n|a_{n+p}|$ and so we have completed the proof of (37). This completes the proof of Theorem 3. □

The following result is related to the functions involving derivatives.

Theorem 4. *If j of the form (1) satisfies condition (22), then*

$$\Re\left(\frac{\left(D_q^{(p)}j\right)(\zeta)}{\left(D_q^{(p)}j_m\right)(\zeta)}\right)\geq 1-\frac{[j+p]_q}{\rho_{j+1}}\qquad(\forall\zeta\in\mathbb{U}) \tag{43}$$

and

$$\Re\left(\frac{\left(D_q^{(p)}j_m\right)(\zeta)}{\left(D_q^{(p)}j\right)(\zeta)}\right)\geq\frac{\rho_{j+1}}{\rho_{j+1}+[j+p]_q}\qquad(\forall\zeta\in\mathbb{U}), \tag{44}$$

where ρ_m is given by (38).

Proof. The proof of Theorem 4 is similar to that of Theorem 3; we here choose to omit the analogous details. □

4. Analytic Functions with Negative Coefficients

First, we give a new subclass of starlike functions having negative coefficients. Let $\mathcal{T}\subset\mathcal{A}$, which consists of functions having negative coefficients, i.e.,

$$j(\zeta)=\zeta^p-\sum_{n=1}^{\infty}|a_{n+p}|\zeta^{n+p}. \tag{45}$$

Now, we state and prove the distortion results for subclass $\mathcal{TS}^*(N,L,q,k,p)$ of $\mathcal{T}$.

Theorem 5. *If $j\in\mathcal{TS}^*(N,L,q,k,p)$, then*

$$r^p-\frac{|L-N|(1+q)}{\Lambda(2,k,N,L,q,p)}r^{1+p}\leq|j(\zeta)|\leq r^p+\frac{|L-N|(1+q)}{\Lambda(2,k,N,L,q,p)}r^{1+p},\quad(|\zeta|=r)$$

where $\Lambda(2,k,N,L,q,p)$ is given by (23).

Proof. We note that the following inequality follows from Theorem 1:

$$\Lambda(2,k,N,L,q,p)\sum_{n=1}^{\infty}|a_{n+p}| \leq \sum_{n=1}^{\infty}\Lambda(n,k,N,L,q,p)|a_{n+p}| < |L-N|(1+q),$$

which yields

$$|j(\zeta)| \leq r^p + \sum_{n=1}^{\infty}|a_{n+p}|r^{n+p} \leq r^p + r^{1+p}\sum_{n=1}^{\infty}|a_{n+p}| \leq r^p + \frac{|L-N|(1+q)}{\Lambda(2,k,N,L,q,p)}r^{1+p},$$

Also, we have

$$|j(\zeta)| \geq r^p - \sum_{n=1}^{\infty}|a_{n+p}|r^{n+p} \geq r^p - r^{1+p}\sum_{n=1}^{\infty}|a_{n+p}| \geq r^p - \frac{|L-N|(1+q)}{\Lambda(2,k,N,L,q,p)}r^{1+p}.$$

Therefore, the proof is completed. □

In the special case that

$$k=0, \; N=1-2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad L=-1, p=1,$$

and if we let $q \to 1-$, Theorem 5 reduces to the following known result.

Corollary 5 ([28]). *If $j \in \mathcal{TS}^*(\alpha)$, then*

$$r - \frac{1-\alpha}{2-\alpha}r^2 \leq |j(\zeta)| \leq r + \frac{1-\alpha}{2-\alpha}r^2, \quad (|\zeta|=r).$$

The proof of the following theorem is similar to the proof of Theorem 5; therefore, it is omitted.

Theorem 6. *If $j \in \mathcal{TS}^*(N,L,q,k,p)$, then*

$$1 - \frac{2|L-N|}{\Lambda(2,k,N,L,q,p)}r^p \leq |j'(\zeta)| \leq 1 + \frac{2|L-N|}{\Lambda(2,k,N,L,q,p)}r^p, \quad (|\zeta|=r)$$

where $\Lambda(2,k,N,L,q,p)$ is given by (23).

In the special case that

$$k=0, \quad N=1-2\alpha, \quad (0 \leq \alpha < 1) \quad L=-1, \text{and } p=1,$$

and if we let $q \to 1-$, Theorem 6 reduces to the known result given in [28].

5. Concluding Remarks and Observations

In this paper, we have systematically used the conic domain and the celebrated Janowski functions with higher-order q-derivatives. We have defined a new subclass of q-starlike functions. We have then obtained several remarkable results, such as sufficient conditions and some coefficient estimates. We have also given some specific cases of our main results in the form of remarks and corollaries.

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