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Applications of the q-Sălăgean Differential Operator Involving Multivalent Functions

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Abstract: In this article we explore several applications of q-calculus in geometric function theory. Using the method of differential subordination, we obtain interesting univalence properties for the q-Sălăgean differential operator. Sharp subordination results are obtained by using functions with remarkable geometric properties as subordinating functions and considering the conditions of expressions involving the q-Sălăgean differential operator and a convex combination using it.

Keywords: analytic functions; multivalent function; q-derivative; q-analogue of the Sălăgean differential operator; differential subordination; best dominant

MSC: 30C45

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1. Introduction

Since q-calculus has numerous applications in physics, mathematics and engineering sciences, it became attractive for many researchers. Jackson ([1,2]) gave the first application of q-calculus by defining the q-derivative and q-integral. In 1989 [3], Srivastava set the basic context for using q-calculus in geometric function theory, and in 1990, Ismail et al. [4] introduced and studied an extension of the class of starlike functions using the notions of q-calculus in this domain. The class of q-starlike functions was further extended. Agrawal and Sahoo [5] studied starlike functions of the α order using q-calculus aspects, and later, the Hankel and Toeplitz determinants were obtained for a subclass of starlike functions of the α order [6]. Coefficient inequalities for a subclass of q-starlike functions associated with a conic domain were obtained in [7]. A certain subclass of q-starlike functions associated with the Janowski functions was introduced and studied in [8]. Multivalent q-starlike functions were studied in connection with the circular domain in [9], and by using certain higher-order q-derivatives, the subclasses of multivalent q-starlike functions were introduced and investigated in [10]. Special functions have also been associated with q-calculus, such as the famous Mittag–Leffler function [11–13].

Interpreting geometrically the q-analysis is accomplished by introducing and studying numerous q-analogue differential operators. Srivastava showed in a comprehensive review paper in 2020 [14] the applications of q-calculus, mentioning various q-operators introduced up to that date by researchers using fractional calculus and convolution. In 2014 [15], the q-analogue of the Ruscheweyh differential operator was introduced. Certain q-integral operators of p-valent functions can be seen in [16], and a q-analogue of the Ruscheweyh-type operator for multivalent functions was introduced in [17]. In 2017 [18], the q-analogue of the Sălăgean differential operator was defined, and it was extended to the class of multivalent functions in 2019 [19]. Using these operators, interesting results were obtained by introducing new classes of analytic functions ([20–22]) and multivalent functions ([19,23,24]).

The results presented in this paper involve the q-analogue of the Sălăgean differential operator applied to multivalent functions. This study was inspired by the investigation presented in [25] for the q-analogue of the Ruscheweyh operator.

The theory of differential subordination initiated by Miller and Mocanu ([26,27]) is used for obtaining the main results of this paper. In the next section, the results obtained by different researchers and used to obtain the original results from this article are presented. Then, in Section 3 of the paper, the new subordination properties regarding the q- Sălăgean differential operator are explored. A sharp subordination is investigated in Theorem 1, and an interesting corollary emerges by using a particular function in Theorem 1. An example is given to show an application of the result. In Theorem 2, a subordination is studied considering the real part of an expression involving the q- Sălăgean differential operator. The best dominant is obtained for this subordination, and an example is also given to illustrate the use of the results. Convolution is involved in the subordination result presented in Theorem 3. Conclusions on the study presented in this paper are given in Section 4, where future directions of study are also suggested.

2. Preliminaries

We now explore the definitions and notations used in this research.

Let $\mathcal{A}(p)$ be the class of analytic and p-valent functions in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \ p \in \mathbb{N}.$$

The analytic function f is subordinate to the analytic function g, written as $f \prec g$, if there is an analytic Schwartz function ω in $\mathcal{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for $z \in \mathcal{U}$.

For the univalent function g in $\mathcal{U}$, the equivalence relation holds, where $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

We also explore the notations and concepts of q-calculus.

For $x \in \mathbb{N}$, $0 < q < 1$, it is noted that

$$[x]_q = \frac{1 - q^x}{1 - q},$$

and

$$[x]_q! = \begin{cases} \prod_{y=1}^{x} [y]_q, \ x \in \mathbb{N}^*, \\ 1, \ x = 0. \end{cases}$$

For a function $f \in \mathcal{A}(1)$, the q-derivative operator $\mathcal{D}_q$ is defined as in ([2]):

$$\mathcal{D}_q(f(z)) = \begin{cases} \frac{f(z) - f(zq)}{z(1-q)}, \ z \neq 0, \\ f'(0), \ \ z = 0. \end{cases}$$

We can see that

$$\lim_{q \to 1} \mathcal{D}_q(f(z)) = \lim_{q \to 1} \frac{f(z) - f(zq)}{z(1-q)} = f'(z),$$

when f is a differentiable function.

When $f(z) = z^n$, we obtain $\mathcal{D}_q(f(z)) = \mathcal{D}_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$.

The Sălăgean differential operator ([28]) has the form $S^m f(z) = z^p + \sum_{j=p+1}^{\infty} j^m a_j z^j$ for $f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \in \mathcal{A}(p)$, where $z \in \mathcal{U}$:

Definition 1 ([19]). *We denote by $\mathcal{S}_{q,p}^n$ the extended q-Sălăgean differential operator*

$$\mathcal{S}_{q,p}^n f(z) = z^p + \sum_{j=p+1}^{\infty} [j-p+1]_q^n a_j z^j,$$

for $f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \in \mathcal{A}(p)$, where $z \in \mathcal{U}$.

We observe that for $p = 1$, the extended q-Sălăgean differential operator $\mathcal{S}_{q,p}^n f$ reduces to the q-analogue of the Sălăgean differential operator $\mathcal{S}_q^n$ ([29]). For $p = 1$ and $q \to 1$, the extended q-Sălăgean differential operator $\mathcal{S}_{q,p}^n f$ reduces to the familiar Sălăgean differential operator $S^n f$.

After a short computation, we deduce that

$$z\mathcal{D}_q\left(\mathcal{S}_{q,p}^n f(z)\right) = [p-1]_q \mathcal{S}_{q,p}^n f(z) + q^{p-1}\mathcal{S}_{q,p}^{n+1} f(z). \quad (1)$$

There are many papers which adapt different lemmas from the classical theory of subordination considering q-calculus aspects. Some lemmas used for the proof of the new results are presented next. They are cited over 70 times and are also used in [25,29]:

Lemma 1 ([27]). *Let $g(z) = 1 + a_1 z + a_2 z^2 + \ldots$ be the analytic in $\mathcal{U}$ and h be a univalent analytic and convex function in $\mathcal{U}$ with $h(0) = 1$. If*

$$\frac{z\mathcal{D}_q(g(z))}{b} + g(z) \prec h(z),\ z \in \mathcal{U},\ b \neq 0,$$

then

$$g(z) \prec \frac{b}{z^b}\int_0^z \frac{h(t)t^b}{t} dt,$$

with $Re(b) \geq 0$.

Lemma 2 ([30]). *Consider θ and ϕ analytic functions in a domain $D \supset q(\mathcal{U})$ such that and $\phi(\omega) \neq 0$, $\omega \in q(\mathcal{U})$ and u is a univalent function in $\mathcal{U}$. Let $Q(z) = z\phi(u(z))\mathcal{D}_q(u(z))$ and $h(z) = \theta(u(z) + Q(z))$, assuming that $Q(z)$ is a univalent starlike function in $\mathbb{U}$ and $Re\left(\frac{z\mathcal{D}_q(h(z))}{Q(z)}\right) =$ $\frac{z\mathcal{D}_q(Q(z))}{Q(z)} + Re\left(\frac{\mathcal{D}_q(\theta(u(z)))}{\phi(u(z))}\right) > 0$, where $z \in \mathcal{U}$.*

When $g(z)$ is an analytic function in $\mathcal{U}$ with the properties $g(0) = u(0)$, $p(\mathcal{U}) \subset D$ and

$$\theta(g(z)) + z\phi(g(z))\mathcal{D}_q(g(z)) \prec \theta(u(z)) + z\phi(u(z))\mathcal{D}_q(u(z)) = h(z),$$

then $g \prec u$, and u is the best dominant.

Lemma 3 ([31]). *The necessary and sufficient condition for the function $f(z) = (1-z)^{\alpha}$, $\alpha \neq 0$ to be univalent in $\mathcal{U}$ is $|\alpha + 1| \leq 1$ or $|\alpha - 1| \leq 1$.*

Lemma 4 ([32]). *Considering f_i, the analytic functions of the form $1 + a_1 z + a_2 z^2 + \ldots$ in $\mathcal{U}$ that verify the inequality $Re(f_i) > \alpha_i$, $0 \leq \alpha_i < 1$ and $i = 1, 2$, we obtain that $f_1 * f_2$ is an analytic function of the form $1 + a_1 z + a_2 z^2 + \ldots$ in $\mathbb{U}$ which verifies the inequality $Re(f_1 * f_2) > 1 - 2(1-\alpha_1)(1-\alpha_2)$.*

Lemma 5 ([33]). *By considering the analytic function $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ that verifies the inequality $Re(f(z)) > \alpha$, $0 \leq \alpha < 1$, then*

$$Re(f(z)) > 2\alpha - 1 + \frac{2(1-\alpha)}{|z|+1},\ \ z \in \mathcal{U}.$$

3. Main Results

Theorem 1. *If $f \in \mathcal{A}(p)$ satisfies*

$$\alpha \frac{\mathcal{S}_{q,p}^{n+1} f(z)}{z^p} + (1-\alpha) \frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} \prec \frac{Nz+1}{Mz+1}, \tag{2}$$

for $-1 \leq M < N \leq 1$, $\alpha > 0$, then

$$Re\left(\left(\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} \right)^{\frac{1}{k}} \right) > \left(\frac{1}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}(Nu-1)}{u(Mu-1)} du \right)^{\frac{1}{k}}, \;\; k \geq 1, \tag{3}$$

where the result is sharp.

Proof. By denoting $g(z) = \frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} = 1 + a_1 z + \ldots$ for $f \in \mathcal{A}(p)$ is analytic in $\mathcal{U}$ and applying the logaritmic q-differentiation, we obtain

$$\mathcal{D}_q(g(z)) = \mathcal{D}_q\left(\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} \right) = \frac{z\mathcal{D}_q\left(\mathcal{S}_{q,p}^{n} f(z) \right) - [p]_q \mathcal{S}_q^n f(z)}{q^p z^{p+1}}$$

In addition, by taking account the relation in Equation (1), we obtain

$$z\mathcal{D}_q(g(z)) = -\frac{1}{q} g(z) + \frac{1}{q} \frac{\mathcal{S}_{q,p}^{n+1} f(z)}{z^p}.$$

We find that

$$\frac{\mathcal{S}_{q,p}^{n+1} f(z)}{z^p} = g(z) + qz\mathcal{D}_q(g(z))$$

and

$$\begin{aligned} \alpha \frac{\mathcal{S}_{q,p}^{n+1} f(z)}{z^p} + (1-\alpha) \frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} &= \alpha\big(qz\mathcal{D}_q(g(z)) + g(z) \big) + (1-\alpha) g(z) \\ &= \alpha qz\mathcal{D}_q(g(z)) + g(z). \end{aligned}$$

We can write the differential subordination in Equation (2) as follows:

$$\alpha qz\mathcal{D}_q(g(z)) + g(z) \prec \frac{Nz+1}{Mz+1},$$

Then, by applying Lemma 1, we obtain

$$g(z) \prec \frac{1}{q\alpha} z^{-\frac{1}{q\alpha}} \int_0^z \frac{t^{\frac{1}{q\alpha}}(Nt+1)}{t(Mt+1)} dt,$$

Alternatively, by using the subordination properties, we obtain

$$\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} = \frac{1}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}(Nu\omega(z)+1)}{u(Mu\omega(z)+1)} du.$$

Since $-1 \leq M < N \leq 1$, we obtain

$$\mathrm{Re}\left(\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} \right) > \frac{1}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}(Nu-1)}{u(Mu-1)} du,$$

where the inequality $\mathrm{Re}\left(x^{\frac{1}{n}} \right) \geq (\mathrm{Re}x)^{\frac{1}{n}}$ is considered for $n \geq 1$ and $\mathrm{Re}x > 0$.

To show the sharpness of Equation (3), we consider $f \in \mathcal{A}(p)$, defined by

$$\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} = \frac{1}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}(Nuz+1)}{u(Muz+1)} du.$$

For the defined function, we can write

$$\alpha \frac{\mathcal{S}_{q,p}^{n+1} f(z)}{z^p} + (1-\alpha) \frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} = \frac{Nz+1}{Mz+1}$$

and

$$\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} \to \frac{1}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}(Nu-1)}{u(Mu-1)} du \ \text{ as } z \to -1,$$

Thus, the proof is complete. □

Remark 1. *For $p = 1$, we are led to similar results to those given in [29].*

Corollary 1. *If $f \in \mathcal{A}(p)$ satisfies*

$$\alpha \frac{\mathcal{S}_{q,p}^{n+1} f(z)}{z^p} + (1-\alpha) \frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p} \prec \frac{(2b-1)z+1}{z+1}, \tag{4}$$

for $\alpha > 0$, $0 \leq b < 1$, then

$$Re\left(\left(\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p}\right)^{\frac{1}{k}}\right) > \left((2b-1) + \frac{2(1-b)}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}}{u(u+1)} du\right)^{\frac{1}{k}}, \ \ k \geq 1.$$

Proof. Following the same steps as in the proof of Theorem 1 for $g(z) = \frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p}$, the differential subordination in Equation (4) becomes

$$\alpha q z \mathcal{D}_q(g(z)) + g(z) \prec \frac{(2b-1)z+1}{z+1}.$$

Therefore, we obtain

$$\text{Re}\left(\left(\frac{\mathcal{S}_{q,p}^{n} f(z)}{z^p}\right)^{\frac{1}{k}}\right) > \left(\frac{1}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}((2b-1)u+1)}{u(u+1)} du\right)^{\frac{1}{k}} =$$

$$\left(\frac{1}{q\alpha} \int_0^1 \left((2b-1) + \frac{2(1-b)}{u+1}\right) \frac{u^{\frac{1}{q\alpha}}}{u} du\right)^{\frac{1}{k}} =$$

$$\left((2b-1) + \frac{2(1-b)}{q\alpha} \int_0^1 \frac{u^{\frac{1}{q\alpha}}}{u(u+1)} du\right)^{\frac{1}{k}}.$$

□

Example 1. *Let $f(z) = z^p + z^{p+1}$, $n = 1$, $\alpha = 2$, $b = \frac{1}{2}$ and $k = 2$. Then, $\mathcal{S}_{q,p}^{1} f(z) = z^p + [2]_q z^{p+1} = (1+q) z^{p+1} + z^p$ and $\mathcal{S}_{q,p}^{2} f(z) = z^p + [2]_q^2 z^{p+1} = (1+q)^2 z^{p+1} + z^p$.*

We have $\alpha \frac{\mathcal{S}_{q}^{n+1} f(z)}{z} + (1-\alpha) \frac{\mathcal{S}_{q}^{n} f(z)}{z} = 2 \frac{\mathcal{S}_{q}^{2} f(z)}{z} - \frac{\mathcal{S}_{q}^{1} f(z)}{z} = (2q^2 + 3q + 1) z^p + z^{p-1}$.

By applying Corollary 1, we obtain

$$\left(2q^2+3q+1\right)z^p+z^{p-1}\prec\frac{1}{z+1},\ z\in\mathcal{U},$$

which induces

$$Re\sqrt{(1+q)z^{p+1}+z^p}>\frac{1}{\sqrt{2q}}\sqrt{\int_0^1\frac{u^{\frac{1}{2q}}}{u(u+1)}du},\ z\in\mathcal{U}.$$

Theorem 2. *Consider $0\le\sigma<1$, and $\mu\neq0$ and is a complex number such that $\left|\frac{2\mu(1-\sigma)}{q}-1\right|\le 1$ or $\left|\frac{2\mu(1-\sigma)}{q}+1\right|\le1$. If $f\in\mathcal{A}(p)$ verifies the inequality*

$$Re\left(\frac{\mathcal{S}_{q,p}^{n+1}f(z)}{\mathcal{S}_{q,p}^{n}f(z)}\right)>\sigma,\ z\in\mathcal{U},$$

then

$$\left(\frac{\mathcal{S}_{q,p}^{n}f(z)}{z^p}\right)^{\mu}\prec\frac{1}{(1-z)^{\frac{2\mu(1-\sigma)}{q}}},\ z\in\mathcal{U},$$

where $\frac{1}{(1-z)^{\frac{2\mu(1-\rho)}{q}}}$ is the best dominant.

Proof. By considering $g(z)=\left(\frac{\mathcal{S}_{q,p}^{n}f(z)}{z^p}\right)^{\mu}$ and applying logarithmic q-differentiation, we obtain

$$\mathcal{D}_q(g(z))=\frac{\mu}{qz}\left(\frac{\mathcal{S}_{q,p}^{n}f(z)}{z^p}\right)^{\mu}\frac{\mathcal{S}_{q,p}^{n+1}f(z)-\mathcal{S}_{q,p}^{n}f(z)}{\mathcal{S}_{q,p}^{n}f(z)}$$

and

$$\frac{z\mathcal{D}_q(g(z))}{g(z)}=\frac{\mu}{q}\left(\frac{\mathcal{S}_{q,p}^{n+1}f(z)}{\mathcal{S}_{q,p}^{n}f(z)}-1\right).$$

From the above, we obtain that

$$\frac{\mathcal{S}_{q,p}^{n+1}f(z)}{\mathcal{S}_{q,p}^{n}f(z)}=\frac{q}{\mu}\frac{z\mathcal{D}_q(g(z))}{g(z)}+1.$$

We can write the inequality $\mathrm{Re}\left(\frac{\mathcal{S}_{q,p}^{n+1}f(z)}{\mathcal{S}_{q,p}^{n}f(z)}\right)>\sigma$ as follows:

$$\frac{\mathcal{S}_{q,p}^{n+1}f(z)}{\mathcal{S}_{q,p}^{n}f(z)}\prec\frac{(1-2\sigma)z+1}{1-z},$$

This is equivalent with

$$\frac{q}{\mu}\frac{z\mathcal{D}_q(g(z))}{g(z)}+1\prec\frac{(1-2\sigma)z+1}{1-z},\ z\in\mathcal{U}.$$

Let us suppose that

$$u(z)=\frac{1}{(1-z)^{\frac{2\mu(1-\sigma)}{q}}},\ \theta(\omega)=1,\ \phi(\omega)=\frac{q}{\mu\omega},$$

Then, we find that $u(z)$ is univalent by Lemma 3. It is easy to prove that u, θ and ϕ satisfy the conditions of Lemma 2. The function $Q(z) = z\phi(u(z))\mathcal{D}_q(u(z)) = \frac{2(1-\sigma)z}{1-z}$ is univalent and starlike in $\mathcal{U}$, and $h(z) = \theta(u(z) + Q(z)) = \frac{(1-2\sigma)z+1}{1-z}$.

Following Lemma 2, we obtain the proof. □

Remark 2. *For $p = 1$, we are led to similar results to those given in [29].*

Example 2. *Let $f(z) = z^p + z^{p+1}$, $n = 1$, $\rho = \frac{1}{2}$, and $\mu = \frac{q}{2}$. Then, $\mathcal{S}^1_{q,p}f(z) = z^p + [2]_q z^{p+1} = (1+q)z^{p+1} + z^p$ and $\mathcal{S}^2_{q,p}f(z) = z^p + [2]_q^2 z^{p+1} = (1+q)^2 z^{p+1} + z^p$.*

By applying Theorem 2, we obtain

$$Re\left(\frac{\mathcal{S}^2_{q,p}f(z)}{\mathcal{S}^1_{q,p}f(z)}\right) = Re\left(\frac{(1+q)^2 z + 1}{(1+q)z+1}\right) > \frac{1}{2},\ z \in \mathcal{U},$$

which induces

$$\sqrt{((1+q)z+1)^q} \prec \frac{1}{\sqrt{1-z}},\ \ z \in \mathcal{U}.$$

Theorem 3. *Consider $-1 \le M_i < N_i \le 1$, $i = 1,2$ and $\alpha < 1$. If the function $f_i \in \mathcal{A}(p)$ satisfies the differential subordination*

$$\alpha\frac{\mathcal{S}^{n+1}_{q,p}f_i(z)}{z^p} + (1-\alpha)\frac{\mathcal{S}^n_{q,p}f_i(z)}{z^p} \prec \frac{N_i z + 1}{M_i z + 1},\ i = 1,2, \tag{5}$$

then

$$\alpha\frac{\mathcal{S}^{n+1}_{q,p}(f_1 * f_2)(z)}{z^p} + (1-\alpha)\frac{\mathcal{S}^n_{q,p}(f_1 * f_2)(z)}{z^p} \prec \frac{(1-2\mu)z+1}{z+1},$$

where $*$ represents the convolution product between f_1 and $f_2$and

$$\mu = 1 - \frac{4(N_1 - M_1)(N_2 - M_2)}{(1-M_1)(1-M_2)}\left(1 - \frac{1}{q\alpha}\int_0^1 \frac{u^{\frac{1}{q\alpha}}}{u(u+1)}du\right).$$

Proof. Considering $h_i(z) = \alpha\frac{\mathcal{S}^{n+1}_{q,p}f_i(z)}{z^p} + (1-\alpha)\frac{\mathcal{S}^n_{q,p}f_i(z)}{z^p}$, we can write the differential subordination in Equation (5) as follows: $\mathrm{Re}(h_i(z)) > \frac{1-N_i}{1-M_i}$, where $i = 1,2$.

By the proof of Theorem 1, we obtain

$$\mathcal{S}^n_{q,p}f_i(z) = \frac{1}{q\alpha}\int_0^1 \frac{h_i(t)t^{\frac{1}{q\alpha}}}{t}dt,\ \ i = 1,2,$$

and

$$\mathcal{S}^n_{q,p}(f_1 * f_2)(z) = \frac{1}{q\alpha}z^{1-\frac{1}{q\alpha}}\int_0^1 \frac{h_0(t)t^{\frac{1}{\alpha q}}}{t}dt,$$

with

$$h_0(z) = \alpha\frac{\mathcal{S}^{n+1}_{q,p}(f_1 * f_2)(z)}{z^p} + (1-\alpha)\frac{\mathcal{S}^n_{q,p}(f_1 * f_2)(z)}{z^p} =$$

$$\frac{1}{q\alpha}z^{1-\frac{1}{q\alpha}}\int_0^1 \frac{(h_1 * h_2)(t)t^{\frac{1}{\alpha q}}}{t}dt.$$

Using Lemma 4, we find that $h_1 * h_2$ is an analytic function in $\mathcal{U}$ of the form $1 + a_1 z + a_2 z^2 + \ldots$ that verifies the relation $\mathrm{Re}(h_1 * h_2) > 1 - 2\Big(1 - \frac{1-N_1}{1-M_1}\Big)(1 - \frac{1-N_2}{1-M_2}) = \psi$.

By applying Lemma 5, we obtain

$$\mathrm{Re}(h_0(z)) = \frac{1}{q\alpha}\int_0^1 \frac{\mathrm{Re}(h_1 * h_2)(uz)u^{\frac{1}{q\alpha}}}{u}du \geq$$

$$\frac{1}{q\alpha}\int_0^1 \left(2\psi - 1 + \frac{2(1-\psi)}{u|z|+1}\right)\frac{u^{\frac{1}{q\alpha}}}{u}du >$$

$$\frac{1}{q\alpha}\int_0^1 \left(2\psi - 1 + \frac{2(1-\psi)}{u+1}\right)\frac{u^{\frac{1}{q\alpha}}}{u}du =$$

$$1 - \frac{4(N_1 - M_1)(N_2 - M_2)}{(1-M_1)(1-M_2)}\left(1 - \frac{1}{q\alpha}\int_0^1 \frac{u^{\frac{1}{q\alpha}}}{u(u+1)}du\right) = \mu,$$

Thus, the proof is completed. □

4. Conclusions

The study presented in this paper followed the line of research set by introducing q-calculus in geometric function theory. The extended q-Sălăgean differential operator given in Definition 1 was previously introduced by Hussain, Khan, Zaighum and Darus [25] and was used mainly for defining and studying new classes of univalent functions. In this paper, we obtained some interesting subordination results involving this operator. In Theorem 1, a sharp subordination result is presented with a corollary obtained using a particular function. An example follows those results. By taking special conditions for the real part of a relation using the q-Sălăgean differential operator, in Theorem 2, the best dominant of a certain differential subordination is obtained, and an associated example is presented. The last theorem gives a property for the q-Sălăgean differential operator applied to a convolution product of functions.

By following the same steps using the differential superordination theory, the dual results could be obtained, and sandwich-type relations could emerge for the q-Sălăgean differential operator as in [34] or [35]. In addition, the condition in Equation (2) from Theorem 1 suggests that a new subclass of p-valent functions could be introduced using the subordination theory. Future studies could be conducted in this regard, as seen in recent papers [36,37].

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