



Article A Fractional-Order SIR-C Cyber Rumor Propagation Prediction Model with a Clarification Mechanism

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Abstract: As communication continues to develop, the high freedom and low cost of the communication network environment also make rumors spread more rapidly. If rumors are not clarified and controlled in time, it is very easy to trigger mass panic and undermine social stability. Therefore, it is important to establish an efficient model for rumor propagation. In this paper, the impact of rumor clarifiers on the spread of rumors is considered and fractional order differentiation is introduced to solve the problem that traditional models do not take into account the "anomalous propagation" characteristics of information. A fractional-order Susceptible-Infected-Removal-Clarify (SIR-C) rumor propagation prediction model featuring the clarification mechanism is proposed. The existence and asymptotic stability conditions of the rumor-free equilibrium point (RFEP) E_0 ; the boundary equilibrium points (BEPs) E_1 and E_2 are also given. Finally, the stability conditions and practical cases are verified by numerical simulations. The experimental results confirm the analysis of the theoretical study and the model fits well with the real-world case data with just minor deviations. As a result, the model can play a positive and effective role in rumor propagation prediction.

Keywords: fractional-order; SIR model; rumor propagation; stability

MSC: 94-10

1. Introduction

Cyber rumors [1] are rumors propagated through the Internet medium that are not based on facts and are offensive and purposeful. As communication technology has steadily developed, large social networking sites, represented by Twitter and Weibo, have become an important source of information reception for people. The China Internet Network Information Center's 50th Statistical Report [2] on the Development Status of the Internet in China states that as of June 2022, there were 1.051 billion Internet users in China, with 99.6% of those users accessing the Internet using mobile devices. This high degree of freedom, low cost of dissemination and the huge scale of users have also led to the "explosive" spread of cyber rumors. If cyber rumors are not clarified and controlled in a timely manner, but are instead spread excessively, they can easily trigger irrational behaviour such as panic and excessive anxiety among the public, disrupting public order, destabilising society and even affecting the credibility of the state. Hence, it is of significant theoretical and practical importance to analyse the mechanisms and laws of rumor propagation and to establish an effective model of cyber rumor propagation.

Research into rumor propagation models began in the 1960s with the DK model [3], which divided the audience into three categories according to rumor propagation: ignorants, spreaders and removers, and used stochastic process analysis to analyse rumor



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Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). propagation. Subsequently, Maki and Thomson [4] proposed the MT model found on the DK model and analysed the model mathematically. These two models produce representative results and serve as the theoretical foundation for rumor propagation models. However, there are still differences in the transmission mechanisms and pathways between rumor propagation and disease propagation, and they are not fully applicable to the propagation of cyber rumors. Since then, in order to obtain a more precise description of the dynamic process and individual characteristics of rumor spread, a number of researchers have started to consider the use of network topology to improve the traditional epidemic model Susceptible-Infected-Removal (SIR), and have contributed significantly by using it to analyze the spread of cyber rumors. Among them, the influence of human behaviour on the rumor spreading process has received extensive attention and research from scholars. Examples include the forgetting mechanism [5–7], hesitation mechanism [8,9], and leader's opinion [10,11], etc. Some studies [12] have also found that the pattern of human activities can have an impact on the spread of information, making it slower.

In recent years, influenced by COVID-19, there has been a renewed interest in infectious disease models and rumor propagation models, both domestically and internationally, and many scholars have extended the models for more in-depth study. The authors of [13] developed a model of information propagation with simultaneous censorship, sharing, collection, and suppression mechanisms. The authors of [14] provide an evolutionary game model that considers how individual choices affect the spread and management of rumors. The authors of [15] contend that the spread of rumors is influenced by the state of scientific knowledge. The authors of [16] provide a model with an influence mechanism, taking into account the likelihood of propagation in rumor transmission is not fixed, but is influenced by the number of current propagations. The authors of [17] address the current gap in research on the propagation power of rumors based on a function of two types of characteristics of content: false rumors and true rumors.

Although the above articles make an important contribution to rumor propagation, there are still problems with these models. Firstly, most of the models are built using the traditional integer-order. However, the integer-order model does not show the "anomalous propagation" of the actual information spread, that is, the rumor spreads explosively fast in the early stages and slows down in the later stages. The fractional-order model is a good solution to this problem. Fractional-order calculus provides a tool to describe the genetic and memory impacts of different materials. The memorability and heritability of fractional-order differentiation is theoretically proven and widely used [18–21]. For the rumor propagation model, memorability is an important feature of immunity. When a rumor spreads, people "remember" the clarifying information once they have received it in response to a successful immune response to the rumor, and this memorability plays an important role in stopping the spread of the rumor. Furthermore, the determination of future states throughout the spread of rumors depends on the historical process of propagation. The way in which a person reacts to external influences also depends on the experience he has accumulated in the past, and fractional-order differentiation is a very natural tool for modelling heritability, and is also considered to be the best method for modelling transmission [22]. Secondly, there are fewer existing rumor models that provide a mathematical analysis of their global asymptotic stable(GAS). Therefore, this study and gives local asymptotic stable(LAS) and GAS conditions for the rumor-free equilibrium point (RFEP) E_0 and the boundary equilibrium points (BEPs) E_1 , E_2 of the fractional-order model. Furthermore, most research make the assumption that rumors spread within a closed system and do not account for the inflow and outflow of the population. In fact, social networks are open platforms that need to take into account the impact of individual flows during the propagation process. Using the analysis just mentioned, this paper investigates a fractional-order Susceptible-Infected-Removal-Clarify (SIR-C) cyber rumor propagation prediction model with a clarification mechanism.

To sum up, the main contributions of this research are fourfold:

- 1. We propose fractional-order differentiation to solve the problem of "anomalous propagation" of cyber rumors;
- 2. We propose a mechanism containing clarification considering the influence of clarifiers in the spread of cyber rumors;
- 3. We propose LAS and GAS conditions for the RFEP (E_0) and the BEPs (E_1 , E_2) of the model;
- 4. We consider the population inflow and outflow of rumor propagation in social networks.

The remainder of the paper is structured as follows. In Section 2, we describe a fractional-order SIR-C cyber rumor propagation prediction model featuring the clarification mechanism, and introduce the mathematical properties of the corresponding fractional-order differential equations. In Section 3, the RFEP (E_0) and the BEPs (E_1 , E_2) are obtained and their stability is discussed. In Section 4, numerical simulations are used to verify the theoretical study and the usefulness of the model is verified by the actual event of "Record low average maths score in 2022 National College Entrance Examnation(NEMT)". A brief conclusion of the paper is given in Section 5.

2. Formulation of Fractional-Order SIR-C Model with Basic Mathematical Properties

In this section, the model's fractional-order differential equations, the structure, and the basic mathematical properties are systematically introduced.

2.1. Conformable Fractional Derivative (CFD)

In the last few decades, fractional-order differential models have been popular in several domains such as engineering and science due to their unique memory effects and genetic properties [23], no longer applied only in pure mathematics. Typical definitions of fractional-order derivatives are the Riemann-Liouville (RL) [24], the Caputo [25] and the CFD [26]. The deficiency of RL and Caputo is that they lose the basic properties such as multiplication and chaining that general derivatives have [27]. In this paper, CFD is used, which satisfies the product, quotient and chain rules for non-linear derivatives and will make the model calculation easier. And it can be converted directly to an equation of integer-order, which facilitates subsequent model comparisons, which facilitates subsequent model comparisons [28]. The authors of [29,30] demonstrate the computational simplicity of CFD compared to other fractional order differentiations. CFD is a new well-behaved and simple definition that has the ability to successfully overcome some of the drawbacks of the traditional definition. So this paper takes advantage of CFD in applied problems.CFD fractional-order derivatives are defined as follows.

Definition 1. For all t > 0, $\alpha \in (0,1]$. Given a function $f : [0,\infty) \to R$. Then the CFD of f of order α is defined by

$$D^{\alpha}f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$
(1)

if f is α *-differentiable in some* $(0, \alpha)$ *, and* $\lim_{t\to 0^+} f^{\alpha}(t)$ *exist, then define*

$$f^{\alpha}(0) = \lim_{t \to 0^+} f^{\alpha}(t).$$
⁽²⁾

Sometimes, we will write $f^{\alpha}(t)$ for $D^{\alpha}f(t)$, to denote the CFD of f of order α .

We should be aware that $D^{\alpha}(t^{p}) = pt^{p-\alpha}$, Further, the definition of CFD is consistent with the classical definition of RL and Caputo on polynomials and has the multiplication and chain laws.

Theorem 1. [26] Let $\alpha \in (0, 1]$, and f be α -differentiable at a point t > 0. If, in addition, f is differentiable. Then,

$$D^{\alpha}f(t) = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}(t),\tag{3}$$

2.2. Formulation of the Model

It is well known that persuasive rumor-clearing messages have a significant effect in curbing the spread of rumors [14,31]. Moreover, social networking is an open-ended interactive platform, with users registering or logging out at any time, and a large number of users are attracted by rumors and flock to social networks to discuss and spread the topic. The rumor propagation model takes into account the inflow and outflow of people on the rumor topic, and combines the memory effect and genetic properties of fractional-order calculus to illustrate the "anomalous propagation" of rumors. Finally, a fractional-order cyber rumor propagation prediction model with a clarification mechanism is formulated. The model is based on the following assumptions:

- In social networking sites, we assume that there is a fixed rate of population inflow and outflow, and the quantity of individuals entering each time unit remains constant, i.e., the inflow rate is Λ and the outflow rate is *d*;
- The total number of transmission subjects N changes over time and is divided into three divisions: *S* (susceptible), which refers to nodes that not received information about the rumor; *I* (infected), which refers to nodes that receive the rumor and propagate it; *C* (clarify), which refers to people who learn the truth and clear up the rumor; and *R* (removal), which refers to nodes that immune to the cyber rumor and not propagate it;
- Suppose that susceptible persons are converted to infected persons (transmitters, clarifiers) by a proportionality factor (spread rate) β as a result of being affected by the spread of the rumor. The setting for the rumor transmission rate is β_1 and the rumor clarification rate is β_2 . The conversion rate of spreaders to clarifiers is set to μ ;
- Infected persons (spreaders, clarifiers) are converted to immune persons by a scaling factor (removal rate) *γ*. The removal rate for spreaders is set to *γ*₁ and the removal rate for clarifiers is set to *γ*₂;
- All parameters are constants with non-negative values between 0 and 1.

Based on the above assumptions, we can obtain the basic scheme of the fractional order SIR-C model, as seen in Figure 1.



Figure 1. The basic scheme of the fractional-order SIR-C model.

The specific details of the S, I, C and R nodes and the transformation relationships of the four nodes are described below:

- S-nodes refer to those that have not received the cyber rumor and are not yet affected by it. Some S-nodes may receive rumor information and be influenced by it, listen to the rumor and transform into I-nodes with spread rate β₁ to spread cyber rumor. Some S-nodes may receive rumor clarification information, learn the truth, and transform into C-nodes at spread rate β₂ to spread the truth about the facts;
- I-nodes refer to those that receive cyber rumors, listen to them and spread them. Some I-nodes are likely to learn the truth and transform to C-nodes with conversion rate

 μ , clarify the rumor and spread it. Some I-nodes may lose interest in the rumor topic as time passes and are transformed to an immune and indifferent state at a removal rate γ_1 ;

- C-nodes refer to those that know the truth and clarify the rumor. When the C-nodes have finished clarifying the rumor, they will be immune to the rumor at a removal rate γ₂ and will no longer pay attention to it;
- R-nodes refer to those that have received the information, lost interest in the rumor topic and will not have any further influence on the rumor topic.

Based on the above analysis of the model node transition rules, the differential equation for the fractional-order SIR-C model can be derived as Equation (4).

where, S(t), I(t), C(t) and R(t) are denoted as the quantities of S, I, R and C at moment t respectively. S(0), I(0), C(0), $R(0) \ge 0$. Λ , d, β_1 , β_2 , μ , γ_1 , $\gamma_2 \in (0, 1)$.

The number of individuals across all social networks is a in social networks is N(t), i.e.,

$$N(t) = S(t) + I(t) + C(t) + R(t),$$
(5)

when, Equation (4) is substituted into Equation (5), we get

$$D^{\alpha}N(t) = \Lambda - dN(t), \tag{6}$$

Thus $N(t) = \frac{\Lambda}{d} + \left(N(0) - \frac{\Lambda}{d}\right)e^{-dt}$. Hence, we have $\lim_{t\to 0^+} N(t) = \frac{\Lambda}{d}$. The positive variable set of the system (4) is $\Phi = \left\{(S, I, C, R) \in R_4^+ : S + I + C + R \leq \frac{\Lambda}{d}\right\}$.

Obviously, R(t) does not affect the first three equations in system (4), and the R state node appears only in the fourth equation, which is determined by *S*, *I* and *C*. The dimensionality of system (4) can be reduced to eliminate the *R* state node in the differential equation. The simplified model is as follows.

$$\begin{cases} D^{\alpha}S(t) = \Lambda - \beta_{1}S(t)I(t) - \beta_{2}S(t)C(t) - dS(t), \\ D^{\alpha}I(t) = \beta_{1}S(t)I(t) - \mu I(t)C(t) - (\gamma_{1} + d)I(t), \\ D^{\alpha}C(t) = \beta_{2}S(t)C(t) + \mu I(t)C(t) - (\gamma_{2} + d)C(t), \end{cases}$$
(7)

Thus, the feasible region of the system (7) can be described as a closed positively invariant set, denoted by $\Omega = \left\{ (S, I, C) \in R_3^+ : S + I + C \leq \frac{\Lambda}{d} \right\}$. A subsequent stability analysis of the fractional order SIR-C model will use system (7).

3. System Equilibrium Points and Stability Analysis

In this part, equilibrium points and basic reproduction number(BRN) of fractionalorder cyber rumor propagation models are computed, as well as the stability of cyber rumor propagation equilibrium states is discussed using Lyapunov's stability theorem and the Routh-Hurwitz criterion. This section focuses on the local and global stability of three equilibrium points, the RFEP (E_0) and the BEPs (E_1 , E_2).

3.1. $RFEP(E_0)$

System (7) has a RFEP $E_0 = (S_0, I_0, C_0)$, where there are neither rumor spreaders nor rumor clarifiers. Let System (7)'s rightmost component be zero. The new equation is expressed as follows:

$$\begin{cases} \Lambda - \beta_1 S(t)I(t) - \beta_2 S(t)C(t) - dS(t) = 0, \\ \beta_1 S(t)I(t) - \mu I(t)C(t) - (\gamma_1 + d)I(t) = 0, \\ \beta_2 S(t)C(t) + \mu I(t)C(t) - (\gamma_2 + d)C(t) = 0, \end{cases}$$
(8)

According to the definition of the RFEP, under the condition $I_0 = 0$, $C_0 = 0$. We obtain the unique RFEP.

$$E_0 = (S_0, I_0, C_0) = (\frac{\Lambda}{d}, 0, 0).$$

Next, using the concept of the BRN R_0 [32] from the classical infectious disease model as an important parameter for determining whether a rumor can be quelled. R_0 is defined as the quantity of people a cyber rumor spreader can convert susceptible people into cyber rumor spreaders in the process of spreading. Subsequent analyses of stability were carried out based on R_0 .

The next generation matrix method [33] is applied to solve for R_0 , which is referred to as the next-generation matrix's spectral radius.

Let $X(t) = (S(t), I(t), C(t))^{T}$, system (7) is rewriteable as

$$D^{\alpha}X(t) = \mathcal{F}(x) + \Psi(x), \qquad (9)$$

where,

$$\mathcal{F}(x) = \begin{bmatrix} 0 \\ \beta_1 S(t) I(t) \\ \beta_2 S(t) C(t) + \mu I(t) C(t) \end{bmatrix}, \Psi(x) = \begin{bmatrix} \beta_1 S(t) I(t) + \beta_2 S(t) C(t) + dS(t) \\ \mu I(t) C(t) + (\gamma_1 + d) I(t) \\ (\gamma_2 + d) C(t) \end{bmatrix}.$$

Further, calculate the Jacobi matrix of $\mathcal{F}(x)$, $\Psi(x)$ at $E_0 = (\frac{\Lambda}{d}, 0, 0)$, repectively, we have

$$J(\mathcal{F}|E_0) = \begin{bmatrix} 0 & 0 & 0\\ 0 & \frac{\beta_1 \Lambda}{d} & 0\\ 0 & 0 & \frac{\beta_2 \Lambda}{d} \end{bmatrix} = \begin{bmatrix} 0 & 0\\ 0 & F \end{bmatrix}, J(\Psi|E_0) = \begin{bmatrix} d & \frac{\beta_1 \Lambda}{d} & \frac{\beta_2 \Lambda}{d}\\ 0 & \gamma_1 + d & 0\\ 0 & 0 & \gamma_2 + d \end{bmatrix} = \begin{bmatrix} A & A_1\\ 0 & V \end{bmatrix}.$$

where,

$$F_0 = \begin{bmatrix} \frac{\beta_1 \Lambda}{d} & 0\\ 0 & \frac{\beta_2 \Lambda}{d} \end{bmatrix}, V_0 = \begin{bmatrix} \gamma_1 + d & 0\\ 0 & \gamma_2 + d \end{bmatrix}.$$

Therefore, the next generation FV^{-1} can be calculated as follows:

$$(F_0 V_0)^{-1} = \begin{bmatrix} \frac{\beta_1 \Lambda}{d(\gamma_1 + d)} & 0\\ 0 & \frac{\beta_2 \Lambda}{d(\gamma_2 + d)} \end{bmatrix}$$

The R_0 of system (7), which is the spectral radius of the matrix FV^{-1} , we obtain

$$R_0 = \rho\left((F_0 V_0)^{-1}\right) = \max\left\{\frac{\beta_1 \Lambda}{d(\gamma_1 + d)}, \frac{\beta_2 \Lambda}{d(\gamma_2 + d)}\right\}.$$
(10)

where,

$$R_{01} = \frac{\beta_1 \Lambda}{d(\gamma_1 + d)}, R_{02} = \frac{\beta_2 \Lambda}{d(\gamma_2 + d)}$$

Theorem 2. For system (7), if $R_{01} < 1$ and $R_{02} < 1$. Then, the RFEP E_0 is LAS.

Proof. The Jacobi matrix of system (7) at $E_0 = (\frac{\Lambda}{d}, 0, 0)$ as follows.

$$J(E_0) = \begin{bmatrix} -d & -\frac{\beta_1\Lambda}{d} & -\frac{\beta_2\Lambda}{d} \\ 0 & \frac{\beta_1\Lambda}{d} - (\gamma_1 + d) & 0 \\ 0 & 0 & \frac{\beta_2\Lambda}{d} - (\gamma_2 + d) \end{bmatrix}$$

Considering the upper triangular matrix's characteristics, the eigenvalues of the matrix $J(E_0)$ can be obtained.

$$\lambda_{1} = -d,$$

$$\lambda_{2} = \frac{\beta_{1}\Lambda}{d} - (\gamma_{1} + d) = (\gamma_{1} + d)(R_{01} - 1),$$

$$\lambda_{3} = \frac{\beta_{2}\Lambda}{d} + (\gamma_{2} + d) = (\gamma_{2} + d)(R_{02} - 1).$$

If $R_{01} < 1$ and $R_{02} < 1$ (i.e., $R_0 < 1$), then $\lambda_1, \lambda_2, \lambda_3 < 0$, and they all have negative real parts. According to the Lyapunov stability theorem [34], the RFEP E_0 is LAS. Conversely, if $R_0 > 1$, then only at least one of λ_2 and λ_3 is positive, then E_0 is unstable. \Box

Theorem 3. For system (7), if $R_{01} < 1$ and $R_{02} < 1$. Then, the RFEP E_0 is GAS.

Proof. Let $X(t) = (S(t), I(t), C(t))^T$, According to the Lyapunov stability theorem, the Lyapunov function V(t) is constructed.

$$V(t) = \left(S(t) - S_0 \ln \frac{S(t)}{S_0}\right) + I(t) + C(t).$$

Next, by taking CFD in time of the function V(t), using the basic arithmetic properties of CFD and the define of BRN, we immediately obtain.

$$\begin{split} D^{\alpha}V(t) &= \left(1 - \frac{S_0}{S(t)}\right) D^{\alpha}S(t) + D^{\alpha}I(t) + D^{\alpha}C(t), \\ &= 2\Lambda - \left(dS(t) + \frac{\Lambda^2}{dS(t)}\right) + \left(\frac{\Lambda\beta_1}{d} - (\gamma_1 + d)\right)I(t) + \left(\frac{\Lambda\beta_2}{d} - (\gamma_2 + d)\right)C(t), \\ &= \Lambda\left(2 - \frac{dS(t)}{\Lambda} - \frac{\Lambda}{dS(t)}\right) + (\gamma_1 + d)(R_{01} - 1) + (\gamma_2 + d)(R_{02} - 1). \end{split}$$

where,

$$dS(t) > 0, \Lambda > 0.$$

from the basic inequality we know that.

$$\left(2 - \frac{dS(t)}{\Lambda} - \frac{\Lambda}{dS(t)}\right) \leq 0.$$

The formula takes the equal sign when and only when $dS(t) = \Lambda$, i.e., $S(t) = S_0 = \frac{\Lambda}{d}$. Therefore, we directly get that

$$D^{\alpha}V(t) \leq (\gamma_1 + d)(R_{01} - 1) + (\gamma_2 + d)(R_{02} - 1).$$

If $R_{01} < 1$ and $R_{02} < 1$, then, for all $X(t) \in \Omega$, when $X(t) \neq E_0$, we have $D^{\alpha}V(t) < 0$. According to the LaSalle invariant principle [35], E_0 is GAS. \Box

3.2. BEP E₁

System (7) has a BEP $E_1 = (S_1, I_1, C_1)$, where there are no rumor spreaders, but exist rumor clarifiers.

According to the definition of E_1 , under the condition $I_0 = 0$, brought into Equation (8). We obtain E_1 .

$$E_1 = (S_1, I_1, C_1) = \left(\frac{\gamma_2 + d}{\beta_2}, 0, \frac{\beta_2 - d(\gamma_2 + d)}{\beta_2(\gamma_2 + d)}\right) = \left(\frac{S_0}{R_{02}}, 0, \frac{d(R_{02} - 1)}{\beta_2}\right).$$
(11)

Futher, calculate the Jacobi matrix of $\mathcal{F}(x)$, $\Psi(x)$ at E_1 , repectively, we have

$$J(\mathcal{F}|E_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta_1 S_1 & 0 \\ \beta_2 C_1 & \mu C_1 & \beta_2 S_1 \end{bmatrix}, J(\Psi|E_1) = \begin{bmatrix} \beta_2 C_1 + d & \beta_1 S_1 & \beta_2 S_1 \\ 0 & \mu C_1 + (\gamma_1 + d) & 0 \\ 0 & 0 & \gamma_2 + d \end{bmatrix}.$$

where,

$$F_1 = [\beta_1 S_1], V_1 = [\mu C_1 + (\gamma_1 + d)].$$

The BRN R_1 of system (7), which is the spectral radius of the matrix $(F_1V_1)^{-1}$, we obtain

$$R_1 = \rho\left((F_1 V_1)^{-1}\right) = \frac{\beta_1 S_1}{\mu C_1 + (\gamma_1 + d)}.$$
(12)

Theorem 4. For system (7), if $R_1 < 1$. Then, the BEP E_1 is LAS.

Proof. The Jacobi matrix of system (7) at $E_1 = (S_1, I_1, C_1) = \left(\frac{S_0}{R_{02}}, 0, \frac{d(R_{02}-1)}{\beta_2}\right)$ as follows.

$$J(E_1) = \begin{bmatrix} -\beta_2 C_1 - d & -\beta_1 S_1 & -\beta_2 S_1 \\ 0 & \beta_1 S_1 - \mu C_1 - (\gamma_1 + d) & 0 \\ \beta_2 C_1 & \mu C_1 & \beta_2 S_1 - (\gamma_2 + d) \end{bmatrix}$$

The characteristic equation of the matrix $J(E_1)$ can be written as

$$(\lambda - \beta_1 S_1 + \mu C_1 + (\gamma_1 + d))(\lambda + \beta_2 C_1 + d)(\lambda - \beta_2 S_1 - (\gamma_2 + d)) + \beta_2 S_1 \beta_2 C_1 = 0,$$

through simplification, we obtain

$$(\lambda - \beta_1 S_1 + \mu C_1 + (\gamma_1 + d)) \left(\lambda^2 + (m_a + m_b) \lambda + m_c m_d \right) = 0,$$
(13)

where,

$$\begin{split} m_{a} &= \beta_{2}C_{1} + d, \\ m_{b} &= -\beta_{2}S_{1} + (\gamma_{2} + d), \\ m_{c} &= \beta_{2}S_{1}, \\ m_{d} &= \beta_{2}C_{1}. \end{split}$$

from the Equation (13), we can easily obtain the eigenvalues λ_1 .

$$\lambda_1 = \beta_1 S_1 - \mu C_1 - (\gamma_1 + d).$$

If $R_1 < 1$, i.e., $\left(\frac{\beta_1 S_1}{\mu C_1 + (\gamma_1 + d)} < 1\right)$, the eigenvalues $\lambda_1 < 0$. The remaining two eigenvalues λ_2 ans λ_3 satisfy the following equation.

$$\left(\lambda^2 + (m_a + m_b)\lambda + m_c m_d\right) = 0, \tag{14}$$

Taking $E_1 = (S_1, 0, C_1)$ into the third line of the formula in Equation (8) we get

$$\beta_2 S_1 - (\gamma_2 + d) = 0.$$

thus,

$$m_a + m_b = \beta_2 C_1 + d - \beta_2 S_1 + (\gamma_2 + d) = \beta_2 C_1 + d > 0$$

$$m_c m_d = (\beta_2)^2 S_1 C_1 > 0,$$

According to the Routh-Hurwiz criterion [36], two eigenvalues of Equation (14) with $\lambda_2, \lambda_3 < 0$ can be easily obtained. If $R_1 < 1$, then $\lambda_1, \lambda_2, \lambda_3 < 0$, according to the Lyapunov stability theorem, the BEP E_1 is LAS. Conversely, if $R_1 > 1$. Then, $\lambda_1 > 0$ and the E_1 is unstable. \Box

Theorem 5. For system (7), if $R_{02} > 1$ and $R_1 < 1$. Then, the BEP E_1 exists and is GAS.

Proof. Let $X(t) = (S(t), I(t), C(t))^{T}$, According to the Lyapunov stability theorem, the Lyapunov function $V_1(t)$ is constructed.

$$V_1(t) = \left(S(t) - S_1 \ln \frac{S(t)}{S_1}\right) + I(t) + \left(C(t) - C_1 \ln \frac{C(t)}{C_1}\right).$$

Next, by taking CFD in time of the function $V_1(t)$, using the basic arithmetic properties of CFD and the define of BRN, we immediately obtain.

$$\begin{split} D^{\alpha}V_{1}(t) &= \left(1 - \frac{S_{1}}{S(t)}\right)D^{\alpha}S(t) + D^{\alpha}I(t) + \left(1 - \frac{C_{1}}{C(t)}\right)D^{\alpha}C(t), \\ &= \Lambda\left(1 - \frac{S_{1}}{S(t)}\right) + I(t)(\beta_{1}S_{1} - (\gamma_{1} + d) - \mu C_{1}) + dS_{1} - dS(t) - \beta_{2}C_{1}S(t) + C_{1}(\gamma_{2} + d), \\ &= \beta_{2}S_{1}C_{1}\left(2 - \frac{S_{1}}{S(t)} - \frac{S(t)}{S_{1}}\right) + dS_{1}\left(2 - \frac{S_{1}}{S(t)} - \frac{S(t)}{S_{1}}\right) + I(t)(\beta_{1}S_{1} - (\gamma_{1} + d) - \mu C_{1}), \\ &= (\beta_{2}S_{1}C_{1} + dS_{1})\left(2 - \frac{S_{1}}{S(t)} - \frac{S(t)}{S_{1}}\right) + I(t)(\beta_{1}S_{1} - (\gamma_{1} + d) - \mu C_{1}), \end{split}$$

where,

$$S(t) > 0, S_1 > 0, (\beta_2 S_1 C_1 + dS_1) > 0.$$

from the basic inequality we know that.

$$\left(2-\frac{S_1}{dS(t)}-\frac{dS(t)}{S_1}\right)\leq 0.$$

The formula takes the equal sign when and only when $S_1 = S(t)$. Therefore, we directly get that

> $D^{\alpha}V_{1}(t) \leq I(t)(\beta_{1}S_{1} - (\gamma_{1} + d) - \mu C_{1}),$ $\leq I(t)(\mu C_{1} + (\gamma_{1} + d))(R_{1} - 1).$

where,

$$I(t) > 0, (\mu C_1 + (\gamma_1 + d)) > 0$$

If $R_1 < 1$, for all $X(t) \in \Omega$, when $X(t) \neq E_1$, we have $D^{\alpha}V_1(t) < 0$. According to the LaSalle invariant principle, E_1 is GAS. From Equation (11), the E_1 exists if $\frac{d(R_{02}-1)}{\beta_2} > 0$, i.e., $R_{02} > 1$.

Overall, if $R_{02} > 1$ and $R_1 < 1$. Then, the BEP E_1 exists and is GAS. \Box

3.3. BEP E₂

System (7) has a BEP $E_2 = (S_2, I_2, C_2)$, where there are no rumor clarifiers, but exist rumor spreaders.

According to the definition of E_2 , under the condition $C_0 = 0$, brought into Equation (8). We obtain the E_2 .

$$E_2 = (S_2, I_2, C_2) = \left(\frac{\gamma_1 + d}{\beta_1}, \frac{\Lambda - d(\gamma_1 + d)}{\beta_1(\gamma_1 + d)}, 0\right) = \left(\frac{S_0}{R_{01}}, \frac{d(R_{01} - 1)}{\beta_1}, 0\right).$$
(15)

Futher, calculate the Jacobi matrix of $\mathcal{F}(x)$, $\Psi(x)$ at E_2 , repectively, we have

$$J(\mathcal{F}|E_2) = \begin{bmatrix} 0 & 0 & 0\\ \beta_1 I_2 & \beta_1 S_2 & 0\\ 0 & 0 & \beta_2 S_2 + \mu I_2 \end{bmatrix}, J(\Psi|E_2) = \begin{bmatrix} \beta_1 I_2 + d & \beta_1 S_2 & \beta_2 I_2\\ 0 & \gamma_1 + d & \mu I_2\\ 0 & 0 & \gamma_2 + d \end{bmatrix}.$$

where,

$$F_2 = [\beta_2 S_2 + \mu I_2], V_2 = [\gamma_2 + d].$$

The BRN R_2 of system (7), which is the spectral radius of the matrix $(F_2V_2)^{-1}$, we obtain

$$R_2 = \rho\left((F_2 V_2)^{-1}\right) = \frac{\beta_2 S_2 + \mu I_2}{(\gamma_2 + d)}.$$
(16)

Theorem 6. For system (7), if $R_2 < 1$. Then, the BEP E_2 is LAS.

Proof. The Jacobi matrix of system (7) at $E_2 = (S_2, I_2, C_2) = \left(\frac{S_0}{R_{01}}, \frac{d(R_{01}-1)}{\beta_1}, 0\right)$ as follows.

$$J(E_2) = \begin{bmatrix} -\beta_1 I_2 - \beta_2 C_2 - d & -\beta_1 S_2 & -\beta_2 S_2 \\ \beta_1 I_2 & \beta_1 S_2 - \mu C_2 - (\gamma_1 + d) & -\mu I_2 \\ \beta_2 C_2 & \mu C_2 & \beta_2 S_2 + \mu I_2 - (\gamma_2 + d) \end{bmatrix}.$$
$$= \begin{bmatrix} -\beta_1 I_2 - d & -\beta_1 S_2 & -\beta_2 S_2 \\ \beta_1 I_2 & \beta_1 S_2 - (\gamma_1 + d) & -\mu I_2 \\ 0 & 0 & \beta_2 S_2 + \mu I_2 - (\gamma_2 + d) \end{bmatrix}$$

The characteristic equation of the matrix $J(E_2)$ can be obtained as

$$(\lambda - \beta_2 S_2 + \mu I_2 + (\gamma_2 + d))(\lambda + \beta_1 I_2 + d)(\lambda - \beta_1 S_2 - (\gamma_1 + d)) + \beta_1 S_2 \beta_1 I_2 = 0,$$

Through simplification, we obtain

$$(\lambda - \beta_2 S_2 - \mu I_2 + (\gamma_2 + d)) \left(\lambda^2 + (n_a + n_b)\lambda + n_c n_d\right) = 0,$$
(17)

where,

$$n_{a} = \beta_{1}I_{2} + d,$$

$$n_{b} = -\beta_{1}S_{2} + (\gamma_{1} + d),$$

$$n_{c} = \beta_{1}S_{2},$$

$$n_{d} = \beta_{1}I_{2}.$$

from the Equation (17), we can easily obtain the eigenvalues λ_1 .

$$\lambda_1 = \beta_2 S_2 + \mu I_2 - (\gamma_2 + d).$$
 If $R_2 < 1$, i.e., $\left(\frac{\beta_2 S_2 + \mu I_2}{(\gamma_2 + d)} < 1\right)$, the eigenvalues $\lambda_1 < 0$.

The remaining two eigenvalues λ_2 ans λ_3 satisfy the following equation.

$$\left(\lambda^2 + (n_a + n_b)\lambda + n_c n_d\right) = 0, \tag{18}$$

Taking $E_2 = (S_2, I_2, 0)$ into the second line of the formula in Equation (8) we get

$$\beta_1 S_2 - (\gamma_1 + d) = 0$$

Thus,

$$n_a + n_b = \beta_1 I_2 + d - \beta_1 S_2 + (\gamma_1 + d) = \beta_1 I_2 + d > 0,$$

 $n_c n_d = (\beta_1)^2 S_2 I_2 > 0,$

According to the Routh-Hurwiz criterion, two eigenvalues of Equation (18) with $\lambda_2, \lambda_3 < 0$ can be easily obtained. If $R_2 < 1$, then $\lambda_1, \lambda_2, \lambda_3 < 0$, according to the Lyapunov stability theorem, the BEP E_2 is LAS. Conversely, if $R_2 > 1$. Then, $\lambda_1 > 0$ the E_2 is unstable. \Box

Theorem 7. For system (7), if $R_{01} > 1$ and $R_2 < 1$. Then, the BEP E_2 exists and is GAS.

Proof. Let $X(t) = (S(t), I(t), C(t))^{T}$, According to the Lyapunov stability theorem, the Lyapunov function $V_2(t)$ is constructed.

$$V_2(t) = \left(S(t) - S_2 \ln \frac{S(t)}{S_2}\right) + \left(I(t) - I_2 \ln \frac{I(t)}{I_2}\right) + C(t).$$

Next, by taking CFD in time of the function $V_2(t)$, using the basic arithmetic properties of CFD and the define of BRN, we immediately obtain.

$$\begin{split} D^{\alpha}V_{2}(t) &= \left(1 - \frac{S_{2}}{S(t)}\right)D^{\alpha}S(t) + \left(1 - \frac{I_{2}}{I(t)}\right)D^{\alpha}I(t) + D^{\alpha}C(t),\\ &= \Lambda\left(1 - \frac{S_{2}}{S(t)}\right) + C(t)(\beta_{2}S_{2} - (\gamma_{2} + d) + \mu I_{2}) + dS_{2} - dS(t) - \beta_{1}I_{2}S(t) + I_{2}(\gamma_{1} + d),\\ &= \beta_{1}S_{2}I_{2}\left(2 - \frac{S_{2}}{S(t)} - \frac{S(t)}{S_{2}}\right) + dS_{2}\left(2 - \frac{S_{2}}{S(t)} - \frac{S(t)}{S_{2}}\right) + C(t)(\beta_{2}S_{2} - (\gamma_{2} + d) + \mu I_{2}),\\ &= (\beta_{1}S_{2}I_{2} + dS_{2})\left(2 - \frac{S_{2}}{S(t)} - \frac{S(t)}{S_{2}}\right) + C(t)(\beta_{2}S_{2} - (\gamma_{2} + d) + \mu I_{2}), \end{split}$$

where,

$$S(t) > 0, S_2 > 0, (\beta_1 S_2 I_2 + dS_2) > 0.$$

from the basic inequality we know that.

$$\left(2-\frac{S_2}{dS(t)}-\frac{dS(t)}{S_2}\right)\leq 0.$$

The formula takes the equal sign when and only when $S_2 = S(t)$. Therefore, we directly get that

$$D^{\alpha}V_{2}(t) \leq C(t)(\beta_{2}S_{2} - (\gamma_{2} + d) + \mu I_{2}),$$

$$\leq C(t)(\gamma_{2} + d)(R_{2} - 1).$$

where,

$$C(t) > 0, (\gamma_2 + d) > 0.$$

If $R_2 < 1$, for all $X(t) \in \Omega$, when $X(t) \neq E_2$, we have $D^{\alpha}V_2(t) < 0$. According to the LaSalle invariant principle, the BEP E_2 is GAS. From Equation (15), the E_2 exists if $\frac{d(R_{01}-1)}{\beta_1} > 0$, i.e., $R_{01} > 1$.

Overall, if $R_{01} > 1$ and $R_2 < 1$. Then, the BEP E_2 exists and is GAS. \Box

4. Numerical Simulations and Discussions

In this part, Matlab is used as the simulation platform to numerically simulate the System (7) to verify the results of the RFEP (E_0) and the BEPs (E_1 , E_2) are in agreement with Theorems 3, 5 and 7. The prediction results of the fractional-order SIR-C model and the traditional SIR model are compared to validate the feasibility of the model using the actual cyber rumor "Record low average maths score in 2022 NEMT". The descriptions of the key parameters mentioned in this section are specified in Table 1.

Table 1. Description of the parameters used in numerical siumlations.

| Parameter | Description |
|------------|--|
| Λ | Unit time inflow rate of users |
| β_1 | Unit time rumor spread rate |
| β_2 | Unit time rumor clarification rate |
| d | Unit time outflow rate of users |
| μ | Unit time conversion rate of spreaders to clarifiers |
| γ_1 | Unit time removal rate for spreaders |
| γ_2 | Unit time removal rate for clarifiers |
| α | Differential order |

4.1. Stability Simulation and Analysis of RFEP E₀

Using the data in Table 2, the global asymptotic stability of E_0 was simulated. where the initial value of the propagation density for each node in the system is set as: S(0) = 0.7, I(0) = 0.2, C(0) = 0.1, R(0) = 0. The different differential orders $\alpha \in \{0.7, 1, 1.2\}$ were set to verify the effect of different α on the fractional-order SIR-C rumor propagation model. The simulation results are shown in Figure 2.

Table 2. Parameter for the global asymptotic stability of E_0 .

| Λ | β_1 | β_2 | d | μ | γ_1 | γ_2 |
|-----|-----------|-----------|-----|-----|------------|------------|
| 0.2 | 0.35 | 0.1 | 0.2 | 0.1 | 0.01 | 0.2 |



Figure 2. (a) Global stability of the rumor-free equilibrium point (RFEP) $E_0 = (1, 0, 0)$, for system (7). (b) Consider the effect of different differential orders of $\alpha \in \{0.7, 1, 1.2\}$ on the spread of cyber rumors, for system (7).

According to the data in Table 2, it can be obtained that , $R_{01} = 0.833 < 1$, and $R_{02} = 0.667 < 1$. From Figure 2a, the system (7) will gradually converge to the $E_0 = (1, 0, 0)$ as time goes on, i.e., E_0 is GAS. This is in agreement with Theorem 3.

Meanwhile, we observe that over time, *S* tends to 1 and *I* and *C* tend to 0. This means that the density of nodes spreading rumors will eventually shift to zero and cease to fluctuate, and the rumors will eventually be eliminated. That is, system (7) reaches a steady state where there are no rumor spreaders and clarifiers, only susceptible people. At the same time, it can be found that the fewer rumor spreaders there are, the fewer rumor clarifiers there will be, which is consistent with the actual rumor propagation process in social networking sites. The authors of [37] have analyzed the actual rumor spreading process of "Does KFC sell rat?" in social networking sites. The analysis of the rumor messages and counter-rumor messages volumes obtained according to the conclusions of [37] is in agreement with the results of the theoretical variation of rumor spreaders and rumor clarifiers in this paper. The veracity of the results of this numerical simulation is further verified.

With Figure 2b, we observe that the fractional-order model is equivalent to that of the traditional integer-order system when the value of α is taken as 1. As the value of α decreases, the curve of rumor propagation converges more slowly. From the perspective of public opinion propagation, its interpretation suggests that such rumors take longer to eradicate.

4.2. Stability Simulation and Analysis of BEP E_1

Using the data in Table 3, the global asymptotic stability of E_1 was simulated. where the initial value of the propagation density for each node in the system is set as: S(0) = 0.8, I(0) = 0.1, C(0) = 0.1, R(0) = 0. The different differential orders $\alpha \in \{0.6, 0.8, 1.1\}$ were set to verify the effect of different α on the fractional-order SIR-C rumor propagation model. The simulation results are shown in Figure 3.

Table 3. Parameter for the global asymptotic stability of E_1 .



Figure 3. (a) Global stability of the boundary equilibrium point (BEP) $E_1 = (0.525, 0, 0.45)$, for system (7). (b) Consider the effect of different differential orders of $\alpha \in \{0.6, 0.8, 1.1\}$ on the spread of cyber rumors, for system (7).

According to the data in Table 3, it can be obtained that , $R_{02} = 1.90 > 1$, and $R_1 = 0.152 < 1$. From Figure 3a, the system (7) will gradually converge to the $E_1 = (0.525, 0, 0.45)$ as time goes on, i.e., E_1 is GAS. This is in agreement with Theorem 5.

Meanwhile, we observe that over time, *S* and *C* converge to a constant and *I* converges to 0. This means that the density of nodes spreading rumors will eventually shift to 0

and cease to fluctuate, and the rumors will eventually be eliminated. This means that system (7) reaches a steady state where there are no rumor spreaders, only susceptibles and clarifiers. It can also be found that the more rumor clarifiers there will be, the fewer rumor spreaders there will be, which is consistent with the actual rumor propagation process in social networking sites.

Through Figure 3b, we observe that the curve of rumor propagation converges more slowly as the value of α decreases, a situation that may affect the longer duration of rumor propagation.

4.3. Stability Simulation and Analysis of BEP E₂

Table 4. Parameter for the global asymptotic stability of *E*₂.

Using the data in Table 4, the global asymptotic stability of E_2 was simulated. where the initial value of the propagation density for each node in the system is set as: S(0) = 0.8, I(0) = 0.1, C(0) = 0.1, R(0) = 0. The different differential orders $\alpha \in \{0.9, 1, 1.3\}$ were set to verify the effect of different α on the fractional-order SIR-C rumor propagation model. The simulation results are shown in Figure 4.





Figure 4. (a) Global stability of the BEP $E_2 = (0.6, 0.38, 0)$, for system (7). (b) Consider the effect of different differential orders of $\alpha \in \{0.9, 1, 1.3\}$ on the spread of cyber rumors, for system (7).

According to the data in Table 4, it can be obtained that , $R_{01} = 1.667 > 1$, and $R_2 = 0.687 < 1$. From Figure 4a, the system (7) will gradually converge to $E_2 = (0.6, 0.38, 0)$ as time goes on, i.e., E_2 is GAS. This is in agreement with Theorem 7.

Meanwhile, we observe that over time, *S* and *I* converge to a constant and *C* converges to 0. This means that the density of nodes clarifying rumors will eventually shift to 0 and cease to fluctuate and rumors are continuously spread. That is, the system (7) reaches a steady state where there are no rumor clarifiers, only susceptibles and spreaders. Under this parameter condition, the rumor-clarifying mechanism cannot suppress rumors in a timely and effective manner.

Through Figure 4b, we observe that the curve of rumor propagation converges more slowly as the value of α decreases, a situation that may affect the longer duration of rumor propagation.

4.4. Experimental Simulation of Actual Cyber Rumor Propagation

Based on the fractional-order SIR-C model, the experimental simulation of the propagation process of "Record low average maths score in 2022 NEMT", an actual Internet rumor, was carried out, and the experiments were compared with the traditional integer-order SIR model to verify the effectiveness of the model. The data for this experiment are the cyber rumors about "Record low average maths score in 2022 NEMT" forwarded on social network platforms from 08:00 on 22 June to 20:00 on 24 June 2022. Since this event involves keywords that are easily noticed by the society, such as education and examination, it has caused more controversies and concerns, so it was chosen as the validation experiment for this model.

Considering the obtained data, the initial value of the propagation density for each node in the system is assumed to be: S(0) = 0.98, I(0) = 0.01, C(0) = 0.01, R(0) = 0. Also, in order to obtain the best fit, the least squares method is used for data fitting in this paper. Fitting curve based on the actual data, the parameter values of each node can be calculated as shown in Table 5.

Table 5. Parameter for the SIR-C model.

| Λ | β_1 | β_2 | d | μ | γ1 | γ_2 |
|------|-----------|-----------|------|-----|-----|------------|
| 0.01 | 0.55 | 0.4 | 0.01 | 0.3 | 0.1 | 0.13 |

The actual propagation process of rumors in social network platforms is usually "anomalous propagation", but the integer-order model cannot predict "anomalous propagation". Therefore, the fractional-order SIR-C model is proposed to split the prediction time, i.e., different differential orders are used to represent the propagation speed of rumors in the early, middle and late stages of propagation, so that the prediction curve is closer to the real data curve and the experimental results are more realistic.

According to the characteristics of each stage of the cyber rumor spreading process, the dichotomous search method is used to set different a values in different time stages. When $t \in [0, 10]$, $\alpha = 1.2$ is set to satisfy the accelerated spread of rumors in the early stage of rumor spread; when $t \in (10, 20)$, $\alpha = 1$ is set to satisfy the middle stage of rumor spread, where the speed of rumor propagation is reduced due to the presence of rumor clarifiers. When $t \in (20, 60)$, $\alpha = 0.9$ is set to satisfy the slowdown of rumor spread in the late stage of rumor spread as time passes and people lose interest in the rumor. The simulation diagram of the fractional-order SIR-C model results is shown in Figure 5a.



Figure 5. (a) The simulation diagram of the fractional-order SIR-C model. (b) Consider the effect of different differential orders of $\alpha \in \{0.9, 1, 1.3\}$ on the spread of cyber rumors, for system (7).

In Figure 5b, for the traditional SIR model, rumor clarifiers do not exist in the whole rumor propagation process because there is no rumor clarification mechanism. However, in the model established in this paper, although there is a peak in the node density of cyber rumor propagation over time, it is significantly smaller than the peak in the traditional SIR model. Finally, with the action of rumor clarification mechanism and the passage of time, the rumor will be cleared eventually. We can find that the fractional-order SIR-C model proposed in this paper outperforms the existing SIR models. The model offers a more accurate approximation to actual data, and experimental results fit better with less error.

5. Conclusions

In this paper, based on the traditional SIR model, the rumor clarification mechanism is added and a fractional-order differential equation is introduced to establish a fractionalorder SIR-C rumor spread model that takes into account "anomalous propagation" of information. Next, the LAS and GAS of the RFEP (E_0) and the BEPs (E_1 , E_2) are analyzed by using the Routh-Hurwitz criterion and Lyapunov's stability theorem. Next, the validity of the above theoretical analysis is verified by numerical simulations, and a comparison with the existing SIR rumor propagation model is completed with actual rumor propagation examples to demonstrate that the model can play a positive and effective role in rumor propagation prediction. The model is optimized by constantly adjusting the parameters through the acquisition of the initial node densities of S, I, C and R in social networking sites at the time of rumor spread, as well as the real-time number of rumors spreading in the early stages. The model can eventually predict the trend of rumor propagation in the future.

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Abbreviations

The following abbreviations are used in this manuscript:

| SIR-C | Susceptible-Infected-Removal-Clarify |
|-------|--------------------------------------|
| RFEP | Rumor-free equilibrium point |
| BEP | Boundary equilibrium point |
| SIR | Susceptible-Infected-Removal |
| GAS | Global asymptotic stable |
| LAS | Local asymptotic stable |
| NEMT | National College Entrance Examnation |
| CFD | Conformable fractional derivative |
| RL | Riemann-Liouville |
| BRN | Basic reproduction number |
| | |

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