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Some New Integral Inequalities for Generalized Preinvex Functions in Interval-Valued Settings

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Abstract: In recent years, there has been a significant amount of research on the extension of convex functions which are known as preinvex functions. In this paper, we have used this approach to generalize the preinvex interval-valued function in terms of $(£_1, £_2)$-preinvex interval-valued functions because of its extraordinary applications in both pure and applied mathematics. The idea of $(£_1, £_2)$-preinvex interval-valued functions is explained in this work. By using the Riemann integral operator, we obtain Hermite-Hadamard and Fejér-type inequalities for $(£_1, £_2)$-preinvex interval-valued functions. To discuss the validity of our main results, we provide non-trivial examples. Some exceptional cases have been discussed that can be seen as applications of main outcomes.

Keywords: interval-valued functions; fuzzy Riemann integrals; $(£_1, £_2)$-preinvex interval-valued functions; Hermite-Hadamard inequalities

MSC: 26A33; 26A51; 26D10

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1. Introduction

Set-valued analysis, which is the study of sets in the context of mathematics and general topology, is a subset of interval analysis. A well-known example of interval enclosure is the Archimedean method, which includes calculating a circle's circumference. The interval uncertainty that is present in many computational and mathematical models of deterministic real-world systems is addressed by this theory. This method studies interval variables rather than point variables and expresses computing results as intervals, avoiding inaccuracies that lead to inaccurate conclusions. One of the initial objectives of the interval-valued analysis was to take into account the error estimates of the numerical solutions for finite state machines. One of the essential methods in numerical analysis is interval analysis, which Moore initially introduced in his well-known work [1]. Due to this, it has found use in a variety of areas, including computer graphics [2], differential equations for intervals [3], neural network output optimization [4], automatic error analysis [5], and many more. We recommend [6–17] to readers who are interested in results and applications.

Particularly those associated with the Jensen, Ostrowski, Hermite-Hadamard, Bullen, Simpson, and Opial inequalities have a considerable impact on mathematics. Some renowned scholars have lately extended many of these inequalities to interval-valued functions (I·V·Fs) (see, for instance, [18–28]), and many have also studied the Hermite-

Hadamard inequality ($H{\cdot}H$-inequality) for convex functions. The $H{\cdot}H$-inequality for convex mapping $\mathfrak{G} : \mathfrak{O} \to \mathbb{R}$ on an interval $\mathfrak{O} = [\varsigma, \mathfrak{q}]$ is all $\varsigma, \mathfrak{q} \in \mathfrak{O}$.

$$\mathfrak{G}\left(\frac{\varsigma+\mathfrak{q}}{2}\right) \leq \frac{1}{\mathfrak{q}-\varsigma} \int_{\varsigma}^{\mathfrak{q}} \mathfrak{G}(\mathfrak{x})d\mathfrak{x} \leq \frac{\mathfrak{G}(\varsigma)+\mathfrak{G}(\mathfrak{q})}{2}, \tag{1}$$

For more information, see [29–38] and the references therein.

On the other hand, a powerful tool for solving a wide range of problems in applied analysis and nonlinear analysis, including many concerns in mathematical physics, is the generalized convexity of mappings. Recently, extensive study has been done on a number of generalizations of convex functions. It is intriguing to explore integral inequalities from a mathematical analytic perspective. Inequalities and other extended convex mappings have been thought to be related to the study of differential and integral equations. They have made a significant contribution to a number of fields, including electrical networks, symmetry analysis, operations research, finance, decision-making, numerical analysis, and equilibrium, see [39–49]. We explore the possibility of encouraging the subjective features of convexity by employing a variety of basic integral inequalities.

Several types of convexity are related to the Hermite-Hadamard inequality; for several instances, see [50–62]. The concept of harmonic convexity and several associated Hermite Hadamard type inequalities were introduced by Iscan [63] in 2014, and 2015 saw the first description of harmonic $\hbar$-convex functions and certain related Hermite-Hadamard inequalities by the authors of [64]. In recent years, numerous studies have explored the relationship between integral inequalities and interval-valued functions, yielding many important results. Using the extended Hukuhara derivative, Chalco Cano [65] researched the Ostrowski-type inequalities, while Roman-Flores [66] established the Minkowski type inequalities and the Beckenbach type inequalities. Costa [67] introduced the Opial-type inequalities. Zhao et al. [68] recently built on this notion by incorporating interval-valued coordinated convex functions and generating related $H{\cdot}H$ type inequalities. It was also used to support the $H{\cdot}H$- and Fejér-type inequalities for the preinvex function [69,70] and convex interval-valued function for n-polynomials [71]. The idea of interval-valued preinvex functions, first proposed by Lai et al. [72], has recently been extended to include interval-valued coordinated preinvex functions. The $H{\cdot}H$ inequality was expanded to include interval $\hbar$-convex functions [73], interval harmonic $\hbar$-convex functions [74], interval $(\hbar_1, \hbar_2)$-convex functions [75], interval $(\hbar_1, \hbar_2)$-convex functions, and interval harmonically $(\hbar_1, \hbar_2)$-convex functions [76], when interval analysis was combined. The authors in [77] used the definition of the $\hbar$-Godunova-Levin function to account for this inequality. Additionally, the authors of [78] developed a Jensen-type inequality for $(\hbar_1, \hbar_2)$ interval-nonconvex functions, whereas the author of [78] published a fuzzy Jensen-type integral inequality for fuzzy interval-valued functions. For more information, see [79–98] and the references therein.

Our investigation was inspired by the substantial body of literature and the targeted studies [76,77]. Interval-valued $(\pounds_1, \pounds_2)$-preinvex functions are introduced, and new $H{\cdot}H$-type inequalities are constructed for the previously covered topic. The following is how the paper is set up: Section 2 provides the introduction and the mathematical context. The scenario and our key findings are covered in Section 3. Section 4 contains the conclusion and future scope.

2. Preliminaries

Let $\mathfrak{E}_C$ be the collection of all closed and bounded intervals of $\mathbb{R}$ that is $\mathfrak{E}_C = \{[\mathcal{Z}_*, \mathcal{Z}^*] : \mathcal{Z}_*, \mathcal{Z}^* \in \mathbb{R} \text{ and } \mathcal{Z}_* \leq \mathcal{Z}^*\}$. If $\mathcal{Z}_* \geq 0$, then $[\mathcal{Z}_*, \mathcal{Z}^*]$ is named as a positive interval. The set of all positive intervals is denoted by $\mathfrak{E}_C^+$ and defined as $\mathfrak{E}_C^+ = \{[\mathcal{Z}_*, \mathcal{Z}^*] : \mathcal{Z}_*, \mathcal{Z}^* \in \mathfrak{E}_C \text{ and } \mathcal{Z}_* \geq 0\}$.

If $[\mathfrak{A}_*, \mathfrak{A}^*], [\mathcal{Z}_*, \mathcal{Z}^*] \in \mathfrak{E}_C$ and $s \in \mathbb{R}$, then arithmetic operations are defined by

$$[\mathfrak{A}_*, \mathfrak{A}^*] + [\mathcal{Z}_*, \mathcal{Z}^*] = [\mathfrak{A}_* + \mathcal{Z}_*, \mathfrak{A}^* + \mathcal{Z}^*], \tag{2}$$

$$[\mathfrak{A}_*,\ \mathfrak{A}^*] \times [\mathcal{Z}_*,\ \mathcal{Z}^*] = [\min\{\mathfrak{A}_*\mathcal{Z}_*, \mathfrak{A}^*\mathcal{Z}_*, \mathfrak{A}_*\mathcal{Z}^*, \mathfrak{A}^*\mathcal{Z}^*\}, \max\{\mathfrak{A}_*\mathcal{Z}_*, \mathfrak{A}^*\mathcal{Z}_*, \mathfrak{A}_*\mathcal{Z}^*, \mathfrak{A}^*\mathcal{Z}^*\}] \tag{3}$$

$$\mathrm{s}.[\mathfrak{A}_*,\ \mathfrak{A}^*] = \begin{cases} [\mathrm{s}\mathfrak{A}_*,\ \mathrm{s}\mathfrak{A}^*]\ if\ \mathrm{s} > 0 \\ \{0\}\ if\ \mathrm{s} = 0, \\ [\mathrm{s}\mathfrak{A}^*, \mathrm{s}\mathfrak{A}_*]\ if\ \mathrm{s} < 0. \end{cases} \tag{4}$$

For $[\mathfrak{A}_*,\ \mathfrak{A}^*],\ [\mathcal{Z}_*,\ \mathcal{Z}^*] \in \mathfrak{E}_C$, the inclusion " $\subseteq$ " is defined by

$$[\mathfrak{A}_*,\ \mathfrak{A}^*] \subseteq [\mathcal{Z}_*,\ \mathcal{Z}^*],\ \text{if and only if},\ \mathcal{Z}_* \leq \mathfrak{A}_*,\ \mathfrak{A}^* \leq \mathcal{Z}^*. \tag{5}$$

Theorem 1 ([1]). *If* $\mathfrak{G} : [\varsigma,\ \mathfrak{q}] \subset \mathbb{R} \to \mathfrak{E}_C$ *is an* $\mathcal{I}\text{-}\mathcal{V}\text{-}\mathcal{F}$ *such that* $\mathfrak{G}(\omega) = [\mathfrak{G}_*(\omega),\ \mathfrak{G}^*(\omega)]$, *then* $\mathfrak{G}$ *is Riemann integrable over* $[\varsigma, \mathfrak{q}]$ *if and only if,* $\mathfrak{G}_*(\omega)$ *and* $\mathfrak{G}^*(\omega)$ *are both Riemann integrable over* $[\varsigma, \mathfrak{q}]$ *such that*

$$(IR)\int_{\varsigma}^{\mathfrak{q}} \mathfrak{G}(\omega)d\omega = \left[(R)\int_{\varsigma}^{\mathfrak{q}} \mathfrak{G}_*(\omega)d\omega,\ (R)\int_{\varsigma}^{\mathfrak{q}} \mathfrak{G}^*(\omega)d\omega\right] \tag{6}$$

where $\mathfrak{G}_*, \mathfrak{G}^* : [\varsigma,\ \mathfrak{q}] \to \mathbb{R}$.

The collection of all Riemann integrable real valued functions and Riemann integrable $\mathcal{I}\text{-}\mathcal{V}\text{-}\mathcal{F}\,s$ is denoted by $\mathcal{R}_{[\varsigma,\mathfrak{q}]}$ and $\mathfrak{T}\mathcal{R}_{[\varsigma,\mathfrak{q}]}$, respectively.

Definition 1 ([84]). *Let* $\mathfrak{O}$ *be an invex set. Then, I·V·F* $\mathfrak{G} : \mathfrak{O} \to \mathfrak{E}_C$ *is said to be preinvex on* $\mathfrak{O}$ *with respect to* $\mathfrak{w}$ *if*

$$\mathfrak{G}(x + (1-\mathfrak{o})\mathfrak{w}(y,x)) \supseteq \mathfrak{o}\mathfrak{G}(x) + (1-\mathfrak{o})\mathfrak{G}(y), \tag{7}$$

for all $x,\ y \in \mathfrak{O},\ \mathfrak{o} \in [0,\ 1],\ \mathfrak{w} : \mathfrak{O} \times \mathfrak{O} \to \mathbb{R}$. *If* $\mathfrak{G}$ *is preincave I·V·F, then* $-\mathfrak{G}$ *is preinvex I·V·F.*

Definition 2 ([83]). *Let* $\mathfrak{O}$ *be an invex set and* $\pounds : [0,\ 1] \subseteq \mathfrak{O} \to \mathbb{R}$ *such that* $\pounds \not\equiv 0$. *Then, I·V·F* $\mathfrak{G} : \mathfrak{O} \to \mathfrak{E}_C$ *is said to be* $\pounds$ *-preinvex on* $\mathfrak{O}$ *with respect to* $\mathfrak{w}$ *if*

$$\mathfrak{G}(x + (1-\mathfrak{o})\mathfrak{w}(y,x)\ ) \supseteq \pounds(\mathfrak{o})\mathfrak{G}(x) + \pounds(1-\mathfrak{o})\mathfrak{G}(y), \tag{8}$$

for all $x,\ y \in \mathfrak{O},\ \mathfrak{o} \in [0,\ 1]$, *where* $\mathfrak{G}(x) \geq 0$ *and* $\mathfrak{w} : \mathfrak{O} \times \mathfrak{O} \to \mathbb{R}$.

Definition 3 ([84]). *Let* $\mathfrak{O}$ *be an invex set and* $\pounds_1,\ \pounds_2 : [0,\ 1] \subseteq \mathfrak{O} \to \mathbb{R}$ *such that* $\pounds_1,\ \pounds_2 \not\equiv 0$. *Then, I·V·F* $\mathfrak{G} : \mathfrak{O} \to \mathfrak{E}_C$ *is said to be:*

- $(\pounds_1,\ \pounds_2)$*-preinvex on* $\mathfrak{O}$ *with respect to* $\mathfrak{w}$ *if*

$$\mathfrak{G}(x + (1-\mathfrak{o})\mathfrak{w}(y,x)\ ) \supseteq \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}(x) + \pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}(y), \tag{9}$$

  for all $x, y \in \mathfrak{O}, \mathfrak{o} \in [0,\ 1]$, *where* $\mathfrak{G}(x) \geq 0$ *and* $\mathfrak{w} : \mathfrak{O} \times \mathfrak{O} \to \mathbb{R}$.
- $(\pounds_1,\ \pounds_2)$*-preincave on* $\mathfrak{O}$ *with respect to* $\mathfrak{w}$ *if inequality (13) is reversed.*

Remark 1. *If* $\pounds_2(\mathfrak{o}) \equiv 1$ *, then* $(\pounds_1,\ \pounds_2)$*-preinvex I·V·F, we acquire the definition of* $\pounds_1$*-preinvex I·V·F.*

If $\pounds_1(\mathfrak{o}) = \mathfrak{o}^s, \pounds_2(\mathfrak{o}) \equiv 1$, then from the definition of $(\pounds_1,\ \pounds_2)$-preinvex I·V·F becomes s-preinvex I·V·F in the second sense, that is

$$\mathfrak{G}(x + (1-\mathfrak{o})\mathfrak{w}(y,x)) \supseteq \mathfrak{o}^s\mathfrak{G}(x) + (1-\mathfrak{o})^s\mathfrak{G}(y),\ \forall\ x,\ y \in \mathfrak{O},\ \mathfrak{o} \in [0,\ 1]. \tag{10}$$

If $\pounds_1(\mathfrak{o}) = \mathfrak{o}, \pounds_2(\mathfrak{o}) \equiv 1$, then $(\pounds_1,\ \pounds_2)$-preinvex I·V·F becomes preinvex I·V·F.

If $\pounds_1(\mathfrak{o}) = \pounds_2(\mathfrak{o}) \equiv 1$, then from the definition of $(\pounds_1,\ \pounds_2)$-preinvex I·V·F, we acquire the definition of P I·V·F, that is, see [83]:

$$\mathfrak{G}(x + (1-\mathfrak{o})\mathfrak{w}(y,x)) \supseteq \mathfrak{G}(x) + \mathfrak{G}(y),\ \forall\ x,\ y \in \mathfrak{O},\ \mathfrak{o} \in [0,\ 1]. \tag{11}$$

Proposition 1. *Let $\mathfrak{O}$ be an invex set and non-negative real-valued function $\pounds_1,\ \pounds_2 : [0,\ 1] \subseteq \mathfrak{O} \to \mathbb{R}$ such that $\pounds_1,\ \pounds_2 \not\equiv 0$. Let $\mathfrak{G} : \mathfrak{O} \to \mathfrak{E}_C$ be an I·V·F with $\mathfrak{G}(x) \geq 0$, such that*

$$\mathfrak{G}(x) = [\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)],\ \forall\ x \in \mathfrak{O}. \tag{12}$$

for all $x \in \mathfrak{O}$. Then, $\mathfrak{G}$ is $(\pounds_1,\ \pounds_2)$-preinvex I·V·F on $\mathfrak{O}$, if and only if, $\mathfrak{G}_*(x)$ is $(\pounds_1, \pounds_2)$-preinvex function and $\mathfrak{G}^*(x)$ is $(\pounds_1,\ \pounds_2)$-preincave function.

Proof. The proof of this proposition is similar to the Theorem 3.7, see [73]. □

Proposition 2. *Let $\mathfrak{O}$ be an invex set and non-negative real-valued function $\pounds_1,\ \pounds_2 : [0,\ 1] \subseteq \mathfrak{O} \to \mathbb{R}$ such that $\pounds_1,\ \pounds_2 \not\equiv 0$. Let $\mathfrak{G} : \mathfrak{O} \to \mathfrak{E}_C^+$ be an I·V·F with $\mathfrak{G}(x) \geq 0$, such that*

$$\mathfrak{G}(x) = [\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)],\ \forall\ x \in \mathfrak{O}. \tag{13}$$

for all $x \in \mathfrak{O}$. Then, $\mathfrak{G}$ is $(\pounds_1,\ \pounds_2)$-preincave I·V·F on $\mathfrak{O}$, if and only if, $\mathfrak{G}_*(x)$ and $\mathfrak{G}^*(x)$ are $(\pounds_1,\ \pounds_2)$-preincave and preinvex functions, respectively.

Proof. The demonstration of the proof is analogous to Proposition 1. □

3. Results

In this section, using the Riemann integral operator, we achieve various modifications of the Hermite-Hadamard type inequality. We add a few quality and interesting remarks for the readers. The next step is to present a crucial $H{\cdot}H$-inequality for $(\pounds_1,\ \pounds_2)$-preinvex I·V·Fs via Riemann integrals.

Theorem 2. *Let $\mathfrak{G} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathfrak{E}_C^+$ be a $(\pounds_1,\ \pounds_2)$-preinvex I·V·F with non-negative real-valued functions $\pounds_1,\ \pounds_2 : [0,\ 1] \to \mathbb{R}$ and $\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right) \neq 0$, such that $\mathfrak{G}(x) = [\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)]$ for all $x \in [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)]$. If $\mathfrak{G} \in \mathcal{IR}_{([\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)])}$, then*

$$\begin{aligned} \frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\ \mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) &\supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x)dx \\ &\supseteq [\mathfrak{G}(\varsigma) + \mathfrak{G}(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}. \end{aligned} \tag{14}$$

Let $\mathfrak{G} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathfrak{E}_C^+$ be a $(\pounds_1,\ \pounds_2)$-preincave I·V·F. Then, we have

$$\begin{aligned} \frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\ \mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) &\subseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x)dx \\ &\subseteq [\mathfrak{G}(\varsigma) + \mathfrak{G}(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}. \end{aligned} \tag{15}$$

Proof. Let $\mathfrak{G} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathfrak{E}_C^+$ be a $(\pounds_1,\ \pounds_2)$-preinvex I·V·F. Then, by hypothesis, we have

$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \supseteq \mathfrak{G}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma)) + \mathfrak{G}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)).$$

Therefore, we have

$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \leq \mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\,\varsigma))+\mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma)),$$
$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \geq \mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\,\varsigma))+\mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma)).$$

Then,

$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\int_0^1 \mathfrak{G}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right)d\mathfrak{o} \leq \int_0^1 \mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o}+\int_0^1 \mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o},$$
$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\int_0^1 \mathfrak{G}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right)d\mathfrak{o} \geq \int_0^1 \mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o}+\int_0^1 \mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o}.$$

and it follows that

$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \leq \frac{2}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}_*(x)dx,$$
$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \geq \frac{2}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}^*(x)dx.$$

That is

$$\frac{1}{\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\left[\mathfrak{G}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right),\,\mathfrak{G}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right)\right] \supseteq \frac{2}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\left[\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}_*(x)dx,\,\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}^*(x)dx\right].$$

Thus,

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\,(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}(x)dx. \tag{16}$$

In a similar way as above, we have

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\,(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}(x)dx \supseteq [\mathfrak{G}(\varsigma)+\mathfrak{G}(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}. \tag{17}$$

Combining (16) and (17), we have

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\,(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}(x)dx \supseteq [\mathfrak{G}(\varsigma)+\mathfrak{G}(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}. \tag{18}$$

This completes the proof. □

Note that, if $\mathfrak{G}(x)$ is $(\pounds_1,\,\pounds_2)$-preincave I·V·F, then integral inequality (Equation (14)) is reversed.

Remark 2. *If $\pounds_2(\mathfrak{o}) \equiv 1$, then Theorem 2 reduces to the result for $\pounds_1$-preinvex I·V·F, see [83]:*

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)}\mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\,(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}(x)dx \supseteq [\mathfrak{G}(\varsigma)+\mathfrak{G}(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})d\mathfrak{o}.$$

If $\pounds_1(\mathfrak{o}) = \mathfrak{o}^s$ and $\pounds_2(\mathfrak{o}) \equiv 1$, then Theorem 2 reduces to the result for s-preinvex I·V·F, see [83]:

$$2^{s-1}\,\mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\,(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}(x)dx \supseteq \frac{1}{s+1}[\mathfrak{G}(\varsigma)+\mathfrak{G}(\mathfrak{q})]. \tag{19}$$

If $\pounds_1(\mathfrak{o}) = \mathfrak{o}$ and $\pounds_2(\mathfrak{o}) \equiv 1$, then Theorem 2 reduces to the result for preinvex I·V·F, see [84]:

$$\mathfrak{G}\left(\frac{2\varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x)dx \supseteq \frac{\mathfrak{G}(\varsigma) + \mathfrak{G}(\mathfrak{q})}{2}. \tag{20}$$

If $\pounds_1(\mathfrak{o}) = \pounds_2(\mathfrak{o}) \equiv 1$, then Theorem 2 reduces to the result for P-I·V·F, see [83]:

$$\frac{1}{2}\mathfrak{G}\left(\frac{2\varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x)dx \supseteq \mathfrak{G}(\varsigma) + \mathfrak{G}(\mathfrak{q}). \tag{21}$$

If $\mathfrak{G}_*(x) = \mathfrak{G}^*(x)$, then we from (14) obtain the classical integral inequality for $(\pounds_1, \pounds_2)$-preinvex functions.

If $\mathfrak{G}_*(x) = \mathfrak{G}^*(x)$, $\pounds_2(\mathfrak{o}) \equiv 1$, then from (14) we obtain the classical integral inequality for $(\pounds_1, \pounds_2)$-preinvex functions.

Note that, if $\mathfrak{w}(\mathfrak{q}, \varsigma) = \mathfrak{q} - \varsigma$, then the above integral inequalities reduce to classical ones.

Example 1. *We consider* $\pounds_1(\mathfrak{o}) = \mathfrak{o},\ \pounds_2(\mathfrak{o}) \equiv 1$, *for* $\mathfrak{o} \in [0,\ 1]$ *and the I·V·F* $\mathfrak{G} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] = [0,\ \mathfrak{w}(2,\ 0)] \to \mathfrak{E}_C^+$ *defined by* $\mathfrak{G}(x) = \left[2x^2,\ 4x\right]$. *Since end point functions* $\mathfrak{G}_*(x) = 2x^2$, $\mathfrak{G}^*(x) = 4x$ *are* $(\pounds_1, \pounds_2)$*-preinvex functions with respect to* $\mathfrak{w}(\mathfrak{q},\ \varsigma) = \mathfrak{q} - \varsigma$, *then* $\mathfrak{G}(x)$ *is* $(\pounds_1, \pounds_2)$*-preinvex I·V·F with respect to* $\mathfrak{w}(\mathfrak{q},\ \varsigma) = \mathfrak{q} - \varsigma$. *Now we compute the following*

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\ \mathfrak{G}_*\left(\frac{2\varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \le \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ \int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)dx \le [\mathfrak{G}_*(\varsigma) + \mathfrak{G}_*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}.$$

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\ \mathfrak{G}_*\left(\frac{2\varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) = \mathfrak{G}_*(1) = 2,$$

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ \int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)dx = \frac{1}{2}\int_0^2 2x^2dx = \frac{8}{3},$$

$$[\mathfrak{G}_*(\varsigma) + \mathfrak{G}_*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o} = 4,$$

which means

$$2 \le \frac{8}{3} \le 4.$$

Similarly, it can be easily show that

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\ \mathfrak{G}^*\left(\frac{2\varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \ge \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ \int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)dx \ge [\mathfrak{G}^*(\varsigma) + \mathfrak{G}^*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o},$$

such that

$$\frac{1}{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}\ \mathfrak{G}^*\left(\frac{2\varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) = \mathfrak{G}_*(1) = 4,$$

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ \int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)dx = \frac{1}{2}\int_0^2 4xdx = 4,$$

$$[\mathfrak{G}^*(\varsigma) + \mathfrak{G}^*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o} = 4,$$

that is

$$[2,\ 4] \supseteq \left[\frac{8}{3},\ 4\right] \supseteq [4,\ 4].$$

Hence, the Theorem 2 is verified.

Theorem 3. *(The second H·H-Fejér inequality for $(£_1,\ £_2)$. -preinvex I·V·Fs). Let $\mathfrak{G} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathfrak{E}_C^+$ be a $(£_1,\ £_2)$-preinvex I·V·F with $\varsigma < \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)$ and non-negative real-valued functions $£_1,\ £_2 : [0,\ 1] \to \mathbb{R}$, such that $\mathfrak{G}(x) = [\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)]$ for all $x \in [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)]$. If $\mathfrak{G} \in \mathcal{IR}_{([\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)])}$ and $\mathcal{C} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathbb{R}$, $\mathcal{C}(x) \geq 0$, symmetric with respect to $\varsigma + \frac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)$, then*

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x)\mathcal{C}(x)dx \supseteq [\mathfrak{G}(\varsigma) + \mathfrak{G}(\mathfrak{q})]\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o}. \tag{22}$$

Proof. Let $\mathfrak{G}$ be a $(£_1,\ £_2)$-preinvex I·V·F. Then, we have

$$\begin{gathered}\mathfrak{G}_*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\\ \leq (£_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathfrak{G}_*(\varsigma) + £_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathfrak{G}_*(\mathfrak{q}))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)),\\ \mathfrak{G}^*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\\ \geq (£_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathfrak{G}^*(\varsigma) + £_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathfrak{G}^*(\mathfrak{q}))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)).\end{gathered} \tag{23}$$

We also have

$$\begin{gathered}\mathfrak{G}_*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)) \leq \begin{pmatrix} £_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathfrak{G}_*(\varsigma)\\ +£_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathfrak{G}_*(\mathfrak{q})\end{pmatrix}\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)),\\ \mathfrak{G}^*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)) \geq \begin{pmatrix} £_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathfrak{G}^*(\varsigma)\\ +£_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathfrak{G}^*(\mathfrak{q})\end{pmatrix}\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)).\end{gathered} \tag{24}$$

After adding (23) and (24), and integrating over $[0,\ 1]$, we get

$$\begin{gathered}\int_0^1 \mathfrak{G}_*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ +\int_0^1 \mathfrak{G}_*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ \leq \int_0^1 \begin{bmatrix} \mathfrak{G}_*(\varsigma)\{£_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) + £_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\}\\ +\mathfrak{G}_*(\mathfrak{q})\{£_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) + £_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\}\end{bmatrix}d\mathfrak{o},\\ \int_0^1 \mathfrak{G}^*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ +\int_0^1 \mathfrak{G}^*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ \geq \int_0^1 \begin{bmatrix} \mathfrak{G}^*(\varsigma)\{£_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) + £_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\}\\ +\mathfrak{G}^*(\mathfrak{q})\{£_1(1-\mathfrak{o})£_2(\mathfrak{o})\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) + £_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\}\end{bmatrix}d\mathfrak{o}.\end{gathered}$$

$$\begin{gathered}= 2\mathfrak{G}_*(\varsigma)\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ +2\mathfrak{G}_*(\mathfrak{q})\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o},\\ = 2\mathfrak{G}^*(\varsigma)\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ +2\mathfrak{G}^*(\mathfrak{q})\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})C(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o}.\end{gathered}$$

Since $\mathcal{C}$ is symmetric, then

$$\begin{gathered}= 2[\mathfrak{G}_*(\varsigma) + \mathfrak{G}_*(\mathfrak{q})]\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})C(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o},\\ = 2[\mathfrak{G}^*(\varsigma) + \mathfrak{G}^*(\mathfrak{q})]\int_0^1 £_1(\mathfrak{o})£_2(1-\mathfrak{o})C(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o}.\end{gathered} \tag{25}$$

Since

$$\begin{gathered}\int_0^1 \mathfrak{G}_*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o}\\ = \int_0^1 \mathfrak{G}_*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} = \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)\mathcal{C}(x)dx,\\ \int_0^1 \mathfrak{G}^*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o}\\ = \int_0^1 \mathfrak{G}^*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o} = \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)\mathcal{C}(x)dx.\end{gathered} \tag{26}$$

From (25) and (26), we have

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}_*(x)\mathcal{C}(x)dx \leq [\mathfrak{G}_*(\varsigma)+\mathfrak{G}_*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})C(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o},$$

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}^*(x)\mathcal{C}(x)dx \geq [\mathfrak{G}^*(\varsigma)+\mathfrak{G}^*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})C(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o},$$

that is

$$\left[\frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}_*(x)\mathcal{C}(x)dx,\ \frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}^*(x)\mathcal{C}(x)dx\right] \supseteq [\mathfrak{G}_*(\varsigma)+ \mathfrak{G}_*(\mathfrak{q}),\ \mathfrak{G}^*(\varsigma)+\mathfrak{G}^*(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})C(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o},$$

hence

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\,\varsigma)}\,(IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\,\varsigma)}\mathfrak{G}(x)\mathcal{C}(x)dx \supseteq [\mathfrak{G}(\varsigma)+\mathfrak{G}(\mathfrak{q})]\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{C}(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\,\varsigma))d\mathfrak{o},$$

then we complete the proof. □

The following assumption is required to prove the next result regarding the bi-function $\mathfrak{w}:\mathfrak{O}\times\mathfrak{O}\to\mathbb{R}$ which is known as:

Condition C. *Let $\mathfrak{O}$ be an invex set with respect to $\mathfrak{w}$. For any $\varsigma,\ \mathfrak{q}\in\mathfrak{O}$ and $\mathfrak{o}\in[0,\ 1]$,*

$$\mathfrak{w}(\mathfrak{q},\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma)) = (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),$$

$$\mathfrak{w}(\varsigma,\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma)) = -\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma).$$

Clearly for $\mathfrak{o} = 0$, we have $\mathfrak{w}(\mathfrak{q},\varsigma) = 0$ if and only if, $\mathfrak{q}=\varsigma$, for all $\varsigma,\ \mathfrak{q}\in\mathfrak{O}$. For the applications of Condition C, see [79–84].

Theorem 4. *(The first H·H- Fejér inequality for $(\pounds_1,\ \pounds_2)$-preinvex I·V·Fs). Let $\mathfrak{G}:[\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)]\to\mathfrak{E}_C^+$ be a $(\pounds_1,\ \pounds_2)$-preinvex I·V·F with $\varsigma<\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)$ and non-negative real-valued functions $\pounds_1,\ \pounds_2:[0,\ 1]\to\mathbb{R}$, such that $\mathfrak{G}(x)=[\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)]$ for all $x\in[\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)]$. If $\mathfrak{G}\in\mathcal{IR}_{([\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)])}$ and $\mathcal{C}:[\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)]\to\mathbb{R}$, $\mathcal{C}(x)\geq 0$, symmetric with respect to $\varsigma+\frac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)$, and $\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathcal{C}(x)dx>0$, and Condition C holds for $\mathfrak{w}$, the,*

$$\mathfrak{G}\left(\varsigma+\frac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) \supseteq \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathcal{C}(x)dx}\,(IR)\int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathfrak{G}(x)\mathcal{C}(x)dx. \tag{27}$$

Proof. Using Condition C, we can write

$$\varsigma+\frac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)=\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\frac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma)).$$

Since $\mathfrak{G}$ is a $(\pounds_1,\ \pounds_2)$-preinvex then, we have

$$\begin{aligned}
\mathfrak{G}_*\left(\varsigma+\tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) &= \mathfrak{G}_*\left(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\tfrac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\right)\\
&\leq \pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)(\mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))+\mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))),\\
\mathfrak{G}^*\left(\varsigma+\tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) &= \mathfrak{G}^*\left(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\tfrac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\right)\\
&\geq \pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)(\mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))+\mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma)))
\end{aligned} \tag{28}$$

By multiplying (28) by $\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) = \mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))$ and integrating it by $\mathfrak{o}$ over $[0,\ 1]$, we obtain

$$\begin{aligned} &\mathfrak{G}_*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right)\int_0^1 \mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} \\ &\leq \pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\int_0^1 \mathfrak{G}_*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} \\ +\int_0^1 \mathfrak{G}_*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o}\end{array}\right), \\ &\mathfrak{G}^*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right)\int_0^1 \mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} \\ &\geq \pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\int_0^1 \mathfrak{G}^*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} \\ +\int_0^1 \mathfrak{G}^*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o}\end{array}\right), \end{aligned} \tag{29}$$

since

$$\begin{aligned} &\int_0^1 \mathfrak{G}_*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o} \\ &= \int_0^1 \mathfrak{G}_*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} \\ &= \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)\mathcal{C}(x)dx \\ &\int_0^1 \mathfrak{G}^*(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\mathcal{C}(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} \\ &= \int_0^1 \mathfrak{G}^*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))\mathcal{C}(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma))d\mathfrak{o} \\ &= \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)\mathcal{C}(x)dx. \end{aligned} \tag{30}$$

From (29) and (30), we have

$$\begin{aligned} \mathfrak{G}_*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) &\leq \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathcal{C}(x)dx}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)\mathcal{C}(x)dx, \\ \mathfrak{G}^*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) &\geq \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathcal{C}(x)dx}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)\mathcal{C}(x)dx, \end{aligned}$$

from which we have

$$\begin{aligned} &\left[\mathfrak{G}_*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right),\ \mathfrak{G}^*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right)\right] \\ &\supseteq \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathcal{C}(x)dx}\left[\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)\mathcal{C}(x)dx,\ \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)\mathcal{C}(x)dx\right], \end{aligned}$$

that is

$$\mathfrak{G}\left(\varsigma + \frac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) \supseteq \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathcal{C}(x)dx}\,(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x)\mathcal{C}(x)dx,$$

And this completes the proof. □

Remark 3. *If $\pounds_2(\mathfrak{o}) \equiv 1$, $\mathfrak{o} \in [0,\ 1]$, then inequalities in Theorems 3 and 4 reduce for $\pounds_1$-preinvex I·V·Fs, see* [83].

If $\pounds_1(\mathfrak{o}) = \mathfrak{o}$ and $\pounds_2(\mathfrak{o}) \equiv 1$, $\mathfrak{o} \in [0,\ 1]$, then inequalities in Theorems 3 and 4 reduce for preinvex I·V·Fs, see [84].

If in the Theorems 3 and 4 $\pounds_2(\mathfrak{o}) \equiv 1$ and $\mathfrak{w}(\mathfrak{q},\ \varsigma) = \mathfrak{q} - \varsigma$, then we obtain the appropriate theorems for $\pounds_1$-convex I·V·Fs, see [73].

If in the Theorems 3 and 4 $\pounds_1(\mathfrak{o}) = \mathfrak{o}$, $\pounds_2(\mathfrak{o}) \equiv 1$ and $\mathfrak{w}(\mathfrak{q},\ \varsigma) = \mathfrak{q} - \varsigma$, then we obtain the appropriate theorems for convex I·V·Fs, see [73].

If $\mathfrak{G}_*(x) = \mathfrak{G}^*(x)$ with $\pounds_2(\mathfrak{o}) \equiv 1$, then Theorems 3 and 4 reduce to classical first and second $H{\cdot}H$-Fejér inequalities for £-preinvex function, see [80].

If in the Theorems 3 and 4 $\mathfrak{G}_*(x) = \mathfrak{G}^*(x)$ with $\pounds_2(\mathfrak{o}) \equiv 1$ and $\mathfrak{w}(\mathfrak{q},\ \varsigma) = \mathfrak{q} - \varsigma$, then we obtain the appropriate theorems for £-convex function.

If $\mathcal{C}(x) = 1$, then combining Theorems 3 and 4, we get Theorem 2.

Example 2. *We consider* $\pounds_1(\mathfrak{o}) = \mathfrak{o},\ \pounds_2(\mathfrak{o}) = 1$ *for* $\mathfrak{o} \in [0,\ 1]$ *and the I·V·F* $\mathfrak{G} : [1,\ 1 + \mathfrak{w}(4,\ 1)] \to \mathfrak{E}_C^+$ *defined by,* $\mathfrak{G}(x) = \left[\frac{1}{x}, x\right]$. *Since end point functions* $\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)$. *are* $(\pounds_1,\ \pounds_2)$*-preinvex functions* $\mathfrak{w}(y, x) = y - x$, *then* $\mathfrak{G}(x)$ *is* $(\pounds_1,\ \pounds_2)$*-preinvex I·V·F. If*

$$\mathcal{C}(x) = \begin{cases} x - 1,\ \sigma \in \left[1, \frac{5}{2}\right], \\ 4 - x,\ \sigma \in \left(\frac{5}{2},\ 4\right], \end{cases}$$

then we have

$$\begin{aligned} &\tfrac{1}{\mathfrak{w}(4,\ 1)} \textstyle\int_1^{1+\mathfrak{w}(4,\ 1)} \mathfrak{G}_*(x)\mathcal{C}(x)dx = \tfrac{1}{3} \int_1^4 \mathfrak{G}_*(x)\mathcal{C}(x)dx = \tfrac{1}{3} \int_1^{\frac{5}{2}} \mathfrak{G}_*(x)\mathcal{C}(x)dx + \tfrac{1}{3} \int_{\frac{5}{2}}^4 \mathfrak{G}_*(x)\mathcal{C}(x)dx, \\ &\tfrac{1}{\mathfrak{w}(4,\ 1)} \textstyle\int_1^{1+\mathfrak{w}(4,\ 1)} \mathfrak{G}^*(x)\mathcal{C}(x)dx = \tfrac{1}{3} \int_1^4 \mathfrak{G}^*(x)\mathcal{C}(x)dx = \tfrac{1}{3} \int_1^{\frac{5}{2}} \mathfrak{G}^*(x)\mathcal{C}(x)dx + \tfrac{1}{3} \int_{\frac{5}{2}}^4 \mathfrak{G}^*(x)\mathcal{C}(x)dx, \\ &\quad = \tfrac{1}{3} \textstyle\int_1^{\frac{5}{2}} \frac{1}{x}(x-1)dx + \frac{1}{3} \int_{\frac{5}{2}}^4 \frac{1}{x}(4-x)dx = \frac{1}{3}\left(4log\left(\frac{8}{5}\right) + log\left(\frac{5}{2}\right)\right), \\ &\qquad = \tfrac{1}{3} \textstyle\int_1^{\frac{5}{2}} x(x-1)dx + \frac{1}{3} \int_{\frac{5}{2}}^4 x(4-x)dx = \frac{15}{8}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} &[\mathfrak{G}_*(\varsigma) + \mathfrak{G}_*(\mathfrak{q})] \textstyle\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})C(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} = \frac{5}{4}\left[\int_0^{\frac{1}{2}} 3\mathfrak{o}^2 dx + \int_{\frac{1}{2}}^1 \mathfrak{o}(3 - 3\mathfrak{o})d\mathfrak{o}\right] = \frac{15}{32} \\ &[\mathfrak{G}^*(\varsigma) + \mathfrak{G}^*(\mathfrak{q})] \textstyle\int_0^1 \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})C(\varsigma + \mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))d\mathfrak{o} = 5\left[\int_0^{\frac{1}{2}} 3\mathfrak{o}^2 dx + \int_{\frac{1}{2}}^1 \mathfrak{o}(3 - 3\mathfrak{o})d\mathfrak{o}\right] = \frac{15}{8}. \end{aligned} \tag{32}$$

From (31) and (32), we have

$$\left[\frac{1}{3}\left(4log\left(\frac{8}{5}\right) + log\left(\frac{5}{2}\right)\right),\ \frac{15}{8}\right] \supseteq \left[\frac{15}{32},\ \frac{15}{8}\right].$$

Hence, Theorem 3 is verified.
For Theorem 4, we have

$$\begin{aligned} \mathfrak{G}_*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) = \tfrac{2}{5}, \\ \mathfrak{G}^*\left(\varsigma + \tfrac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)\right) = \tfrac{5}{2}, \end{aligned} \tag{33}$$

$$\begin{aligned} \textstyle\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathcal{C}(x)dx = \int_1^{\frac{5}{2}} (x-1)dx + \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} (4-x)dx = \frac{9}{4}, \\ \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathcal{C}(x)dx} \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}_*(x)\mathcal{C}(x)dx = \frac{4}{9}\left(4log\left(\frac{8}{5}\right) + log\left(\frac{5}{2}\right)\right), \\ \frac{2\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)}{\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathcal{C}(x)dx} \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}^*(x)\mathcal{C}(x)dx = \frac{5}{2}. \end{aligned} \tag{34}$$

From (33) and (34), we have

$$\left[\frac{2}{5},\ \frac{5}{2}\right] \supseteq \left[\frac{4}{9}\left(4log\left(\frac{8}{5}\right) + log\left(\frac{5}{2}\right)\right),\ \frac{5}{2}\right].$$

Hence, Theorem 4 is verified.

Theorem 5. *Let* $\mathfrak{G}, \mathcal{J} : [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathfrak{E}_C^+$ *be two* $(\pounds_1,\ \pounds_2)$*-preinvex I·V·Fs with non-negative real-valued functions* $\pounds_1,\ \pounds_2 : [0,\ 1] \to \mathbb{R}$, *such that* $\mathfrak{G}(x) = [\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)]$ *and* $\mathcal{J}(x) = [\mathcal{J}_*(x),\ \mathcal{J}^*(x)]$ *for all* $x \in [\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)]$. *If* $\mathfrak{G}(x) \times \mathcal{J}(x) \in \mathcal{IR}_{([\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)])}$, *then*

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR) \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x) \times \mathcal{J}(x)dx \supseteq \mathcal{M}(\varsigma, \mathfrak{q}) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} + \mathcal{N}(\varsigma, \mathfrak{q}) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}$$

where $\mathcal{M}(\varsigma, \mathfrak{q}) = \mathfrak{G}(\varsigma) \times \mathcal{J}(\varsigma) + \mathfrak{G}(\mathfrak{q}) \times \mathcal{J}(\mathfrak{q}), \mathcal{N}(\varsigma, \mathfrak{q}) = \mathfrak{G}(\varsigma) \times \mathcal{J}(\mathfrak{q}) + \mathfrak{G}(\mathfrak{q}) \times \mathcal{J}(\varsigma)$ with $\mathcal{M}(\varsigma, \mathfrak{q}) = [\mathcal{M}_*((\varsigma, \mathfrak{q})),\ \mathcal{M}^*((\varsigma, \mathfrak{q}))]$ and $\mathcal{N}(\varsigma, \mathfrak{q}) = [\mathcal{N}_*((\varsigma, \mathfrak{q})),\ \mathcal{N}^*((\varsigma, \mathfrak{q}))]$.

Proof. Since $\mathfrak{G}$ and $\mathcal{J}$ both are $(\pounds_1,\ \pounds_2)$-preinvex I·V·Fs on $[\varsigma,\ \varsigma + \mathfrak{w}(\mathfrak{q},\ \varsigma)]$, then we have

$$\begin{aligned} \mathfrak{G}_*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q}, \varsigma)) \le \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}_*(\varsigma) + \pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}_*(\mathfrak{q}), \\ \mathfrak{G}^*(\varsigma + (1-\mathfrak{o})\mathfrak{w}(\mathfrak{q}, \varsigma)) \ge \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}^*(\varsigma) + \pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}^*(\mathfrak{q}). \end{aligned}$$

We also have

$$\mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \leq \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{J}_*(\varsigma)+\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathcal{J}_*(\mathfrak{q}),$$
$$\mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \geq \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{J}^*(\varsigma)+\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathcal{J}^*(\mathfrak{q}).$$

From the definition of $(\pounds_1,\ \pounds_2)$-preinvex I·V·Fs, it follows that $\mathfrak{G}(x) \geq 0$ and $\mathcal{J}(x) \geq 0$, so

$$\begin{gathered}
\mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \times \mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \\
\leq \begin{pmatrix} \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}_*(\varsigma) \\ +\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}_*(\mathfrak{q}) \end{pmatrix} \begin{pmatrix} \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{J}_*(\varsigma) \\ +\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathcal{J}_*(\mathfrak{q}) \end{pmatrix} \\
= \mathfrak{G}_*(\varsigma) \times \mathcal{J}_*(\varsigma)[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 + \mathfrak{G}_*(\mathfrak{q}) \times \mathcal{J}_*(\mathfrak{q})[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 \\
+\mathfrak{G}_*(\varsigma)\mathcal{J}_*(\mathfrak{q})\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o}) \\
+\mathfrak{G}_*(\mathfrak{q}) \times \mathcal{J}_*(\varsigma)\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o}), \\
\mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \times \mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \\
\geq \begin{pmatrix} \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}^*(\varsigma) \\ +\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}^*(\mathfrak{q}) \end{pmatrix} \begin{pmatrix} \pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{J}^*(\varsigma) \\ +\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathcal{J}^*(\mathfrak{q}) \end{pmatrix} \\
= \mathfrak{G}^*(\varsigma) \times \mathcal{J}^*(\varsigma)[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 + \mathfrak{G}^*(\mathfrak{q}) \times \mathcal{J}^*(\mathfrak{q})[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 \\
+\mathfrak{G}^*(\varsigma) \times \mathcal{J}^*(\mathfrak{q})\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o}) \\
+\mathfrak{G}^*(\mathfrak{q}) \times \mathcal{J}^*(\varsigma)\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o}).
\end{gathered}$$

Integrating both sides of above inequality over [0, 1] we get

$$\begin{gathered}
\int_0^1 \mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \times \mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) = \frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)} \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)} \mathfrak{G}_*(x) \times \mathcal{J}_*(x)dx \\
\leq (\mathfrak{G}_*(\varsigma) \times \mathcal{J}_*(\varsigma) + \mathfrak{G}_*(\mathfrak{q}) \times \mathcal{J}_*(\mathfrak{q})) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} \\
+(\mathfrak{G}_*(\varsigma) \times \mathcal{J}_*(\mathfrak{q}) + \mathfrak{G}_*(\mathfrak{q}) \times \mathcal{J}_*(\varsigma)) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}, \\
\int_0^1 \mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) \times \mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma)) = \frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)} \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)} \mathfrak{G}^*(x) \times \mathcal{J}^*(x)dx \\
\geq (\mathfrak{G}^*(\varsigma) \times \mathcal{J}^*(\varsigma) + \mathfrak{G}^*(\mathfrak{q}) \times \mathcal{J}^*(\mathfrak{q})) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} \\
+(\mathfrak{G}^*(\varsigma) \times \mathcal{J}^*(\mathfrak{q}) + \mathfrak{G}^*(\mathfrak{q}) \times \mathcal{J}^*(\varsigma)) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}.
\end{gathered}$$

It follows that

$$\begin{gathered}
\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)} \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)} \mathfrak{G}_*(x) \times \mathcal{J}_*(x)dx \leq \mathcal{M}_*((\varsigma,\mathfrak{q})) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} \\
+\mathcal{N}_*((\varsigma,\mathfrak{q})) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}, \\
\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)} \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)} \mathfrak{G}^*(x) \times \mathcal{J}^*(x)dx \geq \mathcal{M}^*((\varsigma,\mathfrak{q})) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} \\
+\mathcal{N}^*((\varsigma,\mathfrak{q})) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o},
\end{gathered}$$

that is

$$\begin{gathered}
\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)} \left[\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)} \mathfrak{G}_*(x) \times \mathcal{J}_*(x)dx,\ \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)} \mathfrak{G}^*(x) \times \mathcal{J}^*(x)dx\right] \supseteq [\mathcal{M}_*((\varsigma,\mathfrak{q})),\ \mathcal{M}^*((\varsigma,\mathfrak{q}))] \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} \\
+[\mathcal{N}_*((\varsigma,\mathfrak{q})),\ \mathcal{N}^*((\varsigma,\mathfrak{q}))] \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}.
\end{gathered}$$

Thus,

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR) \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x) \times \mathcal{J}(x)dx \supseteq \mathcal{M}(\varsigma,\mathfrak{q}) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o} + \mathcal{N}(\varsigma,\mathfrak{q}) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}, \tag{35}$$

and the theorem has been established. □

Theorem 6. *Let $\mathfrak{G}, \mathcal{J}\ :[\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)] \to \mathfrak{E}_C^+$ be two $(\pounds_1,\ \pounds_2)$-preinvex I·V·Fs with non-negative real valued functions $\pounds_1,\ \pounds_2 : [0,\ 1] \to \mathbb{R}$ and $\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right) \neq 0$, such that $\mathfrak{G}(x) = [\mathfrak{G}_*(x),\ \mathfrak{G}^*(x)]$ and $\mathcal{J}(x) = [\mathcal{J}_*(x),\ \mathcal{J}^*(x)]$ for all $x \in [\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)]$. If $\mathfrak{G}(x) \times \mathcal{J}(x) \in \mathcal{IR}_{([\varsigma,\ \varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)])}$, and Condition C hold for $\mathfrak{w}$, then*

$$\begin{gathered}
\frac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2} \mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \times \mathcal{J}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right) \supseteq \frac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ (IR) \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)} \mathfrak{G}(x) \times \mathcal{J}(x)dx + \mathcal{M}(\varsigma,\mathfrak{q}) \int_0^1 \pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o} \\
+\mathcal{N}(\varsigma,\mathfrak{q}) \int_0^1 [\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2 d\mathfrak{o},
\end{gathered} \tag{36}$$

where $\mathcal{M}(\varsigma,\mathfrak{q}) = \mathfrak{G}(\varsigma) \times \mathcal{J}(\varsigma) + \mathfrak{G}(\mathfrak{q}) \times \mathcal{J}(\mathfrak{q})$, $\mathcal{N}(\varsigma,\mathfrak{q}) = \mathfrak{G}(\varsigma) \times \mathcal{J}(\mathfrak{q}) + \mathfrak{G}(\mathfrak{q}) \times \mathcal{J}(\varsigma)$, and $\mathcal{M}(\varsigma,\mathfrak{q}) = [\mathcal{M}_*((\varsigma,\mathfrak{q})),\ \mathcal{M}^*((\varsigma,\mathfrak{q}))]$ and $\mathcal{N}(\varsigma,\mathfrak{q}) = [\mathcal{N}_*((\varsigma,\mathfrak{q})),\ \mathcal{N}^*((\varsigma,\mathfrak{q}))]$.

Proof. Using Condition C, we can write

$$\varsigma+\frac{1}{2}\mathfrak{w}(\mathfrak{q},\ \varsigma)=\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\frac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma)).$$

By hypothesis, we have

$$\begin{aligned}
&\mathfrak{G}_*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\times\mathcal{J}_*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\\
&\mathfrak{G}^*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\times\mathcal{J}^*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\\
&=\mathfrak{G}_*\left(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\tfrac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\right)\\
&\times\mathcal{J}_*\left(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\tfrac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\right),\\
&=\mathfrak{G}^*\left(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\tfrac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\right)\\
&\times\mathcal{J}^*\left(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma)+\tfrac{1}{2}\mathfrak{w}(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\varsigma),\ \varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\varsigma))\right),\\
&\leq\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix}\\
&+\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix},\\
&\geq\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix}\\
&+\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix},\\
&\leq\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix}\\
&+\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}(\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}_*(\varsigma)+\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}_*(\mathfrak{q}))\\\times(\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{J}_*(\varsigma)+\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{J}_*(\mathfrak{q}))\\+(\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}_*(\varsigma)+\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}_*(\mathfrak{q}))\\\times(\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{J}_*(\varsigma)+\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{J}_*(\mathfrak{q}))\end{bmatrix},\\
&\geq\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix}\\
&+\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}(\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}^*(\varsigma)+\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}^*(\mathfrak{q}))\\\times(\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{J}^*(\varsigma)+\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{J}^*(\mathfrak{q}))\\+(\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{G}^*(\varsigma)+\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{G}^*(\mathfrak{q}))\\\times(\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathfrak{J}^*(\varsigma)+\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})\mathfrak{J}^*(\mathfrak{q}))\end{bmatrix},\\
&=\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}_*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix}\\
&+2\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{M}_*((\varsigma,\mathfrak{q}))\\+[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2\mathcal{N}_*((\varsigma,\mathfrak{q}))\end{bmatrix},\\
&=\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathfrak{G}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+(1-\mathfrak{o})\mathfrak{w}(\mathfrak{q},\ \varsigma))\\+\mathfrak{G}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\times\mathcal{J}^*(\varsigma+\mathfrak{o}\mathfrak{w}(\mathfrak{q},\ \varsigma))\end{bmatrix}\\
&+2\left[\pounds_1\left(\tfrac{1}{2}\right)\pounds_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})\mathcal{M}^*((\varsigma,\mathfrak{q}))\\+[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2\mathcal{N}^*((\varsigma,\mathfrak{q}))\end{bmatrix},
\end{aligned}$$

integrating over $[0,\ 1]$, we have

$$\begin{aligned}
&\tfrac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2}\ \mathfrak{G}_*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\times\mathcal{J}_*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\leq\tfrac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ \int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathfrak{G}_*(x)\times\mathcal{J}(x)_*(x)dx\\
&+\mathcal{M}_*((\varsigma,\mathfrak{q}))\int_0^1\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}\\
&+\mathcal{N}_*((\varsigma,\mathfrak{q}))\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o},\\
&\tfrac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2}\ \mathfrak{G}^*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\times\mathcal{J}^*\left(\tfrac{2\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}{2}\right)\geq\tfrac{1}{\mathfrak{w}(\mathfrak{q},\ \varsigma)}\ \int_{\varsigma}^{\varsigma+\mathfrak{w}(\mathfrak{q},\ \varsigma)}\mathfrak{G}^*(x)\times\mathcal{J}^*(x)dx\\
&+\mathcal{M}^*((\varsigma,\mathfrak{q}))\int_0^1\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}\\
&+\mathcal{N}^*((\varsigma,\mathfrak{q}))\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o},
\end{aligned}$$

from which we have

$$\frac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2}\left[\mathfrak{G}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)\times\mathcal{J}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right),\ \mathfrak{G}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)\times\mathcal{J}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)\right]$$
$$\supseteq\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)}\left[\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}\mathfrak{G}_*(x)\times\mathcal{J}_*(x)dx,\ \int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}\mathfrak{G}^*(x)\times\mathcal{J}^*(x)dx\right]$$
$$+\int_0^1\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}[\mathcal{M}_*((\varsigma,\mathfrak{q})),\ \mathcal{M}^*((\varsigma,\mathfrak{q}))]$$
$$+[\mathcal{N}_*((\varsigma,\mathfrak{q})),\ \mathcal{N}^*((\varsigma,\mathfrak{q}))]\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o},$$

that is

$$\frac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)\times\mathcal{J}\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)$$
$$\supseteq\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)}(IR)\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}\mathfrak{G}(x)\widetilde{\times}\mathcal{J}(x)dx+\mathcal{M}(\varsigma,\mathfrak{q})\int_0^1\pounds_1(\mathfrak{o})\pounds_2(\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(1-\mathfrak{o})d\mathfrak{o}$$
$$+\mathcal{N}(\varsigma,\mathfrak{q})\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o},\tag{37}$$

hence, the required result. □

Remark 4. *If $\pounds_2(\mathfrak{o})\equiv 1$, $\mathfrak{o}\in[0,\ 1]$, then Theorems 5 and 6 reduce for $\pounds_1$-preinvex I·V·Fs, see [73].*

If $\pounds_1(\mathfrak{o})=\mathfrak{o}$ and $\pounds_2(\mathfrak{o})\equiv 1$, $\mathfrak{o}\in[0,\ 1]$, then Theorems 5 and 6 reduce for preinvex I·V·Fs, see [73].

If in the above theorem $\mathfrak{G}_*(x)=\mathfrak{G}^*(x)$, then we obtain the appropriate Theorems 5 and 6 for $(\pounds_1,\ \pounds_2)$-preinvex functions, see [79].

If in Theorems 5 and 6 $\mathfrak{G}_*(x)=\mathfrak{G}^*(x)$ with $\pounds_2(\mathfrak{o})\equiv 1$, $\mathfrak{o}\in[0,\ 1]$, then we obtain the appropriate theorems for $\pounds_1$-preinvex functions, see [80].

If in Theorems 5 and 6 $\mathfrak{G}_*(x)=\mathfrak{G}^*(x)$, $\mathfrak{w}(y,x)=y-x$, $\pounds_1(\mathfrak{o})=\mathfrak{o}$ and $\pounds_2(\mathfrak{o})\equiv 1$, $\mathfrak{o}\in[0,\ 1]$, then we obtain the appropriate theorems for convex functions, see [81].

If in Theorems 5 and 6 $\mathfrak{G}_*(x)=\mathfrak{G}^*(x)$, $\mathfrak{w}(y,x)=y-x$, $\pounds_1(\mathfrak{o})=\mathfrak{o}^s$ and $\pounds_2(\mathfrak{o})\equiv 1$, $\mathfrak{o}\in[0,\ 1]$, $s\in[0,\ 1]$, then we obtain the appropriate theorems for s-convex functions in the second sense, see [81].

Example 3. *We consider $\pounds_1(\mathfrak{o})=\mathfrak{o}$, $\pounds_2(\mathfrak{o})\equiv 1$, for $\mathfrak{o}\in[0,\ 1]$, and the I·V·Fs defined by $\mathfrak{G}(x)=\left[2x^2,\ 4x\right]$ and $\mathcal{J}(x)=[x,\ 2x]$. Since end point functions $\mathfrak{G}_*(x)=2x^2$ and $\mathfrak{G}^*(x)=4x$ both are $(\pounds_1,\ \pounds_2)$-preinvex functions, and $\mathcal{J}_*(x)=x$, and $\mathcal{J}^*(x)=2x$ both are also $(\pounds_1,\ \pounds_2)$-preinvex functions with respect to same $\mathfrak{w}(y,\ x)=y-x$, then $\mathfrak{G}$ and $\mathcal{J}$ both are $(\pounds_1,\ \pounds_2)$-preinvex I·V·Fs, respectively. Since $\mathfrak{G}_*(x)=2x^2$ and $\mathfrak{G}^*(x)=4x$, and $\mathcal{J}_*(x)=x$, and $\mathcal{J}^*(x)=2x$, then*

$$\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}\mathfrak{G}_*(x)\times\mathcal{J}_*(x)dx=\int_0^1\left(2x^2\right)(x)dx=\frac{1}{2},$$
$$\frac{1}{\mathfrak{w}(\mathfrak{q},\varsigma)}\int_\varsigma^{\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}\mathfrak{G}^*(x)\times\mathcal{J}^*(x)dx=\int_0^1\left(4\varsigma^2\right)(2\varsigma)dx=\frac{8}{3},$$

$$\mathcal{M}_*((\varsigma,\mathfrak{q}))\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o}=\frac{2}{3},$$
$$\mathcal{M}^*((\varsigma,\mathfrak{q}))\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o}=\frac{8}{3},$$

$$\mathcal{N}_*((\varsigma,\mathfrak{q}))\int_0^1\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})d\mathfrak{o}=0$$
$$\mathcal{N}^*((\varsigma,\mathfrak{q}))\int_0^1\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})d\mathfrak{o}=0,$$

which means

$$\frac{1}{2}\leq\frac{2}{3}+0=\frac{2}{3},$$
$$\frac{8}{3}=\frac{8}{3}+0=\frac{8}{3},$$

Hence, Theorem 5 is verified.

For Theorem 6, we have

$$\frac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)\times\mathcal{J}_*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)=\frac{1}{2},$$
$$\frac{1}{2\left[\pounds_1\left(\frac{1}{2}\right)\pounds_2\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)\times\mathcal{J}^*\left(\frac{2\varsigma+\mathfrak{w}(\mathfrak{q},\varsigma)}{2}\right)=4,$$

$$\mathcal{M}_*((\varsigma,\mathfrak{q}))\int_0^1\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})d\mathfrak{o}=\frac{1}{3},$$
$$\mathcal{M}^*((\varsigma,\mathfrak{q}))\int_0^1\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})\pounds_1(1-\mathfrak{o})\pounds_2(\mathfrak{o})=\frac{4}{3},$$

$$\mathcal{N}_*((\varsigma,\mathfrak{q}))\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o}=0,$$
$$\mathcal{N}^*((\varsigma,\mathfrak{q}))\int_0^1[\pounds_1(\mathfrak{o})\pounds_2(1-\mathfrak{o})]^2d\mathfrak{o}=0,$$

which means

$$\left[\frac{1}{2}, 4\right] \supseteq \left[\frac{5}{6}, 4\right].$$

Hence, Theorem 6 is demonstrated.

4. Conclusions

A useful method for introducing uncertainty into prediction processes is to use interval-valued functions. In order to establish the Hermite-Hadamard and Pachpatt-type inequalities, we first introduced a new idea of interval-valued harmonic convexity, i.e., a harmonically interval valued (h_1, h_2)-preinvex function. Many of the definitions that already existed in the literature were generalized by our new concept. Thus, we contributed to the set-valued setting's extension of several classical integral inequalities. To further explain the findings, some numerical examples were given.

This innovative approach could be applied to future presentations of various inequalities, such as those of the Hermite-Hadamard, Ostrowski, Hadamard-Mercer, Simpson, Fejér, and Bullen kinds. Numerous interval-valued LR convexities, fuzzy interval convexities, and CR convexities can be used to illustrate the aforementioned inequalities. Additionally, these results will be used to fractional calculus, coordinated interval-valued functions, quantum calculus, and other areas. Many mathematicians will be interested in examining how various types of interval-valued analyses may be applied to integral inequalities because these are the most active areas of research in the field of integral inequalities.

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