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Some New Generalized Inequalities of Hardy Type Involving Several Functions on Time Scale Nabla Calculus

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Abstract: In this article, we establish several new generalized Hardy-type inequalities involving several functions on time-scale nabla calculus. Furthermore, we derive some new multidimensional Hardy-type inequalities on time scales nabla calculus. The main results are proved by applying Minkowski's inequality, Jensen's inequality and Arithmetic Mean–Geometric Mean inequality. As a special case of our results, when $\mathbb{T} = \mathbb{R}$, we obtain refinements of some well-known continuous inequalities and when $\mathbb{T} = \mathbb{N}$, the results which are essentially new.

Keywords: Hardy-type inequality; time scales nabla calculus; weighted functions; inequalities; arithmetic mean–geometric mean inequality.

MSC: 26D10; 26D15; 34N05; 42B25; 42C10; 47B38

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1. Introduction

In [1], Hardy proved that

$$\sum_{l=1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l}a(i)\right)^{q} \leq \left(\frac{q}{q-1}\right)^{q}\sum_{l=1}^{\infty}a^{q}(l), \quad q>1, \tag{1}$$

where $a(l) \geq 0$ for $l \geq 1$ and $\sum_{l=1}^{\infty} a^{q}(l) < \infty$.

In [2], Hardy proved the continuous case of (1) in the following form

$$\int_{0}^{\infty}\left(\frac{1}{\theta}\int_{0}^{\theta}f(\tau)d\tau\right)^{q}d\theta \leq \left(\frac{q}{q-1}\right)^{q}\int_{0}^{\infty}f^{q}(\theta)d\theta, \quad q>1, \tag{2}$$

where $f \geq 0$ and integrable over any finite interval $(0,\theta)$, $\theta \in (0,\infty)$, $f \in L^{q}(0,\infty)$ and the constant $(q/(q-1))^{q}$ in (1) and (2) is sharp.

In [3], Kaijser et al. established that if Φ is a convex function on $\mathbb{R}^{+}$, then

$$\int_{0}^{\infty}\Phi\left(\frac{1}{\lambda}\int_{0}^{\lambda}\varpi(\eta)d\eta\right)\frac{d\lambda}{\lambda} \leq \int_{0}^{\infty}\Phi(\varpi(\lambda))\frac{d\lambda}{\lambda}, \tag{3}$$

where $\varpi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ is a locally integrable positive function. In [4], Čižmešija et al. generalized (3) in the following form

$$\int_{0}^{r}\varkappa(\lambda)\Phi\left(\frac{1}{\lambda}\int_{0}^{\lambda}\varpi(\eta)d\eta\right)\frac{d\lambda}{\lambda} \leq \int_{0}^{r}\omega(\lambda)\Phi(\varpi(\lambda))\frac{d\lambda}{\lambda},$$

where $\varkappa : (0, r) \to \mathbb{R}$, $0 < r \le \infty$, is a non-negative function, such that $\lambda \to \varkappa(\lambda)/\lambda^2$ is locally integrable, Φ is a convex function, $\varpi : (0, r) \to \mathbb{R}\ \forall \lambda \in (0, r)$ and

$$\omega(\eta) = \eta \int_\eta^r \frac{\varkappa(\lambda)}{\lambda^2} d\lambda, \quad \text{for} \quad \eta \in (0, r).$$

In [5], Kaijser et al. explicated that if $\varkappa : (0, r) \to \mathbb{R}$, $\varrho : (0, r) \times (0, r) \to \mathbb{R}$, $0 < r \le \infty$ are positive functions, such that $0 < \Lambda(\eta) = \int_0^\eta \varrho(\eta, \vartheta) d\vartheta < \infty$, $\eta \in (0, r)$, Φ is a convex function on $I \subseteq \mathbb{R}$, and

$$\omega(\lambda) = \lambda \int_\lambda^r \varkappa(\eta) \frac{\varrho(\eta, \lambda)}{\Lambda(\eta)} \frac{d\eta}{\eta} < \infty, \quad \lambda \in (0, r),$$

then

$$\int_0^r \varkappa(\lambda) \Phi(A_\varrho \varpi(\lambda)) \frac{d\lambda}{\lambda} \le \int_0^r \omega(\lambda) \Phi(\varpi(\lambda)) \frac{d\lambda}{\lambda}, \tag{4}$$

where $\varpi : (0, r) \to \mathbb{R}$ is a function with values in I, and

$$A_\varrho \varpi(\lambda) = \frac{1}{\Lambda(\lambda)} \int_0^\lambda \varrho(\lambda, \vartheta) \varpi(\vartheta) d\vartheta,\ \lambda \in (0, r).$$

Additionally, in [5] it is established that if $1 < p \le q < \infty$, $s \in (1, p)$ and $0 < r < \infty$, $\varrho : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a non-negative kernel, $\varkappa(\lambda) \ge 0$ and $\omega(\lambda) \ge 0$ are weighted functions, and

$$\left(\int_0^r [\Phi(A_\varrho \varpi(\lambda))]^q \varkappa(\lambda) \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}} \le C \left[\int_0^r \Phi^p(\varpi(\lambda)) \omega(\lambda) \frac{d\lambda}{\lambda} \right]^{\frac{1}{p}}, \tag{5}$$

holds for all $\varpi(\lambda) \ge 0$, $\lambda \in [0, r]$ and $C > 0$, if

$$A(s) = \sup_{0 < \vartheta < r} [\Omega(\vartheta)]^{\frac{s-1}{p}} \left(\int_\vartheta^r \left(\frac{\varrho(\lambda, \vartheta)}{\Lambda(\lambda)} \right)^q [\Omega(\lambda)]^{\frac{q(p-s)}{p}} \varkappa(\lambda) \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}} < \infty,$$

where

$$\Omega(\vartheta) = \int_0^\vartheta [\omega(\eta)]^{\frac{-1}{p-1}} \eta^{\frac{1}{p-1}} d\eta.$$

In the last few decades, researchers discovered the time-scale calculus which unifies the continuous and discrete calculus. A time scale $\mathbb{T}$ is an arbitrary, non-empty closed subset of the real numbers $\mathbb{R}$. Many authors established some new dynamic inequalities on $\mathbb{T}$; see the books [6,7] and the papers [8–12].

In [13], Özkan et al. demonstrated that if $0 \le r < y \le \infty$, $u \in C_{rd}([r, y), \mathbb{R})$ is a non-negative function, such that $\int_t^y \frac{u(\lambda)}{(\lambda - r)(\sigma(\lambda) - r)} \Delta\lambda$ exists as a finite number, Φ is continuous and convex, $f \in C_{rd}([r, y), \mathbb{R})$ and

$$v(t) = (t - r) \int_t^y \frac{u(\lambda)}{(\lambda - r)(\sigma(\lambda) - r)} \Delta\lambda, \quad t \in [r, y),$$

then

$$\int_r^y u(\lambda) \Phi\left(\frac{1}{\sigma(\lambda) - r} \int_r^{\sigma(\lambda)} f(t) \Delta t \right) \frac{\Delta\lambda}{\lambda - r} \le \int_r^y v(\lambda) \Phi(f(\lambda)) \frac{\Delta\lambda}{\lambda - r}. \tag{6}$$

They also proved that if $u \in C_{rd}([y, \infty), \mathbb{R})$ is a non-negative function, and

$$v(t) = \frac{1}{t} \int_b^t u(\lambda) \Delta\lambda, \quad t \in [y, \infty),$$

then

$$\int_r^y u(\lambda)\Phi\left(\frac{1}{\sigma(\lambda)-r}\int_r^{\sigma(\lambda)} f(t)\Delta t\right)\frac{\Delta\lambda}{\lambda-r} \le \int_r^y v(\lambda)\Phi(f(\lambda))\frac{\Delta\lambda}{\lambda-r},$$

holds for all $f \in C_{rd}([y,\infty),\mathbb{R})$.

In [14], the authors proved the time-scale version of (4) as follows. Let $k(\lambda,\theta) \in C_{rd}([r,y)\times[r,y),\mathbb{R})$, $u \in C_{rd}([r,y),\mathbb{R})$ be non-negative functions, $f \in C_{rd}([r,y),\mathbb{R})$, Φ is a continuous and convex function, and

$$v(t) = (t-r)\int_t^y \frac{k(\lambda,t)}{K(\sigma(\lambda),\lambda)}u(\lambda)\frac{\Delta\lambda}{\lambda-r}, \quad t \in [r,y).$$

Then,

$$\int_r^y u(\lambda)\Phi(A_k f(\sigma(\lambda),\lambda))\frac{\Delta\lambda}{\lambda-r} \le \int_r^y v(\lambda)\Phi(f(\lambda))\frac{\Delta\lambda}{\lambda-r}, \tag{7}$$

where

$$A_k f(t,\beta) = \frac{1}{K(t,\beta)}\int_r^t k(\beta,\theta)f(\theta)\Delta\theta, \quad K(t,\beta) := \int_r^t k(\beta,\theta)\Delta\theta.$$

Our aim in this study is to generalize (4) on time-scale nabla calculus of power $\eta \ge 1$ in the form

$$\int_r^y \chi^\eta\left(A_\varrho\varpi(\zeta)\right)\frac{\varkappa(\zeta)}{\rho(\zeta)-r}\nabla\zeta \le \left(\frac{B}{A}\right)^\eta\left(\int_r^y \chi(\varpi(\vartheta))\frac{\omega(\vartheta)}{\rho(\vartheta)-r}\nabla\vartheta\right)^\eta,$$

where A, B are positive constants. We will also establish the last inequality for several functions. Furthermore, we will prove the last inequality in multidimensions on time-scales nabla calculus.

The paper proceeds as follows. In Section 2, we state some properties concerning the time-scales nabla calculus needed in Section 3, where we prove the main results. Our main results when $\mathbb{T} \to \mathbb{R}$, we obtain (4) proved by Kaijser et al. [5] and when $\mathbb{T} \to \mathbb{N}$, we obtain a new discrete inequality.

2. Preliminaries and Basic Lemmas

For a time scale $\mathbb{T}$, we define the backward jump operator as $\rho(\gamma) = \sup\{s \in \mathbb{T} : s < \gamma\}$. Additionally, we define a mapping $\nu : \mathbb{T} \to \mathbb{R}^+$ by $\nu(\gamma) = \gamma - \rho(\gamma)$, such that if ϖ is nabla differentiable at γ, then $\nu(\gamma)\varpi^\nabla(\gamma) = \varpi(\gamma) - \varpi^\rho(\gamma)$. For more details about $\mathbb{T}$ calculus, see ([6,7]).

The nabla derivative of $\varkappa\omega$ and $\varkappa/\omega$ (where $\omega(\gamma)\omega^\rho(\gamma) \neq 0$) are given by

$$\begin{aligned}(\varkappa\omega)^\nabla(\gamma) &= \varkappa^\nabla(\gamma)\omega(\gamma) + \varkappa^\rho(\gamma)\omega^\nabla(\gamma) \\ &= \varkappa(\gamma)\omega^\nabla(\gamma) + \varkappa^\nabla(\gamma)\omega^\rho(\gamma),\end{aligned}$$

and

$$\left(\frac{\varkappa}{\omega}\right)^\nabla(\gamma) = \frac{\varkappa^\nabla(\gamma)\omega(\gamma) - \varkappa(\gamma)\omega^\nabla(\gamma)}{\omega(\gamma)\omega^\rho(\gamma)}.$$

Definition 1 ([6]). *A function $F : \mathbb{T} \to \mathbb{R}$ is a nabla antiderivative of $\varpi : \mathbb{T} \to \mathbb{R}$ if $F^\nabla(t) = \varpi(t)$ holds $\forall t \in \mathbb{T}$. Hence, we have*

$$\int_r^t \varpi(\gamma)\nabla\gamma = F(t) - F(r) \quad \forall t \in \mathbb{T}.$$

Theorem 1 ([6]). *0If $r, y \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and ϖ, λ are ld-continuous functions, then*

(1) $\int_r^y [\varpi(\gamma) + \lambda(\gamma)]\nabla\gamma = \int_r^y \varpi(\gamma)\nabla\gamma + \int_r^y \lambda(\gamma)\nabla\gamma$;
(2) $\int_r^y \alpha\varpi(\gamma)\nabla\gamma = \alpha\int_r^y \varpi(\gamma)\nabla\gamma$;
(3) $\int_r^r \varpi(\gamma)\nabla\gamma = 0$.

The integration by parts formula on time scales nabla calculus [6] is

$$\int_r^y \varkappa(\gamma)\omega^{\nabla}(\gamma)\nabla\gamma = [\varkappa(\gamma)\omega(\gamma)]_r^y - \int_r^y \varkappa^{\nabla}(\gamma)\omega^{\rho}(\gamma)\nabla\gamma. \tag{8}$$

The Arithmetic Mean–Geometric Mean inequality is given by

$$[\lambda_1(\zeta)\lambda_2(\zeta)...\lambda_n(\zeta)]^{\frac{1}{n}} \leq \frac{\sum_{k=1}^n \lambda_k(\zeta)}{n}. \tag{9}$$

where $\lambda_1(\zeta), ..., \lambda_n(\zeta)$, $n \geq 1$ are non-negative functions.

In 2008, Ferreira et al. [15] proved Minkowski's inequality on diamond alpha time scales. As a special case of this inequality (when $\alpha = 0$), we get Minkowski's inequality on time-scale nabla calculus as follows.

Lemma 1 ([15]). *Let $r, y \in \mathbb{T}$ and f, g be non-negative functions. Then,*

$$\left(\int_r^y (f(\gamma) + g(\gamma))^p \nabla\gamma\right)^{\frac{1}{p}} \leq \left(\int_r^y f^p(\gamma)\nabla\gamma\right)^{\frac{1}{p}} + \left(\int_r^y g^p(\gamma)\nabla\gamma\right)^{\frac{1}{p}}, \tag{10}$$

for $p \geq 1$.

Lemma 2 ([16]). *Let $\varkappa, \omega$ and ϖ be non-negative functions on Ω, Υ and $\Omega \times \Upsilon$, respectively. If $\alpha \geq 1$, then*

$$\left(\int_\Omega \varkappa(\lambda)\left(\int_\Upsilon \varpi(\lambda,\vartheta)\omega(\vartheta)\nabla\vartheta\right)^{\alpha}\nabla\lambda\right)^{\frac{1}{\alpha}}$$
$$\leq \int_\Upsilon \omega(\vartheta)\left(\int_\Omega \varpi^{\alpha}(\lambda,\vartheta)\varkappa(\lambda)\nabla\lambda\right)^{\frac{1}{\alpha}}\nabla\vartheta. \tag{11}$$

Theorem 2 ([16]). *Let $\varepsilon_i, \varsigma_i \in \mathbb{T}$, $i = 1, 2, .., m$, $\gamma \geq 1$, $\varkappa : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{R}$ and $w, h : \mathbb{T}^m \to \mathbb{R}$ be non-negative rd-continuous functions. Then,*

$$\left[\int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} w(\mathbf{y})\left(\int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} h(\mathbf{z})\varkappa(\mathbf{y},\mathbf{z})\nabla\mathbf{z}\right)^{\gamma}\nabla\mathbf{y}\right]^{\frac{1}{\gamma}}$$
$$\leq \int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} h(\mathbf{z})\left(\int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} w(\mathbf{y})\varkappa^{\gamma}(\mathbf{y},\mathbf{z})\nabla\mathbf{y}\right)^{\frac{1}{\gamma}}\nabla\mathbf{z}, \tag{12}$$

where $\nabla\mathbf{y} = \nabla y_1 ... \nabla y_m$, $\varkappa(\mathbf{y},\mathbf{z}) = \varkappa(y_1, ..., y_m, z_1, ..., z_m)$, $w(\mathbf{z}) = w(z_1, ..., z_m)$ and $h(\mathbf{z}) = h(z_1, ..., z_m)$.

In [17], Jensen's inequality is proved for the diamond$-\alpha$ time scale. In the case, $\alpha = 0$, this inequality can be written in nabla time-scale calculus as follows.

Lemma 3 ([17]). *Let $r, y \in \mathbb{T}$, $h \in C_{ld}([r,y]_{\mathbb{T}}, \mathbb{R})$, $u : [r,y]_{\mathbb{T}} \to (c,d)$, $c, d \in \mathbb{R}$ be ld-continuous and Φ be continuous and convex. Then,*

$$\Phi\left(\frac{1}{\int_r^y h(\vartheta)\nabla\vartheta}\int_r^y h(\gamma)u(\gamma)\nabla\gamma\right) \leq \frac{1}{\int_r^y h(\vartheta)\nabla\vartheta}\int_r^y h(\gamma)\Phi(u(\gamma))\nabla\gamma. \tag{13}$$

If Φ is a concave function, then (13) will be reversed.

Theorem 3 ([17])**.** *Let $\varepsilon_i, \varsigma_i \in \mathbb{T}$, $i = 1, 2, ..., m$, $g : \mathbb{T}^m \to (c, d)$, $c, d \in \mathbb{R}$ be ld-continuous and Φ be continuous and convex. Then,*

$$\Phi\left(\frac{1}{\int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} \varrho(\mathbf{y}, \mathbf{z})\nabla\mathbf{z}} \int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} \varrho(\mathbf{y}, \mathbf{z}) g(\mathbf{z})\nabla\mathbf{z}\right)$$
$$\leq \frac{1}{\int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} \varrho(\mathbf{y}, \mathbf{z})\nabla\mathbf{z}} \int_{\varepsilon_1}^{\varsigma_1} ... \int_{\varepsilon_m}^{\varsigma_m} \varrho(\mathbf{y}, \mathbf{z})\Phi(g(\mathbf{z}))\nabla\mathbf{z}, \tag{14}$$

where $\nabla\mathbf{z} = \nabla z_1 ... \nabla z_m$, $\varrho(\mathbf{y}, \mathbf{z}) = \varrho(y_1, ..., y_m, z_1, ..., z_m)$ and $g(\mathbf{z}) = g(z_1, ..., z_m)$.

3. Main Results

Throughout this section, we will assume that the functions (without mention) are non-negative ld-continuous functions and the integrals in the statements of the theorems are convergent. We define the general Hardy operator A_ϱ as follows

$$A_\varrho\varpi(\lambda) = \frac{1}{\Lambda(\lambda)} \int_r^y \varrho(\lambda, \vartheta)\varpi(\vartheta)\nabla\vartheta, \quad \Lambda(\lambda) = \int_r^y \varrho(\lambda, \vartheta)\nabla\vartheta,$$

where $\lambda > r$ and $\varpi \in C_{ld}([r, y]_{\mathbb{T}}, \mathbb{R}^+)$ and $\varrho(\lambda, \vartheta) \in C_{ld}([r, y]_{\mathbb{T}} \times [r, y]_{\mathbb{T}}, \mathbb{R}^+)$.

Now, we state and prove our main results.

Theorem 4. *Let $r, y \in \mathbb{T}$, $\eta \geq 1$ and $\varkappa, \omega$ be weighted functions, such that*

$$\omega(\vartheta) = (\rho(\vartheta) - r)\left(\int_r^y \left(\frac{\varrho(\lambda, \vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda\right)^{\frac{1}{\eta}}. \tag{15}$$

Furthermore, assume that χ, ξ defined on (c, d), $-\infty < c < d < \infty$ and ξ is a convex function, such that

$$A\xi(\lambda) \leq \chi(\lambda) \leq B\xi(\lambda), \quad c < \lambda < d, \tag{16}$$

where A, B are positive constants; then

$$\int_r^y \chi^\eta\left(A_\varrho\varpi(\lambda)\right)\frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda \leq \left(\frac{B}{A}\right)^\eta \left(\int_r^y \chi(\varpi(\vartheta))\frac{\omega(\vartheta)}{\rho(\vartheta) - r}\nabla\vartheta\right)^\eta, \tag{17}$$

holds for the non-negative function ϖ.

Proof. Using (16) and applying (13) (where ξ is convex), we have

$$\int_r^y \chi^\eta\left(A_\varrho\varpi(\lambda)\right)\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda) - r}$$
$$= \int_r^y \chi^\eta\left(\frac{1}{\Lambda(\lambda)}\int_r^y \varrho(\lambda, \vartheta)\varpi(\vartheta)\nabla\vartheta\right)\frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda$$
$$\leq B^\eta \int_r^y \xi^\eta\left(\frac{1}{\Lambda(\lambda)}\int_r^y \varrho(\lambda, \vartheta)\varpi(\vartheta)\nabla\vartheta\right)\frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda$$
$$\leq B^\eta \int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda, \vartheta)\xi(\varpi(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda. \tag{18}$$

Applying (11) on the term

$$\int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda, \vartheta)\xi(\varpi(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda,$$

with $\eta \geq 1$, we see that

$$\left(\int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda,\vartheta)\xi(\varpi(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}$$
$$\leq \int_r^y \xi(\varpi(\vartheta))\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta,$$

then

$$\begin{aligned}
&\int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda,\vartheta)\xi(\varpi(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda \\
&\leq \left[\int_r^y \xi(\varpi(\vartheta))\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta\right]^\eta \\
&= \left[\int_r^y \xi(\varpi(\vartheta))\frac{1}{\rho(\vartheta)-r}(\rho(\vartheta)-r)\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta\right]^\eta. \qquad (19)
\end{aligned}$$

Substituting (19) into (18), we have from (15) that

$$\begin{aligned}
&\int_r^y \chi^\eta(A_\varrho\varpi(\lambda))\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda)-r} \\
&\leq B^\eta\left[\int_r^y \xi(\varpi(\vartheta))\frac{1}{\rho(\vartheta)-r}(\rho(\vartheta)-r)\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta\right]^\eta \\
&= B^\eta\left[\int_r^y \xi(\varpi(\vartheta))\frac{1}{\rho(\vartheta)-r}\omega(\vartheta)\nabla\vartheta\right]^\eta,
\end{aligned}$$

and then, we get from (16) that

$$\int_r^y \chi^\eta(A_\varrho\varpi(\lambda))\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda)-r} \leq \left(\frac{B}{A}\right)^\eta\left[\int_r^y \chi(\varpi(\vartheta))\frac{1}{\rho(\vartheta)-r}\omega(\vartheta)\nabla\vartheta\right]^\eta,$$

which is (17). □

Corollary 1. *If $A = B$ and $\eta = 1$, then*

$$\int_r^y \chi(A_\varrho\varpi(\lambda))\frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda \leq \left(\int_r^y \chi(\varpi(\vartheta))\frac{\omega(\vartheta)}{\rho(\vartheta)-r}\nabla\vartheta\right),$$

where

$$\omega(\vartheta) = (\rho(\vartheta)-r)\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)\frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right).$$

Remark 1. *If $\mathbb{T} = \mathbb{N}$, $r = 0$, then $\rho(n) = n-1$ and (17) reduces to*

$$\sum_{n=1}^N \chi\left(\frac{1}{\sum_{m=1}^n \varrho(n,m)}\sum_{m=1}^n \varrho(n,m)\varpi(m)\right)\frac{\varkappa(n)}{n-1}$$
$$\leq \left[\sum_{n=1}^N \chi(\varpi(n))\frac{\omega(n)}{n-1}\right], \quad \textit{for} \quad N \in \mathbb{N}.$$

Remark 2. *If $\mathbb{T} = \mathbb{R}$, $r = 0$, then $\rho(\zeta) = \zeta$, and we have*

$$\int_r^y \chi(A_\varrho\varpi(\lambda))\frac{\varkappa(\lambda)}{\lambda}d\lambda \leq \left(\int_r^y \chi(\varpi(\vartheta))\frac{\omega(\vartheta)}{\vartheta}d\vartheta\right),$$

where

$$\omega(\vartheta) = \vartheta\bigg(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)\frac{\varkappa(\lambda)}{\lambda}d\lambda\bigg).$$

Remark 3. *If* $\varrho(\lambda,\vartheta) = \begin{cases} 0 & , \lambda \in [r,\vartheta), \\ f(\lambda,\vartheta), & \lambda \in [\vartheta, y]. \end{cases}$ *, we get the inequality (4) proved by Kaijser et al. [5].*

The following theorem is proved for several functions.

Theorem 5. *Let* $r, y \in \mathbb{T}$, $\eta \geq 1$ *and* $\varkappa, \omega$ *be as in Theorem 4, such that*

$$\omega(\vartheta) = (\rho(\vartheta) - r)\bigg(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^{\eta}\frac{\varkappa(\lambda)}{\rho(\lambda) - r}\nabla\lambda\bigg)^{\frac{1}{\eta}}. \tag{20}$$

Furthermore, assume that $\varpi_k, k = 1, 2, ..., n$ *and* χ, ξ *are as in Theorem 4, such that*

$$A\xi(\lambda) \leq \chi(\lambda) \leq B\xi(\lambda), \tag{21}$$

where A, B *are positive constants, then*

$$\begin{aligned} &\int_r^y \left[\Pi_{k=1}^n \chi\big(A_\varrho \varpi_k(\lambda)\big)\right]^{\frac{\eta}{n}} \varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda) - r} \\ &\leq \left(\frac{B}{nA}\right)^{\eta}\left(\sum_{k=1}^{n}\left[\int_r^y \chi(\varpi_k(\vartheta))\frac{1}{\rho(\vartheta) - r}\omega(\vartheta)\nabla\vartheta\right]\right)^{\eta}, \end{aligned} \tag{22}$$

holds for $n \geq 1$.

Proof. Applying (Arithmetic Mean–Geometric Mean) inequality (9), we see that

$$\begin{aligned} &\left[\Pi_{k=1}^n \chi\big(A_\varrho \varpi_k(\lambda)\big)\right]^{\frac{1}{n}} \\ &= \left[\chi\big(A_\varrho \varpi_1(\lambda)\big)\chi\big(A_\varrho \varpi_2(\lambda)\big)...\chi\big(A_\varrho \varpi_n(\lambda)\big)\right]^{\frac{1}{n}} \\ &\leq \frac{\sum_{k=1}^n \chi\big(A_\varrho \varpi_k(\lambda)\big)}{n}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\int_r^y \left[\Pi_{k=1}^n \chi\big(A_\varrho \varpi_k(\lambda)\big)\right]^{\frac{\eta}{n}} \varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda) - r} \\ &\leq \int_r^y \left(\frac{\sum_{k=1}^n \chi\big(A_\varrho \varpi_k(\lambda)\big)}{n}\right)^{\eta} \varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda) - r} \\ &= \left(\frac{1}{n}\right)^{\eta}\int_r^y \left(\sum_{k=1}^{n} \chi\big(A_\varrho \varpi_k(\lambda)\big)\right)^{\eta} \varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda) - r}. \end{aligned} \tag{23}$$

By applying (10) (where $\eta \geq 1$), we observe that

$$\begin{aligned}
&\left(\int_r^y \left(\sum_{k=1}^n \chi\left(A_\varrho \varpi_k(\lambda)\right)\right)^\eta \varkappa(\lambda) \frac{\nabla\lambda}{\rho(\lambda)-r}\right)^{\frac{1}{\eta}} \\
&= \left(\int_r^y \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\left[\chi\left(A_\varrho \varpi_1(\lambda)\right)+\ldots+\chi\left(A_\varrho \varpi_n(\lambda)\right)\right]^\eta \nabla\lambda\right)^{\frac{1}{\eta}} \\
&\leq \left(\int_r^y \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\chi^\eta\left(A_\varrho \varpi_1(\lambda)\right)\nabla\lambda\right)^{\frac{1}{\eta}}+\ldots+\left(\int_r^y \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\chi^\eta\left(A_\varrho \varpi_n(\lambda)\right)\nabla\lambda\right)^{\frac{1}{\eta}} \\
&= \sum_{k=1}^n \left(\int_r^y \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\chi^\eta\left(A_\varrho \varpi_k(\lambda)\right)\nabla\lambda\right)^{\frac{1}{\eta}},
\end{aligned}$$

and then

$$\begin{aligned}
&\int_r^y \left(\sum_{k=1}^n \chi\left(A_\varrho \varpi_k(\lambda)\right)\right)^\eta \varkappa(\lambda) \frac{\nabla\lambda}{\rho(\lambda)-r} \\
&\leq \left[\sum_{k=1}^n \left(\int_r^y \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\chi^\eta\left(A_\varrho \varpi_k(\lambda)\right)\nabla\lambda\right)^{\frac{1}{\eta}}\right]^\eta.
\end{aligned} \tag{24}$$

Substituting (24) into (23), we have that

$$\begin{aligned}
&\int_r^y \left[\Pi_{k=1}^n \chi\left(A_\varrho \varpi_k(\lambda)\right)\right]^{\frac{\eta}{n}} \varkappa(\lambda) \frac{\nabla\lambda}{\rho(\lambda)-r} \\
&\leq \left(\frac{1}{n}\right)^\eta \left[\sum_{k=1}^n \left(\int_r^y \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\chi^\eta\left(A_\varrho \varpi_k(\lambda)\right)\nabla\lambda\right)^{\frac{1}{\eta}}\right]^\eta.
\end{aligned} \tag{25}$$

Using (21) and applying (13), we get

$$\begin{aligned}
&\int_r^y \chi^\eta\left(A_\varrho \varpi_k(\lambda)\right)\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda)-r} \\
&= \int_r^y \chi^\eta\left(\frac{1}{\Lambda(\lambda)}\int_r^y \varrho(\lambda,\vartheta)\varpi_k(\vartheta)\nabla\vartheta\right)\frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda \\
&\leq B^\eta \int_r^y \xi^\eta\left(\frac{1}{\Lambda(\lambda)}\int_r^y \varrho(\lambda,\vartheta)\varpi_k(\vartheta)\nabla\vartheta\right)\frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda \\
&\leq B^\eta \int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda,\vartheta)\xi(\varpi_k(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda.
\end{aligned} \tag{26}$$

Applying (11) on the term

$$\int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda,\vartheta)\xi(\varpi_k(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda,$$

with $\eta \geq 1$, we see that

$$\begin{aligned}
&\left(\int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda,\vartheta)\xi(\varpi_k(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}} \\
&\leq \int_r^y \xi(\varpi_k(\vartheta))\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta,
\end{aligned}$$

then

$$\begin{aligned}
&\int_r^y \frac{1}{\Lambda^\eta(\lambda)}\left(\int_r^y \varrho(\lambda,\vartheta)\xi(\varpi_k(\vartheta))\nabla\vartheta\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda \\
&\le \left[\int_r^y \xi(\varpi_k(\vartheta))\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta\right]^\eta \\
&= \left[\int_r^y \xi(\varpi_k(\vartheta))\frac{1}{\rho(\vartheta)-r}(\rho(\vartheta)-r)\right. \\
&\left.\times\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta\right]^\eta. \qquad (27)
\end{aligned}$$

Substituting (27) into (26) and using (20), we get

$$\begin{aligned}
&\int_r^y \chi^\eta(A_\varrho\varpi_k(\lambda))\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda)-r} \\
&\le B^\eta\left[\int_r^y \xi(\varpi_k(\vartheta))\frac{1}{\rho(\vartheta)-r}(\rho(\vartheta)-r)\right. \\
&\left.\times\left(\int_r^y \left(\frac{\varrho(\lambda,\vartheta)}{\Lambda(\lambda)}\right)^\eta \frac{\varkappa(\lambda)}{\rho(\lambda)-r}\nabla\lambda\right)^{\frac{1}{\eta}}\nabla\vartheta\right]^\eta \\
&= B^\eta\left[\int_r^y \xi(\varpi_k(\vartheta))\frac{1}{\rho(\vartheta)-r}\omega(\vartheta)\nabla\vartheta\right]^\eta.
\end{aligned}$$

From (21), we obtain

$$\begin{aligned}
&\int_r^y \chi^\eta(A_\varrho\varpi_k(\lambda))\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda)-r} \\
&\le \left(\frac{B}{A}\right)^\eta\left[\int_r^y \chi(\varpi_k(\vartheta))\frac{1}{\rho(\vartheta)-r}\omega(\vartheta)\nabla\vartheta\right]^\eta.
\end{aligned}$$

Substituting the last inequality into (25), we get

$$\begin{aligned}
&\int_r^y \left[\Pi_{k=1}^n\chi(A_\varrho\varpi_k(\lambda))\right]^{\frac{\eta}{n}}\varkappa(\lambda)\frac{\nabla\lambda}{\rho(\lambda)-r} \\
&\le \left(\frac{B}{nA}\right)^\eta\left(\sum_{k=1}^n\left[\int_r^y \chi(\varpi_k(\vartheta))\frac{1}{\rho(\vartheta)-r}\omega(\vartheta)\nabla\vartheta\right]\right)^\eta,
\end{aligned}$$

which is (22). □

Remark 4. *If $n=1$, we get Theorem 4.*

Multidimensional Inequalities on Time Scales

In the following section, we define

$$A_\varrho\varpi(\mathbf{y}) = \frac{1}{\Lambda(\mathbf{y})}\int_{\varepsilon_1}^{\epsilon_1}\ldots\int_{\varepsilon_m}^{\epsilon_m}\varrho(\mathbf{y},\mathbf{z})\varpi(\mathbf{z})\nabla\mathbf{z},\ \ \Lambda(\mathbf{y}) = \int_{\varepsilon_1}^{\epsilon_1}\ldots\int_{\varepsilon_m}^{\epsilon_m}\varrho(\mathbf{y},\mathbf{z})\nabla\mathbf{z},$$

where $\varrho(\mathbf{y},\mathbf{z}) = \varrho(y_1,\ldots,y_m,z_1,\ldots,z_m)$, $\nabla\mathbf{y} = \nabla y_1\ldots\nabla y_m$ and $\varpi(\mathbf{z}) = \varpi(z_1,\ldots,z_m)$.

Theorem 6. *Let $\varepsilon_i, \epsilon_i \in \mathbb{T}$, $i = 1,2,...,m$, $\eta \geq 1$ and $\varkappa, \omega$ be as in Theorem 4, such that*

$$\omega(\mathbf{z}) = (\rho(z_1) - \varepsilon_1)...(\rho(z_m) - \varepsilon_m) \times \left(\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \left(\frac{\varrho(\mathbf{y}, \mathbf{z})}{\Lambda(\mathbf{y})} \right)^{\eta} \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y} \right)^{\frac{1}{\eta}}. \tag{28}$$

In addition, assume that χ, ξ are as in Theorem 4, such that

$$A\xi(\mathbf{y}) \leq \chi(\mathbf{y}) \leq B\xi(\mathbf{y}), \tag{29}$$

where A, B are positive constants, then

$$\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \chi^{\eta} (A_{\varrho}\varpi(\mathbf{y})) \varkappa(\mathbf{y}) \frac{\nabla \mathbf{y}}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \leq \left(\frac{B}{A} \right)^{\eta} \left[\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \chi(\varpi(\mathbf{z})) \frac{1}{(\rho(z_1) - \varepsilon_1)...(\rho(z_m) - \varepsilon_m)} \omega(\mathbf{z}) \nabla \mathbf{z} \right]^{\eta}, \tag{30}$$

holds for the non-negative function ϖ.

Proof. Using (29) and applying (14), we get

$$\begin{aligned} &\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \chi^{\eta} (A_{\varrho}\varpi(\mathbf{y})) \varkappa(\mathbf{y}) \frac{\nabla \mathbf{y}}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \\ &= \int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \chi^{\eta} \left(\frac{1}{\Lambda(\mathbf{y})} \int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z}) \varpi(\mathbf{z}) \nabla \mathbf{z} \right) \\ &\times \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y} \\ &\leq B^{\eta} \int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \xi^{\eta} \left(\frac{1}{\Lambda(\mathbf{y})} \int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z}) \varpi(\mathbf{z}) \nabla \mathbf{z} \right) \\ &\times \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y} \\ &\leq B^{\eta} \int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \frac{1}{\Lambda^{\eta}(\mathbf{y})} \left(\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z}) \xi(\varpi(\mathbf{z})) \nabla \mathbf{z} \right)^{\eta} \\ &\times \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y}. \end{aligned} \tag{31}$$

Applying (12) on the term

$$\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \frac{1}{\Lambda^{\eta}(\mathbf{y})} \left(\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z}) \xi(\varpi(\mathbf{z})) \nabla \mathbf{z} \right)^{\eta} \times \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y},$$

with $\eta \geq 1$, we see that

$$\begin{aligned} &\left[\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \frac{1}{\Lambda^{\eta}(\mathbf{y})} \left(\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z}) \xi(\varpi(\mathbf{z})) \nabla \mathbf{z} \right)^{\eta} \times \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y} \right]^{\frac{1}{\eta}} \\ &\leq \int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \xi(\varpi(\mathbf{z})) \left(\int_{\varepsilon_1}^{\epsilon_1} ... \int_{\varepsilon_m}^{\epsilon_m} \left(\frac{\varrho(\mathbf{y}, \mathbf{z})}{\Lambda(\mathbf{y})} \right)^{\eta} \frac{\varkappa(\mathbf{y})}{(\rho(y_1) - \varepsilon_1)...(\rho(y_m) - \varepsilon_m)} \nabla \mathbf{y} \right)^{\frac{1}{\eta}} \nabla \mathbf{z}, \end{aligned}$$

then

$$\begin{aligned}
&\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\frac{1}{\Lambda^{\eta}(\mathbf{y})}\left(\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\varrho(\mathbf{y},\mathbf{z})\xi(\varpi(\mathbf{z}))\nabla\mathbf{z}\right)^{\eta}\\
&\times\frac{\varkappa(\mathbf{y})}{(\rho(y_1)-\varepsilon_1)\dots(\rho(y_m)-\varepsilon_m)}\nabla\mathbf{y}\\
&\leq\Bigg[\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\xi(\varpi(\mathbf{z}))\\
&\times\left(\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\left(\frac{\varrho(\mathbf{y},\mathbf{z})}{\Lambda(\mathbf{y})}\right)^{\eta}\frac{\varkappa(\mathbf{y})}{(\rho(y_1)-\varepsilon_1)\dots(\rho(y_m)-\varepsilon_m)}\nabla\mathbf{y}\right)^{\frac{1}{\eta}}\nabla\mathbf{z}\Bigg]^{\eta}. \qquad (32)
\end{aligned}$$

Substituting (32) into (31), we have from (28) that

$$\begin{aligned}
&\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\chi^{\eta}(A_{\varrho}\varpi(\mathbf{y}))\varkappa(\mathbf{y})\frac{\nabla\mathbf{y}}{(\rho(y_1)-\varepsilon_1)\dots(\rho(y_m)-\varepsilon_m)}\\
&\leq B^{\eta}\Bigg[\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\xi(\varpi(\mathbf{z}))\\
&\times\left(\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\left(\frac{\varrho(\mathbf{y},\mathbf{z})}{\Lambda(\mathbf{y})}\right)^{\eta}\frac{\varkappa(\mathbf{y})}{(\rho(y_1)-\varepsilon_1)\dots(\rho(y_m)-\varepsilon_m)}\nabla\mathbf{y}\right)^{\frac{1}{\eta}}\nabla\mathbf{z}\Bigg]^{\eta}\\
&=B^{\eta}\left[\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\xi(\varpi(\mathbf{z}))\frac{1}{(\rho(z_1)-\varepsilon_1)\dots(\rho(z_m)-\varepsilon_m)}\omega(\mathbf{z})\nabla\mathbf{z}\right]^{\eta},
\end{aligned}$$

and then we have from (29) that

$$\begin{aligned}
&\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\chi^{\eta}(A_{\varrho}\varpi(\mathbf{y}))\varkappa(\mathbf{y})\frac{\nabla\mathbf{y}}{(\rho(y_1)-\varepsilon_1)\dots(\rho(y_m)-\varepsilon_m)}\\
&\leq\left(\frac{B}{A}\right)^{\eta}\left[\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\chi(\varpi(\mathbf{z}))\frac{1}{(\rho(z_1)-\varepsilon_1)\dots(\rho(z_m)-\varepsilon_m)}\omega(\mathbf{z})\nabla\mathbf{z}\right]^{\eta},
\end{aligned}$$

which is (30). □

Remark 5. *If $\mathbb{T}=\mathbb{R}$, $\rho(\vartheta)=\vartheta$ and $A=B=1$, then*

$$\begin{aligned}
&\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\chi^{\eta}(A_{\varrho}\varpi(\mathbf{y}))\frac{\varkappa(\mathbf{y})}{(y_1-\varepsilon_1)\dots(y_m-\varepsilon_m)}d\mathbf{y}\\
&\leq\left(\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\chi(\varpi(\mathbf{z}))\frac{\omega(\mathbf{z})}{(z_1-\varepsilon_1)\dots(z_m-\varepsilon_m)}d\mathbf{z}\right)^{\eta},
\end{aligned}$$

where $\varpi(\mathbf{z})=\varpi(z_1,z_2,\dots,z_m)$, $\varrho(\mathbf{y},\mathbf{z})=\varrho(y_1,\dots,y_m,z_1,\dots,z_m)$ and

$$A_{\varrho}\varpi(\mathbf{y})=\frac{1}{\Lambda(\mathbf{y})}\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\varrho(\mathbf{y},\mathbf{z})\varpi(\mathbf{z})d\mathbf{z},\quad\Lambda(\mathbf{y})=\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\varrho(\mathbf{y},\mathbf{z})d\mathbf{z},$$

with

$$\begin{aligned}
\omega(\mathbf{z})&=(z_1-\varepsilon_1)\dots(z_m-\varepsilon_m)\\
&\times\left(\int_{\varepsilon_1}^{\epsilon_1}\dots\int_{\varepsilon_m}^{\epsilon_m}\left(\frac{\varrho(\mathbf{y},\mathbf{z})}{\Lambda(\mathbf{y})}\right)^{\eta}\frac{\varkappa(\mathbf{y})}{(y_1-\varepsilon_1)\dots(y_m-\varepsilon_m)}d\mathbf{y}\right)^{\frac{1}{\eta}}.
\end{aligned}$$

Remark 6. *If $\mathbb{T} = \mathbb{N}$, $\rho(\vartheta) = \vartheta - 1$ and $A = B = 1$, then*

$$\sum_{\varepsilon_1}^{\epsilon_1} \dots \sum_{\varepsilon_m}^{\epsilon_m} \chi^{\eta}\left(A_{\varrho}\varpi(\mathbf{y})\right) \frac{\varkappa(\mathbf{y})}{(y_1 - \varepsilon_1 - 1)\dots(y_m - \varepsilon_m - 1)}$$
$$\leq \left[\sum_{\varepsilon_1}^{\epsilon_1} \dots \sum_{\varepsilon_m}^{\epsilon_m} \chi(\varpi(\mathbf{z})) \frac{1}{(z_1 - \varepsilon_1 - 1)\dots(z_m - \varepsilon_m - 1)} \omega(\mathbf{z})\right]^{\eta},$$

where $\varpi(\mathbf{z}) = \varpi(z_1, z_2, \dots, z_m)$, $\varrho(\mathbf{y}, \mathbf{z}) = \varrho(y_1, \dots, y_m, z_1, \dots, z_m)$ and

$$A_{\varrho}\varpi(\mathbf{y}) = \frac{1}{\Lambda(\mathbf{y})} \sum_{\varepsilon_1}^{\epsilon_1} \dots \sum_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z})\varpi(\mathbf{z}), \quad \Lambda(\mathbf{y}) = \sum_{\varepsilon_1}^{\epsilon_1} \dots \sum_{\varepsilon_m}^{\epsilon_m} \varrho(\mathbf{y}, \mathbf{z}),$$

with

$$\omega(\mathbf{z}) = (z_1 - \varepsilon_1 - 1)\dots(z_m - \varepsilon_m - 1)$$
$$\times \left(\sum_{\varepsilon_1}^{\epsilon_1} \dots \sum_{\varepsilon_m}^{\epsilon_m} \left(\frac{\varrho(\mathbf{y}, \mathbf{z})}{\Lambda(\mathbf{y})}\right)^{\eta} \frac{\varkappa(\mathbf{y})}{(y_1 - \varepsilon_1 - 1)\dots(y_m - \varepsilon_m - 1)}\right)^{\frac{1}{\eta}}.$$

4. Conclusions

In this research, we generalize some new inequalities on time-scale nabla calculus. We will also establish some dynamic inequalities for several functions. Furthermore, we will establish these inequalities in multiple dimensions on time-scales nabla calculus. All of these inequalities can be proved by applying Minkowski's inequality, Jensen's inequality and Arithmetic Mean–Geometric Mean inequality. In the future, we hope to study these dynamic inequalities via conformable nabla fractional calculus on time scales.

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