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Diagrammatic and Modal Dimensions of the Syllogisms of Hegel and Peirce

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Abstract: While in his *Science of Logic*, Hegel employed neither diagrams nor formulae, his reinterpretation of Aristotle's syllogistic logic in the "Subjective Logic" of Book III strongly suggests a diagrammatic dimension. Significantly, an early diagram depicting a "triangle of triangles" found among his papers after his death captures the organization of categories to be found in *The Science of Logic*. Features of this diagram help us understand Hegel's logical project as an attempt to retrieve features of Plato's thinking that are implicit within Aristotle's syllogistic logic. It is argued that parallels between Hegel's modification of Aristotle's syllogistic figures and Peirce's functional alignment of those syllogistic figures with his three inference forms—deduction, induction, and abduction—suggest modifications of the traditional "square of opposition" into a logical hexagon as found in recent discussions. However, Hegel had conceived of Aristotle's syllogism as a distorted version of the "syllogism" thought by Plato to bind the parts of the cosmos into a unity as described in the dialogue *Timaeus*. In accord with this, it is argued that seen in the light of Hegel's platonistic reconstruction of Aristotle's logic, such logical hexagons should be understood as two-dimensional projections of a logical polyhedron.

Keywords: Hegel; Peirce; Aristotle; logical hexagon; dialectical logic; diagrammatic logic; syllogistic logic; modal logic

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1. Introduction

After Hegel's death in 1831, Karl Rosenkranz, who had been assigned to edit Hegel's manuscripts and papers, found a single page seemingly written sometime in 1800–1801, containing a diagram meant to represent the "beautiful bond" which Plato, in the *Timaeus*, had claimed linked the parts of the "cosmic animal" into a coherent whole [1]. It depicts a "triangle of triangles", showing an inverted equilateral triangle embedded within another (Figure 1, left). The embedded triangle has sides half the length of the larger, such that a further three smaller triangles are generated inside the first with the same orientation. Clearly, the process can be iterated indefinitely in each of the component upright triangles in a "fractal" way as now found in a "Sierpinski triangle" [2] (Figure 1, right).

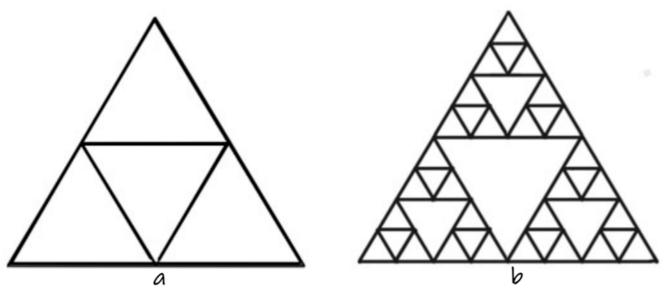

Figure 1. (**a**) Hegel's "Triangle of Triangles" (adapted from [1] (p. 149)). (**b**) A Sierpinski Triangle, with three levels of iteration.

As Hegel would surely have been aware, such triangular designs were to be found in tiling patterns in medieval cathedrals [3], the triangle itself a commonly used symbol for the Christian Trinity with the potentially infinite iteration designed to draw the viewer into the pattern, as it were, inducing a feeling of the infinity of the absolute being.

It is often noted that during his early years, Hegel had been attracted to such mystical or "theosophical" elements within medieval Christianity, although by 1800 he is thought to have been moving in more philosophical directions [4] (p. 184). While Hegel's diagram may still have reflected such mystical interests, it also, it has been claimed, reflected an interest in "geometric logic" [1] (p. 139). Consonant with this, it is known that around that time Hegel had embarked upon an intensive reading of Euclid's *Elements* [5]. With this interest in ancient Greek geometry in mind, it is easy to recognize the likeness of his triangle of triangle to the ancient Pythagorean *tetraktys* a figure in which ten dots are arranged in a triangular sequence of rows of one, two, three and four, as in Figure 2 below. The *tetraktys* had been invoked in relation to Plato's world-soul only a year or so before by an associate of Hegel's friend, F. W. Schelling, the Christian nature-philosopher, Franz von Baader [6]. In ancient times, the *tetraktys* had been used to represent the number theory applied by the Pythagoreans in music and cosmology and, following them, Plato in his account of the structure of the cosmos in the *Timaeus*. A quarter of a century after drawing his diagram, in his lectures on the history of Greek philosophy at the University of Berlin, Hegel would portray Plato's arithmetically structured "beautiful bond", binding the parts of the cosmos into a whole, as a "syllogism" and as the source and underlying nature of Aristotle's syllogistic logic [7] (p. 210).

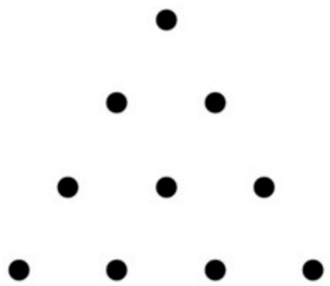

Figure 2. The Pythagorean *tetraktys*.

In this paper, I pursue parallels between the ways that Hegel and, later and seemingly independently, Charles Sanders Peirce, would attempt to modify Aristotle's syllogistic logic in ways that exploited issues of order, direction and spatial orientation in quasi-diagrammatic ways. Like many later commentators on Aristotle's syllogistic logic, Hegel understood it to be based on Plato's speculative thought. However, Aristotle had practiced philosophy "as a thoughtful observer of the world who attends to all aspects of the universe" [7] (p. 232), and so from this perspective, judgments and syllogisms were primarily intended to be adequate to the world as observed. In relation to this, Hegel would note how Plato's own syllogism, when understood as itself instantiated in nature as in the *Timaeus*, exhibited an important difference to Aristotle's. While the middle terms of Plato's syllogisms were described as doubled or "split", those of Aristotle's formal syllogism were single and univocal [7] (p. 210). Hegel would thus modify Aristotle's in an attempt to restore its "speculative" or "rational" Platonic foundations.

In his reading of the *Timaeus*, Hegel linked Plato's dual or split middle term to the three-dimensionality of the cosmos, as encoded in the Pythagorean *tetraktys* which, among other things, was meant to represent the spatial dimensions of the universe: one dot representing a *point*; two dots representing the two points that define a *line*; three, those defining a *plane*; and four, those defining a *volume*. However, the Pythagoreans had organized numerical quantities into ratios and proportions, or *ratios* of ratios, such that three points of a plane, for example, were to be thought of as a pair of ratios sharing a common "middle term" that divided the "extremes". As in the fourth row of the *tetraktys*, corresponding to the three dimensions of space, the extremes were mediated by two, not one, middle terms [8] (pp. 55–63; 109–115). Effectively, increasing the number of "middle terms" for Hegel increased the dimensionality of the "logical space" inhabited by judgments.

Although Hegel never subsequently employed diagrams in relation to logic, I will argue that his modification of Aristotle's syllogistic logic, like Peirce's, strongly suggests the type of "hexagonal" expansions of the traditional square of opposition as pursued by a number of logicians from the middle of the twentieth century to the present to give a proper place to the *modal* dimensions of Aristotle's syllogistic [9,10]. Modal logics differ in their inferential properties from non-modal ones, and comprehensive logics meant to capture both types of logic are sometimes described as "two-dimensional" [11]. However, as Plato's "syllogism" was *three*-dimensional, Hegel seems to have aspired to capture the unity of these two different, modal and nonmodal, dimensions in a way that presupposed a *third* dimension. Accordingly, I will suggest, for Hegel, any such logical hexagons would be *better* understood as two-dimensional projections of some form of Platonic logical *polygon*.

2. Original Presentation

Hegel's *Science of Logic* had appeared in two volumes. Volume One, titled "Objective Logic" and published in 1812, consisted of two books, "The Doctrine of Being" and "The Doctrine of Essence", while Volume Two, "The Subjective Logic" published in 1816, contained only one, "The Doctrine of the Concept" [12]. It is in Book One that Hegel's discussion of mathematics is to be found, while Book Three contains his own reinterpretation of Aristotle's syllogistic logic. While it is a commonplace of Hegel interpretation that Hegel's "logic" is not, like modern logic, "mathematical" [13] (p. 161), this simple denial fails to capture the complexity of the relation of logic to mathematics in Hegel's approach.

Hegel eschews any axiomatic presentations in ways that rely on establishing fixed self-evident truths as a basis on which other truths can be built or in some way derived. Thus, when starting the Objective Logic with the category of "Being", a concept thought to be unproblematically attributed to *anything*, this concept will show itself to be "contradictory" and hence not a concept that can be simply presupposed by others. As Being has no features that distinguish it from its apparent opposite, Nothing, the distinction collapses, and the two components of the opposition have to be rethought, now as the internal "moments" of a third concept, Becoming [12] (pp. 59–60). The series of categorical triads that come to be progressively unpacked throughout *The Science of Logic* from this apparently simple starting point are meant to be understood as having been implicit within the original category, Being, the unfolding structure having clear parallels in the way triads of triangles had been unpacked within the original triangle representing God, the absolute being, in Hegel's early diagram.

Hegel calls this process in which categories that were initially affirmed but which become negated in a way that allows them to be redefined as elements of a new holistically conceived totality, as in the passage of Being, through Nothing, to Becoming, "*Aufhebung*", usually translated as "sublation" and taken as having dimensions of both negation and preservation. This methodology ensures that configurations of concepts that appear in Books One and Two are likely to reappear as "sublated" [*aufgehoben*] in new contexts in Book Three. We should expect, then, that the mathematical structures found in Book One will have analogues in the "Subjective Logic", but not in such a way that the logic could be understood as *grounded* in mathematics. The relation will be reversed. While mathematical structures will be *reflected* in logical ones, it is the later *logic* that provides the perspective from which the earlier mathematics is meant to be understood, not vice versa.

While Hegel's interest in mathematics seems primarily to have been directed to *Greek* mathematics, it was not entirely limited to it. For example, one finds in the discussion of "Quantity" in Book I of *The Science of Logic*, discussions of the metaphysical problems surrounding the modern use of infinitesimals. However, such modern topics are standardly dealt with in their relation to the earlier Greek approaches from which they developed—in this case, the "method of exhaustion" that had been used by Archimedes to approximate a numerical value for the ratio of the circumference of a circle to its radius (the magnitude π), Hegel actually having possessed a copy of Archimedes' text, *Measurement of a Circle* [14] (p. 670).

While acknowledging the advances of modern science, Hegel was generally critical of the idea that the modern sciences could simply *replace* the ancient thought upon which they drew. For example, he supported Kepler's *geometric* approach to celestial mechanics over Newton's more "analytic" approach linked to modern calculus [15] (§ 270 remark), an approval he extended to Kepler's invocation of Plato's cosmology of the *Timaeus* [16]. Going against the grain of the modern reception of Newton, he seemed to think that Newtonian mechanics, at least as developed *algebraically* in continental Europe, had, in eliminating the continuous magnitudes of geometry, deprived itself of the capacity to be empirically applied to the world. Hegel's interest in geometry had even extended to contemporary movements such as the revival of *descriptive* or *synthetic* geometry in France at the turn of the nineteenth century, which led into the development of *projective* geometry. In fact, Hegel had possessed a work in which the hero of the French Revolutionary wars, the military engineer, Lazare Carnot, had reintroduced the fundamentals of projective geometry together with its distinctive double ratio later called the "harmonic cross-ratio" [14,17]. Projective geometry, a form of non-Euclidean geometry found in late antiquity, had been briefly, but unsuccessfully, revived in the Seventeenth Century by Girard Desargues [18] and would be developed throughout the nineteenth century in ways that integrated geometry with new forms of algebra [19]. It would later influence the approach to logic of Peirce.

Hegel was an omnivorous reader interested in, among many other things, the history of mathematics but was himself no mathematician. Charles Sanders Peirce was very different. The scientific polymath son of the distinguished American mathematician, Benjamin Peirce, C. S. Peirce had thought of himself principally as a "logician", understanding logic in the strict mathematical sense initiated in the middle of the nineteenth century by George Boole and August de Morgan. With a strong philosophical bent, Peirce was aware of the clear parallels between the triadic structuring of Hegel's logical categories and that of his own [20] (chs. 8, 9, and 10). However, he was critical of the lack of any actual mathematics or formalization in Hegel's work, as well as what he perceived as Hegel's lack of competence in the area. Nevertheless, he recognized deep connections binding the ideas of both of them to geometry and the issue of the irreducibility of the continuum to discrete quantities. Thus, in 1891 he would write: "Had I more space, I now ought to show how important for philosophy is the mathematical conception of continuity. Most of what is true in Hegel is a darkling glimmer of a conception which the mathematicians had long before made pretty clear, and which recent researchers have still further illustrated" [21] (p. 296).

Peirce was active from the mid eighteen sixties until his death in 1914, and his thinking about logic was, while drawing on the algebraic approach of Boole and de Morgan, deeply indebted to recent developments in geometry including the projective geometry of which Hegel had caught an early glimpse in the work of Carnot. Via the work of his father, Peirce was particularly influenced by the type of vector theory developed by Hermann Grassmann as "linear extension theory" [22] and that would become known as "linear algebra"—a staple of mathematics education today. Grassmann had been a student at the University of Berlin during Hegel's time there and had read Hegel's logic, and while his new approach to mathematics has been linked to Hegel's logic [23], it would seem that he had followed the different but overlapping "dialectics" of Friedrich Schleiermacher [24]. In short, despite Peirce's criticisms of Hegel, their attitudes to the relation of mathematics to logic may not have been as different as Peirce had assumed. While Peirce had advocated a strongly *mathematized* logic, he was deeply opposed to the *arithmetization* of geometry progressing in the discipline of "analysis" during this period and central to the modern "classical logic" of Frege, Peano, and Russell. Moreover, Peirce would devote much time to the development of new species of logical graphs that drew on the new forms of geometry.

To try to illuminate some of the mathematical dimensions of Hegel's logic that were similar to those employed by Peirce, I will turn to Hegel's interpretation of the Greek approach to ratios and proportions that he addresses in Book I of *The Science of Logic*. There we see important structures that will be "*aufgehoben*" into his treatment of Aristotelian logic in Book III. Crucial here will be the roles of *directionality* and *inversion* allowed by diagrammatic logical representations and reflecting the dimensionality of space itself.

3. Hegel's Account of the Role of Ratio in Greek Mathematics at the Time of Plato's Academy

The section "Magnitude" in Book I of *The Science of Logic* is the longest section of that work, testifying to the importance of mathematics for Hegel. In it, mathematical ideas are presented as evolving from simple procedures of counting and measuring, this development portrayed as a conceptual one that in some respects runs parallel with an historical account from the Greeks up to Hegel's present. Hegel's ambition was to unearth the philosophical assumptions underlying basic practices within first-order calculations. In Greek mathematics, these had especially concerned assumptions about the nature of number and the relation of number to the continuous magnitudes of geometry—issues that would be similarly central to Peirce in his opposition to the modern analytic reduction of continuous to discrete magnitudes.

The history of Greek mathematics is still a heavily contested realm [25], but read in the light of standard accounts as found, for example, in the still relevant work of Thomas Heath [26] or, more recently, those of Wilfrid Knorr or Árpád Szabó [27–29], Hegel seems to have been well informed and his analyses relatively secure. He seems particularly to have been influenced by the work of Proclus (412–485 CE) [5] and other late neo-Platonic and neo-Pythagorean thinkers such as Nicomachus of Gerasa (c. 60–120 CE) and Iamblichus (c. 245–325 CE) [14], who still today remain, along with the writings of Plato and Aristotle, important sources for the reconstruction of the earlier "golden age" of Greek geometry. In relation to his treatment of Aristotle's syllogistic in Book III, perhaps the most relevant parts of Hegel's treatment of quantity concern those devoted to the category of "ratio" (*Verhältnis*), which he treats as developing through the phases of "direct ratio", "inverse ratio" and the "ratio of powers". In the twentieth century, the classicist Benedict Einarson would show Aristotle's heavy dependence on the vocabulary of this form of mathematics that the Pythagoreans had developed in relation to both music and cosmology [30], with Aristotle discussing the constituent divisions of syllogisms into judgments in ways that paralleled the discussion of the division of musical intervals in the work *Sectio Canonis* [30] (p. 158).

By "direct ratio" Hegel means ratios of the natural numbers as recognized by early Pythagoreans, but, as the story is standardly told, at some point in the development of Greek mathematics, this conception of magnitude had been threatened by the discovery of the phenomenon of the incommensurability between some continuous magnitudes and others, such as that between the side and diagonal of a square. Were the side of a square deemed to measure one unit, the length of the diagonal was *irrational* in the sense of unable to be represented by a ratio of natural numbers, this idea seeming to have challenged the very idea of "rational" thought being able to properly "measure" the world. A similar scepticism, as found in the Sophists, for example, had been challenged by Plato, while in the context of mathematics, this apparent threat to rationality was addressed in the early Academy by mathematicians such as Theaetetus and Eudoxus of Cnidus.

In the *Theaetetus*, Plato portrays the geometer Theodorus of Cyrene as offering a proof to Theaetetus and another young Athenian of the irrationality of a series of non-square numbers up to 17 [31] (147d-148c]. In the account, Theaetetus himself had gone on to discover a general proof for the irrationality of all non-square numbers and to develop a classification of different kinds of irrational numbers [32]. This rendered them utterable or expressible [*rhetai*] and law-like and, *in that sense*, rational [*loga*] despite not being expressible in ratios [*logoi*] of natural numbers [25] (pp. 162–3). Later, Eudoxus had taken this approach further when he discovered a way of taking the ratio between incommensurable line segments as determinate *despite* their being unable to be specified by pairs of whole numbers, a discovery that has been described as an anticipation of Dedekind's specification of the real number series at the end of the nineteenth century [33]. Such developments are generally thought to have liberated Greek geometry from the constraints of the *arithmeticism* of the early Pythagoreans, limited as it was by their conception of number in which there was no place for zero or for negative numbers, let alone "irrational" ones.

Hegel's *inverse* ratios draw upon Euclid's definition 12 of Book V, which states that "inverse ratio means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent" [34]. That is, if a:b :: c:d, then a:c :: b:d. As Book V, believed to have been written by Eudoxus, is concerned with ratios of continuous magnitudes, Hegel's transition from direct to inverse ratios would appear to coincide with that between discrete and continuous magnitudes. We might expect, then, that in line with his idea of the two stages of negation involved in *Aufhebung*, features of both direct and inverse ratio should be found as somehow redetermined in his *ratio of powers* (*Potenzenverhältniss*).

Hegel's discussion of this ratio is far from clear, but we would expect the construction of squares to have a role. The Greek word for power, *dynamis*, had been taken from the word for square, [29] (pt. 1.2)—an etymology reflected in the fact that in modern languages such as English and German, a number multiplied by itself is described as "squared" (in German, "*quadriert*") or as raised to the "power" ("*Potenz*") of 2. The Greeks, however, had interpreted such arithmetical "squares" *geometrically*, identifying the square of a number n (n^2) with the *area* of a square constructed on a line segment of length n. This introduced the problem of *incommensurability* among magnitudes, as it was realized that the length of sides of squares with certain areas, for example, an area of 3 square units, were incommensurable with the length of the sides of a square with area 1 square unit (in modern terms, that the number $\sqrt{3}$ is irrational).

Hegel was critical of the tendency in modern thought, exemplified, for example, by Descartes' analytic geometry as well as modern calculus, to disregard these "qualitative" differences by reducing continuous magnitudes to discrete numerical values. The ratio of powers was meant to represent a quantum in which this incommensurability among different *kinds* of magnitudes is *aufgehoben* rather than ignored. Thus, he describes the ratio of powers as a quantum that posits itself "as self-identical in its otherness". In "determining its own movement of self-surpassing", this ratio "has come to be a being-for-itself" and is "posited in the potency of having returned into itself; it is immediately itself and also its otherness" [12] (p. 278). Hegel's verbiage may be opaque, but there is a clear sense of some type of relation that brings otherwise incommensurable magnitudes into some form of unity. His descriptions, I suggest, can be illuminated by Plato's "beautiful bond" in which he took so much interest.

In Nicomachus of Gerasa's *Introduction to Arithmetic*, a version of which was also in Hegel's possession [14] (p. 672–673), Nicomachus had identified Plato's beautiful bond as a peculiar double-ratio called the "*harmonia*" or "musical *tetraktys*". This arithmetical structure, mentioned in the work, *Epinomis*, attributed to Plato but probably written by one of his followers [31] (991b), was represented by the numbers 6, 8, 9, and 12, and had been drawn from Pythagorean music theory [35]. What was significant about this sequence for the Pythagoreans was the way it was regarded as uniting *geometric*, *arithmetic*, and *harmonic* means—the latter two being incommensurable with the former. This triadically articulated unity involving incommensurable magnitudes, would fit perfectly the needs of Hegel's categorical triads.

In the *harmonia*, the extremes 6 and 12 are taken as belonging to an extended geometric ratio (n, 2n, 4n . . .), and taken *musically* as representing a full octaval interval. The number 9 is their *arithmetic mean*, computed for extremes a and b as half the sum, $\frac{a+b}{2}$, which from a musical point of view represents the main *consonant* interval within the octave—the perfect *fifth* (as in C to G). The harmonic mean, which had earlier been known as the "subcontrary", had been renamed by the Pythagorean mathematician, statesman, and apparent friend of Plato, Archytas of Tarentum, who described it such that "the part of the third by which the middle term exceeds the third is the same as the part of the first by which the first exceeds the second" [36] (p. 42). Consonant with its *earlier* name which implies a type of inversion, the harmonic mean is the *reciprocal* of the arithmetic mean, such that for the arithmetic mean $\frac{a+b}{2}$, the corresponding harmonic mean is calculated as $\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1}$. Musically, the harmonic mean determined the interval considered to be the next most consonant interval, now known as the perfect *fourth*, as in C to F. Simple arithmetic reveals the arithmetic mean of 6 and 12 as 9 and the corresponding harmonic mean as 8.

In Greek geometry, a "subcontrary" section of a triangle ABC produces a smaller triangle DEC within the first that is similar to it with ∠CAB = ∠CDE and ∠CBA = ∠CED, as in Figure 3 below. However, the new triangle is a *reflected* version of the first, as if rotated through 180 degrees on an axis through A, that is, rotated through a third dimension perpendicular to the plane of the triangle. As a *logical* relation, the idea of subcontrariety would become familiar in Aristotelian logic as that holding between propositions that could not be false together and represented as holding between the lower two vertices of the "square of opposition". Understood geometrically, such sub-contrary triangles instantiated what Kant had, prior to Hegel, described for three-dimensional structures as "incongruent counterparts" [37] (p. 370) as in right- and left-handed gloves. Such a relation of geometric subcontrariety would be manifest in Peirce's account of Aristotle's syllogisms, and the idea of a kind of identity of otherwise *inverted* and opposed structures would, I suggest, be especially central for Hegel's logic.

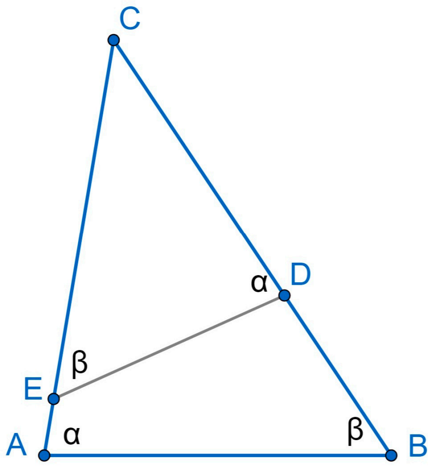

Figure 3. The "Subcontrary" Section" of a Triangle. Triangle DEC is subcontrary to ABC.

In what is usually considered to be the most significant text on geometry from the late Hellenistic period—*Collection*, by the 4th century CE mathematician, Pappus of Alexandria, a diagram (Figure 4 below) shows the role played by this subcontrary inversion of a triangle in an attempt to demonstrate the relation among the three musical means, the geometric, arithmetic and harmonic [38] (p. 569). E is the centre of a circle of which AC is a diagonal, with ABC a right-angle triangle constructed, B being a point on the circumference. A line BD is dropped to meet AC at a right angle, and DZ is drawn perpendicular to EB, making the triangle BDZ subcontrary to BED, which is itself similar to ACB. The point E, being equidistant from A and C, is the arithmetic mean of these two extremes. The length of BD is the geometric mean of the lengths AD and DC, while the length of BZ is their harmonic mean.

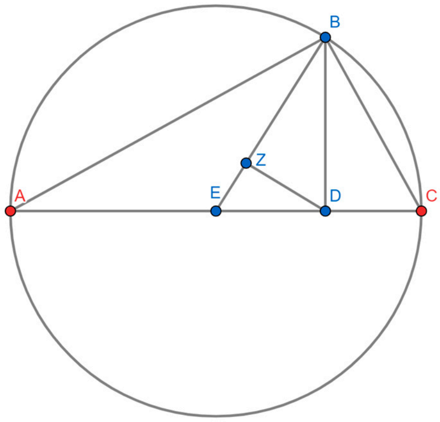

Figure 4. Pappus' "Theory of Means". Based on [38] (p. 569). Interval BZ, as side of triangle BZD, which is subcontrary to ABC, shows the role of subcontraraiety in the construction of the harmonic mean of AD and DC.

The inverse or subcontrary relation between the arithmetic and harmonic means is reflected in musical theory in that the interval from C to F, a fourth, is able to be understood as an inversion of the *fifth*, F to C, in which the C is now below rather than above the root note F. The significance of this structure for Hegel is clear. Of the sequence 6, 8, 9, 12, the extremes can be understood as qualitatively different instantiations of the same note, while 8 and 9, while dividing different intervals, can be considered the same when the sequence is conceived as running in inverse directions. In Hegel's rhetoric, these two double ratios of 6, 9, 12 and 6, 8, 12 can be conceived as "self-identical" in their "otherness", thereby binding their extremes into a "unity" that, nevertheless, does not deny their difference—an "identity in difference" analogous to that between two notes located an octave apart.

This is consistent with what Hegel says about Plato's beautiful bond in the *Lectures on the History of Philosophy*, where he insists that in relation to the ratios structuring Plato's cosmic animal, its three-dimensionality required a "doubled middle term", "*gedoppelt Mitte*" [7] (p. 211) for the unification of its parts, and in contrast with which he had suggested the middle term of Aristotle's formal syllogism to be simple or univocal. This structure, with these inversely related double ratios, would, I suggest, be reflected in his interpretations of Aristotle's syllogisms in Book III, but, as chance would have it, this same inverse double ratio was also to be found in the type of geometry being revived in the nineteenth century that would be significant for the logic of Peirce. Thus, the double-ratio to be called the "harmonic cross-ratio"—the name reflecting the ancients' "*harmonia*"—was a generalization of the inverted double ratio holding among the numbers 6, 8, 9 and 12 for the Pythagoreans. For four points, A, B, C, and D, lying on one line, the harmonic cross-ratio is said to hold when $AB/BC = -AD/DC$.

After its reintroduction by Carnot, it would be elaborated throughout the century and later named the "harmonic cross-ratio" by the British mathematician/philosopher William Kingdon Clifford, with whom Peirce would be familiar and by whom he would be influenced, especially in relation to his new style of logical graphs [39] (p. 40).

4. Common Features of Hegel's and Peirce's Reconstructions of Aristotle's Syllogistic Figures

For over two thousand years Aristotle's syllogisms had been taught as "logic" but this eventually ended with the development of modern mathematical logic, especially as developed by Frege and Russell, who decisively broke with the Aristotelianism that Boole had interpreted algebraically. Now, when the notion of a *syllogism* is mentioned, what typically comes to mind are inferences of the type "all animals are mortal, all humans are animals, therefore all humans are mortal" or "all humans are mortal, Socrates is a human, therefore Socrates is mortal". Neither of these, however, unproblematically instantiate what Aristotle described as a syllogism in *Prior Analytics*.

As for the former, Aristotle tended *not* to order the component sentences in the standard subject-predicate order as this suggests. To try to capture why syllogisms were valid, he invoked the idea of the transitivity of *containment relations* apparent for either of the two "perfect" syllogisms that occur in the first figure and to which all syllogisms in the other two figures could be reduced: "When three terms are so related to one another that the last is wholly contained in the middle and the middle is wholly contained in or excluded from the first, the extremes must admit of perfect syllogism" [40] (25b32–37). Here, that the more general "containing" term is the first, and the more particular "contained" the last, is captured by Aristotle's alternate wording: "For if A is predicated of all B, and B of all C, A must necessarily be predicated of all C" (25b38–40). Aristotle was therefore thinking of the constitutive sentences of this first figure syllogism as having a predicate-subject order, rather than the subject-predicate order that is in fact normal for both English and ancient Greek, and that is assumed in the familiar "syllogisms" above. Hegel would restore the subject-predicate order in syllogisms but, unlike Aristotle, he would acknowledge the significance of inverting the orderings of terms in this way.

For *its* part, the status of the syllogism about Socrates *as a syllogism* is questionable [41] (p. 1). Although Aristotle does give some examples using singular terms, as Patzig points out he was "obviously inclined to exclude them" [42] (p. 4–5). The terms of Aristotle's syllogisms are routinely called upon to play both roles of subjects and predicates in the constitutive judgments and Aristotle believed that singular terms could not play the role of predicates. Reflecting this, in the first paragraphs of *Prior Analytics*, he describes a premise as "a sentence that affirms or denies something of something, and this is either universal or particular or indeterminate" [40] (24a16–18), noticeably omitting *singular* judgments, that is, judgments with singular terms. Limiting the "quantities" of judgments to universal and particular in this way would become codified in the traditional "square of opposition" introduced later by Apuleius in the second century CE on the basis of Aristotle's earlier descriptions. To overcome this obvious limitation, Medieval nominalist logicians would treat singular terms *as* universals, the justification being that with an assertion such as "Socrates is mortal", mortality is being attributed to *all*, not *part of*, Socrates. To this, Leibniz would add the alternate strategy of treating singulars *as* particulars, as when Socrates is referred to as "some philosopher", the "some" used in the quantity of particularity as not *excluding* just one [43].

These expedients, however, face difficulties in *modal* contexts where, as Kripke would argue in the mid-twentieth century [44], proper names have different modal properties to those definite descriptions that uniquely pick out the bearers of those names. This distinction had been insisted upon by Hegel in terms of the categories of *singularity* [*Einzelheit*] and *particularity* [*Besonderheit*], the former correlated with demonstrative phrases such as "this rose", "this house", treated clearly as singular terms, the latter with general concepts that "subsume" or are true of a plurality of individuals. Thus, in the *Encyclopedia Logic*, he describes the syllogism "in its truth" and in contrast to "the meaning it has in the old, formal logic", as "that determination in virtue of which the particular is supposed to be the middle that joins the extremes of the universal and the singular together. This form of syllogistic inference is a universal form of all things. Everything is something particular that joins itself as something universal with the singular" [45] (§ 24, add. 2). In contrast, rather than treating "the determinateness of the three terms to each other", Aristotle's syllogism expresses "the *equal relation* of inherence of the one extreme to the middle term, and then again of this [middle term] to the other extreme" [12] (p. 591).

Aristotle had, seemingly on the model of the way Greek geometers labelled their diagrams, labelled his syllogisms with letters such as A, B, and C [A, B, and Γ], meant to stand as meaningless placeholders for terms playing the role of subject or predicate in the component sentences. Thus, the sequence ABC was meant to stand for the two premises AB and BC from which the conclusion AC could be deduced. However, according to Hegel the formal syllogism's employment of a single or univocal middle term would mean that its extremes "have the value of independent characteristics", failing to render "the extremes one in the highest degree" [7] (pp. 210 and 261). In relation to this, while Aristotle's syllogism strictly restricted quantities of terms to particular and universal, Hegel would use the abbreviations E, B, A for *Einzelheit* (singularity), *Besonderheit* (particularity), and *Allgemeinheit* (universality).

Strictly distinguishing singularity and particularity in this way, in *The Science of Logic* Hegel distinguishes two judgment-forms on the basis of the type of relation holding between subject and predicate [12] (p. 555). In simple "qualitative" types of judgment such as perceptual ones, the predicate is said to "inhere" in a concrete singular subject, reflecting the way that, say, the colour of some specific rose picked out as "*this* rose" will be *perceived* as inhering *in* that rose. In contrast, in a type of "quantitative" or "reflective" judgment, the intelligible predicate is said to "subsume" the subject. Effectively, a predicate "subsumes" a subject in the sense of being instantiated by or being true of that subject.

Hegel's predicative distinction had actually drawn on the one alluded to by Aristotle, with his "predicated of" and "is in" expressions of the perfect syllogism, and the same sort of distinction would become part of Boole's logic after Hegel. Thus, we may think of Boole's "primary propositions", as in "the sun shines", as equivalent to Hegel's predications of *inherence* and his "secondary propositions", as in "it is true that the sun shines" [46] (p. 38), as involving the predication of "true" to complex subjects that are the properly *propositional* analogues of those primary propositions, these latter being equivalent to Hegel's judgments of *subsumption*. From such a distinction we should expect Hegel's method to involve a transition from both of these opposed judgment forms to some third form in which each form is negated but retained in the sense of *aufgehoben*, and we find this when both these *inherence* and *subsuming* structures are incorporated into a new judgment form in which a new and more complex concrete subject, a type of Aristotelian *secondary substance* or *genus*, becomes its subject. While the earlier simple judgment of inherence may have been about *this* individual rose, as in the perceptually based judgment "the rose is red", the new judgment might be about *the genus*, rose, as in "the rose is a plant" [12] (p. 576).

The subjects of this new judgment form will be understood as combining the inherence and subsuming predicates of the earlier judgments in that while understanding what it is *to be* a rose will involve understanding some universal laws about roses which involve "subsuming judgments", one must also understand that such kinds or secondary substances are themselves *made up* of the individual "primary substances" that are the subjects of "inherence judgments": "Things are singularities [*einzelne*] ... the lion in general does not exist" [15] (§246, add.). While inherence and subsuming judgments are logically different, this difference is in some sense maintained *within* the new identity in which each component is considered as *inverse* of the other. This will ultimately be shown in the structure of the syllogism SPU, in which two premises SP and PU—the former a judgment with a singular subject and inhering predicate, the latter, one in which a universal predicate subsumes a particular subject, are shown to be unified in the conclusion, SU.

Given Hegel's claim that Plato's syllogism provides the true underlying form of Aristotle's, we should not be surprised that this structure formally repeats the ancient "harmonic" double ratio that Nicomachus had identified with Plato's "beautiful bond". Echoing his description of the *ratio of powers*, Hegel says that "in the syllogism of reason a subject or a content is represented as joining itself with itself through the other and in the other" [7] (p. 210). The rational syllogism "constitutes the syllogism's authentic form and nature", and in *The Science of Logic* Hegel attempts to reveal this reality behind Aristotle's formal syllogism via showing its development through its three figures.

In his formal classification, Aristotle had divided syllogisms into first, second and third "figures" (*schemata*, the term used by Greek geometers for their diagrams) by the position of the "middle term" in each. Clearly the two perfect syllogisms contained in the first figure were considered as the paradigm of a syllogism, and concordantly, the middle term (B), understood as the term that contains the last (C) and is contained by the first (A), is represented as standing *between them*, as in ABC.

From the first figure, Aristotle had generated second and third figures in a quasi-mechanical way by describing the movement of the *middle term*. Thus, he writes that in the second figure, "the middle is placed outside the extremes and is first in position" [40] (26b37–38) while in the third, "the middle is placed outside the extremes, and is last by position" (28a14–15). Within each of these figures, different "moods" show different "ways" or "manners" in which the figures might be realized when the different quantities of "all" and "some" are assigned to the subject terms of the sentences, together with the opposing "qualities" of affirmation or negation. Arithmetically, each figure could be realized in 64 ways but most of these are ruled out by counterexamples. Across the three figures only 14 such combinations remain as genuine syllogisms. In contrast, both Peirce and Hegel would attempt to generate Aristotle's classification into figures and moods in more principled ways, with Hegel's account of the syllogistic figures in *The Science of Logic* sharing important characteristics with Peirce's *functional* treatment of the syllogistic figures, in which he aligned the three figures with three different kinds of inference: deduction, induction and abduction.

In an early paper from 1878, "Deduction, Induction, and Hypothesis", in which "hypothesis" refers to what he would later call "abduction" [21] (pp. 186–199), Peirce links the three syllogistic figures to the three forms of inferential reasoning. For Peirce, rather than Aristotle's perfect syllogisms in the first figure as paradigms of inference to which syllogisms in the second and third are to be reduced, perfect syllogisms involve "nothing but the application of a rule" that, being laid down in the major premise, can be applied to a case stated in the minor to produce a result. Inductive and hypothetical (abductive) reasoning, however, "being something more than the mere application of a general rule to a particular case, can never be reduced to this form" (p. 187).

Aristotle had invoked a form of inference, *epagoge*, standardly translated as "induction", in which the inference went in reverse direction from a particular *result* of a syllogism to its *principle*. Peirce believed, however, that Aristotle had distinguished a similar but different *non*-deductive inference that he would call *abduction* [47] (p. 205). If one started with the conclusion of a syllogism, one should be able to infer from it and its minor premise to the major premise (induction), *or* from the conclusion and the major premise to the minor premise (abduction). He described this as "rowing up the current of deductive sequence" in either of two ways (p. 188). In this paper he attempts to show Aristotle's second figure as the structure of abduction and the third as that of induction. The subcontrary relation is clearly present in Peirce's imagery, as in Figure 5 below.

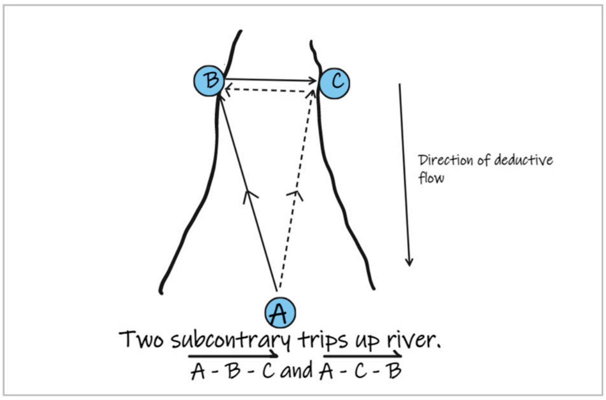

Figure 5. Peirce's subcontrary ways of rowing up the deductive stream. The path from A to B to C is subcontrary to that from A to C to B.

Striking parallels to Peirce's functional interpretation of Aristotle's syllogistic figures can be found in Hegel's own interpretation of Aristotle's formal syllogistic. Hegel introduces the syllogism as an expansion of a type of value judgment, the "judgment of the concept" [12] (pp. 581–587) that has features like those of the perceptual judgments, the implicit conceptual structure of which Peirce believes is unpacked in abductive inference [47] (p. 227). Once introduced, however, the syllogism passes through a series of developmental cycles similar to those traversed earlier by forms of judgment. Just as the most immediate judgment type had been the first judgment of inherence labelled the "positive judgment of existence" [12] (pp. 557–561], the first syllogistic type is the "qualitative" syllogism of existence (p. 589) that transitions into the *quantitative* syllogism of reflection (p. 609) which in its turn transitions into the syllogism of necessity (p. 617). Each of these syllogisms is analysed has having three syllogistic components: those of the syllogism of existence are the traditional three *figures*.

Given the complexities introduced by Hegel's treatment of the syllogism via the peculiar "dialectical" triadic patterns within which his analysis unfolds, it might be thought ambitious to recover much in common between Hegel's treatment and the approaches of Aristotle and Peirce. Nevertheless, enough commonality can be observed to suggest a convergence between Peirce and Hegel here based on Hegel's explicit disambiguation of Aristotle's treatment of singularity and particularity [48] (ch. 10). It can be appreciated that Hegel's three syllogisms internal to the syllogism of reflection—the syllogisms of allness, induction and analogy—correspond closely to Peirce's functional rendering of Aristotle's three figures as deductive, inductive and abductive. It is the role of order and directionality among points on a two-dimensional plane *implicit* in Aristotle's original geometric framing of his syllogistic but made explicit by Peirce that, I suggest, enables this. Hegel will use the idea of moving Aristotle's "middle term", with its dual senses of "middle", in either of the two directions defined by the "extremes" to capture something like the rearrangements that Peirce envisages as "rowing up" the deductive stream along two different routes, one which takes reasons from the conclusion and minor premise to the major (induction) and one that takes reasons from the conclusion and the major premise to the minor (Peirce's abduction, Hegel's inference by analogy). While a number of factors such as the numbering of the syllogisms need to be ignored as resulting from the complications of the different word orderings appealed to as well as the different attitudes to the role of singular terms, the convergence between Peirce and Hegel is striking.

Hegel's *second* subsyllogism of the reflective syllogism, the "syllogism of induction", reflecting the "PSU" ordering of his second figure, has "singularity for its middle term". This, however, is "not *abstract* singularity but singularity as *completed*, that is to say, posited with its opposite determination, that of universality" [12] (p. 612). The first premise PS (or, in natural word order, S is P, the singular is particular) is the *"abstract* singularity" referred to which relies on Leibniz's resolution of the traditional problem of representing singulars, as in "Socrates is some (a) man"; the second SU, stating "the singular is universal", *qua* "completed singular" is the Medieval resolution where Socrates can be understood as an individual *real essence* [49]. Hegel goes on: "The one extreme is some predicate or other which is common to all these singulars; its connection with them makes up the kind of immediate premises, of which one was supposed to be the conclusion in the preceding syllogism". This aligns with Peirce's account of induction treated as reversal of the immediately preceding first-figure deduction.

Here, Hegel is assuming induction in the modern sense as a way of testing hypothesized theories and this in turn needs some way of generating the hypotheses *to be* tested. Clearly Peirce's *abduction* was meant to provide this starting point and Hegel turns to "inference by analogy" to perform this task. "The truth of the syllogism of induction is therefore a syllogism that has for its middle term a singularity which is immediately in itself universality. This is the syllogism of analogy" [12] (p. 614).

There is clearly *something* of Peirce's abductively arrived at hypothesis in Hegel's syllogism of analogy. As in Hegel's example, arguing *from* "*The earth* has inhabitants" and "the moon is *an earth*" (that is, is a thing of basically the same kind as *the* earth) to the conclusion "Therefore, the moon has inhabitants" is to hypothesize [12] (p. 614). Unlike the situation in induction which relied on the nominalist treatment of a singular term as a universal to bring out its logical properties, this one relies on the alternative introduced by

Leibniz, in which the substitution between quantities is between singular and *particular*. An analogy between the moon and the earth converts "the earth" from a proper to a common name. The moon is now considered as, like the earth, "an earth"—an instance of the type *planet*, revolving around a bigger body.

5. Logical Hexagons and the Dimensions of Logical Space

In the 1960s, a French logician, Robert Blanché, proposed an expansion of the traditional square of opposition into a logical hexagon [9]. Blanché's hexagon was actually an extension of a *modal* interpretation of the traditional square of opposition proposed by the Polish logician Jan Łukasiewicz [50] (Figure 6 and Table 1 below). In the traditional interpretation, the top A and E corners of the square represent universal affirmative and universal negative judgments, All Fs are G and No Fs are G, while the lower I and O corners represent particular affirmative and particular negatives, Some Fs are G and Some Fs are not G. At the A, E, I, and O corners, however, Łukasiewicz located the adverbially qualified propositions "necessarily *p*", "impossibly *p*", "not impossibly *p*" and "not necessarily *p*", reflecting the tendency in modern logic to treat the contents of judgments as complete propositions in the style of the Stoics rather than as combinations of subject and predicate terms as in Aristotelian logic.

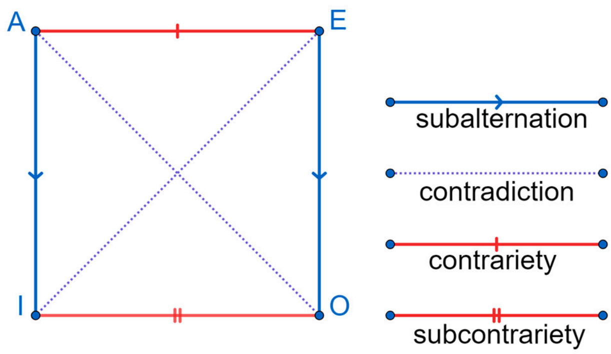

Figure 6. Square of Opposition, illustrating logical relations among propositions A, E, I, and O.

Table 1. Traditional and Modal Interpretations of the Square of Opposition.

	Traditional Square	Modal Square (Propositional)
A	Universal positive: All Fs are G	Necessarily *p*
E	Universal negative: No Fs are G	Necessarily *not p*
I	Particular positive: Some Fs are G	Not impossibly *p*
O	Particular negative: Some Fs are not G	Not necessarily *p*

Blanché exploited the subcontrariety relation between I- and O-judgments across the bottom of the square to add the modal category of *contingency*, the sentence, "*contingently p*" being understood as equivalent to the *conjunction* of both "*not impossibly p*" and "*not necessarily p*". Because a conjunction implies *both* individual conjuncts, a lower vertex, Y, could be added to the modal square such that the Y-judgment is understood as implying each of the I- and O-judgments, adding to their subalternation implications from the A- and E-judgments, respectively. Similarly, an upper vertex, U, can be added, representing the judgment "*not* contingently *p*" (and so, the contradictory of Y as represented by its opposition across a diagonal), because "not contingently *p*" can in turn be understood as the *disjunction* of the A-judgment, "necessarily *p*", and the E-judgment, "impossibly *p*", because implied by both (Figure 7 and Table 2).

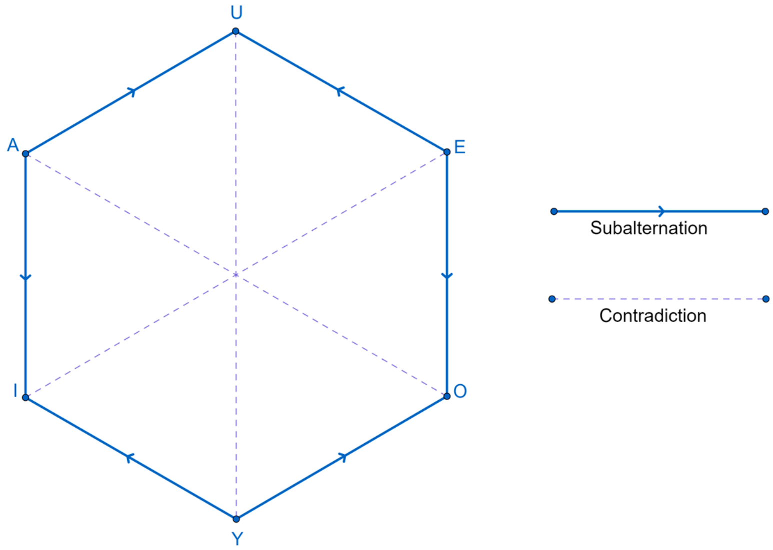

Figure 7. Blanché's Hexagon, showing subalternation and contradiction among propositions located at vertices A, E, I, O, Y, and U.

Table 2. The Interpretation of Blanché's Y and U nodes.

Node	Interpretation
Y	Equivalent to I *and* O such that Y →I and Y →O
U	Equivalent to A *or* E, such that A → U and E → U

Blanché's hexagonal extension of the traditional square has been relatively widely discussed in recent decades in relation to a range of phenomena. For example, in a special double issue of the journal *Logica Universalis* devoted to the "hexagon of opposition" [51], Blanché is discussed in eight of the ten articles included. However, a similar hexagonal extension proposed in 1955 by the Polish logician Tadeusz Czezowski [10] has, with a few exceptions, e.g., [52,53], been largely ignored. (Thus, he is discussed in none of the articles in the issue of *Logica Universalis* noted above and mentioned in the bibliography of only one.) In his case, the relevant "square" to be expanded was not Łukasiewicz's *modal* version of the traditional square, which preserves the basically *propositional* nature of modern modal logic as established by C. I. Lewis, but the type of *singular square* to which, as pointed out recently by Laurence Horn [54], Aristotle had alluded in *Prior Analytics* [40] (bk. 1, ch. 46), and in which the nodes of the square are instantiated as in Table 3 below. (To stress the singularity of the subject I have used the demonstrative "this" rather than the usual "it").

Table 3. Aristotle's singular square.

	Singular Square
A	This is a white log.
E	This is a non-white log.
I	This is not a non-white log.
O	This is not a white log.

Important for our purposes is that Czezowski, like Hegel but unlike Aristotle or modern quantified predicate calculus, specifically employs the distinction between *singular* and *particular* propositions: "The name, 'This S' in the subject of a singular proposition I regard to be a proper name denoting a given individual from the extension of the S term, just as 'Francis Bacon' denotes one of the members of the Bacon family" [10] (p. 392). In fact, Czezowski's hexagon will bring out the implicit ambiguity between singular and particular in Aristotle's original square of opposition in that, as we will see, his hexagon tends to fracture into a triad of overlapping squares. Reflecting this ambiguity, in Aristotle's square, while the verticals and horizontal represent relations within a *term logic*, the diagonals, representing relations of contradiction, tend to construe the corners as the propositions of Stoic or modern *propositional* logic.

As in Figure 8 below (reoriented so as to parallel Blanché's), Czezowski adds opposite U and Y vertices like Blanché, but while for Blanché the U node represents the disjunction of A and E nodes, for Czezowski it represents an affirmative singular proposition of the type "This S is P" (Hegel's "positive judgment of existence"), while its diagonally opposite Y node represents the *contradictory* of that singular proposition. The nodes of Czezowski's hexagon thus give representation to the following six judgment types: A: universal positive, E: universal negative, U: singular positive, Y: singular negative, I: particular positive, O: particular negative.

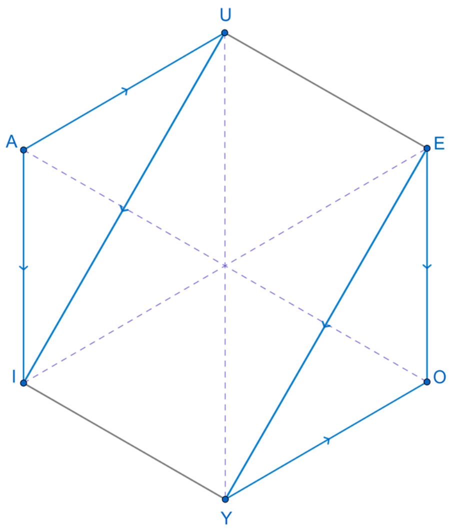

Figure 8. Czezowski's Hexagon, showing subalternation and contradiction among propositions located at vertices A, E, I, O, Y, and U.

Comparing Blanché and Czezowski's hexagons, it can be seen that while both have inferences A → U, A → I, E → O and Y → O, Blanché's E → U and Y → I are missing from Czezowski's, and Czezowski's U → I and E → Y are missing from Blanché's. These graphically displayed differences are paralleled by differences within the corresponding algebraic interpretations [53].

Not surprisingly, Czezowski's unique inference types capture some of the important inferential relations found in Hegel's account of syllogisms in *The Science of Logic*. For example, while Czezowski's U-judgment, "this F is G", will, like Blanché's, U, be implied by the A-judgment "all Fs are G", Czezowski, *but not Blanché*, has I-judgments inferred from U-judgments, repeating Hegel's abstractive inference from the singular judgment "this F is G" to the less determinate particular judgment "some F is G". On the other hand, Hegel's subsuming judgments are like proper propositional judgments, the negations of which are *contradictories*, and one might expect his logic to also incorporate the inferences found in Blanché's square but not Czezowski's. This is a point to which we will return.

There is a further and deeper way in which Czezowski's account of logical relations echoes Hegel in that for Czezowski, it emerges that these relations among singular, particular and universal propositions can no longer be considered *univocal* such that they will be defined differently depending on the contextually determined interpretation of the sentences being related. Czezowski puts this down to the fact that singular propositions are, from a formal point of view, *hybrids*: in certain contexts, they behave logically like universal propositions, in others they behave like particular propositions [52].

This "hybridicity" anticipates recent talk of "two-dimensional logics", but most cases of two-dimensional logic are conceived in a reductive way. Typically, the modal calculus or "language" is regarded "as a fragment of first- or second-order classical logic", that is, the nonmodal language [55] (p. xiv), although Arthur Prior, the inventor of "tense logic" as a type of modal logic, had first conceived of this relation as operating in the reverse way. "It is not that modal logic or tense logic is an artificially truncated uniform monadic first-order predicate calculus; the latter, rather is an artificially expanded modal logic or tense logic" [56] (p. 56). Later, however, Prior would steer towards his own "hybrid" logic [57], more like Czezowski's, seemingly attempting to keep *both* modal and classical dimensions somehow in play. While not often acknowledged, the influence of Hegel can be perceived in Prior's work in that, while he was in no sense directly influenced by Hegel, he acknowledged the influence on his logic of John N. Findlay [58] (pp. 1; 13–15), whose attitude to logic had indeed been deeply influenced by Hegel [59].

When the duality or "hybridicity" of judgment forms is taken seriously, Czezowski's logical hexagon fractures into three different but related logical squares. While Czezowski's three different squares do not directly correspond to Hegel's three syllogistic figures, given the Peircean "functional" interpretation I have adopted here, they are like Hegel's in being unable to be smoothly integrated into some two-dimensional logical figure such as a square or a hexagon. As we have seen, Hegel had appealed to the properly *three-dimensional* nature of Plato's syllogism that had to be considered the underlying true form of Aristotle's, implicitly construed as *two-dimensional.* We could ask: Might it not make sense to think of the three squares of Czezowski's hexagon as faces of a cube that, as a three-dimensional logic diagram, could give representation to different ways of navigating between these differently interpretable terms? After all, it is obvious when looking at these logical hexagons that one could be looking at a type of transparent *logical cube*. Additionally, further, while three of the faces might correspond to different *deductive* pathways, perhaps the remaining three faces could accommodate the corresponding types of *non-deductive* inference we have seen in Hegel and Peirce. Drawing on parallels in projective geometry, Peirce himself had suggested this type of attitude to logical diagrams when discussing the "metadiagrammatic" aspects of abductive inference [60].

This idea of a logical cube is clearly suggestive for Hegel, given his appeal to Plato's three-dimensional syllogism as the underlying true form of Aristotle's syllogism. Hegel attempts to give a *deeper* account of the unity holding between the components of "hybrid" logics like those of Prior and Czezowski than Prior or Czezowski themselves. Hegel's solutions here typically employ mystical-sounding expressions such as "the identity of identity and difference", that can be easily dismissed as mere wordplay. However, the mathematical models we have examined suggest ways of filling out such locutions with genuine content. I have suggested that with such formulae Hegel has in mind something like that identity in difference holding between the fourth and fifth divisions of an octave when these double-ratios are considered as having opposed orientations, a structure that would be discussed in more general ways in nineteenth-century projective geometry as the "harmonic cross-ratio". Indeed, it is not difficult to consider how a type of "identity in difference" might be thought to hold between typically modal and nonmodal judgments such as "the sun shines" and "It is true that the sun shines". In one sense such sentences say *the same thing*, an insight that is behind the so-called "redundancy theory of truth" in which it is argued that the *truth* predicated of the proposition "the sun shines" in the latter is simply redundant. However, while saying "the same thing" they clearly do so in different ways. If we qualified those mathematical "relational structures" referred to in the following quote from a contemporary text on modal logic with the word "worldly", something close to Hegel's duality of predication would result: "although both modal and classical languages talk about relational structures, they do so very differently. Whereas modal languages take an internal perspective, classical languages, with their quantifiers and variable binding, are the prime example of how to take an external perspective on relational structures" [55] (p. xiii).

The "external perspective" on the world is what is often referred to as a "God's-eye view", a view many treat as inapplicable for humans, given the worldly constraints (physical, biological, cultural) on our finite cognitive capacities. That is, they treat humans as essentially constrained to the internal perspective and incapable of achieving the external. In his theology, however, Hegel did not accept any such implied radical dichotomy between the divine and the earthly implicit in the standard distinction between "subjective" and "objective", and this affected how both sides were considered. It is in God's nature to *become* human, as given in a type of pictorial way in the story of Christ, and so God's "omniscience" must then *include* the type of limited, intra-worldly perspectival knowledge typical of humans. For our part, our knowledge, despite its "internal" limitations, must somehow be capable of achieving the *external* perspective, as long as this is not thought of as simply *opposed* to the internal. This is reflected in the way that inverse logical forms, like those between the constituents of modal and nonmodal logics, must be tied together in ways that nevertheless preserve their incommensurability—the structure we have seen modelled on the "ratio of powers". Moreover, if it makes sense to conceive of a type of "identity" between these different internal and external judgment structures, might not the same be said about the different types of inferences we employ—deductive, inductive and abductive? Abduction and induction clearly correspond to inferences made from some "internal" perspective while, as classically conceived, deduction corresponds to the "external" perspective. Once more, each must be understood as related to the other; separated from its relation to induction and abduction, pure deduction collapses into meaninglessness for both Hegel [12] (p. 611) and Peirce [21] (p. 187). In this way, the dichotomy between logic considered as either about thought *or* as about the world itself collapses. At the end of Hegel's treatment of the syllogism, he describes it as having "stepped forth out of subjectivity into *objectivity*" [12] (p. 590).

6. Conclusions

Despite his massive *Science of Logic*, Hegel is not standardly taken as contributing to the "science of logic" in the modern sense. I have argued that, while restricted in relation to the technical apparatuses with which modern logicians work, Hegel's logic might nevertheless be recognized as a contribution to the types of "two-dimensional" and "hybrid" logics discussed today in relation to the contrast between modal and nonmodal logics, and in a way that employs the resources of a "geometric logic". However, accepting a certain incommensurability of these component logics, Hegel goes further than existing two-dimensional and hybrid logics in the claim for a higher *dialectical* unity holding across this incommensurability, a unity in difference conceived on the model of the Pythagorean unity of the "musical means". Could this correspond to anything in modern logic beyond the types of contributions we have mentioned so far?

It would be rash to rule out possibilities here. Since the 1980s, there has developed a type of logic, "linear logic" [61], so called because it draws explicitly upon the linear algebra started by Grassmann that had influenced Peirce's logic. Significantly, linear logic adds a further "exponential" connective to the "additive" and "multiplicative" ones of Boolean logic, giving it an internally triadic structure. Just as "multiplication" underlies Boolean logical conjunction, and "addition" logical disjunction, exponentiation provides an algebraic analogue for implication when it is not taken as defined in terms of negation and either of the other logical relations. In linear logic, this third "arithmetical" operation is meant to unify the other two. Significantly, in his account of Greek number theory, Hegel had described the complete determination of the notion of "number" as made up of the various connections between the three arithmetic operations of addition, multiplication and the raising to a power, taken together with their inverses [12] (pp. 175–176). Once more, we should expect this structure to be found somehow *aufgehoben* in the later discussion of the syllogism.

In instituting modern quantified predicate calculus as *the* logic for philosophy, Russell had effectively excommunicated Hegel from modern logic [62] (intro.), but he also excommunicated both existing approaches to modal logic as found, for example, in the work of Hugh MacColl [63], as well as the various nineteenth-century approaches that, in the wake of Boole, preserved aspects of Aristotle's *geometric* approach. The current revivals of modal logic on the one hand and geometric forms of logic inspired by the traditional square of opposition on the other may provide a context within which Hegel's status *as* a logician will be reassessed.

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