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Generalized k-Fractional Chebyshev-Type Inequalities via Mittag-Leffler Functions

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Abstract: Mathematical inequalities have gained importance and popularity due to the application of integral operators of different types. The present paper aims to give Chebyshev-type inequalities for generalized k-integral operators involving the Mittag-Leffler function in kernels. Several new results can be deduced for different integral operators, along with Riemann–Liouville fractional integrals by substituting convenient parameters. Moreover, the presented results generalize several already published inequalities.

Keywords: Chebyshev inequality; fractional integrals; Mittag-Leffler function

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1. Introduction

Integral operators play a very important role in the field of mathematical inequalities. A large number of integral inequalities exist in the literature for different types of integral operators [1–9]. Due to the extensions and generalizations of integral operators, it becomes possible to obtain extensions and generalizations of classical inequalities. From classical inequalities, the Chebyshev inequality is studied extensively by using such extensions and generalizations (for details, see [3,5,10–16]).

Inspired by this latest research, the aim of the present paper is to establish Chebyshev-type inequalities for generalized k-integral operators containing the Mittag-Leffler function in their kernels, which produce many well-known integral operators. The results in this paper provide generalizations of various inequalities published in the literature of fractional integral inequalities. Next, we give the definition of Riemann–Liouville integral operators, the classical Chebyshev inequality, Chebyshev inequalities for Riemann–Liouville integral operators, and definitions of generalized integral operators containing the Mittag-Leffler function.

The Riemann–Liouville integral operators are defined as follows:

Definition 1. *Let $\zeta \in L_1[\sigma_1, \sigma_2]$. Then, Riemann–Liouville integral operators of order $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ are defined by:*

$$\left(\xi^{\mu}_{\sigma_1^+}\zeta\right)(x) = \frac{1}{\Gamma(\mu)}\int_{\sigma_1}^{x}(x-\tau)^{\mu-1}\zeta(\tau)d\tau, \quad x > \sigma_1, \tag{1}$$

$$\left(\xi^{\mu}_{\sigma_2^-}\zeta\right)(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{\sigma_2}(\tau-x)^{\mu-1}\zeta(\tau)d\tau, \quad x < \sigma_2, \tag{2}$$

where $\Gamma(.)$ is the gamma function defined as: $\Gamma(\mu) = \int_0^\infty \tau^{\mu-1}e^{-\tau}d\tau$.

For more details and results related to the fractional integrals (1) and (2), we refer the readers to [11,12,17–19]. The Chebyshev inequality [20] is given as follows:

$$\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\zeta_1(\tau)\zeta_2(\tau)d\tau \geq \left(\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\zeta_1(\tau)d\tau\right)\left(\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\zeta_2(\tau)d\tau\right), \quad (3)$$

where ζ_1 and ζ_2 are two integrable and synchronous functions over the interval $[\sigma_1, \sigma_2]$. Two functions are called synchronous on $[\sigma_1, \sigma_2]$ if the following inequality holds:

$$(\zeta_1(\psi)-\zeta_1(\phi))(\zeta_2(\psi)-\zeta_2(\phi)) \geq 0, \qquad \forall\ \psi,\phi \in [\sigma_1,\sigma_2].$$

Many researchers have introduced various generalizations and extensions of inequality (3) for different integral operators. In [11], Belarbi and Dahmani proved the following Chebyshev-type inequalities for Riemann–Liouville integral operators.

Theorem 1. *Let $\zeta_1, \zeta_2 : [0,\infty) \to \mathbb{R}$ be two integrable functions of same monotonicity. Then, for Riemann–Liouville integral operators, we have*

$$\left(\xi^{\mu}_{0^+}\zeta_1\zeta_2\right)(x) \geq \frac{\Gamma(\mu+1)}{x^{\mu}}\left(\xi^{\mu}_{0^+}\zeta_1\right)(x)\left(\xi^{\mu}_{0^+}\zeta_2\right)(x).$$

Theorem 2. *Assume that the conditions given in Theorem 1 are valid. Then*

$$\begin{aligned}\frac{x^{\mu}}{\Gamma(\mu+1)}\left(\xi^{\nu}_{0^+}\zeta_1\zeta_2\right)(x) &+ \frac{x^{\nu}}{\Gamma(\nu+1)}\left(\xi^{\mu}_{0^+}\zeta_1\zeta_2\right)(x)\\ &\geq \left(\xi^{\mu}_{0^+}\zeta_1\right)(x)\left(\xi^{\nu}_{0^+}\zeta_2\right)(x) + \left(\xi^{\nu}_{0^+}\zeta_1\right)(x)\left(\xi^{\mu}_{0^+}\zeta_2\right)(x).\end{aligned}$$

Theorem 3. *Let $(\zeta_i)_{i=1,\ldots\ldots,n}$ be n positive increasing functions on $[0,\infty)$. Then*

$$\left(\xi^{\mu}_{0^+}\prod_{i=1}^{n}\zeta_i\right)(x) \geq \left((\xi^{\mu}_{0^+}1)(x)\right)^{1-n}\prod_{i=1}^{n}\left(\xi^{\mu}_{0^+}\zeta_i\right)(x).$$

Theorem 4. *Let ζ_1 and ζ_2 be two functions defined on $[0,\infty)$, such that ζ_1 is increasing, ζ_2 is differentiable and $m := inf_{x\in[0,\infty)}\zeta_2'(x)$. Then*

$$\begin{aligned}\left(\xi^{\mu}_{0^+}\zeta_1\zeta_2\right)(x) \geq \left((\xi^{\mu}_{0^+}1)(x)\right)^{-1}&\left(\xi^{\mu}_{0^+}\zeta_1\right)(x)\left(\xi^{\mu}_{0^+}\zeta_2\right)(x)\\ &-\frac{mx}{\mu+1}\left(\xi^{\mu}_{0^+}\zeta_1\right)(x) + m\left(\xi^{\mu}_{0^+}x\zeta_1\right)(x).\end{aligned}$$

Several integral operators containing the Mittag-Leffler function have been defined by various authors (for details, see [21–24]). Recently, Chebyshev-type inequalities for operators involving Mittag-Leffler functions and other operators have been established in [25–30]. Next, we give the generalized fractional integral operators defined by Andrić et al. [31], as follows:

Definition 2. *Let $\zeta : [\sigma_1, \sigma_2] \to \mathbb{R}$, $0 < \sigma_1 < \sigma_2$ be an integrable function. Furthermore, let $\lambda, \omega, \mu, \delta, \alpha, \eta \in \mathbb{C}$, $\Re(\omega), \Re(\mu), \Re(\delta) > 0$, $\Re(\eta) > \Re(\alpha) > 0$ with $q \geq 0$, $\varsigma > 0$ and $0 < \rho \leq \varsigma + \Re(\omega)$. Then, for $x \in [\sigma_1, \sigma_2]$, the generalized integral operators are defined by:*

$$\left(\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta\right)(x;q) = \int_{\sigma_1}^{x}(x-\tau)^{\mu-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta}(\lambda(x-\tau)^{\omega};q)\zeta(\tau)d\tau, \quad (4)$$

$$\left(\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_2^-}\zeta\right)(x;q) = \int_{x}^{\sigma_2}(\tau-x)^{\mu-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta}(\lambda(\tau-x)^{\omega};q)\zeta(\tau)d\tau, \quad (5)$$

where $E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta}(\tau;q)$ *is the generalized Mittag-Leffler function defined by:*

$$E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta}(\tau;q) = \sum_{n=0}^{\infty} \frac{B_q(\alpha+n\rho,\eta-\alpha)}{B(\alpha,\eta-\alpha)} \frac{(\eta)_{n\rho}}{\Gamma(\omega n+\mu)} \frac{\tau^n}{(\delta)_{n\varsigma}},$$

$B_q(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}e^{-\frac{q}{\tau(1-\tau)}}d\tau$ *and* $(\eta)_{n\rho} = \frac{\Gamma(\eta+n\rho)}{\Gamma(\eta)}$.

Recently, Zhang et al. introduced the generalized k-integral operators involving the Mittag-Leffler function in ([32] Definition 4). It is noted that in ([32] Definition 4) some conditions of convergence of the Mittag-Leffler function were misprinted, we state it with correct conditions as follows:

Definition 3. *Let* $\zeta, \gamma : [\sigma_1, \sigma_2] \to \mathbb{R}$, $0 < \sigma_1 < \sigma_2$ *be the functions such that* ζ *be a positive and integrable and* γ *be a differentiable and strictly increasing. Furthermore, let* $\lambda, \mu, \delta, \alpha, \eta \in \mathbb{C}$, $\Re(\mu), \Re(\delta) > 0$, $\Re(\eta) > \Re(\alpha) > 0$ *with* $q \geq 0, \omega, \varsigma > 0$, $0 < \rho \leq \varsigma + \omega$ *and* $k > 0$. *Then for* $x \in [\sigma_1, \sigma_2]$ *the integral operators are defined by:*

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta\right)(x;q) = \int_{\sigma_1}^{x} (\gamma(x)-\gamma(\tau))^{\frac{\mu}{k}-1} E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x)-\gamma(\tau))^{\frac{\omega}{k}};q)\zeta(\tau)d(\gamma(\tau)), \quad (6)$$

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_2^-}\zeta\right)(x;q) = \int_{x}^{\sigma_2} (\gamma(\tau)-\gamma(x))^{\frac{\mu}{k}-1} E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(\tau)-\gamma(x))^{\frac{\omega}{k}};q)\zeta(\tau)d(\gamma(\tau)), \quad (7)$$

where $E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\tau;q)$ *is the modified Mittag-Leffler function defined by:*

$$E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\tau;q) = \sum_{n=0}^{\infty} \frac{B_q(\alpha+n\rho,\eta-\alpha)}{B(\alpha,\eta-\alpha)} \frac{(\eta)_{n\rho}}{k\Gamma_k(\omega n+\mu)} \frac{\tau^n}{(\delta)_{n\varsigma}}.$$

Remark 1. *Many new integral operators containing the Mittag-Leffler function can be deduced from (6) and (7) (for details, see [32] Remark 1). Furthermore, the integral operators (6) and (7), reproduce various well-known integral operators (for details, see [32] Remark 2).*

In [32], Zhang et al. proved the following formulas for constant function, which we will use in our results:

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}1\right)(x;q) = k(\gamma(x)-\gamma(\sigma_1))^{\frac{\mu}{k}} E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu+k,\delta,k}(\lambda(\gamma(x)-\gamma(\sigma_1))^{\frac{\omega}{k}};q) := \chi^{\mu}_{\sigma_1^+}(x;q) \quad (8)$$

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_2^-}1\right)(x;q) = k(\gamma(\sigma_2)-\gamma(x))^{\frac{\mu}{k}} E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu+k,\delta,k}(\lambda(\gamma(\sigma_2)-\gamma(x))^{\frac{\omega}{k}};q) := \chi^{\mu}_{\sigma_2^-}(x;q). \quad (9)$$

In the upcoming section, we give Chebyshev-type inequalities for generalized k-integral operators containing the Mittag-Leffler function in kernels. Furthermore, we give generalizations of Chebyshev-type inequalities for well-known integral operators proved in [11,13–16], and some new fractional versions of Chebyshev inequalities can be deduced for integral operators given in [32] (Remark 1).

2. Main Results

In first theorem, we prove the Chebyshev-type inequality by using the k-integral operator and the same monotonicity of functions.

Theorem 5. *Let* $\zeta_1, \zeta_2 : [0,\infty) \to \mathbb{R}$ *be two integrable functions of same monotonicity. Then, for* k*-integral operator (6), we have*

$$\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q) \geq \left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q), \tag{10}$$

provided $\chi^{\mu}_{\sigma_1^+}(x;q) \neq 0$.

Proof. As we know the functions ζ_1 and ζ_2 are increasing or decreasing simultaneously, then for all $\psi, \phi \geq 0$, we have

$$(\zeta_1(\psi) - \zeta_1(\phi))(\zeta_2(\psi) - \zeta_2(\phi)) \geq 0. \tag{11}$$

This gives the following inequality:

$$\zeta_1(\psi)\zeta_2(\psi) + \zeta_1(\phi)\zeta_2(\phi) \geq \zeta_1(\psi)\zeta_2(\phi) + \zeta_1(\phi)\zeta_2(\psi). \tag{12}$$

Multiplying (12) with $(\gamma(x) - \gamma(\psi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\psi))^{\omega};q)\gamma'(\psi)$ and integrating with respect to ψ over $[\sigma_1, x]$, we have

$$\begin{aligned}
&\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\psi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\psi))^{\omega};q)\gamma'(\psi)\zeta_1(\psi)\zeta_2(\psi)d\psi \\
&+ \zeta_1(\phi)\zeta_2(\phi)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\psi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\psi))^{\omega};q)\gamma'(\psi)d\psi \\
&\geq \zeta_2(\phi)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\psi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\psi))^{\omega};q)\gamma'(\psi)\zeta_1(\psi)d\psi \\
&+ \zeta_1(\phi)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\psi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\psi))^{\omega};q)\gamma'(\psi)\zeta_2(\psi)d\psi.
\end{aligned} \tag{13}$$

By using (6) and (8), we obtain

$$\begin{aligned}
&\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q) + \zeta_1(\phi)\zeta_2(\phi)\chi^{\mu}_{\sigma_1^+}(x;q) \\
&\geq \zeta_2(\phi)\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q) + \zeta_1(\phi)\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q).
\end{aligned} \tag{14}$$

Now, multiplying (14) with $(\gamma(x) - \gamma(\phi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\phi))^{\omega};q)\gamma'(\phi)$ and integrating with respect to ϕ over $[\sigma_1, x]$, we have

$$\begin{aligned}
&\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\phi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\phi))^{\omega};q)\gamma'(\phi)d\phi \\
&+ \chi^{\mu}_{\sigma_1^+}(x;q)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\phi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\phi))^{\omega};q)\gamma'(\phi)\zeta_1(\phi)\zeta_2(\phi)d\phi \\
&\geq \left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\phi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\phi))^{\omega};q)\gamma'(\phi)\zeta_2(\phi)d\phi \\
&+ \left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q)\int_{\sigma_1}^{x}(\gamma(x) - \gamma(\phi))^{\frac{\mu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,k}(\lambda(\gamma(x) - \gamma(\phi))^{\omega};q)\gamma'(\phi)\zeta_1(\phi)d\phi.
\end{aligned} \tag{15}$$

Again, by using k-integral operator (6), the required inequality (10) is obtained. □

Remark 2. *Several new Chebyshev-type inequalities can be deduced from Theorem 5 for integral operators given in [32] (Remark 1) with the help of the substitution of parameters. Furthermore, Theorem 5 reproduces the Chebyshev-type inequalities for well-known integral operators. For example, for* $\lambda = q = 0$ *and* $\gamma(x) = \frac{(x-\sigma_1)^z}{z}$*, we obtain the first inequality of ([13] Theorem 3.1) (it is explained in Corollary 2), for* $k = 1$, $\lambda = q = \sigma_1 = 0$ *and* $\gamma(x) = \frac{x^z}{z}$*, we obtain ([16]*

Theorem 5), for $k = 1$, $\lambda = q = \sigma_1 = 0$ and $\gamma(x) = \frac{x^{y+z}}{y+z}$, we obtain ([14] Theorem 2.1), for $\lambda = q = \sigma_1 = 0$, we obtain ([15] Theorem 4.1).

Corollary 1. *For $\gamma(x) = x$, $k = 1$ and $\lambda = q = 0$, we obtain the following result for the Riemann–Liouville fractional integral:*

$$\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x) \geq \frac{\Gamma(\mu+1)}{x^{\mu}}\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x)\left(\xi^{\mu}_{\sigma_1^+}\zeta_2\right)(x).$$

Remark 3. *It can be noted that if $\sigma_1 \to 0$, then one can obtain Theorem 1.*

Corollary 2. *The following result holds for a k-fractional conformable integral:*

$$\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x) \geq \left(\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}1\right)(x)\right)^{-1}\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x)\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_2\right)(x). \tag{16}$$

Proof. For $\gamma(x) = \frac{(x-\sigma_1)^z}{z}$, $z \in \mathbb{R} - \{0\}$ and $\lambda = q = 0$ in (6), we obtain the definition of left k-fractional conformable integral $\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\cdot\right)(x)$ given in [13]. Therefore, by using these substitutions in the proof of the above theorem, the inequality (16) is obtained. □

Theorem 6. *Assume that the conditions given in Theorem 5 are valid. Then*

$$\begin{aligned}
&\chi^{\nu}_{\sigma_1^+}(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q) + \chi^{\mu}_{\sigma_1^+}(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,l,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q) \\
&\geq \left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,l,\lambda,\sigma_1^+}\zeta_2\right)(x;q) + \left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,l,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q).
\end{aligned} \tag{17}$$

Proof. Multiplying (14) with $(\gamma(x)-\gamma(\phi))^{\frac{\nu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,\delta,k}(\lambda(\gamma(x)-\gamma(\phi))^{\omega};q)\gamma'(\phi)$ and integrating with respect to ϕ over $[\sigma_1, x]$, we have

$$\begin{aligned}
&\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q)\int_{\sigma_1}^{x}(\gamma(x)-\gamma(\phi))^{\frac{\nu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,\delta,k}(\lambda(\gamma(x)-\gamma(\phi))^{\omega};q)\gamma'(\phi)d\phi \\
&+\chi^{\mu}_{\sigma_1^+}(x;q)\int_{\sigma_1}^{x}(\gamma(x)-\gamma(\phi))^{\frac{\nu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,\delta,k}(\lambda(\gamma(x)-\gamma(\phi))^{\omega};q)\gamma'(\phi)\zeta_1(\phi)\zeta_2(\phi)d\phi \\
&\geq \left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\int_{\sigma_1}^{x}(\gamma(x)-\gamma(\phi))^{\frac{\nu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,\delta,k}(\lambda(\gamma(x)-\gamma(\phi))^{\omega};q)\gamma'(\phi)\zeta_2(\phi)d\phi \\
&+\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q)\int_{\sigma_1}^{x}(\gamma(x)-\gamma(\phi))^{\frac{\nu}{k}-1}E^{\alpha,\varsigma,\rho,\eta}_{\omega,\nu,\delta,k}(\lambda(\gamma(x)-\gamma(\phi))^{\omega};q)\gamma'(\phi)\zeta_1(\phi)d\phi.
\end{aligned} \tag{18}$$

By using k-integral operator (6), the required inequality (17) is obtained. □

Remark 4. *Several new Chebyshev-type inequalities can be deduced from Theorem 6 for integral operators given in [32] (Remark 1) with the help of the substitution of parameters. Furthermore, Theorem 6 reproduces the Chebyshev-type inequalities for well-known integral operators. For example, for $\lambda = q = 0$ and $\gamma(x) = \frac{(x-\sigma_1)^z}{z}$, we obtain the second inequality of ([13], Theorem 3.1]) (explained in Corollary 2), for $k = 1$, $\lambda = q = \sigma_1 = 0$ and $\gamma(x) = \frac{x^z}{z}$, we obtain ([16] Theorem 6), for $k = 1$, $\lambda = q = \sigma_1 = 0$ and $\gamma(x) = \frac{x^{y+z}}{y+z}$, we obtain ([14] Theorem 2.2), for $\lambda = q = \sigma_1 = 0$, we obtain ([15] Theorem 4.5).*

Corollary 3. *For $\gamma(x) = x$, $k = 1$ and $\lambda = q = 0$, we obtain the following result for the Riemann–Liouville fractional integral:*

$$\frac{x^{\mu}}{\Gamma(\mu+1)}\left(\xi^{\nu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x)+\frac{x^{\nu}}{\Gamma(\nu+1)}\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x)$$
$$\geq\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x)\left(\xi^{\nu}_{\sigma_1^+}\zeta_2\right)(x)+\left(\xi^{\nu}_{\sigma_1^+}\zeta_1\right)(x)\left(\xi^{\mu}_{\sigma_1^+}\zeta_2\right)(x).$$

Remark 5. *It can be noted that if $\sigma_1 \to 0$, then one can obtain Theorem 2. Furthermore, from Theorem 6 for $\mu = \nu$, one can obtain Theorem 5.*

Corollary 4. *The following result holds for a k-fractional conformable integral:*

$$\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x)\left({}^{k}_{z}\xi^{\nu}_{\sigma_1^+}1\right)(x)+\left({}^{k}_{z}\xi^{\nu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x)\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}1\right)(x)$$
$$\geq\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x)\left({}^{k}_{z}\xi^{\nu}_{\sigma_1^+}\zeta_2\right)(x)+\left({}^{k}_{z}\xi^{\nu}_{\sigma_1^+}\zeta_1\right)(x)\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_2\right)(x). \tag{19}$$

Proof. For $\gamma(x) = \frac{(x-\sigma_1)^z}{z}$, $z \in \mathbb{R} - \{0\}$ and $\lambda = q = 0$ in (6), we obtain the definition of the left k-fractional conformable integral $\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\cdot\right)(x)$ given in [13]. Therefore, by using these substitutions in the proof of the above theorem, the inequality (19) is obtained. □

Remark 6. *It can be noted that for $\mu = \nu$ in (19), one can obtain Corollary 2.*

Theorem 7. *Let $(\zeta_i)_{i=1,\ldots,n}$ be n positive increasing functions on $[0,\infty)$. Then*

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\prod_{i=1}^{n}\zeta_i\right)(x;q)\geq\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{1-n}\prod_{i=1}^{n}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_i\right)(x;q), \tag{20}$$

provided $\chi^{\mu}_{\sigma_1^+}(x;q) \neq 0$.

Proof. Clearly, for $n = 1$, we have an equality. For $n \geq 2$ we use mathematical induction. For $n = 2$, (20) holds true by using Theorem 5, as follows:

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q)\geq\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q).$$

Suppose that (20) holds true for $n-1$

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\prod_{i=1}^{n-1}\zeta_i\right)(x;q)\geq\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{2-n}\prod_{i=1}^{n-1}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_i\right)(x;q). \tag{21}$$

Since $(\zeta_i)_{i=1,\ldots,n}$ are positive and increasing functions, it is easy to see that $\left(\prod_{i=1}^{n-1}\zeta_i\right)(x;q)$ is an increasing function. Hence, by applying Theorem 5 to the functions $\prod_{i=1}^{n-1}\zeta_i = \zeta_1^*$ and $\zeta_n = \zeta_2^*$, we obtain

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\prod_{i=1}^{n}\zeta_i\right)(x;q)=\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1^*\zeta_2^*\right)(x;q) \tag{22}$$
$$\geq\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1^*\right)(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2^*\right)(x;q)$$
$$=\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\prod_{i=1}^{n-1}\zeta_i\right)(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_n\right)(x;q).$$

Using supposition (21) in (22), we obtain

$$\begin{aligned}&\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\prod_{i=1}^{n}\zeta_i\right)(x;q)\\&\geq\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left(\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{2-n}\prod_{i=1}^{n-1}\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_i\right)(x;q)\right)\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_n\right)(x;q).\end{aligned}\tag{23}$$

Hence (20) holds true for n. □

Remark 7. *Several new Chebyshev-type inequalities can be deduced from Theorem 7 for integral operators given in [32] (Remark 1) with the help of the substitution of parameters. Furthermore, Theorem 7 reproduces the Chebyshev-type inequalities for well-known integral operators. For example, for $k=1$, $\lambda=q=\sigma_1=0$ and $\gamma(x)=\frac{x^z}{z}$, we obtain ([16] Theorem 7), for $k=1$, $\lambda=q=\sigma_1=0$ and $\gamma(x)=\frac{x^{y+z}}{y+z}$, we obtain ([14] Theorem 2.3), for $\lambda=q=\sigma_1=0$, we obtain ([15] Theorem 4.9).*

Corollary 5. *For $\gamma(x)=x$, $k=1$ and $\lambda=q=0$, we obtain the following result for the Riemann–Liouville fractional integral:*

$$\left(\xi^{\mu}_{\sigma_1^+}\prod_{i=1}^{n}\zeta_i\right)(x)\geq\left((\xi^{\mu}_{\sigma_1^+}1)(x)\right)^{1-n}\prod_{i=1}^{n}\left(\xi^{\mu}_{\sigma_1^+}\zeta_i\right)(x).$$

Remark 8. *It can be noted that if $\sigma_1\to 0$, then one can obtain Theorem 3.*

Corollary 6. *The following result holds for a k-fractional conformable integral:*

$$\left({}_{z}^{k}\xi^{\mu}_{\sigma_1^+}\prod_{i=1}^{n}\zeta_i\right)(x)\geq\left(({}_{z}^{k}\xi^{\mu}_{\sigma_1^+}1)(x)\right)^{1-n}\prod_{i=1}^{n}\left({}_{z}^{k}\xi^{\mu}_{\sigma_1^+}\zeta_i\right)(x).\tag{24}$$

Proof. For $\gamma(x)=\frac{(x-\sigma_1)^z}{z}$, $z\in\mathbb{R}-\{0\}$ and $\lambda=q=0$ in (6), we obtain the definition of the left k-fractional conformable integral $\left({}_{z}^{k}\xi^{\mu}_{\sigma_1^+}\cdot\right)(x)$ given in [13]. Therefore, by using these substitutions in the proof of the above theorem, the inequality (24) is obtained. □

Theorem 8. *Let ζ_1 and ζ_2 be two functions defined on $[0,\infty)$, such that ζ_1 is increasing, ζ_2 is differentiable and $m:=inf_{x\in[0,\infty)}\zeta_2'(x)$. Then*

$$\begin{aligned}\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q)\geq&\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q)\\&-m\left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}Id\right)(x;q)\\&+m\left({}_{\gamma}^{k}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}Id.\zeta_1\right)(x;q),\end{aligned}\tag{25}$$

provided $\chi^{\mu}_{\sigma_1^+}(x;q)\neq 0$, where Id is the identity function.

Proof. We consider a function as follows:

$$h(x):=\zeta_2(x)-mId(x),$$

where $Id(x)=x$. Since ζ_2 is a differentiable and increasing function, it is clear that h is differentiable and it is increasing on $[0,\infty)$. Then, using Theorem 5, we can write

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1 h\right)(x;q) \geq \left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}h\right)(x;q)$$
$$= \left(\chi^{\mu}_{\sigma_1^+}(x;q)\right)^{-1}\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\right)(x;q)\left[\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_2\right)(x;q) - m\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}Id\right)(x;q)\right]. \tag{26}$$

Since

$$\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1 h\right)(x;q) = \left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}\zeta_1\zeta_2\right)(x;q) - m\left({}^{k}_{\gamma}\xi^{\alpha,\varsigma,\rho,\eta}_{\omega,\mu,\delta,\lambda,\sigma_1^+}Id.\zeta_1\right)(x;q). \tag{27}$$

Now, by using (27) in (26), the required inequality (25) is obtained. □

Remark 9. *Several new Chebyshev-type inequalities can be deduced from Theorem 8 for integral operators given in [32] (Remark 1) with the help of the substitution of parameters. Furthermore, Theorem 5 reproduces the Chebyshev-type inequalities for well-known integral operators. For example, for $k = 1$, $\lambda = q = \sigma_1 = 0$ and $\gamma(x) = \frac{x^z}{z}$, we obtain ([16] Theorem 8), for $k = 1$, $\lambda = q = \sigma_1 = 0$ and $\gamma(x) = \frac{x^{y+z}}{y+z}$, we obtain ([14] Theorem 2.4), for $\lambda = q = \sigma_1 = 0$, we obtain ([15] Theorem 4.13).*

Corollary 7. *For $\gamma(x) = x$, $k = 1$ and $\lambda = q = 0$, we obtain the following result for the Riemann–Liouville fractional integral:*

$$\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x) \geq \left((\xi^{\mu}_{\sigma_1^+}1)(x)\right)^{-1}\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x)\left(\xi^{\mu}_{\sigma_1^+}\zeta_2\right)(x) - \frac{mx}{\mu+1}\left(\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x) + m\left(\xi^{\mu}_{\sigma_1^+}x\zeta_1\right)(x).$$

Remark 10. *It can be noted that if $\sigma_1 \to 0$, then one can obtain Theorem 4.*

Corollary 8. *The following result holds for a k-fractional conformable integral:*

$$\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\zeta_2\right)(x) \geq \left(({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}1)(x)\right)^{-1}\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x)\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_2\right)(x) - \frac{mx}{\mu+1}\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\zeta_1\right)(x) + m\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}x\zeta_1\right)(x). \tag{28}$$

Proof. For $\gamma(x) = \frac{(x-\sigma_1)^z}{z}$, $z \in \mathbb{R} - \{0\}$ and $\lambda = q = 0$ in (6), we obtain the definition of the left k-fractional conformable integral $\left({}^{k}_{z}\xi^{\mu}_{\sigma_1^+}\cdot\right)(x)$ given in [13]. Therefore, by using these substitutions in the proof of the above theorem, the inequality (28) is obtained. □

3. Conclusions

In this paper, we obtained Chebyshev-type inequalities for generalized k-integral operators via the same monotonicity of functions. The outcomes of this paper provide generalizations of Chebyshev-type inequalities for various well-known integral operators. Several new Chebyshev-type inequalities can be deduced from the established results with the help of the substitution of the parameters given in [32] (Remark 1). We leave this for interested readers.

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