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Some Korovkin-Type Approximation Theorems Associated with a Certain Deferred Weighted Statistical Riemann-Integrable Sequence of Functions

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Abstract: Here, in this article, we introduce and systematically investigate the ideas of deferred weighted statistical Riemann integrability and statistical deferred weighted Riemann summability for sequences of functions. We begin by proving an inclusion theorem that establishes a relation between these two potentially useful concepts. We also state and prove two Korovkin-type approximation theorems involving algebraic test functions by using our proposed concepts and methodologies. Furthermore, in order to demonstrate the usefulness of our findings, we consider an illustrative example involving a sequence of positive linear operators in conjunction with the familiar Bernstein polynomials. Finally, in the concluding section, we propose some directions for future research on this topic, which are based upon the core concept of statistical Lebesgue-measurable sequences of functions.

Keywords: Riemann and Lebesgue integrals; statistical Riemann and Lebesgue integral; deferred weighted Riemann summability; Banach space; Bernstein polynomials; positive linear operators; Korovkin-type approximation theorems; Lebesgue-measurable sequences of functions

MSC: 40A05; 40G15; 33C45; 41A36

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1. Introduction and Motivation

The relatively more familiar theory of ordinary convergence is one of the most important topics of study of sequence spaces. It has indeed gradually progressed to a very high level of development. Two prominent researchers, Fast [1] and Steinhaus [2], independently created a new idea in the theory of sequence spaces, which is known as *statistical* convergence. This fruitful concept is extremely valuable for studies in various areas of pure and applied mathematical sciences. It is remarkably more powerful than the traditional convergence and has provided a vital area of research in recent years. Furthermore, such a concept is closely related to the study of Real Analysis, Analytic Probability theory and Number theory, and so on. For some recent related developments on this subject, the reader can see, for example, the works in [3–18].

Suppose that $\mathfrak{E} \subseteq \mathbb{N}$. Moreover, let

$$\mathfrak{E}_k = \{\eta : \eta \leqq k \quad \text{and} \quad \eta \in \mathfrak{E}\}.$$

Then, the natural (or asymptotic) density $d(\mathfrak{E})$ of $\mathfrak{E}$ is

$$d(\mathfrak{E}) = \lim_{k\to\infty} \frac{|\mathfrak{E}_k|}{k} = \tau,$$

where τ is a real and finite number, and $|\mathfrak{E}_k|$ is the cardinality of $\mathfrak{E}_k$.

A sequence (u_n) is said to be statistically convergent to α if, for each $\epsilon > 0$,

$$\mathfrak{E}_\epsilon = \{\eta : \eta \in \mathbb{N} \quad \text{and} \quad |u_\eta - \alpha| \geqq \epsilon\}$$

has zero natural density (see [1,2]). Thus, for every $\epsilon > 0$,

$$d(\mathfrak{E}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{E}_\epsilon|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} u_k = \alpha.$$

For a closed and bounded interval $\mathcal{I} := [a,b] \subset \mathbb{R}$, we define the partition of $[a,b]$ as an ordered set that is finite and we denote it as follows:

$$\mathfrak{P} := \{(r_0, r_1, \cdots, r_k) : a = r_0 < r_1 < \cdots < r_k = b\}.$$

We now divide the interval $[a,b]$ into the following non-overlapping subintervals:

$$\mathcal{I}_1 := [r_0, r_1], \quad \mathcal{I}_2 := [r_1, r_2], \quad \cdots, \quad \mathcal{I}_k := [r_{k-1}, r_k].$$

The resulting partition $\mathfrak{P}$ is then given by

$$\mathfrak{P} := \{[r_{i-1}, r_i] : i = 1, 2, 3, \cdots, k\}.$$

Next, in order to find the norm of the partition $\mathfrak{P}$, we have

$$\|\mathfrak{P}\| := \max\{r_1 - r_0, r_2 - r_1, r_3 - r_2, \cdots, r_k - r_{k-1}\}.$$

Let $\gamma_i \quad (i = 1, 2, 3, \cdots, k)$ be a point that is chosen arbitrarily from each of the subintervals $(\mathcal{I})_{i=1}^k$. We refer to these points as the tags of the subintervals. We also call the subintervals associated with the tags the tagged partitions of $\mathcal{I}$. We denote it as follows:

$$\mathcal{P} := \{([r_{i-1}, r_i]; \gamma_i) : i = 1, 2, 3, \cdots, k\}.$$

Let $[a,b] \subset \mathbb{R}$. Suppose that, for each $i \in \mathbb{N}$, there is a function $h_i : [a,b] \to \mathbb{R}$. We thus construct the sequence $(h_i)_{i\in\mathbb{N}}$ of functions over the closed interval $[a,b]$.

We now define a subsequence $(h_i)_i^k$ of functions with respect to the Riemann sum associated with a tagged partition $\mathcal{P}$ as follows:

$$\delta(h_i; \mathcal{P}) := \sum_{i=1}^{k} h(\gamma_i)(r_i - r_{i-1}).$$

We next recall the definition of the Riemann integrability.

A sequence $(h_k)_{k\in\mathbb{N}}$ of functions is Riemann-integrable to h on $[a,b]$ if, for each $\epsilon > 0$, there exists $\sigma_\epsilon > 0$ such that, for any tagged partition $\mathcal{P}$ of $[a,b]$ with $\|\mathcal{P}\| < \sigma_\epsilon$, we have

$$|\delta(h_k; \mathcal{P}) - h| < \epsilon.$$

The definition of statistically Riemann-integrable functions is given as follows.

Definition 1. *A sequence $(h_k)_{k\in\mathbb{N}}$ of functions is statistically Riemann-integrable to h on $[a,b]$ if, for every $\epsilon > 0$ and for each $x \in [a,b]$, there exists $\sigma_\epsilon > 0$, and for any tagged partition $\mathcal{P}$ of $[a,b]$ with $\|\mathcal{P}\| < \sigma_\epsilon$, the set*

$$\mathfrak{E}_\epsilon = \{\eta : \eta \in \mathbb{N} \quad and \quad |\delta(h_\eta;\mathcal{P}) - h| \geqq \epsilon\}$$

has zero natural density. That is, for every $\epsilon > 0$,

$$d(\mathfrak{E}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{E}_\epsilon|}{k} = 0.$$

We write

$$stat_{\mathrm{Rie}} \lim_{k\to\infty} \delta(h_k;\mathcal{P}) = h.$$

By making use of Definition 1, we first establish an inclusion theorem as Theorem 1 below.

Theorem 1. *If a sequence of functions (h_k) is Riemann-integrable to h over $[a,b]$, then (h_k) is statistically Riemann-integrable to the same function h over $[a,b]$.*

Proof. Given $\epsilon > 0$, there exists $\sigma_\epsilon > 0$. Suppose that $\mathcal{P}$ is any tagged partition of $[a,b]$ such that $\|\mathcal{P}\| < \sigma_\epsilon$. Then

$$|\delta(h_k;\mathcal{P}) - h| < \epsilon.$$

Since, for each $\epsilon > 0$, $\mathcal{P}$ is any tagged partition of $[a,b]$ such that $\|\mathcal{P}\| < \sigma_\epsilon$, so we have

$$\lim_{k\to\infty} \frac{1}{k}|\{\eta : \eta \in \mathbb{N} \quad \text{and} \quad |\delta(h_k;\mathcal{P}) - h| \geqq \epsilon\}| \leqq \lim_{k\to\infty} |\delta(h_k;\mathcal{P}) - h| < \epsilon.$$

Consequently, by Definition 1, we get

$$\mathrm{stat}_{\mathrm{Rie}} \lim_{k\to\infty} \delta(h_k;\mathcal{P}) = h,$$

which completes the proof of Theorem 1. □

Remark 1. *In order to demonstrate that the converse of Theorem 1 is not true, we consider Example 1 below.*

Example 1. *Let $h_k : [0,1] \to \mathbb{R}$ be a sequence of functions defined by*

$$h_k(x) = \begin{cases} \frac{1}{2} & (x \in \mathbb{Q} \cap [0,1];\ k = j^2,\ j \in \mathbb{N}) \\ \\ \frac{1}{n} & (\text{otherwise}). \end{cases} \tag{1}$$

It is easily seen that the sequence (h_k) of functions is statistically Riemann-integrable to 0 over the closed interval $[0,1]$, but it is not Riemann-integrable (in the usual sense) over $[0,1]$.

Motivated mainly by the above-mentioned investigations and developments, we introduce and study the ideas of deferred weighted statistical Riemann integrability and statistical deferred weighted Riemann summability of sequences of real-valued functions. We first prove an inclusion theorem connecting these two potentially useful concepts. We then state and prove two Korovkin-type approximation theorems with algebraic test functions based on the methodologies and techniques that we have adopted here. Furthermore, we consider an illustrative example involving a positive linear operator in conjunction with the familiar Bernstein polynomials, which shows the effectiveness of our findings. Finally, based upon the core concept of statistical Lebesgue-measurable sequences of functions, we suggest some possible directions for future research on this topic in the concluding section of our study.

2. Deferred Weighted Statistical Riemann Integrability

Let (ϕ_k) and (φ_k) be sequences of non-negative integers with the regularity conditions given

$$\phi_k < \varphi_k \qquad \text{and} \qquad \lim_{k\to\infty} \varphi_k = +\infty.$$

Moreover, let (p_i) be a sequence of non-negative real numbers with

$$P_k = \sum_{i=\phi_k+1}^{\varphi_k} p_i.$$

We then define the deferred weighted summability mean for $\circ(h_k;\mathcal{P})$ associated with tagged partition $\mathcal{P}$ as follows:

$$\mathcal{W}(\delta(h_k;\mathcal{P})) = \frac{1}{P_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} p_\varrho \delta(h_\varrho;\mathcal{P}). \tag{2}$$

We now present the following definitions for our proposed study.

Definition 2. *A sequence $(h_k)_{k\in\mathbb{N}}$ of functions is said to be deferred weighted statistically Riemann-integrable to h on $[a,b]$ if, for all $\epsilon > 0$, there exists $\sigma_\epsilon > 0$, and for any tagged partition $\mathcal{P}$ of $[a,b]$ with $\|\mathcal{P}\| < \sigma_\epsilon$, the following set*

$$\{\eta : \eta \leqq P_k \quad \text{and} \quad p_\eta |\delta(h_\eta;\mathcal{P}) - h| \geqq \epsilon\}$$

has zero natural density. Thus, for every $\epsilon > 0$, we have

$$\lim_{k\to\infty} \frac{|\{\eta : \eta \leqq P_k \quad \text{and} \quad p_\eta |\delta(h_\eta;\mathcal{P}) - k| \geqq \epsilon\}|}{P_k} = 0.$$

We write

$$DWR_{\text{stat}} \lim_{k\to\infty} \delta(h_k;\mathcal{P}) = h.$$

Definition 3. *A sequence $(h_k)_{k\in\mathbb{N}}$ of functions is said to statistically deferred weighted Riemann summable to h on $[a,b]$ if, for all $\epsilon > 0$ $\exists$ $\sigma_\epsilon > 0$ and for any tagged partition $\mathcal{P}$ of $[a,b]$ with $\|\mathcal{P}\| < \sigma_\epsilon$, the set*

$$\{\eta : \eta \leqq k \quad \text{and} \quad |\mathcal{W}(\delta(h_\eta;\mathcal{P})) - h| \geqq \epsilon\}$$

has zero natural density. Thus, for all $\epsilon > 0$, we have

$$\lim_{k\to\infty} \frac{|\{\eta : \eta \leqq k \quad \text{and} \quad |\mathcal{W}(\delta(h_\eta;\mathcal{P})) - h| \geqq \epsilon\}|}{k} = 0.$$

We write

$$stat_{\text{DWR}} \lim_{k\to\infty} \delta(h_k;\mathcal{P}) = h.$$

An inclusion theorem between the two new potentially useful notions in Definitions 2 and 3 is now given by Theorem 2 below.

Theorem 2. *If the sequence $(h_k)_{k\in\mathbb{N}}$ of functions is deferred weighted statistically Riemann-integrable to a function h over $[a,b]$, then it is statistically deferred weighted Riemann summable to the same function h over $[a,b]$, but not conversely.*

Proof. Suppose that the sequence $(h_k)_{k\in\mathbb{N}}$ is deferred weighted statistically Riemann-integrable to a function h on $[a,b]$. Then, by Definition 2, we have

$$\lim_{k\to\infty} \frac{|\{\eta : \eta \leqq P_k \quad \text{and} \quad p_\eta|\delta(h_\eta;\mathcal{P}) - h| \geqq \epsilon\}|}{P_k} = 0.$$

Now, if we choose the two sets as follows,

$$\mathcal{O}_\epsilon = \{\eta : \eta \leqq P_k \qquad \text{and} \qquad p_\eta|\delta(h_\eta;\mathcal{P}) - h| \geqq \epsilon\}$$

and

$$\mathcal{O}_\epsilon^c = \{\eta : \eta \leqq P_k \qquad \text{and} \qquad p_\eta|\delta(h_\eta;\mathcal{P}) - h| < \epsilon\},$$

then we have

$$\begin{aligned}
|\mathcal{W}(\delta(h_k;\mathcal{P})) - h| &= \left| \frac{1}{P_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} p_\varrho \delta(h_\varrho;\mathcal{P}) - h \right| \\
&\leqq \left| \frac{1}{P_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} p_\varrho [\delta(h_\varrho;\mathcal{P}) - h] \right| + \left| \frac{1}{P_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} p_\varrho h - h \right| \\
&\leqq \frac{1}{P_k} \sum_{\substack{\varrho=\phi_k+1 \\ (\eta\in\mathcal{O}_\epsilon)}}^{\varphi_k} p_\varrho|\delta(h_\varrho;\mathcal{P}) - h| + \frac{1}{P_k} \sum_{\substack{\varrho=\phi_k+1 \\ (\eta\in\mathcal{O}_\epsilon^c)}}^{\varphi_k} p_\varrho|\delta(h_\varrho;\mathcal{P}) - h| \\
&\quad + |h| \left| \frac{1}{P_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} p_\varrho - 1 \right| \\
&\leqq \frac{1}{P_k}|\mathcal{O}_\epsilon| + \frac{1}{P_k}|\mathcal{O}_\epsilon^c|.
\end{aligned}$$

We thus obtain

$$|\mathcal{W}(\delta(h_k;\mathcal{P})) - h| < \epsilon.$$

Hence, clearly, the sequence of functions (h_k) is statistically deferred weighted Riemann-summable to h over $[a,b]$. □

The following example shows that the converse statement of Theorem 2 is not true.

Example 2. *Let $h_k : [0,1] \to \mathbb{R}$ be a sequence of functions of the form given by*

$$h_k(x) = \begin{cases} 0 & (x \in \mathbb{Q}\cap[0,1];\ k \text{ is even}) \\ \\ 1 & (x \in \mathbb{R}-\mathbb{Q}\cap[0,1];\ k \text{ is odd}), \end{cases} \tag{3}$$

where

$$\phi_k = 2k \quad \varphi_k = 4k \quad \text{and} \quad p_k = 1.$$

The above-specified sequence (h_k) of functions trivially indicates that it is neither Riemann-integrable nor deferred weighted statistically Riemann-integrable. However, as per our proposed mean (2), it is easy to see that

$$\begin{aligned}
\mathcal{W}(\delta(h_k;\mathcal{P})) &= \frac{1}{\varphi_k - \phi_k} \sum_{\varrho=\phi_k+1}^{\varphi_k} \delta(h_\varrho;\mathcal{P}) \\
&= \frac{1}{2k} \sum_{m=2k+1}^{4k} \delta(h_\varrho;\mathcal{P}) = \frac{1}{2}.
\end{aligned}$$

Thus, clearly, the sequence (h_k) *of functions has deferred weighted Riemann sum* $\frac{1}{2}$ *under the tagged partition* $\mathcal{P}$*. Therefore, the sequence* (h_k) *of functions is statistically deferred weighted Riemann-summable to* $\frac{1}{2}$ *over* $[0,1]$*, but it is not deferred weighted statistically Riemann-integrable over* $[0,1]$*.*

3. Korovkin-Type Approximation Theorems via the $\mathcal{W}(\delta(h_k;\mathcal{P}))$-Mean

Many researchers have worked toward extending (or generalizing) the approximation-theoretic aspects of the Korovkin-type approximation theorems in several different areas of mathematics, such as (for example) probability space, measurable space, sequence spaces, and so on. In Real Analysis, Harmonic Analysis and other related fields, this notion is immensely useful. In this regard, we have chosen to refer the interested reader to the recent works (see, for example, [19–28]).

Let $C[0,1]$ be the space of all continuous real-valued functions defined on $[0,1]$. Suppose also that it is a Banach space with the norm $\|.\|_\infty$. Then, for $h \in C[0,1]$, the norm of h is given by

$$\|h\|_\infty = \sup\{|h(\rho)| \ : \ 0 \leqq \rho \leqq 1\}.$$

We say that $\mathfrak{G}_j : C[0,1] \to C[0,1]$ is a sequence of positive linear operators, if

$$\mathfrak{G}_j(h;\rho) \geqq 0 \quad \text{as} \quad h \geqq 0.$$

Now, in view of our above-proposed definitions, we state and prove the following Korovkin-type approximation theorems.

Theorem 3. *Let* $\mathfrak{G}_j : C[0,1] \to C[0,1]$ *be a sequence of positive linear operators. Then, for* $h \in C[0,1]$*,*

$$\mathrm{DWR_{stat}} \lim_{j\to\infty} \|\mathfrak{G}_j(h;\rho) - h(\rho)\|_\infty = 0 \tag{4}$$

if and only if

$$\mathrm{DWR_{stat}} \lim_{j\to\infty} \|\mathfrak{G}_j(1;\rho) - 1\|_\infty = 0, \tag{5}$$

$$\mathrm{DWR_{stat}} \lim_{j\to\infty} \|\mathfrak{G}_j(\rho;\rho) - \rho\|_\infty = 0 \tag{6}$$

and

$$\mathrm{DWR_{stat}} \lim_{j\to\infty} \|\mathfrak{G}_j(\rho^2;\rho) - \rho^2\|_\infty = 0. \tag{7}$$

Proof. Since each of the following functions

$$h_0(\rho) = 1, \quad h_1(\rho) = 2\rho \quad \text{and} \quad h_2(\rho) = 3\rho^2$$

belongs to $C[0,1]$ and is continuous on $[0,1]$, the implication given by (4) obviously implies (5) to (7).

In order to complete the proof of Theorem 3, we first assume that the conditions (5) to (7) hold true. If $h \in C[0,1]$, then there exists a constant $\mathcal{L} > 0$ such that

$$|h(\rho)| \leqq \mathcal{L} \quad (\forall\, \rho \in [0,1]).$$

We thus find that

$$|h(r) - h(\rho)| \leqq 2\mathcal{L} \quad (r,\rho \in [0,1]). \tag{8}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(r) - f(\rho)| < \epsilon \tag{9}$$

whenever

$$|r - \rho| < \delta \quad \text{for all} \quad r, \rho \in [0, 1].$$

If we now choose

$$\mu_1 = \mu_1(r, \rho) = (2r - 2\rho)^2.$$

If

$$|r - \rho| \geqq \delta,$$

then we obtain

$$|h(r) - h(\rho)| < \frac{2\mathcal{L}}{\theta^2}\mu_1(r, \rho). \tag{10}$$

Thus, from Equations (9) and (10), we get

$$|h(r) - h(\rho)| < \epsilon + \frac{2\mathcal{L}}{\theta^2}\mu_1(r, \rho),$$

which implies that

$$-\epsilon - \frac{2\mathcal{L}}{\theta^2}\mu_1(r, \rho) \leqq h(r) - h(\rho) \leqq \epsilon + \frac{2\mathcal{L}}{\theta^2}\mu_1(r, \rho). \tag{11}$$

Now, since $\mathfrak{G}_m(1; \rho)$ is monotone and linear, by applying the operator $\mathfrak{G}_m(1; \rho)$ to the inequality (11), we get

$$\begin{aligned}\mathfrak{G}_m(1; \rho)\left(-\epsilon - \frac{2\mathcal{L}}{\theta^2}\mu_1(r, \rho)\right) &\leqq \mathfrak{G}_m(1; \rho)\big(h(r) - h(\rho)\big) \\ &\leqq \mathfrak{G}_m(1; \rho)\left(\epsilon + \frac{2\mathcal{L}}{\theta^2}\mu_1(r, \rho)\right).\end{aligned}$$

We note that ρ is fixed, and so $h(\rho)$ is a constant number. Therefore, we have

$$\begin{aligned}-\epsilon\mathfrak{G}_m(1; \rho) - \frac{2\mathcal{L}}{\theta^2}\mathfrak{G}_m(\mu_1; \rho) &\leqq \mathfrak{G}_m(h; \rho) - h(\rho)\mathfrak{G}_m(1; \rho) \\ &\leqq \epsilon\mathfrak{G}_m(1; \rho) + \frac{2\mathcal{L}}{\theta^2}\mathfrak{G}_m(\mu_1; \rho).\end{aligned} \tag{12}$$

We also know that

$$\mathfrak{G}_m(h; \rho) - h(\rho) = [\mathfrak{G}_m(h; \rho) - h(\rho)\mathfrak{G}_m(1; \rho)] + h(\rho)[\mathfrak{G}_m(1; \rho) - 1]. \tag{13}$$

Thus, by using (12) and (13), we obtain

$$\mathfrak{G}_m(h; \rho) - h(\rho) < \epsilon\mathfrak{G}_m(1; \rho) + \frac{2\mathcal{L}}{\theta^2}\mathfrak{G}_m(\mu_1; \rho) + h(\rho)[\mathfrak{G}_m(1; \rho) - 1]. \tag{14}$$

We now estimate $\mathfrak{G}_m(\mu_1; \rho)$ as follows:

$$\begin{aligned}\mathfrak{G}_m(\mu_1; \rho) &= \mathfrak{G}_m((2r - 2\rho)^2; \rho) = \mathfrak{G}_m(2r^2 - 8\rho r + 4\rho^2; \rho) \\ &= \mathfrak{G}_m(4r^2; \rho) - 8t\mathfrak{G}_m(r; \rho) + 4\rho^2\mathfrak{G}_m(1; \rho) \\ &= 4[\mathfrak{G}_m(r^2; \rho) - \rho^2] - 8t[\mathfrak{G}_m(r; \rho) - \rho] \\ &\quad + 4\rho^2[\mathfrak{G}_m(1; \rho) - 1],\end{aligned}$$

so that, in view of (14), we obtain

$$\begin{aligned}
\mathfrak{G}_m(h;\rho) - h(\rho) &< \epsilon\mathfrak{G}_m(1;\rho) + \frac{2\mathcal{L}}{\theta^2}\{4[\mathfrak{G}_m(r^2;\rho) - \rho^2] \\
&\quad - 8\rho[\mathfrak{G}_m(r;\rho) - \rho] + 4\rho^2[\mathfrak{G}_m(1;\rho) - 1]\} \\
&\quad + h(\rho)[\mathfrak{G}_m(1;\rho) - 1]. \\
&= \epsilon[\mathfrak{G}_m(1;\rho) - 1] + \epsilon + \frac{2\mathcal{L}}{\theta^2}\{4[\mathfrak{G}_m(r^2;\rho) - \rho^2] \\
&\quad - 8\rho[\mathfrak{G}_m(r;\rho) - \rho] + 4\rho^2[\mathfrak{G}_m(1;\rho) - 1]\} \\
&\quad + h(\rho)[\mathfrak{G}_m(1;\rho) - 1].
\end{aligned}$$

Furthermore, since $\epsilon > 0$ is arbitrary, we can write

$$\begin{aligned}
|\mathfrak{G}_m(h;\rho) - h(\rho)| &\leqq \epsilon + \left(\epsilon + \frac{8\mathcal{L}}{\theta^2} + \mathcal{L}\right)|\mathfrak{G}_m(1;\rho) - 1| \\
&\quad + \frac{16\mathcal{L}}{\theta^2}|\mathfrak{G}_m(r;\rho) - \rho| + \frac{8\mathcal{L}}{\theta^2}|\mathfrak{G}_m(r^2;\rho) - \rho^2| \\
&\leqq \mathcal{A}(|\mathfrak{G}_m(1;\rho) - 1| + |\mathfrak{G}_m(r;\rho) - \rho| \\
&\quad + |\mathfrak{G}_m(r^2;\rho) - \rho^2|),
\end{aligned} \tag{15}$$

where

$$\mathcal{A} = \max\left(\epsilon + \frac{8\mathcal{L}}{\theta^2} + \mathcal{L}, \frac{16\mathcal{L}}{\theta^2}, \frac{8\mathcal{L}}{\theta^2}\right).$$

Now, for a given $\omega > 0$, there exists $\epsilon > 0$ $(\epsilon < \omega)$ such that

$$\mathfrak{T}_m(\rho;\omega) = \{m : m \leqq P_k \quad \text{and} \quad p_m|\mathfrak{G}_m(h;\rho) - h(\rho)| \geqq \omega\}.$$

Furthermore, for $\nu = 0, 1, 2$, we have

$$\mathfrak{T}_{\nu,m}(\rho;\omega) = \left\{m : m \leqq P_k \quad \text{and} \quad p_m|\mathfrak{G}_m(h;\rho) - h_\nu(\rho)| \geqq \frac{\omega - \epsilon}{3\mathcal{A}}\right\},$$

so that

$$\mathfrak{T}_m(\rho;\omega) \leqq \sum_{\nu=0}^{2} \mathfrak{T}_{\nu,m}(\rho;\omega).$$

Clearly, we obtain

$$\frac{\|\mathfrak{T}_m(\rho;\omega)\|_{C[0,1]}}{P_k} \leqq \sum_{\nu=0}^{2} \frac{\|\mathfrak{T}_{\nu,m}(\rho;\omega)\|_{C[0,1]}}{P_k}. \tag{16}$$

Now, using the above assumption about the implications in (5) to (7) and by Definition 2, the right-hand side of (16) tends to zero as $n \to \infty$. Consequently, we get

$$\lim_{k\to\infty} \frac{\|\mathfrak{T}_m(\rho;\omega)\|_{C[0,1]}}{P_k} = 0 \ (\delta, \omega > 0).$$

Therefore, the implication (4) holds true. □

Theorem 4. *Let* $\mathfrak{G}_j : C[0,1] \to C[0,1]$ *be a sequence of positive linear operators. Then, for* $h \in C[0,1]$,

$$\text{stat}_{\text{DWR}} \lim_{j\to\infty} \|\mathfrak{G}_j(h;\rho) - h(\rho)\|_\infty = 0 \tag{17}$$

if and only if

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{j\to\infty} \|\mathfrak{G}_j(1;\rho) - 1\|_\infty = 0, \tag{18}$$

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{j\to\infty} \|\mathfrak{G}_j(\rho;\rho) - \rho\|_\infty = 0 \tag{19}$$

and

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{j\to\infty} \|\mathfrak{G}_j(\rho^2;\rho) - \rho^2\|_\infty = 0. \tag{20}$$

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. Therefore, we choose to skip the details involved. □

In view of Theorem 4, here, we consider an illustrative example. In this connection, we now recall the following operator:

$$\rho(1+\rho D) \qquad \left(D = \frac{d}{d\rho}\right), \tag{21}$$

which was used by Al-Salam [29] and, more recently, by Viskov and Srivastava [30].

Example 3. *Consider the Bernstein polynomials $\mathfrak{B}_n(h;\beta)$ on $C[0,1]$ given by*

$$\mathfrak{B}_k(h;\beta) = \sum_{\varrho=0}^{k} f\left(\frac{\varrho}{k}\right)\binom{k}{\varrho}\beta^{\varrho}(1-\beta)^{k-\varrho} \quad (\beta\in[0,1];k=0,1,\cdots). \tag{22}$$

Here, in this example, we introduce the positive linear operators on $C[0,1]$ under the composition of the Bernstein polynomials and the operators given by (21) as follows:

$$\mathfrak{G}_\varrho(h;\beta) = [1+h_\varrho]\beta(1+\beta D)\mathfrak{B}_\varrho(h;\beta) \quad (\forall\, h\in C[0,1]), \tag{23}$$

where (h_ϱ) is the same as mentioned in Example 2.

We now estimate the values of each of the testing functions 1, β and β^2 by using our proposed operators (23) as follows:

$$\mathfrak{G}_\varrho(1;\beta) = [1+h_\varrho]\beta(1+\beta D)1 = [1+h_\varrho]\beta,$$

$$\mathfrak{G}_\varrho(t;\beta) = [1+h_\varrho]\beta(1+\beta D)\beta = [1+h_\varrho]\beta(1+\beta)$$

and

$$\begin{aligned}\mathfrak{G}_\varrho(t^2;\beta) &= [1+h_\varrho]\beta(1+\beta D)\left\{\beta^2 + \frac{\beta(1-\beta)}{\varrho}\right\}\\ &= [1+h_\varrho]\left\{\beta^2\left(2-\frac{3\beta}{\varrho}\right)\right\}.\end{aligned}$$

Consequently, we have

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{\varrho\to\infty} \|\mathfrak{G}_\varrho(1;\beta) - 1\|_\infty = 0, \tag{24}$$

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{\varrho\to\infty} \|\mathfrak{G}_\varrho(\beta;\beta) - \beta\|_\infty = 0 \tag{25}$$

and

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{\varrho\to\infty} \|\mathfrak{G}_\varrho(\beta^2;\beta) - \beta^2\|_\infty = 0, \tag{26}$$

that is, the sequence $\mathfrak{G}_\varrho(h;\beta)$ satisfies the conditions (18) to (20). Therefore, by Theorem 4, we have

$$\mathrm{stat}_{\mathrm{DWR}} \lim_{\varrho\to\infty} \|\mathfrak{G}_\varrho(h;\beta) - h\|_\infty = 0.$$

Hence, the given sequence (h_k) of functions mentioned in Example 2 is statistically deferred weighted Riemann-summable, but not deferred weighted statistically Riemann-integrable. Therefore, our above-proposed operators defined by (23) satisfy Theorem 4. However, they do not satisfy for statistical versions of deferred weighted Riemann-integrable sequence of functions (see Theorem 3).

4. Concluding Remarks and Directions for Further Research

In this concluding section of our present investigation, we further observe the potential usefulness of our Theorem 4 over Theorem 3 as well as over the classical versions of the Korovkin-type approximation theorems.

Remark 2. *Let us consider the sequence $(h_\varrho)_{\varrho\in\mathbb{N}}$ of functions in Example 2. Suppose also that (h_ϱ) is statistically deferred weighted Riemann-summable, so that*

$$stat_{\mathrm{DWR}} \lim_{\varrho\to\infty} \delta(h_\varrho;\mathcal{P}) = \frac{1}{2} \text{ on } [0,1].$$

We then find that

$$stat_{\mathrm{DWR}} \lim_{k\to\infty} \|\mathfrak{G}_k(h_\nu;\rho) - f_\nu(\rho)\|_\infty = 0 \quad (\nu = 0,1,2). \tag{27}$$

Thus, by Theorem 4, we immediately get

$$stat_{\mathrm{DWR}} \lim_{j\to\infty} \|\mathfrak{G}_k(h;\rho) - h(\rho)\|_\infty = 0, \tag{28}$$

where

$$h_0(\rho) = 1, \quad h_1(\rho) = \rho \quad \textit{and} \quad h_2(\rho) = \rho^2.$$

Now, the given sequence (h_k) of functions is statistically deferred weighted Riemann-summable, but neither deferred weighted statistically Riemann-integrable nor classically Riemann-integrable. Therefore, our Korovkin-type approximation Theorem 4 properly works under the operators defined in the Equation (23), but the classical as well as statistical versions of the deferred weighted Riemann-integrable sequence of functions do not work for the same operators. Clearly, this observation leads us to the fact that our Theorem 4 is a non-trivial extension of Theorem 3 as well as the classical Korovkin-type approximation theorem [31].

Remark 3. *Motivated by some recently published results by Jena et al. [32] and Srivastava et al. [33], we choose to draw the attention of the interested readers toward the potential for further research associated with the analogous notion of statistical Lebesgue-measurable sequences of functions.*

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