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Monotonicity Arguments for Variational–Hemivariational Inequalities in Hilbert Spaces

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Abstract: We consider a variational–hemivariational inequality in a real Hilbert space, which depends on two parameters. We prove that the inequality is governed by a maximal monotone operator, then we deduce various existence, uniqueness and equivalence results. The proofs are based on the theory of maximal monotone operators, fixed point arguments and the properties of the subdifferential, both in the sense of Clarke and in the sense of convex analysis. These results lay the background in the study of various classes of inequalities. We use them to prove existence, uniqueness and continuous dependence results for the solution of elliptic and history-dependent variational–hemivariational inequalities. We also present some iterative methods in solving these inequalities, together with various convergence results.

Keywords: variational–hemivariational inequalities; Clarke subdifferential; convex subdifferential; maximal monotone operator; resolvent; fixed point problem; iterative method

MSC: 47H20; 47H05; 47H09; 49H52; 49J53

Citation: Sofonea, M. Monotonicity Arguments for Variational-Hemivariational Inequalities in Hilbert Spaces. *Axioms* **2022**, *11*, 136. https://doi.org/10.3390/axioms11030136

Academic Editor: Hans J. Haubold

Received: 15 February 2022
Accepted: 14 March 2022
Published: 16 March 2022

1. Introduction

Variational–hemivariational inequalities represent a powerful mathematical tool in the study of nonlinear boundary value problems. Their study is motivated by various applications in Physics, Mechanics and Engineering Sciences, among others. In contrast with variational inequalities (which are governed by convex functions) and hemivariational inequalities (which are governed by nonsmooth locally Lipschitz functions which could be nonconvex), variational–hemivariational inequalities are governed by both convex and locally Lipschitz functions. As a consequence, they have both a convex and nonconvex structure and, therefore, their study is carried out by using arguments on both convex and nonsmooth analysis.

Introduced in the pioneering work of Panagiotopoulos [1], the theory of variational–hemivariational inequalities grew up rapidly, as shown in [2–5] and the references therein. It includes existence, uniqueness and numerical approximation results, obtained in the study of different classes of inequalities, by using various methods and functional arguments. Reference in the field include [6–11]. Among the inequalities studied in these papers we distinguish the class of elliptic, the class of time-dependent and the class of evolutionary variational–hemivariational inequalities. A variational–hemivariational inequality is said to be elliptic if it does not involve the time variable; it is said to be time-dependent if both the data and the solution depend on time but no time-derivatives of the solution appear in its statement; finally, a variational–hemivariational inequality is said to be evolutionary if it is formulated in terms of the derivative of the unknown function.

A first example of elliptic variational-hemivariational inequality is the following: find u such that

$$u \in K, \quad (Au, v-u)_H + \psi(v) - \psi(u) + g^0(u; v-u) \geq (f, v-u)_H \quad \forall\, v \in K. \tag{1}$$

Here and below, H represents a real Hilbert space endowed with the inner product $(\cdot,\cdot)_H$ and the associated norm $\|\cdot\|_H$, K is a nonempty subset of H, $A : H \to H$ is a nonlinear operator, $\psi : H \to \mathbb{R}$ and $g : H \to \mathbb{R}$ are given functions and, finally, $f \in H$. The function ψ is assumed to be convex while the function g is assumed to be locally Lipschitz. Moreover, notation $g^0(u; v)$ represents the Clarke directional derivative of g at the point u, in the direction v.

A second example of elliptic variational-hemivariational inequality is the following: find u such that

$$u \in K, \qquad (Au, v-u)_H + \varphi(u,v) - \varphi(u,u) + j^0(u,u; v-u) \geq (f, v-u)_H \quad \forall v \in K. \tag{2}$$

Note that, in contrast with (1), here the functions φ and j are defined on the product space $H \times H$, that is, $\varphi : H \times H \to \mathbb{R}$ and $j : H \times H \to \mathbb{R}$. The function φ is assumed to be convex with respect to the second variable, j is assumed to be locally Lipschitz with respect to the second argument and notation $j^0(w,u;v)$ represents the Clarke directional derivative of $j(w,\cdot)$ at the point u, in the direction v.

A special case of time-dependent variational-hemivariational inequalities is given by the so-called history-dependent variational-hemivariational inequalities. A typical example is the following: find a function $u : \mathbb{R}_+ \to H$ such that

$$\begin{aligned} u(t) \in K, \quad & (Au(t), v - u(t))_H + \varphi(\mathcal{S}u(t), v) - \varphi(\mathcal{S}u(t), u(t)) \\ & + j^0(\mathcal{R}u(t), u(t); v - u(t)) \geq (f(t), v - u(t))_H \quad \forall v \in K,\ t \in \mathbb{R}_+. \end{aligned} \tag{3}$$

Note that in (3) and below in this paper $\mathcal{S}$ and $\mathcal{R}$ are operators defined on the space of continuous functions defined on $\mathbb{R}_+ = [0, +\infty)$ with values H, denoted in what follows by $C(\mathbb{R}_+; H)$. Moreover, for any function $w \in C(\mathbb{R}_+; H)$ we use the shorthand notation $\mathcal{S}w(t)$ and $\mathcal{R}w(t)$, to represent the value of the functions $\mathcal{S}w$ and $\mathcal{R}w$ at the point $t \in \mathbb{R}_+$, that is, $\mathcal{S}w(t) := (\mathcal{S}w)(t)$ and $\mathcal{R}w(t) := (\mathcal{R}w)(t)$.

Inequality problems of the form (1), (2), arise in the study of mathematical models which describe the equilibrium of elastic body in frictionless and frictional contact with a foundation, respectively. Moreover, inequality problems of the form (3) arise in the study of mathematical models of contact with elastic or viscoelastic materials, in which memory effects are taken into consideration, either in the constitutive law or in the contact conditions. References in the field are the books [4,5] as well as the survey article [2]. Moreover, it is worth noting that variational–hemivariational inequalities arise in the study of complex fluids and history-dependent viscoelastic and elasto-viscoplastic models. A comprehensive reference in the field is the book [12]. There, an introduction to the modeling of complex fluids is provided, up-to-date mathematical and numerical analysis of the corresponding equations can be found, together with several numerical algorithms for the approximation of the solutions. Furthermore, subdifferential operators have been used in [13] in the study of various magnetorheological mixtures composed of a fluid and a solid continuum.

Existence and uniqueness results in the study of elliptic variational–hemivariational inequality have been obtained in many papers, under different assumptions on the data. For instance, a surjectivity result for pseudomonotone multivalued operators was used in [14] in order to obtain the unique solvability of inequality (1). There, the operator A was assumed to be pseudomonotone and strongly monotone and the Clarke subdifferential of the function j was assumed to satisfy a growth condition. The method used in [14] can be used in the study of inequality (2), as shown in [5], for instance. Recently, problem (1) was considered in [15], under the assumption that A a strongly monotone Lipschitz continuous operator and φ is a continuous convex function. The unique solvability of the problem was obtained by using a minimization principle which avoids any pseudomonotonicity argument.

Motivated by the importance of the topic in both pure and applied mathematics, in this paper we introduce a new approach which allows us to prove existence, uniqueness and convergence results for variational–hemivariational inequalities in Hilbert spaces. The novelty of the results we present here arises in the fact that the approach we use is based on arguments of multivalued maximal operators in Hilbert spaces and fixed point. It can be used for various classes of elliptic or history-dependent variational–hemivariational inequalities. Nevertheless, for simplicity, we restrict ourselves to the study of inequalities (1)–(3), which represent three relevant examples. Our results are obtained under assumptions which are slightly different from those used in [5,14,15] and, therefore, they complete the results obtained in these references. For instance, here j is a bifunction, no growth assumption on its subdifferential is assumed and the smallness assumptions involving the constants m_A, α_φ, α_j^1, α_j^2 (related to the data A, φ, j) are relaxed. Relaxing this assumption was possible by using the Browder–Godhe–Kirk fixed point argument instead of the classical Banach fixed point principle. For all these reasons we believe that our results contribute to a better knowledge of the structure of variational–hemivariational inequalities and, in addition, they open the way to the approach of the solution by using various iterative methods.

The rest of the manuscript is organized as follows. In Section 2 we recall some preliminary material. In Section 3 we consider a variational–hemivariational inequality which depends on two parameters. We use arguments of convex and nonsmooth analysis in order to prove that this inequality is governed by a maximal monotone operator. This allows us to obtain various properties for the resolvent of this operator, which have interest in their own. We use these properties in Sections 4–6 in order to deduce existence and uniqueness results for elliptic and history-dependent variational–hemivariational inequalities of the form (1), (2) and (3), respectively. In addition to the properties of the resolvent operator, our proofs are based on equivalence and fixed point arguments. We also introduce several iterative methods in solving these inequalities and deduce various convergence results. Finally, in Section 7 we present some concluding remarks.

2. Preliminaries

The results we present in this section can be found in many books and surveys, including [5,16–19]. For this reason we present them without proofs. Everywhere below H represents a real Hilbert space endowed with the inner product $(\cdot,\cdot)_H$ and the associated norm $\|\cdot\|_H$. We use the symbols "$\to$" and "$\rightharpoonup$" to denote the strong and the weak convergence in the space H and employ the notation H_w for the space H equipped with the weak topology. The limits, lower limits and upper limits are considered as $n \to \infty$, even if we do not mention it explicitly. Moreover, we use *int* M for the interior of the set $M \subset H$, in the strong topology of H. Finally, we denote by J_H the identity map of H, by 0_H the zero element of H and by 2^H the set of parts of H. We start with the following definitions for single-valued operators.

Definition 1. *The operator $A : H \to H$ is said to be:*

(a) *demicontinuous if $u_n \to u$ in H implies $Au_n \rightharpoonup Au$ in H;*
(b) *strongly monotone if there exists constant $m_A > 0$ such that*

$$(Au - Av, u - v)_H \geq m_A \|u - v\|_H^2 \quad \forall\, u, v \in H;$$

(c) *Lipschitz continuous if there exists constant $L_A > 0$ such that*

$$\|Au - Av\|_H \leq L_A \|u - v\|_H \quad \forall\, u, v \in H.$$

Definition 2. *Let $K \subset H$. The operator $A : K \subset H \to H$ is said to be:*

(a) *nonexpansive on K if there exists a constant $k_A \in [0,1]$ such that*

$$\|Au - Av\|_H \leq k_A \|u - v\|_H \quad \forall\, u, v \in K;$$

(b) *a contraction if it is nenexpansive on K with constant $k_A \in [0,1)$.*

In this paper, in addition to the well-known Banach contraction principle we shall use the following Browder–Godhe–Kirk fixed point Theorem, proved in [20], (p. 55).

Theorem 1. *Let K be a nonempty closed bounded convex subset of the Hilbert space H and let $A : K \to K$ be a nonexpansive operator. Then A has at least one fixed point.*

We now proceed with some results concerning multivalued operators defined on the space H. To this end we recall that, given a multivalued operator $T : H \to 2^H$, its domain $D(T)$, range $R(T)$ and graph $Gr(T)$ are the sets defined by

$$D(T) = \{\, v \in H \mid Tv \neq \emptyset \,\},$$
$$R(T) = \{\, f \in H \mid \exists\, v \in D(T) \text{ s.t. } f \in Tv \,\},$$
$$Gr(T) = \{\, (v, v^*) \in H \times H \mid v^* \in Tv \,\}.$$

Definition 3. *The operator $T : H \to 2^H$ is said to be:*

(a) *monotone if*

$$(u_1^* - u_2^*, u_1 - u_2)_H \geq 0 \quad \forall\, (u_1, u_1^*), (u_2, u_2^*) \in Gr(T);$$

(b) *relaxed monotone if there exists constant $\alpha_T > 0$ such that*

$$(u_1^* - u_2^*, u_1 - u_2)_H \geq -\alpha_T \|u_1 - u_2\|_H^2 \quad \forall\, (u_1, u_1^*), (u_2, u_2^*) \in Gr(T); \tag{4}$$

(c) *maximal monotone if it is monotone and, for any v, $v^* \in H$, the following implication holds:*

$$(u^* - v^*, u - v)_H \geq 0 \ \ \forall\, u \in D(T),\ u^* \in Tu \implies v \in D(T) \text{ and } v^* \in Tv.$$

There is a close connection between the property of maximal monotonicity of T and the surjectivity property of the operator $J_H + \lambda T$ with $\lambda > 0$. The fundamental result in this direction is the celebrated theorem of Minty that we recall below.

Theorem 2. *Let $T : H \to 2^H$ be a maximal monotone operator and let $\lambda > 0$. Then $R(J_H + \lambda T) = H$. Moreover, for any $f \in H$ there exists a unique element $u \in D(T)$ such that $u + \lambda Tu \ni f$.*

Theorem 2 allows us to consider the resolvent operator $T_\lambda : H \to D(T)$ defined by

$$T_\lambda f = u \iff u \in D(T) \ \text{ and } \ u + \lambda Tu \ni f \tag{5}$$

for any $f \in H$. In other words, T_λ is the inverse of the operator $J_H + \lambda T$, i.e., $T_\lambda = (J_H + \lambda T)^{-1}$. Note that the resolvent operator exists for each $\lambda > 0$ and is a single valued operator.

Next, we recall two sufficient conditions which guarantee the maximal monotonicity of a multivalued operator.

Proposition 1. *Assume that $T : H \to 2^H$ is a monotone operator such that for every $v \in H$, the set Tv is nonempty convex and weakly closed. Moreover, assume that for all $u, v \in H$, the graph of the mapping $\lambda \mapsto T(\lambda u + (1-\lambda)v)$ is closed in $[0,1] \times H_w$. Then the operator T is maximal monotone.*

Proposition 2. *Let $T_1, T_2 : H \to 2^H$ be two maximal monotone operators such that int $D(T_1) \cap D(T_2) \neq \emptyset$. Then $T_1 + T_2 : H \to 2^H$ is a maximal monotone operator, too.*

We now proceed with the definition and the properties of the Clarke subdifferential of locally Lipschitz functions.

Definition 4. *The Clarke directional derivative of the locally Lipschitz function $j : H \to \mathbb{R}$ at the point $u \in H$ in the direction $v \in H$ is defined by*

$$j^0(u; v) = \limsup_{w \to u, \lambda \downarrow 0} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

The Clarke subdifferential of j is the multivalued operator $\partial j : H \to 2^H$ defined by

$$\partial j(u) = \{\, \xi \in H | j^0(u; v) \geq (\xi, v)_H \quad \forall\, v \in H \,\} \quad \text{for any } u \in H.$$

For the Clarke subdifferential and directional derivative we have the following properties.

Proposition 3. *Let $j : H \to \mathbb{R}$ be a locally Lipschitz function. Then:*

(a) *$\partial j(u)$ is a nonempty convex and weakly compact subset of H, for all $u \in H$;*
(b) *the graph of the Clarke subdifferential ∂j is closed in $H \times H_w$ topology;*
(c) *for all $u, v \in H$, one has*

$$j^0(u; v) = \max\{\, (\xi, v)_H \mid \xi \in \partial j(u) \,\}.$$

We now move to the properties of the subdifferential in the sense of convex analysis.

Definition 5. *The subdifferential of a convex function $\psi : H \to \mathbb{R} \cup \{+\infty\}$ is the multivalued operator $\partial^c \psi : H \to 2^H$ defined by*

$$\partial^c \psi(u) = \{\, \eta \in H | \psi(v) - \psi(u) \geq (\eta, v - u)_H \quad \forall\, v \in H \,\} \quad \text{for any } u \in H.$$

The following result represents an important property of the subdifferential of a convex function.

Proposition 4. *Assume that $\psi : H \to \mathbb{R}$ is a convex lower semicontinuous function. Then the subdifferential operator $\partial^c \psi : H \to 2^H$ is maximal monotone and $D(\partial^c \psi) = H$.*

A relevant example of function defined on H with values on $\mathbb{R} \cup \{+\infty\}$ is the indicator function defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K \end{cases}$$

where $K \subset H$. It is well known that if the subset K is nonempty closed and convex, then the indicator function I_K is proper, convex and lower semicontinuous. Moreover, $D(\partial^c I_K) = K$. In the rest of this paper we shall use notation $\partial^c(\psi + I_K)$ for the subdifferential of the convex function $\psi + I_K$. Moreover, using Propositions 4 and 2 we deduce the following result.

Proposition 5. *Assume that $\psi : H \to \mathbb{R}$ is a proper convex lower semicontinuous function and K is a nonempty closed convex subset of H. Then the operator $\partial^c(\psi + I_K) : H \to 2^H$ is maximal monotone and, moreover, $D(\partial^c(\psi + I_K)) = K$.*

We now recall the following result proved in [21].

Proposition 6. *Let K be a nonempty closed convex subset of H, $u \in K$, K^* a nonempty closed convex bounded subset of H and $\psi : H \to \mathbb{R} \cup \{+\infty\}$ a proper convex lower semicontinuous function. Assume that for each $v \in K$ there exists $u^*(v) \in K^*$ such that*

$$(u^*(v), v - u)_H \geq \psi(u) - \psi(v).$$

Then, there exists $u^* \in K^*$ such that

$$(u^*, v - u)_H \geq \psi(u) - \psi(v) \quad \forall\, v \in K.$$

We end this section by recalling the notion of history-dependent operator. To this end, throughout this paper, for a normed space $(W, \|\cdot\|_W)$ we use the notation $C(\mathbb{R}_+; W)$ for the space of continuous functions on $\mathbb{R}_+$ with values in W. Recall that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a complete metric space. The convergence of a sequence $\{v_n\}$ to an element v, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} v_n \to v \ \text{ in } \ C(\mathbb{R}_+; X) \ \text{ as } \ n \to \infty \ \text{ if and only if} \\ \max\limits_{t \in \mathcal{U}} \|v_n(t) - v(t)\|_X \to 0 \ \text{ as } \ n \to \infty, \\ \text{for any nonempty compact set } \mathcal{U} \subset \mathbb{R}_+. \end{cases}$$

The next definition introduces two important classes of operators defined on spaces of continuous functions.

Definition 6. *Let $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ be two normed spaces. An operator $\mathcal{S} \colon C(\mathbb{R}_+; W_1) \to C(\mathbb{R}_+; W_2)$ is said to be almost history-dependent if for any nonempty compact set $\mathcal{U} \subset \mathbb{R}_+$ there exist $l_{\mathcal{U}}^{\mathcal{S}} \in [0, 1)$ and $L_{\mathcal{U}}^{\mathcal{S}} > 0$ such that*

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_{W_2} \leq l_{\mathcal{U}}^{\mathcal{S}} \|u_1(t) - u_2(t)\|_{W_1} + L_{\mathcal{U}}^{\mathcal{S}} \int_0^t \|u_1(s) - u_2(s)\|_{W_1}\, ds$$

for all $u_1, u_2 \in C(\mathbb{R}_+; W_1)$ and all $t \in \mathcal{U}$. If, in particular, $l_{\mathcal{U}}^{\mathcal{S}} = 0$ for any nonempty compact set $\mathcal{U} \subset \mathbb{R}_+$, then $\mathcal{S}$ is said to be a history-dependent operator.

History-dependent and almost history-dependent operators arise in Functional Analysis, Solid Mechanics and Contact Mechanics, as well. General properties, examples and mechanical interpretations can be found in [5]. In particular, the following fixed point property was proved in [5], (p. 41).

Theorem 3. *Let W be a Banach space and let $\Lambda \colon C(\mathbb{R}_+; W) \to C(\mathbb{R}_+; W)$ be an almost history-dependent operator. Then Λ has a unique fixed point, i.e., there exists a unique element $\eta^* \in C(\mathbb{R}_+; W)$ such that $\Lambda\eta^* = \eta^*$.*

We shall use Theorem 3 in Section 2 in the study of the history-dependent variational hemivariational inequality (3).

3. A Parametric Variational–Hemivariational Inequality

In this section, in addition to the Hilbert space H, we assume that Y and Z are normed spaces endowed with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We also denote by $X = Y \times Z$ the product of the spaces Y and Z. A typical point of X will be denoted by $w = (\eta, \theta)$ where $\theta \in Y$ and $\eta \in Z$.

We now consider the following elliptic variational–hemivariational inequality: given the parameter $w = (\eta, \theta) \in X$, find u such that

$$u \in K, \quad (Au, v-u)_H + \varphi(\eta, v) - \varphi(\eta, u) + j^0(\theta, u; v-u) \geq (f, v-u)_H \quad \forall\, v \in K. \tag{6}$$

In the study of this problem we consider the following assumptions.

$$K \text{ is a nonempty closed convex subset of } H. \tag{7}$$

$$\left\{ \begin{array}{l} A\colon H \to H \text{ is demicontinuous and strongly monotone} \\ \text{with constant } m_A > 0. \end{array} \right. \tag{8}$$

$$\left\{ \begin{array}{l} \varphi : Y \times H \to \mathbb{R} \text{ is such that:} \\ \text{(a) } \varphi(\eta, \cdot) : H \to \mathbb{R} \text{ is convex and lower semicontinuous,} \\ \qquad \text{for any } \eta \in Y. \\ \text{(b) There exists } \alpha_\varphi > 0 \text{ such that} \\ \qquad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \qquad\quad \leq \alpha_\varphi \|\eta_1 - \eta_2\|_Y \|u_1 - u_2\|_H \\ \qquad\qquad \text{for all } \eta_1,\ \eta_2 \in Y,\ \ v_1,\ v_2 \in H. \end{array} \right. \tag{9}$$

$$\left\{ \begin{array}{l} j : Z \times H \to \mathbb{R} \text{ is such that:} \\ \text{(a) } j(\theta, \cdot) \text{ is locally Lipschitz continuous, for any } \theta \in Z. \\ \text{(b) There exist } \alpha_j^1 > 0,\ \alpha_j^2 > 0 \text{ such that} \\ \qquad j^0(\theta_1, v_1; v_2 - v_1) + j^0(\theta_2, v_2; v_1 - v_2) \\ \qquad\quad \leq \alpha_j^1 \|v_1 - v_2\|_H^2 + \alpha_j^2 \|\theta_1 - \theta_2\|_Z \|v_1 - v_2\|_H \\ \qquad\qquad \text{for all } \theta_1,\ \theta_2 \in Z,\ v_1, v_2 \in H. \end{array} \right. \tag{10}$$

$$m_A \geq \alpha_j^1. \tag{11}$$

$$f \in H. \tag{12}$$

Next, for any $w = (\eta, \theta) \in X$ we use the notation $\varphi_\eta : H \to \mathbb{R}$ and $j_\theta : H \to \mathbb{R}$ for the functions defined by

$$\varphi_\eta(v) = \varphi(\eta, v) \quad \forall\, v \in H, \tag{13}$$

$$j_\theta(v) = j(\theta, v) \quad \forall\, v \in H. \tag{14}$$

Moreover, we introduce the operator $S_w : H \to 2^H$ given by

$$S_w u = Au + \partial j_\theta(u) + \partial^c(\varphi_\eta + I_K)(u) \quad \forall\, u \in H \tag{15}$$

and we recall that Propositions 3(a) and 5 guarantee that $D(S_w) = K$. In addition, we have the following comment.

Remark 1. *Assumption* (10)(b) *implies that*

$$j_\theta^0(v_1; v_2 - v_1) + j_\theta^0(v_2; v_1 - v_2) \leq \alpha_j^1 \|v_1 - v_2\|_H^2 \qquad \forall\, v_1,\ v_2 \in H,$$

for any $\theta \in Z$. Then, using Lemma 7 of [5] we deduce that the Clarke subdifferential of the function j_θ satisfies the relaxed monotonicity condition (4) with constant α_j^1.

We now state and prove various results related to the parametric variational-hemivariational inequality (6).

Proposition 7. *Assume (7)–(10) and let $u \in X$, $w = (\eta, \theta) \in X$, $z \in H$. Then u satisfies the inequality*

$$u \in K, \ \ (Au, v-u)_H + \varphi_\eta(v) - \varphi_\eta(u) + j_\theta^0(u; v-u) \geq (z, v-u)_H \quad \forall\, v \in K \tag{16}$$

if and only if

$$z \in S_w u. \tag{17}$$

Proof. Assume that u satisfies the inequality (16). We deduce from Proposition 3(c) that for each $v \in K$ there exists $\xi(v) \in \partial j_\theta(u)$ such that $j_\theta(u, v-u) = (\xi(v), v-u)_H$ and, therefore,

$$(Au + \xi(v) - z, v-u)_H + \varphi_\eta(v) - \varphi_\eta(u) \geq 0.$$

Moreover, from Proposition 3(a), we obtain that the set

$$K^* = \{\, Au + \xi(v) - z \mid v \in K,\ \xi(v) \in \partial j_\theta(u) \,\}$$

is a nonempty closed convex weakly compact subset of H which implies that it is bounded, too. Hence, using Proposition 6 with $u^*(v) = Au + \xi(v) - z$ we see that there exists $\xi \in \partial j_\theta(u)$ which does not depend on v, such that

$$(Au + \xi - z, v-u)_H + \varphi_\eta(v) - \varphi_\eta(u) \geq 0 \quad \forall\, v \in K.$$

So,

$$\varphi_\eta(v) - \varphi_\eta(u) + I_K(v) - I_K(u) \geq (z - Au - \xi, v-u)_H \quad \forall\, v \in H.$$

Next, by the definition of the subdifferential of convex functions and inclusion $\xi \in \partial j_\theta(u)$ we have

$$z \in Au + \partial j_\theta(u) + \partial^c(\varphi_\eta + I_K)(u).$$

This implies that $z \in S_w u$ which shows that (17) holds.

Conversely, assume that (17) holds. Then, the definition (15) of the operator S_w yields

$$z \in Au + \partial j_\theta(u) + \partial^c(\varphi_\eta + I_K)(u).$$

Therefore, there exist $\xi \in \partial j_\theta(u)$ and $\eta \in \partial^c(\varphi_\eta + I_K)(u)$ such that

$$z = Au + \xi + \eta. \tag{18}$$

Moreover, the definitions of the Clarke subdifferential and the subdifferential of a convex function imply that $(\xi, v-u)_H \leq j_\theta^0(u, v-u)$, $(\eta, v-u)_H \leq \varphi_\eta(v) - \varphi_\eta(u)$ for all $v \in K$. Combining these inequalities with equality (18) we deduce that

$$u \in K, \ \ (Au, v-u)_H + \varphi_\eta(v) - \varphi_\eta(u) + j_\theta^0(u; v-u) \geq (z, v-u)_H \quad \forall\, v \in K,$$

which shows that (16) holds and concludes the proof. □

We now focus on the main property of the operator S_w.

Proposition 8. *Assume (7)–(11). Then, for any $w = (\eta, \theta) \in X$ the operator $S_w : H \to 2^H$ is maximal monotone and, moreover, $D(S_w) = K$.*

Proof. Let $w = (\eta, \theta) \in X$ be fixed. The proof is structured in several steps, as follows.

Step (i) The operator $A + \partial j_\theta : H \to 2^H$ is monotone. Indeed, assume that (u_1, u_1^*), $(u_2, u_2^*) \in Gr(A + \partial j_\theta)$. Then, there exist $\xi_1 \in \partial j_\theta(u_1)$ and $\xi_2 \in \partial j_\theta(u_2)$ such that

$$u_1^* = Au_1 + \xi_1, \quad u_2^* = Au_2 + \xi_2$$

and, therefore,

$$(u_1^* - u_2^*, u_1 - u_2)_H = (Au_1 - Au_2, u_1 - u_2)_H + (\xi_1 - \xi_2, u_1 - u_2)_H.$$

We now use the inequalities

$$(Au_1 - Au_2, u_1 - u_2)_H \geq m_A \|u_1 - u_2\|_H^2,$$
$$(\xi_1 - \xi_2, u_1 - u_2)_H \geq -\alpha_j^1 \|u_1 - u_2\|_H^2,$$

guaranteed by assumption (8) and Remark 1, respectively, to see that

$$(u_1^* - u_2^*, u_1 - u_2)_H \geq (m_A - \alpha_j^1)\|u_1 - u_2\|_H^2.$$

Therefore, assumption (11) implies that the multivalued operator $A + \partial j_\theta : H \to 2^H$ is monotone.

Step (ii) The operator $A + \partial j_\theta : H \to 2^H$ is maximal monotone. We start by proving that the mapping $\lambda \mapsto (A + \partial j_\theta)(\lambda u + (1-\lambda)v)$ has a closed graph in $[0,1] \times H_w$. To this end let $u, v \in H$ and assume that $\lambda_n \to \lambda$ in $[0,1]$, $x_n \rightharpoonup x$ in H as $n \to \infty$ and

$$x_n \in (A + \partial j_\theta)(\lambda_n u + (1-\lambda_n)v),$$

for each $n \in \mathbb{N}$. Then,

$$x_n - A(\lambda_n u + (1-\lambda_n)v) \in \partial j_\theta(\lambda_n u + (1-\lambda_n)v)$$

and, since $\lambda_n \to \lambda$, it is obvious to see that

$$\lambda_n u + (1-\lambda_n)v \to \lambda u + (1-\lambda)v \quad \text{in } H. \tag{19}$$

Therefore, using (19), assumption (8) and the convergence $x_n \rightharpoonup x$ in H, we deduce that

$$x_n - A(\lambda_n u + (1-\lambda_n)v) \rightharpoonup x - A(\lambda u + (1-\lambda)v) \quad \text{in } H.$$

We now use the closedness of the graph of ∂j_θ in the product space $H \times H_w$ to see that

$$x - A(\lambda u + (1-\lambda)v) \in \partial j_\theta(\lambda u + (1-\lambda)v),$$

i.e., $x \in (A + \partial j_\theta)(\lambda u + (1-\lambda)v)$.

We conclude from above that the mapping $\lambda \mapsto (A + \partial j_\theta)(\lambda u + (1-\lambda)v)$ has a closed graph in $[0,1] \times H_w$. Moreover, we use Proposition 3(a) to see that for any $v \in H$ the set $Av + \partial j_\theta(v)$ is a nonempty convex and weakly closed subset in H. The maximality of the monotone operator $A + \partial j_\theta : H \to 2^H$ is now a consequence of Proposition 1.

Step (iii) The operator $S_w = A + \partial j_\theta + \partial^c(\varphi + I_K) : H \to 2^H$ is maximal monotone. Indeed, Step (ii) and Proposition 3(a) show that the operator $T_1 = A + \partial j_\theta : H \to 2^H$ is maximal monotone and $D(T_1) = D(A + \partial j_\theta) = H$. Moreover, using (7), (9) and Proposition 5 we deduce that the operator $T_2 = \partial^c(\varphi_\eta + I_K)$ is maximal monotone and $D(T_2) = K$. This implies that $int(D(T_1)) \cap D(T_2) = K \neq \emptyset$. We now use Proposition 2 in order to deduce that the operator $T_1 + T_2 = A + \partial j_\theta + \partial^c(\varphi_\eta + I_K) : H \to 2^H$ is maximal monotone. Now, since (15) shows that $S_w = T_1 + T_2$, it follows that S_w is a maximal monotone operator. Moreover, $D(S_w) = D(T_1) \cap D(T_2) = K$, which concludes the proof. $\square$

Proposition 8 guarantees that, under assumptions (7)–(12), the operator $S_w^f : H \to 2^H$ given by

$$S_w^f u = S_w u - f = Au + \partial j_\theta(u) + \partial^c(\varphi_\eta + I_K)(u) - f \quad \forall\, u \in H \tag{20}$$

is maximal monotone, too. Moreover, $D(S_w^f) = K$. Therefore, for any $\lambda > 0$ we are in a position to define its resolvent, denoted in what follows by $J_{\lambda,w}^f$. We use (5) to see that $J_{\lambda,w}^f : H \to K$ and

$$J_{\lambda,w}^f \sigma = u \iff u \in K \text{ and } u + \lambda S_w^f \ni \sigma, \tag{21}$$

for each $\sigma \in H$, $w \in X$ and $\lambda > 0$. We proceed with the following result.

Proposition 9. *Assume (7)–(12), let $u \in X$, $w = (\eta, \theta) \in X$, $\sigma \in H$, and let $\lambda > 0$. Then*

$$u = J_{\lambda,w}^f(\sigma)$$

if and only if

$$\begin{aligned} u \in K, \quad & (Au, v-u)_H + \varphi_\eta(v) - \varphi_\eta(u) + j_\theta^0(u; v-u) \\ & \geq (f + \frac{\sigma - u}{\lambda}, v-u)_H \quad \forall\, v \in K. \end{aligned}$$

Proof. We use (21), (20) and (15) to see that the following equivalences hold:

$$\begin{aligned} & u = J_{\lambda,w}^f(\sigma) \iff \\ & u + \lambda\big(Au + \partial j_\theta(u) + \partial^c(\varphi_\eta + I_K)(u) - f\big) \ni \sigma \iff \\ & Au + \partial j_\theta(u) + \partial^c(\varphi_\eta + I_K)(u) \ni f + \frac{\sigma - u}{\lambda} \iff \\ & S_w u \ni f + \frac{\sigma - u}{\lambda}. \end{aligned}$$

Proposition 9 is now a direct consequence of Proposition 7, used with the choice $z = f + \frac{\sigma - u}{\lambda}$. □

We now take $\sigma = u$ and obtain the following consequence of Proposition 9.

Corollary 1. *Assume (7)–(12), let $u \in K$, $w = (\eta, \theta) \in X$ and $\lambda > 0$. Then u is a solution of the variational–hemivariational inequality (6) if and only if u is a fixed point of the operator $J_{\lambda,w}^f$, i.e., $J_{\lambda,w}^f(u) = u$.*

The following result represents a Lipschitz continuity result for the resolvent operator $J_{\lambda,w}^f$.

Proposition 10. *Assume (7)–(11), let $w_1 = (\eta_1, \theta_1)$, $w_2 = (\eta_2, \theta_2) \in X$, $f_1, f_2 \in H$, $\sigma_1, \sigma_2 \in H$, $\lambda > 0$ and, for $i = 1, 2$, let $u_i = J_{\lambda,w_i}^{f_i} \sigma_i \in K$. Then, the following inequality holds:*

$$\begin{aligned} & (m_A - \alpha_j^1 + \frac{1}{\lambda})\|u_1 - u_2\|_H \\ & \quad \leq \alpha_\varphi \|\eta_1 - \eta_2\|_Y + \alpha_j^2 \|\theta_1 - \theta_2\|_Z + \|f_1 - f_2\|_H + \frac{1}{\lambda}\|\sigma_1 - \sigma_2\|_H. \end{aligned} \tag{22}$$

Proof. We use Proposition 9 to see that

$$(Au_1, v - u_1)_H + \varphi_{\eta_1}(v) - \varphi_{\eta_1}(u_1) + j^0_{\theta_1}(u_1; v - u_1) \tag{23}$$
$$\geq (f_1 + \frac{\sigma_1 - u_1}{\lambda}, v - u_1)_H \quad \forall\, v \in K,$$

$$(Au_2, v - u_2)_H + \varphi_{\eta_2}(v) - \varphi_{\eta_2}(u_2) + j^0_{\theta_2}(u_2; v - u_2) \tag{24}$$
$$\geq (f_2 + \frac{\sigma_2 - u_2}{\lambda}, v - u_2)_H \quad \forall\, v \in K.$$

Then, we take $v = u_2$ in (23), $v = u_1$ in (24) and add the resulting inequalities to find that

$$\frac{1}{\lambda}\|u_1 - u_2\|^2_H + (Au_1 - Au_2, u_1 - u_2)_H$$
$$\leq \varphi_{\eta_1}(u_2) - \varphi_{\eta_1}(u_1) + \varphi_{\eta_2}(u_1) - \varphi_{\eta_2}(u_2) + j^0_{\theta_1}(u_1; u_2 - u_1) + j^0_{\theta_1}(u_2; u_1 - u_2)$$
$$+(f_1 - f_2, u_1 - u_2)_H + \frac{1}{\lambda}(\sigma_1 - \sigma_2, u_1 - u_2)_H.$$

Next, we use the strong monotonicity of the operator A, notation (13) and (14) and assumptions (9)(b), (10)(b) on the functions φ and j, respectively. In this way we deduce that

$$\frac{1}{\lambda}\, \|u_1 - u_2\|^2_H + m_A\|u_1 - u_2\|^2_H$$
$$\leq \alpha_\varphi\|\eta_1 - \eta_2\|_Y\|u_1 - u_2\|_X + \alpha^1_j\|u_1 - u_2\|^2_X + \alpha^2_j\|\theta_1 - \theta_2\|_Z\|u_1 - u_2\|_X$$
$$+\|f_1 - f_2\|_H\|u_1 - u_2\|_H + \frac{1}{\lambda}\, \|\sigma_1 - \sigma_2\|_H\|u_1 - u_2\|_H.$$

This inequality implies the bound (22), which concludes the proof. □

We now consider the following additional assumptions.

$$K \text{ is a bounded set of } H. \tag{25}$$
$$m_A = \alpha^1_j. \tag{26}$$
$$m_A > \alpha^1_j. \tag{27}$$

We end this section with the following result concerning the parametric variational–hemivariational inequality (6).

Theorem 4. *Assume* (7)–(10), (12). *Then, the following statements hold.*

(a) *Under assumptions* (25) *and* (26) *the variational–hemivariational inequality* (6) *has at least one solution.*

(b) *Under assumption* (27) *the variational–hemivariational inequality* (1) *has a unique solution which depends Lipschitz continuously on f.*

Proof. Let $\lambda > 0$, $w = (\eta, \theta) \in K$, σ_1, $\sigma_2 \in K$ and let $u_1 = J^f_{\lambda,w}(\sigma_1)$, $u_2 = J^f_{\lambda,w}(\sigma_2)$. Note that if (26) or (27) hold, than (11) holds, too. Therefore, Proposition 10 implies that

$$(m_A - \alpha^1_j + \frac{1}{\lambda})\|u_1 - u_2\|_X \leq \frac{1}{\lambda}\, \|\sigma_1 - \sigma_2\|_H. \tag{28}$$

(a) Assume that (25), and (26) hold. Then, using inequality (28) we deduce that the operator $J^f_{\lambda,w} : K \to K$ is non expansive. Therefore, we are in a position to use Theorem 1 to see that $J^f_{\lambda,w}$ has at least one fixed point. We now use Corollary 1 to deduce the solvability of the variational–hemivariational inequality (1).

(b) Assume now (27). Then, it follows from inequality (28) that that the operator $J^f_{\lambda,w} : K \to K$ is a contraction. Therefore, using the Banach fixed point principle we deduce that $J^f_{\lambda,w}$ has a unique fixed point. The unique solvability of the variational–hemivariational inequality (1) is, again, a direct consequence of Corollary 1.

Finally, let $f_1, f_2 \in H$ and let u_1, u_2 denote the solution of inequality (1) for $f_1, f_2 \in H$, respectively. Then $u_1 = J^{f_1}_{\lambda,w}(u_1)$, $u_2 = J^{f_2}_{\lambda,w}(u_2)$ and, using Proposition 10, we deduce that

$$(m_A - \alpha^1_j)\|u_1 - u_2\|_X \leq \|f_1 - f_2\|_H. \tag{29}$$

We now combine inequality (29) with the smallness condition (27) to deduce that the operator $f \mapsto u = u(f) : H \to H$ is Lipschitz continuous, which concludes the proof. □

4. An First Elliptic Variational–Hemivariational Inequaliy

In this section, we study the solvability of the elliptic variational–hemivariational inequality (1). To this end, in addition to assumptions (7), (8) and (12), we consider the following assumptions on the functions ψ and g:

$$\psi : Y \times H \to \mathbb{R} \text{ is convex and lower semicontinuous.} \tag{30}$$

$$\left\{\begin{array}{l} g : H \to \mathbb{R} \text{ is such that:} \\ \text{(a)}\;\; g \text{ is locally Lipschitz continuous.} \\ \text{(b)}\;\; \text{There exist } \alpha_g > 0 \text{ such that} \\ \qquad g^0(v_1; v_2 - v_1) + g^0(v_2; v_1 - v_2) \leq \alpha_g \|v_1 - v_2\|^2_H \;\text{ for all } v_1, v_2 \in H. \end{array}\right. \tag{31}$$

$$m_A = \alpha_g. \tag{32}$$

$$m_A > \alpha_g. \tag{33}$$

Our main result in this section is the following

Theorem 5. *Assume (7), (8), (12), (30) and (31). Then:*

(a) *Under assumptions (25) and (32) the variational–hemivariational inequality (1) has at least one solution.*

(b) *Under assumption (33) the variational–hemivariational inequality (1) has a unique solution which depends Lipschitz continuously on f.*

Proof. Let Y and Z be arbitrary normed spaces and let $X = Y \times Z$. For any $w = (\eta, \theta) \in X$ let $\varphi : Y \times H \to \mathbb{R}$ and $j : Z \times H \to \mathbb{R}$ be the functions defined by

$$\varphi(\eta, u) = \psi(u), \qquad j(\theta, u) = g(u) \qquad \forall\, w = (\eta, \theta) \in X,\ u \in H. \tag{34}$$

First, we see that the functions φ and j satisfy assumption (9) and (10) with $\alpha_\varphi = 0$, $\alpha^1_j = \alpha_g$ and $\alpha^2_j = 0$. Moreover, (13), (14) show that

$$\varphi_\eta(u) = \psi(u), \qquad j_\theta(u) = g(u) \qquad \forall\, w = (\eta, \theta) \in X,\ u \in H \tag{35}$$

and, therefore, with the notation above, inequality (1) can be written in the equivalent form (6). Based on this remark, we use in what follows the results in Section 3.

(a) Assume that (25) and (32) hold and note that this implies that (26) hold, too. Theorem 5(a) is now a direct consequence of Theorem 4(a).

(b) Assume now that (33) and note that this implies that (27) hold, too. Theorem 5(b) is now a direct consequence of Theorem 4(b). □

The proof of Theorem 5 shows that the solution of inequality (1) is a fixed point for the resolvent operator $J^f_{\lambda,w}$, defined for any $\lambda > 0$ and $w \in X$. This suggests us to consider several iterative methods in the solution of this inequality. Details on iterative methods for nonexpansive and contractions operators can be found in [20,22]. Here, we restrict ourselves to present only two examples. Note that, since in the particular case of Theorem 5 the operator $J^f_{\lambda,w}$ does not depend on w, we shall denote it in what follows by J^f_λ.

Example 1. *(Picard iterations.) Under assumptions of Theorem 5* (b)*, let $u_0 \in K$ be arbitrary given and define the sequence $\{u_n\}$ by equality*

$$u_n = J^f_\lambda u_{n-1} \qquad \forall\, n \geq 1. \tag{36}$$

Using Proposition 9 and notation (34), (35), it is easy to see that equality (36) can be written, equivalently, as follows:

$$\begin{aligned} u_n \in K, \quad & (\frac{1}{\lambda}\, u_n + Au_n, v - u_n)_H + \psi(v) - \psi(u_n) + g^0(u_n; v - u_n) \\ & \geq (f + \frac{u_{n-1}}{\lambda}, v - u_n)_H \quad \forall\, v \in K, \quad \forall\, n \geq 1. \end{aligned}$$

We conclude that at each step of this iterative scheme we have to solve an elliptic variational-hemivariational inequality. Now, since the operator $J^f_\lambda : K \to K$ is a contraction, the sequence $\{u_n\}$ converges strongly in H to the fixed point of this operator and, therefore, to the solution u of inequality (1).

Example 2. *(Krasnoselski iterations) Under assumptions of Theorem 5* (a)*, let $\omega \in (0,1)$, $u_0 \in K$ be arbitrary given and define the sequence $\{u_n\}$ by equality*

$$u_n = (1-\omega)u_{n-1} + \omega J^f_\lambda u_{n-1} \qquad \forall\, n \geq 1. \tag{37}$$

Then, since the operator $J^f_\lambda : K \to K$ is nonexpansive, it is well known that the sequence $\{u_n\}$ converge weakly in H to a fixed point of this operator. A proof of this result can be found in ([20], p. 61). Therefore, $u_n \to u$ in H, where u is a solution of inequality (1).

Consider now the particular case when $\omega = \frac{1}{2}$. Then equality (37) becomes

$$2u_n - u_{n-1} = J^f_\lambda u_{n-1} \qquad \forall\, n \geq 1 \tag{38}$$

and, therefore, using notation $x_n = 2u_n - u_{n-1}$, Proposition 9 implies that equality (38) leads to the following iterative scheme: given $u_0 \in K$, the sequence $\{u_n\}$ is determined, recursively, by solving the system

$$\begin{aligned} x_n \in K, \quad & (\frac{1}{\lambda}\, x_n + Ax_n, v - x_n)_H + \psi(v) - \psi(x_n) + g^0(x_n; v - x_n) \\ & \geq (f + \frac{u_{n-1}}{\lambda}, v - x_n)_H \quad \forall\, v \in K, \\ u_n = {} & \frac{x_n + u_{n-1}}{2}, \end{aligned}$$

for all $n \geq 1$.

5. An Second Elliptic Variational–Hemivariational Inequaliy

In this section, we use the results in Section 3 in the study the solvability of the elliptic variatioanal-hemivariational inequality (2). To this end, in addition to assumptions (7), (8) and (12) we consider the following assumptions.

$$\left\{\begin{array}{l} \varphi : H \times H \to \mathbb{R} \text{ is such that:} \\ \text{(a) } \varphi(\eta,\cdot) : H \to \mathbb{R} \text{ is convex and lower semicontinuous,} \\ \qquad \text{for any } \eta \in H. \\ \text{(b) There exists } \alpha_\varphi > 0 \text{ such that} \\ \qquad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \qquad\quad \le \alpha_\varphi \|\eta_1 - \eta_2\|_H \|u_1 - u_2\|_H \quad \text{for all } \eta_1,\ \eta_2\, v_1,\ v_2 \in H. \end{array}\right. \tag{39}$$

$$\left\{\begin{array}{l} j : H \times H \to \mathbb{R} \text{ is such that:} \\ \text{(a) } j(\theta,\cdot) \text{ is locally Lipschitz continuous, for any } \theta \in H. \\ \text{(b) There exist } \alpha_j^1 > 0, \alpha_j^2 > 0 \text{ such that} \\ \qquad j^0(\theta_1, v_1; v_2 - v_1) + j^0(\theta_2, v_2; v_1 - v_2) \\ \qquad\quad \le \alpha_j^1 \|v_1 - v_2\|_H^2 + \alpha_j^2 \|\theta_1 - \theta_2\|_H \|v_1 - v_2\|_H \\ \qquad\qquad \text{for all } \theta_1,\ \theta_2,\ v_1, v_2 \in H. \end{array}\right. \tag{40}$$

$$m_A = \alpha_\varphi + \alpha_j^1 + \alpha_j^2. \tag{41}$$

$$m_A > \alpha_\varphi + \alpha_j^1 + \alpha_j^2. \tag{42}$$

Note that assumptions (39) and (40) represent a particular case of assumptions (9) and (10), respectively, obtained when $Y = Z = H$. Therefore, the preliminary results in Section 3 can be used in the study of inequality (2).

Our main result in this section is the following.

Theorem 6. *Assume* (7), (8), (12), (39), (40). *Then:*

(a) *Under assumptions* (25) *and* (41) *the variational–hemivariational inequality* (2) *has at least one solution.*

(b) *Under assumption* (42) *the variational–hemivariational inequality* (2) *has a unique solution which depends Lipschitz continuously on* f.

Proof. (a) Assume that (25) and (41) hold and recall the equivalence (21). Let $\lambda > 0$ and consider the operator $P_\lambda^f : K \to K$ defined by

$$P_\lambda^f(z) = J_{\lambda,w(z)}^f(z) \qquad \forall\, z \in K,$$

where $w(z) = (z,z) \in H \times H$, for all $z \in H$. Let $z_1,\ z_2 \in K$ and let $u_1 = P_\lambda^f(z_1), u_2 = P_\lambda^f(z_2)$. Then, using Proposition 10 with $X = H \times H$, $\eta_i = \theta_i = z_i = \sigma_i$, $f_i = f$ for $i = 1,2$, we deduce that

$$(m_A - \alpha_j^1 + \frac{1}{\lambda})\|u_1 - u_2\|_H \le (\alpha_\varphi + \alpha_j^2 + \frac{1}{\lambda})\|z_1 - z_2\|_H. \tag{43}$$

We now use assumption (41) and inequality (43) to see that the operator $P_\lambda^f : K \to K$ is nonexpansive. Therefore, we are in a position to use Theorem 1 to see that P_λ^f has at least one fixed point. We now use Corollary 1 to deduce the solvability of the variational–hemivariational inequality (2).

(b) Assume now that (42) hold. Then, inequality (43) shows that the operator $P^f_\lambda : K \to K$ is a contraction and, using the Banach principle we deduce that J^f_λ has a unique fixed point. The unique solvability of the variational–hemivariational inequality (2) is, again, a direct consequence of Corollary 1. Finally, the Lipschitz continuity of the mapping $f \mapsto u : H \to H$ follows from arguments similar to those used in the proof of Theorem 4(b). □

The proof of Theorem 6 shows that the solution of inequality (2) is a fixed point for the resolvent operator $P^f_\lambda : K \to K$, for any $\lambda > 0$. This allows us to consider several iterative methods in the solution of this inequality. In order to avoid repetitions we restrct here to the Picard iterations.

Example 3. *Under assumptions of Theorem 6(b), let $u_0 \in K$ be arbitrary given and define the sequence $\{u_n\}$ by equality*

$$u_n = P^f_\lambda u_{n-1} = J^f_{\lambda,w(u_{n-1})}(u_{n-1}) \qquad \forall\, n \geq 1 \tag{44}$$

where, recall, $w(u_{n-1})$ represents a short hand notation for the pair $(u_{n-1}, u_{n-1}) \in H \times H$. Then, using Proposition 9, it is easy to see that equality (44) can be written, equivalently, as follows:

$$\begin{aligned} u_n \in K, \quad & (\frac{1}{\lambda}\, u_n + Au_n, v - u_n)_H + \varphi(u_{n-1}, v) - \varphi(u_{n-1}, u_n) \\ & + j^0(u_{n-1}, u_n; v - u_n) \geq (f + \frac{u_{n-1}}{\lambda}, v - u_n)_H \quad \forall\, v \in K, \quad \forall\, n \geq 1. \end{aligned} \tag{45}$$

Now, since the operator $P^f_\lambda : K \to K$ is a contraction, it follows that the sequence $\{u_n\}$ converge strongly in H to the solution u of inequality (2).

6. A History-Dependent Variational–Hemivariational Inequaliy

In this section, we use the results in Section 3 in the study the solvability of the history-dependent variatioanal-hemivariational inequality (3). To this end, as usual, we assume that Y and Z are normed spaces endowed with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We also denote by $X = Y \times Z$ the product of the spaces Y and Z and we consider the following assumptions.

$$\mathcal{S} : C(\mathbb{R}_+, H) \to C(\mathbb{R}_+; Y) \text{ is a history-dependent operator.} \tag{46}$$

$$R : C(\mathbb{R}_+, H) \to C(\mathbb{R}_+; Z) \text{ is a history-dependent operator.} \tag{47}$$

$$f \in C(\mathbb{R}_+; H). \tag{48}$$

Our main result in this section is the following.

Theorem 7. *Assume (7)–(10), (27), (46)–(48). Then, the variational-hemivariational inequality (3) has a unique solution $u \in C(\mathbb{R}_+; H)$. Moreover, the solution depends continuously on f.*

Proof. Let $w = (\eta, \theta) \in C(\mathbb{R}_+; X)$, $\lambda > 0$. Then, the arguments in Section 3 allows us to consider the operator $\mathcal{J}^f_{\lambda,w}$ defined as follows:

$$\mathcal{J}^f_{\lambda,w}\sigma(t) = J^{f(t)}_{\lambda,w(t)}\sigma(t) \qquad \forall\, \sigma \in C(\mathbb{R}_+; H),\ t \in \mathbb{R}_+.$$

We claim that $\mathcal{J}^f_{\lambda,w}$ takes values on the space $C(\mathbb{R}_+; H)$, that is $\mathcal{J}^f_{\lambda,w} : C(\mathbb{R}_+; H) \to C(\mathbb{R}_+; H)$. Indeed, let $\sigma \in C(\mathbb{R}_+; H)$ and let $u : \mathbb{R}_+ \to H$ be the function defined by

$$u(t) = J^{f(t)}_{\lambda,w(t)}\sigma(t) \qquad \forall\, t \in \mathbb{R}_+.$$

Then, using inequality (22) we see that that

$$(m_A - \alpha_j^1 + \frac{1}{\lambda})\|u(t_1) - u(t_2)\|_H \le \alpha_\varphi \|\eta(t_1) - \eta(t_2)\|_Y$$
$$+\alpha_j^2 \|\theta(t_1) - \theta(t_2)\|_Z + \|f(t_1) - f(t_2)\|_H + \frac{1}{\lambda}\|\sigma(t_1) - \sigma(t_2)\|_H$$

for all $t_1,\ t_2 \in \mathbb{R}_+$. Next, using the continuity of the functions $t \mapsto \eta(t) : \mathbb{R}_+ \to Y$, $t \mapsto \theta(t) : \mathbb{R}_+ \to Z$, $t \mapsto f(t) : \mathbb{R}_+ \to H$ and $t \mapsto \sigma(t) : \mathbb{R}_+ \to H$, combined with the smallness assumption (27), we deduce that the function $t \mapsto u(t) : \mathbb{R}_+ \to H$ is continuous, too, which proves the claim.

We now consider the operator $\mathcal{T}_\lambda^f : C(\mathbb{R}_+; H) \to C(\mathbb{R}_+; H)$ defined by

$$\mathcal{T}_\lambda^f(z) = \mathcal{J}_{\lambda,w(z)}^f(z) \qquad \forall\, z \in C(\mathbb{R}_+; H)$$

where, here, $w(z)$ represent a short hand notation for the pair $(\mathcal{S}z, \mathcal{R}z) \in C(\mathbb{R}_+; X)$. We now prove that this operator has a unique fixed point. To this end, consider $z_1,\ z_2 \in C(\mathbb{R}_+; H)$ and let $u_1 = \mathcal{T}_\lambda^f(z_1)$, $u_2 = \mathcal{T}_\lambda^f(z_2)$. Moreover, let $\mathcal{U}$ be a nonempty compact subset of $\mathbb{R}_+$ and $t \in \mathcal{U}$. Then, using Proposition 10 with $\eta_i = \theta_i = z_i$, $i = 1,\, 2$, we deduce that

$$(m_A - \alpha_j^1 + \frac{1}{\lambda})\|u_1(t) - u_2(t)\|_H$$
$$\le \alpha_\varphi \|\mathcal{S}z_1(t) - \mathcal{S}z_2(t)\|_Y + \alpha_j^2 \|\mathcal{R}z_1(t) - \mathcal{R}z_2(t)\|_Z + \frac{1}{\lambda}\|z_1(t) - z_2(t)\|_H.$$

We now use assumptions (46) and (47) to deduce that

$$(m_A - \alpha_j^1 + \frac{1}{\lambda})\|u_1(t) - u_2(t)\|_H$$
$$\le \left(\alpha_\varphi L_{\mathcal{U}}^{\mathcal{S}} + \alpha_j^2 L_{\mathcal{U}}^{\mathcal{R}}\right) \int_0^t \|z_1(s) - z_2(s)\|_H\, ds + \frac{1}{\lambda}\|z_1(t) - z_2(t)\|_H$$

where, here and below, $L_{\mathcal{U}}^{\mathcal{S}} > 0$ and $L_{\mathcal{U}}^{\mathcal{R}} > 0$ are the constants which appear in Definition 6 of the history-dependence of the operators $\mathcal{S}$ and $\mathcal{R}$, respectively. Therefore,

$$\|\mathcal{T}_\lambda^f(z_1)(t) - \mathcal{T}_\lambda^f(z_2)(t)\|_H$$
$$\le \frac{\alpha_\varphi L_{\mathcal{U}}^{\mathcal{S}} + \alpha_j^2 L_{\mathcal{U}}^{\mathcal{R}}}{m_A - \alpha_j^1 + \frac{1}{\lambda}} \int_0^t \|z_1(s) - z_2(s)\|_H\, ds + \frac{\frac{1}{\lambda}}{m_A - \alpha_j^1 + \frac{1}{\lambda}}\|z_1(t) - z_2(t)\|_H.$$

We now use the smallness assumption (27) to see that the operator $\mathcal{T}_\lambda^f : C(\mathbb{R}_+; H) \to C(\mathbb{R}_+; H)$ is an almost-history-dependent operator. Then, Theorem 3 shows that $\mathcal{T}_\lambda^f$ has a unique fixed point and, using Corollary 1, we deduce the solvability of the history-dependent variational–hemivariational inequality (3).

Finally, let $f_n, f \in C(\mathbb{R}_+; H)$ and let $u_n, u \in C(\mathbb{R}_+; H)$ denote the solution of inequality (3) for f_n and f, respectively. Then

$$u_n = \mathcal{T}_\lambda^{f_n}(u_n) = \mathcal{J}_{\lambda,w(u_n)}^{f_n}(u_n), \qquad u = \mathcal{T}_\lambda^f(u) = \mathcal{J}_{\lambda,w(u)}^f(u)$$

and, therefore,

$$u_n(t) = J_{\lambda,w(u_n(t))}^{f_n(t)}(u_n(t)), \qquad u(t) = J_{\lambda,w(u(t))}^{f(t)}(u(t))$$

for any $t \in \mathbb{R}_+$. Let $\mathcal{U}$ be a nonempty compact subset of $\mathbb{R}_+$ and let $t \in \mathcal{U}$. Then, using Proposition 10 combined with assumptions (46), (47) and (27) we deduce that

$$\begin{aligned} &\|u_n(t) - u(t)\|_H \\ &\le \frac{\alpha_\varphi l_{\mathcal{U}}^{\mathcal{S}} + \alpha_j^2 l_{\mathcal{U}}^{\mathcal{R}}}{m_A - \alpha_j^1} \int_0^t \|u_n(s) - u(s)\|_H \, ds + \frac{1}{m_A - \alpha_j^1} \|f_n(t) - f(t)\|_H. \end{aligned}$$

We now apply the Gronwall argument to see that there exists two positive constants $c_{\mathcal{U}}$ and $d_{\mathcal{U}}$ which do not depend on n such that

$$\|u_n(t) - u(t)\|_X \le c_{\mathcal{U}} \int_0^t \|f_n(s) - f(s)\|_H \, ds + d_{\mathcal{U}} \|f_n(t) - f(t)\|_H. \tag{49}$$

Assume now that $f_n \to f$ in $C(\mathbb{R}_+; H)$. Then

$$\max_{s \in \mathcal{U}} \|f_n(s) - f(s)\|_H \to 0 \quad \text{as} \quad n \to \infty$$

and, therefore, (49) implies that

$$\max_{s \in \mathcal{U}} \|u_n(s) - u(s)\|_H \to 0 \quad \text{as} \quad n \to \infty. \tag{50}$$

Recall that $\mathcal{U}$ is an arbitrary nonempty compact subset of $\mathbb{R}_+$. Therefore, the convergence (50) implies that $u_n \to u$ in $C(\mathbb{R}_+; H)$ as $n \to \infty$. This shows that the solution of inequality (3) depends continuously on f, which concludes the proof. □

The poof of Theorem 6 shows that the solution of inequality (3) is a fixed point for the resolvent operator $\mathcal{T}_\lambda^f$, for any $\lambda > 0$. Therefore, several iterative methods in the solution of this inequality can be considered.

7. Conclusions

In this paper we considered an elliptic variational–hemivariational inequality depending on two parameters. We proved that this inequality is governed by a maximal monotone operator and its solvability is equivalent with the problem of finding a fixed point of the corresponding resolvent operator. Based on this equivalence we deduced an existence, uniqueness and continuous dependence result for the solution of the parametric variational–hemivariational inequality (Theorem 4). Then, with a conveniend choice of the parameters, we extended these results in the study of elliptic and history-dependent variational–hemivariational inequalities (Theorems 5–7). Moreover, using the above fixed point characterization, we constracted the corresponding Picard and Krasnoselski iterative schemes (Examples 1–3).

The present work shows that, in addition to the classical arguments based on the surjectivity of multivalued pseudomonotone operators, the fixed point methods can be used in the analysis of various classes of variational–hemivariational inequalities. It also gives rise to several open problems that we describe in what follows. First, it would be interesting to extend the results presented in this paper to inequality problems in the framework of reflexive Banach spaces. Second, the use of similar fixed point arguments in the study of evolutionary variational–hemivariational inequalities represents a challenging topic which fully deserves to be considered. Finally, error estimates for the corresponding iterative schemes with applications in the numerical analysis of mathematical models of contact could be investigated in the future. Any progress in the three directions mentioned above will complete our work and will open the way for new advances and ideas.

Funding: This research was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Panagiotopoulos, P.D. Nonconvex energy functions, hemivariational inequalities and substationarity principles. *Acta Mech.* **1983**, *48*, 111–130. [CrossRef]
2. Han, W.; Sofonea, M. Numerical analysis of hemivariational inequalities in contact mechanics. *Acta Numer.* **2019**, *28*, 175–286. [CrossRef]
3. Naniewicz, Z.; Panagiotopoulos, P.D. *Mathematical Theory of Hemivariational Inequalities and Applications*; Marcel Dekker: New York, NY, USA, 1995.
4. Panagiotopoulos, P.D. *Hemivariational Inequalities: Applications in Mechanics and Engineering*; Springer: Berlin, Germany, 1993.
5. Sofonea, M.; Migórski, S. *Variational–Hemivariational Inequalities with Applications*; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2018.
6. Feng, F.; Han, W.; Huang, J. Virtual element method for an elliptic hemivariational inequality with applications to contact mechanics. *J. Sci. Comput.* **2019**, *81*, 2388–2412. [CrossRef]
7. Feng, F.; Wu, B.; Han, W. The virtual element method for general elliptic hemivariational inequalities. *J. Comput. Appl. Math.* **2021**, *389*, 113330.
8. Haslinger, J.; Miettinen, M.; Panagiotopoulos, P.D. *Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications*; Springer Science & Business Media: New York, NY, USA, 2013.
9. Mäkelä, M.M.; Miettinen, M.; Lukšan, L.; Vlček, J. Comparing nonsmooth nonconvex bundle methods in solving hemivariational inequalities. *J. Glob. Optim.* **1999**, *14*, 117–135. [CrossRef]
10. Xu, W.; Huang, Z.; Han, W.; Chen, W.; Wang, C. Numerical analysis of history-dependent variational–hemivariational inequalities with applications in contact mechanics. *J. Comput. Appl. Math.* **2019**, *351*, 364–377. [CrossRef]
11. Zeng, B.; Liu, Z.; Migórski, S. On convergence of solutions to variational-hemivariational inequalities. *Z. Angew. Math. Phys.* **2018**, *69*, 87. [CrossRef]
12. Saramito, P. *Complex Fluids. Modeling and Algorithms*; Mathématiques & Applications 79; Springer: Cham, Switzerland, 2016. [CrossRef]
13. Versaci, M.; Palumbo, A. Magnetorheological Fluids: Qualitative comparison between a mixture model in the extended Irreversible Thermodynamics framework and an Herschel–Bulkley experimental elastoviscoplastic model. *Int. J. Non-Linear Mech.* **2020**, *118*, 103288. [CrossRef]
14. Migórski, S.; Ochal, A.; Sofonea, M. A class of variational–hemivariational in- equalities in reflexive Banach spaces. *J. Elast.* **2017**, *127*, 151–178.
15. Han, W. Minimization principles for elliptic hemivariational inequalities. *Nonlinear Anal. Real World Appl.* **2020**, *54*, 103114. [CrossRef]
16. Clarke, F.H. *Optimization and Nonsmooth Analysis*; SIAM: Philadelphia, PA, USA, 1990. [CrossRef]
17. Denkowski, Z.; Migórski, S.; Papageorgiou, N.S. *An Introduction to Nonlinear Analysis: Applications*; Kluwer Academic/Plenum Publishers: Boston, MA, USA; Dordrecht, The Netherlands; London, UK; New York, NY, USA, 2003. [CrossRef]
18. Kurdila, A.J.; Zabarankin, M. *Convex Functional Analysis*; Birkhäuser: Basel, Switzerland, 2005.
19. Zeidler, E. *Nonlinear Functional Analysis and Its Applications: II/B: Nonlinear Monotone Operators*; Springer Science & Business Media: New York, NY, USA, 2013.
20. Berinde, V. *Iterative Approximation of Fixed Points*; Lecture Notes in Mathematics 1912; Springer: Berlin, Germany, 2007.
21. Giannessi, F.; Khan, A. Regularization of non-coercive quasi variational inequalities. *Control Cybernet.* **2020**, *29*, 91–110.
22. Bauschke, H.H.; Combettes, P.L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*; Springer: New York, NY, USA, 2011; p. 408.