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On the Generalized Bilal Distribution: Some Properties and Estimation under Ranked Set Sampling

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Abstract: The generalized Bilal (GB) distribution can be defined as the distribution of the median of three independent random variables drawn from the Weibull distribution. Its failure rate function can be monotonic (decreasing or increasing) or upside-down bathtub-shaped. In this study, we aim to reveal some important properties of the GB distribution that have not been considered before. The findings are both theoretical and practical. From the theoretical viewpoint, we present explicit expressions for both single and product moments of order statistics from the GB distribution. The L-moments are derived as well. From the practical viewpoint, the parameter estimations are accomplished using the maximum likelihood (ML) method, which is based on two different sampling schemes: simple random sampling (SRS) and ranked set sampling (RSS) schemes. Furthermore, the asymptotic confidence intervals for the SRS and RSS estimators are discussed. For the sake of comparison and illustration, a simulation study and a real data example are presented. Concluding remarks are given at the end.

Keywords: L-moments; generalized bilal (GB) distribution; order statistics; maximum likelihood (ML) method; ranked set sampling (RSS)

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1. Introduction

Abd-Elrahman [1] introduced the Bilal(θ) distribution, a new one-parameter lifetime distribution. He has demonstrated that the Bilal(θ) distribution belongs to the class of new better than average renewal failure rates. As a generalized version of the Bilal(θ) distribution, Abd-Elrahman [2] proposed a new two-parameter lifetime distribution. He named it the generalized Bilal (GB) distribution, or GB(θ, λ) distribution, to mention the involved parameters $\theta > 0$ and $\lambda > 0$. It is defined with the original probability density function (pdf) and cumulative distribution function (cdf) listed as

$$f(x;\theta,\lambda) = \frac{6\lambda}{\theta}\left(\frac{x}{\theta}\right)^{\lambda-1} e^{-2\left(\frac{x}{\theta}\right)^{\lambda}}\left(1-e^{-\left(\frac{x}{\theta}\right)^{\lambda}}\right); \quad x > 0 \tag{1}$$

and

$$F(x;\theta,\lambda) = 1 - e^{-2\left(\frac{x}{\theta}\right)^{\lambda}}\left(3-2e^{-\left(\frac{x}{\theta}\right)^{\lambda}}\right); \quad x > 0, \tag{2}$$

respectively. Henceforth, we refer to X as a random variable with pdf (1). The main features of the GB distribution are described below.

First, its failure rate function can be monotonic (decreasing or increasing), or upside-down bathtub shaped, and it can be utilized for a variety of practical data analysis with scale parameter θ and shape parameter λ. At $\lambda = 1$, the GB(θ, λ) distribution corresponds to the Bilal(θ) distribution.

Abd-Elrahman [2] investigated a variety of statistical properties, including mean, median, mode, variance, skewness, kurtosis, the quantile function, Shanon entropy, the mean residual lifetime, and so on, and used various estimation methods, including the ML method, to estimate the unknown parameters of the GB(θ, λ) distribution. Additional results and applications of the underlying distribution are provided by Abd-Elrahman [3], Chaturvedi et al. [4] and Shi et al. [5].

Order statistics have become more important in recent years since nonparametric conclusions and robust techniques have become more common. The goal of this study is to complete what Abd-Elrahman [2] began by deriving explicit expressions for both single and product moments of order statistics from the GB distribution. The best linear unbiased (BLU) and best linear invariant (BLI) estimators of the scale and location-scale parameters of the GB distribution, as well as BLU and BLI predictors of future unobserved order statistics, might be developed using these findings; see, for example, Balakrishnan and Cohen [6].

On the applied plan, some statistical aspects of the GB distribution remain unexplored and constitute a contribution to this study. More details are given below.

Statistics is the branch of mathematics concerned with collecting, organizing, analyzing data, and drawing inferences from samples of the entire population. There are several sampling schemes available to select a suitable sample from the population. The most fundamental scheme is simple random sampling (SRS) scheme. In this scheme, a sample of size n is chosen from a population of size N, so that all classes of n elements have an equal chance of being included in the sample.

When the measurements of the variable of interest are expensive to measure or difficult to obtain, but easy to rank, McIntyre [7] introduced a new sampling scheme called ranked set sampling (RSS) scheme as an efficient alternative to the SRS scheme for improving the precision and increasing the efficiency of estimation. The mathematical foundation of the RSS scheme was first developed by Dell and Clutter [8] and Takahasi and Wakimoto [9]. For an elaborate treatment on the theory, methods and applications of the RSS scheme, one may refer to Chen et al. [10].

In recent years, several studies have focused on RSS-based parametric estimation for a variety of key real-life distributions. These studies have repeatedly demonstrated that the RSS scheme is more efficient than the SRS scheme, and other traditional sampling schemes. The performance of ML estimation under the RSS scheme has been developed and used in numerous studies. For example, Abu-Dayyeh et al. [11] for the Pareto; Aljohani et al. [12] for the modified Kies exponential; Bantan et al. [13] for the half logistic inverted Topp-Leone; Chen et al. [14] for the Pareto; Esemen and Gürler [15] for the generalized Rayleigh; He et al. [16] for the log-logistic; Sabry and Almetwally [17] for the exponential Pareto; Sabry et al. [18] for the Weibull; Singh and Mehta [19] for the log-logistic and Taconeli and Giolo [20] for the power Lindley and weighted Lindley distributions. Different results regarding parametric estimation based on the RSS scheme, including other estimation methods, are presented by Pedroso et al. [21] and Taconeli and Bonat [22]. For a comprehensive review and extensions of the RSS scheme and its applications, one may refer to Zamanzade et al. [23], Zamanzade and Mahdizadeh [24], Al-Omari and Bouza [25], Bouza and Al-Omari [26], and the references therein.

Abd-Elrahman [2] employed various estimation methods to estimate the unknown parameters of the GB distribution. However, to the best of our knowledge, no published papers address estimating the shape and scale parameters of the GB distribution under the SRS and RSS schemes. Motivated by this, we then provide the results of estimating the parameters of the GB distribution using the ML method and compare the results under these two sampling schemes.

The remainder of the paper is organized as follows. We provide some preliminary on the order statistics in Section 2. We also derive in this section the explicit expressions for single and product moments of order statistics from the GB distribution. L-moments are also derived. Section 3 discusses the ML estimation and asymptotic confidence intervals of the parameters of the GB distribution under the SRS and RSS schemes. A simulation study is also carried out in this section. The real data example is given in Section 4, and some conclusions are offered in Section 5.

2. Moments of Order Statistics

We start with some aspects of the GB distribution moments of order statistics that have not been addressed by Abd-Elrahman [2].

2.1. Preliminaries on Order Statistics

Let n be a positive integer and $X_1, \cdots, X_n$ be a random sample (of size n) from the GB distribution. Thus, $X_1, \cdots, X_n$ are with the pdf $f(x;\theta,\lambda)$ and cdf $F(x;\theta,\lambda)$, given in (1) and (2), respectively. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Then, for any positive integer r such that $1 \leq r \leq n$, the pdf of the rth order statistic $X_{r:n}$, say $f_{r:n}(x)$, is

$$f_{r:n}(x) = C_{r:n}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x); \quad 0 < x < \infty.$$

For any positive integers r and s such that $1 \leq r < s \leq n$, the joint pdf of the rth order statistic ($X_{r:n}$) and sth order statistics ($X_{s:n}$), say $f_{r,s:n}(x,y)$, can be expressed as

$$f_{r,s:n}(x,y) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y); \quad 0 < x < y < \infty,$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} \quad \text{and} \quad C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

These formulae are standard, and the details can be found in Arnold et al. [27]; David and Nagaraja [28].

Next, the pth single moment of $X_{r:n}$ takes the following form:

$$\mu_{r:n}^{(p)} = E(X_{r:n}^p) = \int_0^\infty x^p f_{r:n}(x)dx;\ 1 \leq r \leq n;\ p \in \mathbb{N}, \tag{3}$$

and the (p,q)th product moment of $X_{r:n}$ and $X_{s:n}$ reduces to

$$\mu_{r,s:n}^{(p,q)} = E(X_{r:n}^p X_{s:n}^q) = \int_0^\infty \int_x^\infty x^p y^q f_{r,s:n}(x,y)dydx;\ 1 \leq r < s \leq n;\ p,q \in \mathbb{N}. \tag{4}$$

Furthermore, we use the following integral formulae of Gradshteyn and Ryzhik [29], to prove some results of this paper,

$$\int_0^\infty x^{\nu-1}e^{-\eta x^\omega}dx = \frac{1}{\omega}\eta^{-\frac{\nu}{\omega}}\Gamma\left(\frac{\nu}{\omega}\right);\ \text{Re}\,\eta > 0, \text{Re}\,\nu > 0, \omega > 0, \tag{5}$$

$$\int_u^\infty x^\nu e^{-\eta x^\omega}dx = \frac{\Gamma(\tau,\eta u^\omega)}{\omega\eta^\tau}, \tau = \frac{\nu+1}{\omega};\ u > 0, \text{Re}\,\omega > 0, \text{Re}\,\tau > 0, \text{Re}\,\eta > 0, \tag{6}$$

$$\int_0^\infty x^{\nu-1}e^{-\eta x}\Gamma(\tau,\omega x)dx = \frac{\omega^\tau\Gamma(\nu+\tau)}{\nu(\omega+\eta)^{\nu+\tau}}{}_2F_1\left(1,\nu+\tau;\nu+1;\frac{\eta}{\omega+\eta}\right);$$
$$\text{Re}\,(\omega+\eta) > 0, \text{Re}\,\nu > 0, \text{Re}\,(\nu+\tau) > 0, \tag{7}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1}\mathrm{e}^{-x}dx$, $\Gamma(\nu,u) = \int_u^\infty x^{\nu-1}\mathrm{e}^{-x}dx$ and ${}_2F_1(a,b;c;x) = \sum_{k=0}^\infty [(a)_k(b)_k/(c)_k](x^k/k!)$ are complete gamma, incomplete gamma and Gauss hypergeometric functions, respectively, and $(e)_k = e(e+1)\cdots(e+k+1)$ is the ascending factorial.

2.2. Single Moments

The single moments of order statistics from the GB distribution are presented below.

Theorem 1. *For any positive integer r such that $1 \leq r \leq n-1$ and $p \in \mathbb{N}$, we have*

$$\mu_{r:n}^{(p)} = 6\theta^p \sum_{i=r}^{n} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} (-1)^{i-r+\alpha+\beta} 2^\beta 3^{\alpha-\beta} i \binom{i-1}{r-1} \binom{n}{i} \binom{i-1}{\alpha} \binom{\alpha}{\beta} \times \left[\frac{1}{\{2\alpha+\beta+2\}^{\frac{p}{\lambda}+1}} - \frac{1}{\{2\alpha+\beta+3\}^{\frac{p}{\lambda}+1}} \right] \Gamma\Big(\frac{p}{\lambda}+1\Big). \tag{8}$$

Proof. In view of (3) and the result given by David and Nagaraja ([28], p. 45), we have

$$\mu_{r:n}^{(p)} = \sum_{i=r}^{n} (-1)^{i-r} \binom{i-1}{r-1} \binom{n}{i} \mu_{i:i}^{(p)}, \tag{9}$$

where

$$\mu_{i:i}^{(p)} = i \int_0^\infty x^p [F(x)]^{i-1} f(x) dx = i \sum_{\alpha=0}^{i-1} (-1)^\alpha \binom{i-1}{\alpha} \int_0^\infty x^p [1-F(x)]^\alpha f(x) dx. \tag{10}$$

Using (1) and (2) in (10), we obtain

$$\mu_{i:i}^{(p)} = \frac{6i\lambda}{\theta^\lambda} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} (-1)^{\alpha+\beta} 2^\beta 3^{\alpha-\beta} \binom{i-1}{\alpha} \binom{\alpha}{\beta} \times \left[\int_0^\infty x^{p+\lambda-1} e^{-\left(\frac{2\alpha+\beta+2}{\theta^\lambda}\right) x^\lambda} dx - \int_0^\infty x^{p+\lambda-1} e^{-\left(\frac{2\alpha+\beta+3}{\theta^\lambda}\right) x^\lambda} dx \right].$$

From the integral formula in (5), we obtain

$$\mu_{i:i}^{(p)} = 6i\theta^p \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} (-1)^{\alpha+\beta} 2^\beta 3^{\alpha-\beta} \binom{i-1}{\alpha} \binom{\alpha}{\beta} \times \left[\frac{1}{(2\alpha+\beta+2)^{\frac{p}{\lambda}+1}} - \frac{1}{(2\alpha+\beta+3)^{\frac{p}{\lambda}+1}} \right] \Gamma\Big(\frac{p}{\lambda}+1\Big).$$

Inserting $\mu_{i:i}^{(p)}$ in (9), it follows (8). □

Remark 1. (a) *By setting $n = r = 1$ in (8), we obtain*

$$\mu_{1:1}^{(p)} = \frac{\theta^p}{6^{\frac{p}{\lambda}}} \Big[3^{\frac{p}{\lambda}+1} - 2^{\frac{p}{\lambda}+1} \Big] \Gamma\Big(\frac{p}{\lambda}+1\Big), \tag{11}$$

which is the pth moment of X reported by Abd-Elrahman [2].

Simple expressions for the first four moments of X may be obtained by setting $p = 1, 2, 3,$ and $p = 4$ in (11) (Abd-Elrahman [2]).

(b) *The expressions for the pth moment of the extremum order statistics may be obtained by putting $r = 1$ and $r = n$ in (8), respectively.*

Remark 2. *Setting $\lambda = 1$ in (8), we obtain*

$$\mu_{r:n}^{(p)} = 6p!\theta^p \sum_{i=r}^{n} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} (-1)^{i-r+\alpha+\beta} 2^\beta 3^{\alpha-\beta} i \binom{i-1}{r-1} \binom{n}{i} \binom{i-1}{\alpha} \binom{\alpha}{\beta} \times \left[\frac{1}{(2\alpha+\beta+2)^{p+1}} - \frac{1}{(2\alpha+\beta+3)^{p+1}} \right],$$

which is explicit expression for the pth single moment of the rth order statistic for the Bilal(θ) distribution. Another expression can also be seen in Abd-Elrahman [1].

2.3. Product Moments

The product moments of order statistics from the GB distribution are reported below.

Theorem 2. *For any positive integers r and s such that $1 \le r < s \le n$ and $p, q \in \mathbb{N}$, we have*

$$\begin{aligned}
\mu_{r,s:n}^{(p,q)} = 36\theta^{p+q} &\sum_{i=r}^{s-1} \sum_{j=n-s+i+1}^{n} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-i-1} \sum_{\gamma=0}^{\alpha} (-1)^{j+n-s-r+\alpha+\beta+\gamma+1}\, 2^{\beta+\gamma}\, 3^{j-i+\alpha-\beta-\gamma-1} \\
&\times \binom{i-1}{r-1}\binom{j-i-1}{n-s}\binom{n}{j}\binom{i-1}{\alpha}\binom{j-i-1}{\beta}\binom{\alpha}{\gamma} \frac{\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \frac{\Gamma(\frac{p}{\lambda}+\frac{q}{\lambda}+2)}{(\frac{p}{\lambda}+1)} \\
&\times \left[\frac{{}_2F_1\left(1, \frac{p}{\lambda}+\frac{q}{\lambda}+2; \frac{p}{\lambda}+2; \frac{2\alpha+\gamma+2}{2(j-i+\alpha)+\beta+\gamma+2)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+2\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}} - \frac{{}_2F_1\left(1, \frac{p}{\lambda}+\frac{q}{\lambda}+2; \frac{p}{\lambda}+2; \frac{2\alpha+\gamma+3}{2(j-i+\alpha)+\beta+\gamma+3)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+3\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}} \right. \\
&\left. - \frac{{}_2F_1\left(1, \frac{p}{\lambda}+\frac{q}{\lambda}+2; \frac{p}{\lambda}+2; \frac{2\alpha+\gamma+2}{2(j-i+\alpha)+\beta+\gamma+3)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+3\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}} + \frac{{}_2F_1\left(1, \frac{p}{\lambda}+\frac{q}{\lambda}+2; \frac{p}{\lambda}+2; \frac{2\alpha+\gamma+3}{2(j-i+\alpha)+\beta+\gamma+4)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+4\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}} \right]. \quad (12)
\end{aligned}$$

Proof. In view of (4) and the result given by Arnold et al. ([27], p. 116), we can write

$$\mu_{r,s:n}^{(p,q)} = \sum_{i=r}^{s-1} \sum_{j=n-s+i+1}^{n} (-1)^{j+n-s-r+1} \binom{i-1}{r-1}\binom{j-i-1}{n-s}\binom{n}{j} \mu_{i,i+1:j}^{(p,q)}, \quad (13)$$

where

$$\begin{aligned}
\mu_{i,i+1:j}^{(p,q)} &= \frac{\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \int_0^\infty \int_x^\infty x^p y^q [F(x)]^{i-1} [1-F(y)]^{j-i-1} f(x) f(y)\, dy\, dx \\
&= \frac{\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \sum_{\alpha=0}^{i-1} (-1)^\alpha \binom{i-1}{\alpha} \int_0^\infty x^p [1-F(x)]^\alpha I(x) f(x)\, dx, \quad (14)
\end{aligned}$$

and

$$I(x) = \int_x^\infty y^q [1-F(y)]^{j-i-1} f(y)\, dy.$$

Using (1) and (2), we can write $I(x)$ as

$$\begin{aligned}
I(x) = \frac{6\lambda}{\theta^\lambda} &\sum_{\beta=0}^{j-i-1} (-2)^\beta 3^{j-i-\beta-1} \binom{j-i-1}{\beta} \\
&\times \left[\int_x^\infty y^{q+\lambda-1} e^{-\left(\frac{2j-2i+\beta}{\theta^\lambda}\right) y^\lambda} dy - \int_x^\infty y^{q+\lambda-1} e^{-\left(\frac{2j-2i+\beta+1}{\theta^\lambda}\right) y^\lambda} dy \right].
\end{aligned}$$

Using the integral formula (6), we obtain

$$\begin{aligned}
I(x) = \frac{6}{\theta^\lambda} &\sum_{\beta=0}^{j-i-1} (-2)^\beta 3^{j-i-\beta-1} \binom{j-i-1}{\beta} \\
&\times \left[\frac{\Gamma(\frac{q}{\lambda}+1, (2j-2i+\beta)(\frac{x}{\theta})^\lambda)}{\left(\frac{2j-2i+\beta}{\theta^\lambda}\right)^{\frac{q}{\lambda}+1}} - \frac{\Gamma(\frac{q}{\lambda}+1, (2j-2i+\beta+1)(\frac{x}{\theta})^\lambda)}{\left(\frac{2j-2i+\beta+1}{\theta^\lambda}\right)^{\frac{q}{\lambda}+1}} \right].
\end{aligned}$$

Now, substituting the resultant expression of $I(x)$ in (14) and using (1) and (2), we obtain

$$\mu_{i,i+1:j}^{(p,q)} = \frac{\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \frac{36\lambda}{\theta^{\lambda+1}} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-i-1} \sum_{\gamma=0}^{\alpha} (-1)^{\alpha}(-2)^{\beta+\gamma} 3^{j-i+\alpha-\beta-\gamma-1} \times \binom{i-1}{\alpha}\binom{j-i-1}{\beta}\binom{\alpha}{\gamma} \int_0^{\infty} x^p \left(\frac{x}{\theta}\right)^{\lambda-1} e^{-(2\alpha+\gamma+2)\left(\frac{x}{\theta}\right)^{\lambda}} \left(1-e^{-\left(\frac{x}{\theta}\right)^{\lambda}}\right) \times \left[\frac{\Gamma\left(\frac{q}{\lambda}+1,(2j-2i+\beta)\left(\frac{x}{\theta}\right)^{\lambda}\right)}{\left(\frac{2j-2i+\beta}{\theta^{\lambda}}\right)^{\frac{q}{\lambda}+1}} - \frac{\Gamma\left(\frac{q}{\lambda}+1,(2j-2i+\beta+1)\left(\frac{x}{\theta}\right)^{\lambda}\right)}{\left(\frac{2j-2i+\beta+1}{\theta^{\lambda}}\right)^{\frac{q}{\lambda}+1}}\right] dx. \quad (15)$$

Setting $z=(x/\theta)^{\lambda}$, we can rewrite (15) as

$$\begin{aligned}
\mu_{i,i+1:j}^{(p,q)} &= 36\theta^{p+q} \frac{\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-i-1} \sum_{\gamma=0}^{\alpha} (-1)^{\alpha}(-2)^{\beta+\gamma} 3^{j-i+\alpha-\beta-\gamma-1} \\
&\quad \times \binom{i-1}{\alpha}\binom{j-i-1}{\beta}\binom{\alpha}{\gamma} \int_0^{\infty} z^{\frac{p}{\lambda}} e^{-(2\alpha+\gamma+2)z}\left(1-e^{-z}\right) \\
&\quad \times \left[\frac{\Gamma(\frac{q}{\lambda}+1,(2j-2i+\beta)z}{\{2j-2i+\beta\}^{\frac{q}{\lambda}+1}} - \frac{\Gamma(\frac{q}{\lambda}+1,(2j-2i+\beta+1)z}{\{2j-2i+\beta+1\}^{\frac{q}{\lambda}+1}}\right] dz \\
&= \frac{36\theta^{p+q}\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-i-1} \sum_{\gamma=0}^{\alpha} (-1)^{\alpha}(-2)^{\beta+\gamma} 3^{j-i+\alpha-\beta-\gamma-1} \binom{i-1}{\alpha}\binom{j-i-1}{\beta} \\
&\quad \times \binom{\alpha}{\gamma}\left[\frac{1}{\{2j-2i+\beta\}^{\frac{q}{\lambda}+1}} \int_0^{\infty} z^{\frac{p}{\lambda}} e^{-(2\alpha+\gamma+2)z} \Gamma\left(\frac{q}{\lambda}+1,(2j-2i+\beta)z\right)\right. \\
&\quad - \frac{1}{\{2j-2i+\beta\}^{\frac{q}{\lambda}+1}} \int_0^{\infty} z^{\frac{p}{\lambda}} e^{-(2\alpha+\gamma+3)z} \Gamma\left(\frac{q}{\lambda}+1,(2j-2i+\beta)z\right) \\
&\quad - \frac{1}{\{2j-2i+\beta+1\}^{\frac{q}{\lambda}+1}} \int_0^{\infty} z^{\frac{p}{\lambda}} e^{-(2\alpha+\gamma+2)z} \Gamma\left(\frac{q}{\lambda}+1,(2j-2i+\beta+1)z\right) \\
&\quad \left. + \frac{1}{\{2j-2i+\beta+1\}^{\frac{q}{\lambda}+1}} \int_0^{\infty} z^{\frac{p}{\lambda}} e^{-(2\alpha+\gamma+3)z} \Gamma\left(\frac{q}{\lambda}+1,(2j-2i+\beta+1)z\right)\right].
\end{aligned}$$

Using the integral formula (7), we obtain

$$\begin{aligned}
\mu_{i,i+1:j}^{(p,q)} &= \frac{36\theta^{p+q}\Gamma(j+1)\Gamma(\frac{p}{\lambda}+\frac{q}{\lambda}+2)}{(\frac{p}{\lambda}+1)\Gamma(i)\Gamma(j-i)} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-i-1} \sum_{\gamma=0}^{\alpha} (-1)^{\alpha}(-2)^{\beta+\gamma} 3^{j-i+\alpha-\beta-\gamma-1} \binom{i-1}{\alpha}\binom{j-i-1}{\beta}\binom{\alpha}{\gamma} \\
&\times \left[\frac{{}_2F_1\left(1,\frac{p}{\lambda}+\frac{q}{\lambda}+2;\frac{p}{\lambda}+2;\frac{2\alpha+\gamma+2}{2(j-i+\alpha)+\beta+\gamma+2)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+2\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}} - \frac{{}_2F_1\left(1,\frac{p}{\lambda}+\frac{q}{\lambda}+2;\frac{p}{\lambda}+2;\frac{2\alpha+\gamma+3}{2(j-i+\alpha)+\beta+\gamma+3)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+3\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}}\right. \\
&\left. - \frac{{}_2F_1\left(1,\frac{p}{\lambda}+\frac{q}{\lambda}+2;\frac{p}{\lambda}+2;\frac{2\alpha+\gamma+2}{2(j-i+\alpha)+\beta+\gamma+3)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+3\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}} + \frac{{}_2F_1\left(1,\frac{p}{\lambda}+\frac{q}{\lambda}+2;\frac{p}{\lambda}+2;\frac{2\alpha+\gamma+3}{2(j-i+\alpha)+\beta+\gamma+4)}\right)}{\{2(j-i+\alpha)+\beta+\gamma+4\}^{\frac{p}{\lambda}+\frac{q}{\lambda}+2}}\right].
\end{aligned}$$

Inserting $\mu_{i,i+1:j}^{(p,q)}$ in (13), it follows (12). □

Remark 3. *Setting $\lambda = 1$ in (12), we obtain*

$$
\begin{aligned}
\mu_{r,s:n}^{(p,q)} = {} & 36\theta^{p+q} \sum_{i=r}^{s-1} \sum_{j=n-s+i+1}^{n} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-i-1} \sum_{\gamma=0}^{\alpha} (-1)^{j+n-s-r+\alpha+\beta+\gamma+1} 2^{\beta+\gamma} 3^{j-i+\alpha-\beta-\gamma-1} \\
& \times \binom{i-1}{r-1} \binom{j-i-1}{n-s} \binom{n}{j} \binom{i-1}{\alpha} \binom{j-i-1}{\beta} \binom{\alpha}{\gamma} \frac{\Gamma(j+1)}{\Gamma(i)\Gamma(j-i)} \frac{\Gamma(p+q+2)}{(p+1)} \\
& \times \left[\frac{{}_2F_1\left(1, p+q+2; p+2; \frac{2\alpha+\gamma+2}{2(j-i+\alpha)+\beta+\gamma+2}\right)}{\{2(j-i+\alpha)+\beta+\gamma+2\}^{p+q+2}} - \frac{{}_2F_1\left(1, p+q+2; p+2; \frac{2\alpha+\gamma+3}{2(j-i+\alpha)+\beta+\gamma+3}\right)}{\{2(j-i+\alpha)+\beta+\gamma+3\}^{p+q+2}} \right. \\
& \left. - \frac{{}_2F_1\left(1, p+q+2; p+2; \frac{2\alpha+\gamma+2}{2(j-i+\alpha)+\beta+\gamma+3}\right)}{\{2(j-i+\alpha)+\beta+\gamma+3\}^{p+q+2}} + \frac{{}_2F_1\left(1, p+q+2; p+2; \frac{2\alpha+\gamma+3}{2(j-i+\alpha)+\beta+\gamma+4}\right)}{\{2(j-i+\alpha)+\beta+\gamma+4\}^{p+q+2}} \right],
\end{aligned}
$$

which is explicit expression for the (p,q)th product moment for the Bilal(θ) distribution.

Remark 4. *Setting $p = 1$ in (8), we calculate the means of the order statistics for the GB distribution (for n = 1(1)5) for selected parameter values. These means values are reported in Table 1. It can be noted that the condition $\sum_{r=1}^{n} \mu_{r:n} = nE(X)$ holds (see David and Nagaraja [28]).*

The variance of $X_{r:n}$ $(1 \le r \le n)$ is $V(X_{r:n}) = \mu_{r:n}^{(2)} - \left[\mu_{r:n}^{(1)}\right]^2$, where $\mu_{r:n}^{(1)}$ and $\mu_{r:n}^{(2)}$ can be calculated by setting $p = 1$ and $p = 2$ in (8), respectively. In addition, the covariance of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$, can be found by using the relation

$$
Cov(X_{r:n}, X_{s:n}) = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)} \mu_{s:n}^{(1)},
$$

where $\mu_{r,s:n}^{(1,1)}$ can be obtained by setting $p = q = 1$ in (12). The variances and covariances of the order statistics coming from the GB distribution are computed for selected parameter combinations and $n = 1(1)5$ [for the covariances $n = 2(1)5$] and the variances and covariances are reported in Table 2. Here, it can be seen that the condition $\sum_{r=1}^{n} \sum_{s=1}^{n} \sigma_{r,s:n} = n\sigma^2$ (see David and Nagaraja [28]) is satisfied, where $\sigma_{r,s:n} = Cov(X_{r:n}, X_{s:n})$ and $\sigma^2 = Var(X)$. The R software (R Core Team [30]) is used to compute the means, variances, and covariances.

Table 1. Means of order statistics for the GB(θ, λ) distribution.

n	r	$\theta = 0.25,\ \lambda = 0.25$	$\theta = 0.75,\ \lambda = 2.0$	$\theta = 1.0,\ \lambda = 0.75$	$\theta = 2.0,\ \lambda = 5.0$
1	1	0.97685	0.64248	0.86715	1.84767
2	1	0.11426	0.50942	0.45335	1.68700
	2	1.83945	0.77554	1.28096	2.00833
3	1	0.03547	0.44791	0.31720	1.60373
	2	0.27182	0.63246	0.72566	1.85353
	3	2.62326	0.84709	1.55860	2.08573
4	1	0.01601	0.40993	0.24831	1.54870
	2	0.09387	0.56185	0.52385	1.76884
	3	0.44977	0.70306	0.92748	1.93823
	4	3.34776	0.89509	1.76898	2.13489
5	1	0.00879	0.38323	0.20627	1.50808
	2	0.04488	0.51672	0.41649	1.71118
	3	0.16736	0.62954	0.68488	1.85534
	4	0.63804	0.75208	1.08922	1.99348
	5	4.02519	0.93085	1.93892	2.17025

Table 2. Variances and covariances of order statistics for the GB(θ, λ) distribution.

n	s	r	$\theta = 0.25,\ \lambda = 0.25$	$\theta = 0.75,\ \lambda = 2.0$	$\theta = 1.0,\ \lambda = 0.75$	$\theta = 2.0,\ \lambda = 5.0$
1	1	1	27.80883	0.05597	0.71505	0.08120
2	1	1	0.26160	0.03111	0.166490	0.06040
	2	1	0.74407	0.01771	0.17123	0.02581
		2	53.86792	0.04541	0.92115	0.05037
3	1	1	0.02058	0.02259	0.07503	0.05153
	2	1	0.04362	0.01344	0.07378	0.02380
		2	0.576205	0.00902	0.07707	0.01332
	3	1	0.13526	0.02544	0.23817	0.03652
		2	1.86809	0.01745	0.24943	0.02098
		3	78.60559	0.04003	1.03138	0.03933
4	1	1	0.00367	0.01818	0.04360	0.04633
	2	1	0.00703	0.01095	0.04195	0.02196
		2	0.01491	0.00779	0.04222	0.01347
	3	1	0.04585	0.00563	0.04439	0.00843
		2	0.06677	0.01852	0.11239	0.03080
		3	0.13165	0.01335	0.11340	0.01919
	4	1	0.37800	0.00973	0.11933	0.01214
		2	1.28269	0.02240	0.28249	0.02791
		3	3.22351	0.01657	0.29758	0.01796
		4	102.28030	0.03669	1.10398	0.03346
5	1	1	0.00101	0.01544	0.02896	0.04280
	2	1	0.00183	0.00934	0.02748	0.02052
		2	0.00339	0.00679	0.02720	0.01303
	3	1	0.00700	0.00521	0.02764	0.00894
		2	0.02089	0.00393	0.02915	0.00596
		3	0.01330	0.01488	0.06680	0.02744
	4	1	0.02360	0.01092	0.06628	0.01763
		2	0.04695	0.00843	0.06742	0.01218
		3	0.13516	0.00639	0.07113	0.00816
		4	0.13798	0.01634	0.13755	0.02335
	5	1	0.25899	0.01272	0.14011	0.01629
		2	0.70340	0.00970	0.14788	0.01100
		3	1.95722	0.02044	0.31373	0.02331
		4	4.73286	0.01577	0.33126	0.01595
		5	125.06652	0.03436	1.15715	0.02975

2.4. L-Moments

L-moments are statistics that may be used to summarize the form of a distribution. They allow us to define the L-scale, L-skewness, and L-kurtosis as linear combinations of order statistics that may be used to derive distributional parameters analogous to standard deviation, skewness, and kurtosis. All the information can be found in Hosking [31]. The L-moments can also be used to estimate parameters, and test hypotheses in the model specification. The mth L-moment of a distribution can be defined as

$$\lambda_m = \frac{1}{m}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\mu_{m-j:m}; \quad m \geq 1, \tag{16}$$

where

$$\mu_{i:m} = \frac{m!}{(i-1)!(m-i)!}\int_0^d x[F(x)]^{i-1}[1-F(x)]^{m-i}f(x)dx.$$

The first four L-moments are readily followed by setting $n = 1, 2, 3$ and 4 in (16). The L-moments of the GB distribution can be written as $\lambda_1 = \mu_{1:1}\lambda_2 = \mu_{2:2} - \mu_{1:1}\lambda_3 = 2\mu_{3:3} - 3\mu_{2:2} + \mu_{1:1}$ and $\lambda_4 = 5\mu_{4:4} - 10\mu_{3:3} + 6\mu_{2:2} - \mu_{1:1}$, where

$$\mu_{i:i} = 6i\theta \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} (-1)^{\alpha+\beta} 2^{\beta} 3^{\alpha-\beta} \binom{i-1}{\alpha} \binom{\alpha}{\beta} \\ \times \left[\frac{1}{(2\alpha+\beta+2)^{\frac{1}{\lambda}+1}} - \frac{1}{(2\alpha+\beta+3)^{\frac{1}{\lambda}+1}} \right] \Gamma\left(\frac{1}{\lambda}+1\right).$$

Hosking [31] also introduced some L-moment ratios. They are useful to define the L-coefficient of variation (L-CV) given by λ_2/λ_1, and the L-skewness and L-kurtosis defined by $\tau_3 = \lambda_3/\lambda_2$ and $\tau_4 = \lambda_4/\lambda_2$, respectively.

All the findings on the order statistics above complete the work of Abd-Elrahman [2]. They are the basis for more applications in the extreme value theory, among others things.

3. Estimation of the Parameters

This section provides some practical contributions to the applied study of Abd-Elrahman [2]. It discusses the estimation of the parameters θ and λ of the GB distribution using the ML method under the SRS and RSS schemes. Furthermore, we use a simulation study to investigate the behavior of the estimates.

3.1. Estimation of Parameters under the SRS Scheme

Let n be an integer, $X_1, X_2, \ldots, X_n$ be a SRS scheme (of size n) from the GB(θ, λ) distribution with pdf and cdf as given in (1) and (2), respectively, and $x_1, x_2, \ldots, x_n$ be observations of $X_1, X_2, \ldots, X_n$. We set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then the log-likelihood function, say $\ell(\theta, \lambda; \mathbf{x})$, is given by

$$\ell(\theta, \lambda; \mathbf{x}) = n[\ln(6) + \ln(\lambda) - \ln(\theta)] + (\lambda - 1) \sum_{i=1}^{n} \ln(x_i) - n(\lambda - 1)\ln(\theta) \\ - 2\sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\lambda} + \sum_{i=1}^{n} \ln\left(1 - e^{-\left(\frac{x_i}{\theta}\right)^{\lambda}}\right). \tag{17}$$

The ML estimates (MLEs) of θ and λ are defined by $(\hat{\theta}, \hat{\lambda}) = \operatorname{argmax}_{\theta>0,\lambda>0} \ell(\theta, \lambda; \mathbf{x})$. No closed forms exist for these estimates. However, numerical approach via the partial derivatives of $\ell(\theta, \lambda; \mathbf{x})$ is possible. The partial derivatives of $\ell(\theta, \lambda; \mathbf{x})$ with respect to θ and λ are given by

$$\frac{\partial \ell(\theta, \lambda; \mathbf{x})}{\partial \theta} = -\frac{n\lambda}{\theta} + \frac{2\lambda}{\theta} \sum_{i=1}^{n} x_i \left(\frac{x_i}{\theta}\right)^{\lambda} \left[1 + \sum_{i=1}^{n} \frac{e^{-\left(\frac{x_i}{\theta}\right)^{\lambda}}}{1 - e^{-\left(\frac{x_i}{\theta}\right)^{\lambda}}}\right] \tag{18}$$

and

$$\frac{\partial \ell(\theta, \lambda; \mathbf{x})}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \ln\frac{x_i}{\theta} - 2\sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\lambda} \ln\left(\frac{x_i}{\theta}\right) + \sum_{i=1}^{n} \frac{\ln\left(\frac{x_i}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\lambda} e^{-\left(\frac{x_i}{\theta}\right)^{\lambda}}}{1 - e^{-\left(\frac{x_i}{\theta}\right)^{\lambda}}}, \tag{19}$$

respectively.

By solving the nonlinear equations $\partial\ell(\theta, \lambda; \mathbf{x})/\partial\theta = 0$ and $\partial\ell(\theta, \lambda; \mathbf{x})/\partial\lambda = 0$ with respect to θ and λ, the MLEs could be obtained. The underlying theory on the MLEs can be found in Casella and Berger [32]. The theoretical solutions are often extremely complicated (see (18) and (19)), and in order to get a numerical solution, we applied R-package “bbmle”.

3.2. Estimation of Parameters under the RSS Scheme

The RSS scheme is summarized as follows:

Let c be the total number of cycles and r be the number of sample units chosen in each cycle (fixed size). To obtain a ranked set sample of size $n = rc$, follow the steps below.

- Randomly select r^2 units from the population and allocate these units randomly into r sets of size r.
- Assign ranks to the units in each set based on some accessible and non-expensive ordering criterion.
- To obtain a sample based on the RSS scheme, select the unit ranked at ith position from the ith set, $i = 1, 2, \ldots, r$.
- To obtain a final sample of size $n = rc$, repeat steps 1 to 3 c times.

Let $X_{(ii)j}, i = 1, 2, \ldots, r; j = 1, \ldots, c$, be a RSS scheme drawn from the GB(θ, λ) distribution with sample of size $n = rc$, where r is the set size and c is the number of cycles or cycle size, and $x_{(ii)j}, i = 1, 2, \ldots, r; j = 1, \ldots, c$, be the corresponding observations. We denote by $\mathbf{x}_*$ the vector of these observations. Then the pdf of $X_{(ii)j}$ is given by

$$g_{i:r}(x_{(ii)j}; \theta, \lambda)) = C_{i:r}[F(x_{(ii)j}; \theta, \lambda)]^{i-1} f(x_{(ii)j}; \theta, \lambda) \left[1 - F(x_{(ii)j}; \theta, \lambda)\right]^{r-i}, \tag{20}$$

where $C_{i:r} = \frac{r!}{(i-1)!(r-i)!}$. In view of (20), the likelihood function can be written as

$$\begin{aligned}
L(\theta, \lambda; \mathbf{x}_*) &= \prod_{j=1}^{c}\prod_{i=1}^{r} g_{i:r}(x_{(ii)j}; \theta, \lambda)) \\
&= \prod_{j=1}^{c}\prod_{i=1}^{r} C_{i:r} \frac{6\lambda}{\theta}\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda-1}\left(1 - e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right) e^{-2(r-i+1)\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}} \\
&\quad \times \left[1 - e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\left(3 - 2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right)\right]^{i-1}\left(3 - 2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right)^{(r-i)}.
\end{aligned}$$

In this setting, the log-likelihood function is given by

$$\begin{aligned}
\ell(\theta, \lambda; \mathbf{x}_*) = {}& C + n[\ln(6) + \ln(\lambda) - \lambda\ln(\theta)] + (\lambda - 1)\sum_{j=1}^{c}\sum_{i=1}^{r}\ln\left(x_{(ii)j}\right) \\
&+ \sum_{j=1}^{c}\sum_{i=1}^{r}\ln\left(1 - e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right) - 2\sum_{j=1}^{c}\sum_{i=1}^{r}(r - i + 1)\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda} \\
&+ \sum_{j=1}^{c}\sum_{i=1}^{r}(i - 1)\ln\left[1 - e^{-2\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\left(3 - 2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right)\right] \\
&+ \sum_{j=1}^{c}\sum_{i=1}^{r}(r - i)\ln\left(3 - 2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right),
\end{aligned} \tag{21}$$

where $C = \sum_{j=1}^{c}\sum_{i=1}^{r}\ln C_{i:r}$.

The MLEs of θ and λ are defined by $(\hat{\theta}, \hat{\lambda}) = \mathrm{argmax}_{\theta>0,\lambda>0}\, \ell(\theta, \lambda; \mathbf{x}_*)$. These estimates do not have any closed forms. However, partial derivatives of $l(\theta, \lambda; \mathbf{x}_*)$ can be used to generate a numerical solution. The partial derivatives of $\ell(\theta, \lambda; \mathbf{x}_*)$ associated with unknown parameters can be expressed as

$$\frac{\partial \ell(\theta,\lambda;\mathbf{x}_*)}{\partial \theta} = -\frac{n\lambda}{\theta} + \frac{2\lambda}{\theta}\sum_{j=1}^{c}\sum_{i=1}^{r}(r-i+1)\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}$$
$$-\frac{\lambda}{\theta}\sum_{j=1}^{c}\sum_{i=1}^{r}(i-1)\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}\Delta_{i,j}(\theta,\lambda)e^{-2\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}$$
$$-\frac{\lambda}{\theta}\sum_{j=1}^{c}\sum_{i=1}^{r}e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}\left[\frac{e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}}{1-e^{-2\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}} - \frac{2(r-i)}{3-2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}}\right]$$

and

$$\frac{\partial \ell(\theta,\lambda;\mathbf{x}_*)}{\partial \lambda} = \frac{n}{\lambda} - n\ln(\theta) + \sum_{j=1}^{c}\sum_{i=1}^{r}\ln(x_{(ii)j}) - 2\sum_{j=1}^{c}\sum_{i=1}^{r}(r-i+1)\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}\ln\left(\frac{x_{(ii)j}}{\theta}\right)$$
$$+2\sum_{j=1}^{c}\sum_{i=1}^{r}\ln\left(\frac{x_{(ii)j}}{\theta}\right)\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}$$
$$\times\left[\frac{r-i}{3-2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}} + e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\Delta_{i,j}(\theta,\lambda)\right], \tag{22}$$

where

$$\Delta_{i,j}(\theta,\lambda) = \frac{3-3e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}}{1-e^{-2\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\left(3-2e^{-\left(\frac{x_{(ii)j}}{\theta}\right)^{\lambda}}\right)}.$$

By solving the nonlinear equations $\partial\ell(\theta,\lambda;\mathbf{x}_*)/\partial\theta = 0$ and $\partial\ell(\theta,\lambda;\mathbf{x}_*)/\partial\lambda = 0$ with respect to θ and λ numerically, the MLEs can be obtained. The theoretical solutions are often extremely complicated and in order to get the numerical solutions, we applied the R-package "bbmle".

3.3. Asymptotic Confidence Interval

The Fisher information matrix I of parameters θ, and λ are the negative expectation of the last-second derivative of the log-likelihood function, which is obtained by the Hessian matrix by "bbmle" function. The variance–covariance matrix is the inverse Fisher information matrix. Also as $n \to \infty$, the asymptotic distribution of the MLE $(\hat{\theta},\hat{\lambda})$ is given by

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda} \end{pmatrix} \sim \mathbb{N}\left[\begin{pmatrix} \theta \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & \hat{V}_{22} \end{pmatrix}\right].$$

The asymptotic variance–covariance matrix V of the estimates $\hat{\theta}$ and $\hat{\lambda}$ are obtained by inverting the Hessian matrix. An approximate $100(1-\alpha)\%$ two-sided confidence intervals (CIs) for θ and λ are given by

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}} \quad \text{and} \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}},$$

respectively, where Z_α is the α percentile of the standard normal distribution.

3.4. Simulation Study

This section explains how to use the SRS and RSS schemes to get ML estimates (MLEs) for the unknown parameters θ and λ in the GB distribution. A comparative study is performed based on the biases, mean square errors (MSEs), and relative efficiencies (REs). The performance of the estimates are compared using Monte Carlo simulation in R software (we utilized function "mle2" in R-package "bbmle" (Bolker and R Development Core Team [33])) with 10,000 repetitions for different set sizes, the number of cycles, and selected parameter values. The following algorithm is used to obtain the MLEs and the suggested criteria measures.

Step 1: A random sample of size $n = 4, 7, 10, 15$ and 20 with a set size of r and number of cycles c where $n = rc$ is generated from the GB distribution for each parameter combinations ($\theta = 0.25, \lambda = 0.25$), ($\theta = 0.25, \lambda = 0.75$), ($\theta = 0.25, \lambda = 2$), ($\theta = 0.25, \lambda = 5$), ($\theta = 0.75, \lambda = 0.25$), ($\theta = 0.75, \lambda = 0.75$), ($\theta = 0.75, \lambda = 2$), ($\theta = 0.75, \lambda = 5$), ($\theta = 2$, $\lambda = 0.25$), ($\theta = 2, \lambda = 0.75$), ($\theta = 2, \lambda = 2$), and ($\theta = 2, \lambda = 5$). The pdfs and hazard rate functions (hrfs) of the GB distribution are plotted for these selected parameter values of θ and λ in Figure 1. We recall that $h(x) = h(x;\theta,\lambda) = f(x;\theta,\lambda)/[1 - F(x;\theta,\lambda)]$ is the definition of the hrf.

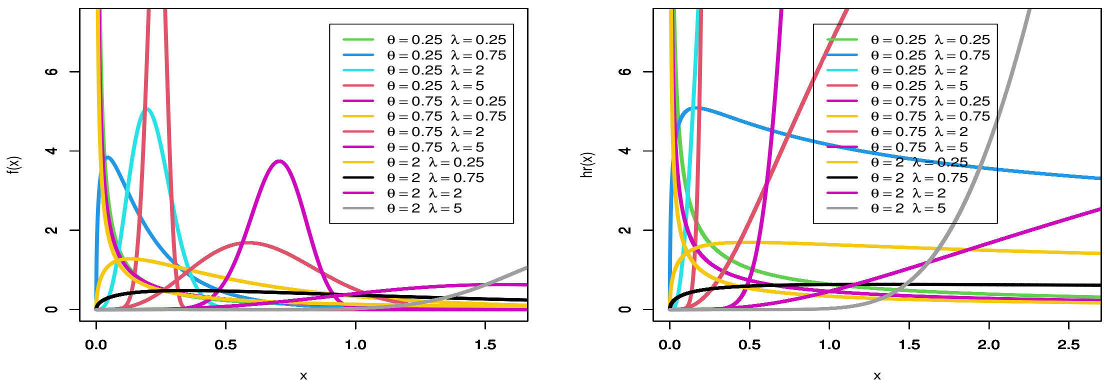

Figure 1. Pdfs and hrfs of the GB(θ, λ) distribution for selected values of θ and λ.

Step 2: The MLEs of the unknown parameters θ and λ are obtained under the SRS and RSS schemes for each n and specified parameter combinations.

Step 3: Repeat steps 1–2, N = 10,000 times. The biases and MSEs are computed using the following formulae:

$$\text{Bias}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta), \quad \text{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2,$$

$$\text{Bias}(\hat{\lambda}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i - \lambda), \quad \text{MSE}(\hat{\lambda}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i - \lambda)^2,$$

respectively, where $\hat{\theta}_i$ and $\hat{\lambda}_i$ denotes the estimate of θ and λ for the ith simulated sample, respectively. The relative efficiencies (REs) are calculated using the following formulae for each simulated scenario and estimation method. More precisely, they are defined by

$$\text{RE}_1(\hat{\theta}) = \frac{\text{MSE}(\hat{\theta})_{\text{SRS}}}{\text{MSE}(\hat{\theta})_{\text{RSS}^{c=1}}}, \quad \text{RE}_2(\hat{\theta}) = \frac{\text{MSE}(\hat{\theta})_{\text{SRS}}}{\text{MSE}(\hat{\theta})_{\text{RSS}^{c=2}}},$$

$$\text{RE}_1(\hat{\lambda}) = \frac{\text{MSE}(\hat{\lambda})_{\text{SRS}}}{\text{MSE}(\hat{\lambda})_{\text{RSS}^{c=1}}}, \quad \text{RE}_2(\hat{\lambda}) = \frac{\text{MSE}(\hat{\lambda})_{\text{SRS}}}{\text{MSE}(\hat{\lambda})_{\text{RSS}^{c=2}}}.$$

The biases, MSEs, and REs for λ are computed in a similar manner.

The findings of the simulation study are reported in Tables 3–5. The findings of the simulation study for confidence intervals and the coverage probabilities (CPs) are reported in Tables 6–8. The larger the sample size, the higher the confidence interval accuracy as the intervals become shorter. We observe that RSS-based bias and MSE values are consistently lower than SRS-based ones in every case. Furthermore, the MSE values based on SRS and RSS schemes decrease when the sample size is increased for all parameters. The CPs for the RSS scheme are better than those for the SRS scheme. We can also observe that when θ is fixed and λ increases, the MSEs for θ decrease, while MSEs for λ increase. Additionally, when λ is fixed and θ increases, the MSEs for θ increase while the MSEs for λ decrease. When the number of cycles is increased, the bias and MSE values decrease for most cases. REs increase with a larger number of cycles in most cases. According to the simulation findings, the RSS scheme outperforms the SRS scheme. We can also deduce that estimating the unknown parameters of the GB distribution using the RSS scheme rather than the SRS scheme is more efficient.

Table 3. Bias, MSE, and RE for the GB(θ,λ) distribution under the SRS and RSS schemes when $\theta = 0.25$.

$\theta = 0.25$			SRS		RSS		RSS 2		RE	
λ	n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
0.25	4	θ	0.10056	0.19554	0.04653	0.09963	0.05762	0.10295	1.96256	1.89933
		λ	0.03811	0.01460	0.03585	0.01342	0.03046	0.01260	1.08814	1.15898
	7	θ	0.04526	0.06025	0.04343	0.04234	0.03655	0.02608	1.42310	2.31015
		λ	0.00617	0.00628	0.00537	0.00586	0.00429	0.00404	1.07321	1.55582
	10	θ	0.03514	0.04235	0.03019	0.01631	0.02174	0.01193	2.59593	3.54922
		λ	−0.00135	0.00470	−0.00115	0.00329	−0.00229	0.00202	1.42844	2.32197
	15	θ	0.03794	0.02684	0.01839	0.00747	0.01457	0.00466	3.59388	5.76114
		λ	−0.01308	0.00312	−0.00693	0.00165	−0.00683	0.00082	1.89221	3.82938
	20	θ	0.00602	0.01483	0.00839	0.00388	0.00975	0.00278	3.81807	5.33176
		λ	−0.01749	0.00238	−0.00571	0.00091	−0.00662	0.00056	2.62979	4.25989
0.75	4	θ	−0.00148	0.00797	−0.00902	0.00602	−0.00505	0.00490	1.32291	1.62674
		λ	0.16129	0.11055	0.15845	0.10940	0.14741	0.10162	1.01053	1.08788
	7	θ	0.00018	0.00362	−0.00291	0.00261	−0.00024	0.00187	1.38868	1.93722
		λ	0.07838	0.03593	0.07204	0.03047	0.06240	0.02506	1.17942	1.43385
	10	θ	0.00004	0.00283	−0.00037	0.00121	−0.00156	0.00096	2.34942	2.94128
		λ	0.06572	0.02829	0.03900	0.01787	0.03878	0.01176	1.58334	2.40503
	15	θ	0.00481	0.00198	−0.00057	0.00058	0.00003	0.00042	3.41952	4.76559
		λ	0.03679	0.01388	0.01443	0.00835	0.01137	0.00461	1.66195	3.01034
	20	θ	−0.00319	0.00142	−0.00187	0.00036	−0.00054	0.00026	3.99485	5.51584
		λ	0.02324	0.00967	0.01398	0.00453	0.00824	0.00278	2.13618	3.47364

Table 3. *Cont.*

$\theta = 0.25$			SRS		RSS		RSS 2		RE	
λ	n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
2	4	θ	−0.00413	0.00108	−0.00633	0.00090	−0.00417	0.00069	1.19711	1.55925
		λ	0.43000	0.78595	0.42519	0.70556	0.40729	0.68976	1.11394	1.13945
	7	θ	−0.00159	0.00050	−0.00232	0.00037	−0.00095	0.00026	1.33903	1.92014
		λ	0.20894	0.25550	0.20453	0.23090	0.16635	0.17815	1.10651	1.43412
	10	θ	−0.00129	0.00039	−0.00070	0.00017	−0.00104	0.00014	2.28691	2.82881
		λ	0.17525	0.20105	0.10394	0.12702	0.10336	0.08357	1.58286	2.40573
	15	θ	0.00091	0.00027	−0.00048	0.00008	−0.00018	0.00006	3.27836	4.60162
		λ	0.09805	0.09871	0.03842	0.05939	0.03026	0.03278	1.66195	3.01088
	20	θ	−0.00187	0.00020	−0.00087	0.00005	−0.00033	0.00004	4.07494	5.68931
		λ	0.06191	0.06878	0.03722	0.03219	0.02233	0.01962	2.13688	3.50477
5	4	θ	−0.00218	0.00018	−0.00299	0.00015	−0.00201	0.00011	1.16128	1.55214
		λ	1.07516	4.91362	0.98564	3.97810	0.91822	3.60870	1.23517	1.36160
	7	θ	−0.00088	0.00008	−0.00111	0.00006	−0.00051	0.00004	1.33101	1.93414
		λ	0.52233	1.59683	0.46129	1.29293	0.41596	1.11371	1.23505	1.43380
	10	θ	−0.00070	0.00006	−0.00037	0.00003	−0.00049	0.00002	2.29067	2.82201
		λ	0.43809	1.25650	0.25983	0.79392	0.25845	0.52244	1.58265	2.40508
	15	θ	0.00024	0.00004	−0.00023	0.00001	−0.00010	0.00001	3.25695	4.58442
		λ	0.24511	0.61696	0.09605	0.37123	0.07566	0.20492	1.66193	3.01072
	20	θ	−0.00085	0.00003	−0.00037	0.00001	−0.00015	0.00001	4.13097	5.79048
		λ	0.15479	0.42981	0.09311	0.20117	0.05585	0.12269	2.13652	3.50330

Table 4. Bias, MSE, and RE for the GB(θ, λ) distribution under the SRS and RSS schemes when $\theta = 0.75$.

$\theta = 0.75$			SRS		RSS		RSS 2		RE	
λ	n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
0.25	4	θ	0.30376	1.75665	0.14183	0.89516	0.16588	0.91976	1.96239	1.90991
		λ	0.04386	0.01386	0.04187	0.01241	0.03951	0.01154	1.11652	1.20094
	7	θ	0.13942	0.54164	0.10677	0.34636	0.09188	0.21885	1.56380	2.47498
		λ	0.01350	0.00537	0.01220	0.00516	0.01101	0.00353	1.04165	1.52141
	10	θ	0.10729	0.38085	0.07435	0.13360	0.04799	0.09963	2.85074	3.82246
		λ	0.00867	0.00418	0.00348	0.00287	0.00337	0.00172	1.45452	2.43040
	15	θ	0.11653	0.24241	0.03638	0.05589	0.03276	0.04017	4.33748	6.03468
		λ	−0.00320	0.00255	−0.00190	0.00128	−0.00273	0.00068	1.98744	3.73125
	20	θ	0.01993	0.13383	0.01430	0.03291	0.01834	0.02333	4.06679	5.73621
		λ	−0.00930	0.00189	−0.00197	0.00075	−0.00271	0.00041	2.52980	4.57054

Table 4. *Cont.*

$\theta = 0.75$			SRS		RSS		RSS 2		RE	
λ	n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
0.75	4	θ	−0.00445	0.07169	−0.02706	0.05419	−0.01508	0.04417	1.32302	1.62323
		λ	0.16127	0.11055	0.15785	0.10593	0.14774	0.10263	1.04356	1.07720
	7	θ	0.00055	0.03258	−0.00874	0.02346	−0.00072	0.01682	1.38868	1.93727
		λ	0.07837	0.03593	0.06920	0.03435	0.06240	0.02506	1.04619	1.43400
	10	θ	0.00011	0.02549	−0.00111	0.01085	−0.00469	0.00867	2.34942	2.94100
		λ	0.06574	0.02828	0.03900	0.01787	0.03879	0.01176	1.58315	2.40603
	15	θ	0.01444	0.01784	−0.00170	0.00522	0.00009	0.00374	3.41971	4.76631
		λ	0.03679	0.01388	0.01443	0.00835	0.01137	0.00461	1.66210	3.01049
	20	θ	−0.00957	0.01277	−0.00562	0.00320	−0.00168	0.00231	3.99416	5.52372
		λ	0.02324	0.00967	0.01398	0.00453	0.00840	0.00276	2.13670	3.50372
2	4	θ	−0.01240	0.00968	−0.01900	0.00808	−0.01252	0.00621	1.19715	1.55920
		λ	0.43006	0.78611	0.42497	0.75545	0.40730	0.69773	1.04058	1.12666
	7	θ	−0.00477	0.00448	−0.00698	0.00335	−0.00286	0.00233	1.33894	1.92012
		λ	0.20896	0.25551	0.20454	0.23091	0.16639	0.17817	1.10656	1.43408
	10	θ	−0.00386	0.00352	−0.00212	0.00154	−0.00314	0.00125	2.28712	2.82879
		λ	0.17530	0.20112	0.10398	0.12704	0.10340	0.08359	1.58314	2.40600
	15	θ	0.00273	0.00241	−0.00145	0.00073	−0.00055	0.00052	3.27776	4.60100
		λ	0.09808	0.09873	0.03846	0.05940	0.03030	0.03279	1.66208	3.01051
	20	θ	−0.00562	0.00184	−0.00261	0.00045	−0.00099	0.00032	4.07439	5.68943
		λ	0.06194	0.06879	0.03727	0.03219	0.02238	0.01963	2.13683	3.50405
5	4	θ	−0.00654	0.00158	−0.00896	0.00136	−0.00602	0.00102	1.16131	1.55209
		λ	1.07517	4.91318	0.98564	3.97022	0.95824	3.61076	1.23751	1.36071
	7	θ	−0.00263	0.00073	−0.00334	0.00055	−0.00152	0.00038	1.33073	1.93409
		λ	0.52247	1.59708	0.50614	1.29317	0.41603	1.11366	1.23502	1.43409
	10	θ	−0.00212	0.00057	−0.00110	0.00025	−0.00146	0.00020	2.29033	2.82176
		λ	0.43832	1.25708	0.25998	0.79417	0.25853	0.52250	1.58288	2.40587
	15	θ	0.00071	0.00038	−0.00070	0.00012	−0.00030	0.00008	3.25697	4.58461
		λ	0.24522	0.61708	0.09621	0.37126	0.07578	0.20499	1.66212	3.01034
	20	θ	−0.00255	0.00030	−0.00112	0.00007	−0.00045	0.00005	4.13039	5.79047
		λ	0.15488	0.42991	0.09320	0.20122	0.05598	0.12269	2.13655	3.50408

Table 5. Bias, MSE, and RE for the GB(θ, λ) distribution under the SRS and RSS schemes when $\theta = 2$.

$\theta = 2$			SRS		RSS		RSS 2		RE	
λ	n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
0.25	4	θ	0.8079	12.4217	0.3810	6.3619	0.4305	6.5093	1.95251	1.90830
		λ	0.0484	0.0133	0.0469	0.0124	0.0452	0.0121	1.07373	1.10571
	7	θ	0.3730	3.8194	0.2483	2.3011	0.2187	1.5370	1.65980	2.48498
		λ	0.0180	0.0049	0.0153	0.0046	0.0147	0.0032	1.08368	1.55359
	10	θ	0.2857	2.7101	0.1711	0.9086	0.1042	0.6708	2.98255	4.04022
		λ	0.0127	0.0040	0.0064	0.0026	0.0063	0.0016	1.51814	2.53252
	15	θ	0.3138	1.7279	0.0798	0.3914	0.0723	0.2732	4.41495	6.32470
		λ	0.0020	0.0022	0.0003	0.0012	−0.0007	0.0006	1.88979	3.48829
	20	θ	0.0573	0.9540	0.0187	0.2261	0.0350	0.1620	4.22029	5.88760
		λ	−0.0038	0.0016	0.0007	0.0006	−0.0007	0.0004	2.42153	4.27447
0.75	4	θ	−0.0119	0.5098	−0.0722	0.3853	−0.0403	0.3139	1.32291	1.62396
		λ	0.1613	0.1105	0.1528	0.1036	0.1577	0.1013	1.06711	1.09168
	7	θ	0.0015	0.2317	−0.0233	0.1668	−0.0019	0.1196	1.38867	1.93715
		λ	0.0784	0.0359	0.0692	0.0324	0.0624	0.0251	1.10792	1.43413
	10	θ	0.0003	0.1813	−0.0030	0.0772	−0.0125	0.0616	2.34955	2.94090
		λ	0.0657	0.0283	0.0390	0.0179	0.0388	0.0118	1.58301	2.40576
	15	θ	0.0385	0.1269	−0.0045	0.0371	0.0002	0.0266	3.41983	4.76643
		λ	0.0368	0.0139	0.0144	0.0084	0.0114	0.0046	1.66198	3.01044
	20	θ	−0.0255	0.0908	−0.0150	0.0227	−0.0045	0.0164	3.99464	5.52447
		λ	0.0232	0.0097	0.0140	0.0045	0.0084	0.0028	2.13696	3.50413
2	4	θ	−0.0331	0.0688	−0.0507	0.0575	−0.0334	0.0441	1.19713	1.55923
		λ	0.4301	0.7861	0.7404	0.7056	0.4730	0.8978	1.11418	0.87563
	7	θ	−0.0127	0.0319	−0.0186	0.0238	−0.0076	0.0166	1.33895	1.92011
		λ	0.2090	0.2555	0.2015	0.2309	0.1664	0.1782	1.10659	1.43404
	10	θ	−0.0103	0.0250	−0.0057	0.0110	−0.0084	0.0089	2.28706	2.82882
		λ	0.1753	0.2011	0.1040	0.1270	0.1034	0.0836	1.58315	2.40591
	15	θ	0.0073	0.0171	−0.0039	0.0052	−0.0015	0.0037	3.27793	4.60098
		λ	0.0981	0.0987	0.0385	0.0594	0.0303	0.0328	1.66209	3.01057
	20	θ	−0.0150	0.0131	−0.0070	0.0032	−0.0026	0.0023	4.07422	5.68928
		λ	0.0619	0.0688	0.0373	0.0322	0.0224	0.0196	2.13657	3.50378
5	4	θ	−0.0174	0.0113	−0.0239	0.0097	−0.0161	0.0073	1.16128	1.55213
		λ	1.0752	4.9135	0.9856	1.9712	0.7824	1.2961	2.49261	3.79094
	7	θ	−0.0070	0.0052	−0.0089	0.0039	−0.0041	0.0027	1.33069	1.93407
		λ	0.5224	1.5970	0.4614	1.2932	0.4160	1.1136	1.23495	1.43408
	10	θ	−0.0056	0.0040	−0.0029	0.0018	−0.0039	0.0014	2.29036	2.82161
		λ	0.4383	1.2571	0.2600	0.7940	0.2585	0.5225	1.58319	2.40588
	15	θ	0.0019	0.0027	−0.0019	0.0008	−0.0008	0.0006	3.25638	4.58419
		λ	0.2452	0.6171	0.0962	0.3713	0.0758	0.2050	1.66202	3.01055
	20	θ	−0.0068	0.0021	−0.0030	0.0005	−0.0012	0.0004	4.13039	5.79057
		λ	0.1549	0.4299	0.0932	0.2012	0.0560	0.1227	2.13672	3.50395

Table 6. Lower, upper bounds of CI and CP for the GB(θ, λ) distribution under the SRS and RSS schemes when $\theta = 0.25$.

$\theta = 0.25$			SRS			RSS			RSS $c = 2$		
λ	n		Lower	Upper	CP	Lower	Upper	CP	Lower	Upper	CP
0.25	4	θ	0.4934	1.1946	27.178%	0.3154	0.9084	39.959%	0.3110	0.9263	40.726%
		λ	0.0633	0.5129	93.726%	0.0699	0.5018	93.779%	0.0687	0.4922	94.085%
	7	θ	0.1776	0.7681	60.209%	0.1008	0.6876	75.615%	0.0218	0.5949	91.241%
		λ	0.1013	0.4111	94.931%	0.1057	0.4050	94.944%	0.1300	0.3786	94.948%
	10	θ	0.1123	0.6826	73.501%	0.0369	0.5234	94.320%	0.0619	0.4816	94.527%
		λ	0.1143	0.3829	94.996%	0.1365	0.3612	94.996%	0.1597	0.3357	94.971%
	15	θ	0.0244	0.6003	90.757%	0.1029	0.4339	94.455%	0.1339	0.3953	94.452%
		λ	0.1304	0.3434	94.334%	0.1646	0.3215	94.657%	0.1888	0.2975	94.302%
	20	θ	0.0177	0.4944	94.972%	0.1374	0.3794	94.789%	0.1582	0.3613	94.594%
		λ	0.1432	0.3218	93.296%	0.1864	0.3022	94.571%	0.1989	0.2879	94.020%
0.75	4	θ	0.0736	0.4234	94.997%	0.0899	0.3920	94.843%	0.1082	0.3817	94.941%
		λ	0.3414	1.4812	91.407%	0.3394	1.4775	91.525%	0.3434	1.4514	91.831%
	7	θ	0.1323	0.3681	95.000%	0.1472	0.3470	94.963%	0.1650	0.3345	95.000%
		λ	0.4901	1.1667	92.607%	0.5104	1.1337	92.617%	0.5273	1.0976	92.868%
	10	θ	0.1457	0.3544	95.000%	0.1816	0.3177	94.999%	0.1877	0.3092	94.971%
		λ	0.5123	1.1192	92.912%	0.5384	1.0396	93.928%	0.5903	0.9873	93.305%
	15	θ	0.1681	0.3416	94.865%	0.2023	0.2966	94.994%	0.2101	0.2900	95.000%
		λ	0.5674	1.0062	93.754%	0.5875	0.9413	94.707%	0.6301	0.8926	94.670%
	20	θ	0.1732	0.3204	94.918%	0.2114	0.2849	94.886%	0.2180	0.2809	94.987%
		λ	0.5859	0.9605	94.320%	0.6350	0.8930	94.482%	0.6561	0.8604	94.714%
2	4	θ	0.1821	0.3096	94.815%	0.1862	0.3011	94.464%	0.1950	0.2966	94.704%
		λ	0.9105	3.9495	91.409%	1.0053	3.8450	90.970%	0.9887	3.8259	91.301%
	7	θ	0.2048	0.2920	94.942%	0.2102	0.2852	94.832%	0.2175	0.2806	94.960%
		λ	1.3068	3.1111	92.608%	1.3523	3.0568	92.430%	1.4060	2.9267	92.869%
	10	θ	0.2100	0.2874	94.952%	0.2237	0.2749	94.967%	0.2260	0.2719	94.910%
		λ	1.3663	2.9842	92.911%	1.4358	2.7721	93.929%	1.5742	2.6325	93.305%
	15	θ	0.2189	0.2829	94.965%	0.2318	0.2672	94.968%	0.2349	0.2648	94.994%
		λ	1.5130	2.6831	93.755%	1.5667	2.5101	94.708%	1.6804	2.3802	94.671%
	20	θ	0.2203	0.2759	94.800%	0.2353	0.2629	94.826%	0.2379	0.2614	94.966%
		λ	1.5624	2.5614	94.322%	1.6932	2.3812	94.484%	1.7513	2.2934	94.701%
5	4	θ	0.2222	0.2735	94.682%	0.2236	0.2704	94.282%	0.2275	0.2685	94.577%
		λ	2.2758	9.8746	91.408%	2.5871	9.3842	91.224%	2.6587	9.1777	91.441%
	7	θ	0.2316	0.2666	94.890%	0.2338	0.2640	94.762%	0.2369	0.2621	94.930%
		λ	3.2670	7.7776	92.608%	3.4243	7.4983	92.715%	3.5150	7.3169	92.868%
	10	θ	0.2338	0.2648	94.910%	0.2394	0.2599	94.945%	0.2403	0.2587	94.879%
		λ	3.4158	7.4604	92.911%	3.5893	6.9303	93.929%	3.9354	6.5815	93.305%
	15	θ	0.2374	0.2630	94.985%	0.2427	0.2568	94.953%	0.2439	0.2559	94.988%
		λ	3.7825	6.7077	93.756%	3.9168	6.2753	94.708%	4.2009	5.9504	94.671%
	20	θ	0.2380	0.2603	94.746%	0.2441	0.2551	94.800%	0.2452	0.2545	94.956%
		λ	3.9062	6.4034	94.321%	4.2332	5.9531	94.483%	4.3781	5.7336	94.701%

Table 7. Lower, upper bounds of CI and CP for the GB(θ, λ) distribution under the SRS and RSS schemes when $\theta = 0.75$.

$\theta = 0.75$			SRS			RSS			RSS $c = 2$		
λ	n		Lower	Upper	CP	Lower	Upper	CP	Lower	Upper	CP
0.25	4	θ	1.4748	3.5824	27.304%	0.9416	2.7253	40.149%	0.9355	2.7673	40.580%
		λ	0.0797	0.5080	93.135%	0.0895	0.4942	93.096%	0.0937	0.4853	93.190%
	7	θ	0.5269	2.3058	60.555%	0.2776	1.9911	77.682%	0.0572	1.7409	91.916%
		λ	0.1223	0.4047	94.597%	0.1234	0.4009	94.660%	0.1465	0.3755	94.592%
	10	θ	0.3339	2.0484	73.693%	0.1229	1.5258	94.505%	0.1865	1.4095	94.729%
		λ	0.1331	0.3843	94.790%	0.1486	0.3583	94.952%	0.1724	0.3344	94.924%
	15	θ	0.0711	1.8041	90.832%	0.3286	1.2442	94.722%	0.3952	1.1703	94.685%
		λ	0.1480	0.3456	94.954%	0.1780	0.3183	94.968%	0.1963	0.2983	94.874%
	20	θ	0.0540	1.4859	94.966%	0.4099	1.1187	94.929%	0.4711	1.0655	94.833%
		λ	0.1574	0.3240	94.449%	0.1946	0.3015	94.940%	0.2078	0.2868	94.794%
0.75	4	θ	0.2208	1.2703	94.997%	0.2698	1.1761	94.843%	0.3241	1.1458	94.941%
		λ	0.3414	1.4812	91.408%	0.3500	1.4657	91.410%	0.3406	1.4549	91.853%
	7	θ	0.3968	1.1043	95.000%	0.4415	1.0410	94.963%	0.4951	1.0035	95.000%
		λ	0.4901	1.1667	92.607%	0.4822	1.1562	93.125%	0.5273	1.0975	92.867%
	10	θ	0.4372	1.0631	95.000%	0.5447	0.9531	94.999%	0.5631	0.9276	94.971%
		λ	0.5123	1.1191	92.910%	0.5384	1.0396	93.928%	0.5903	0.9872	93.303%
	15	θ	0.5042	1.0247	94.865%	0.6068	0.8898	94.994%	0.6302	0.8700	95.000%
		λ	0.5674	1.0062	93.754%	0.5875	0.9413	94.707%	0.6301	0.8926	94.670%
	20	θ	0.5197	0.9612	94.918%	0.6341	0.8547	94.886%	0.6541	0.8425	94.986%
		λ	0.5859	0.9606	94.321%	0.6350	0.8930	94.482%	0.6567	0.8601	94.700%
2	4	θ	0.5463	0.9289	94.815%	0.5587	0.9033	94.463%	0.5850	0.8899	94.703%
		λ	0.9104	3.9497	91.408%	0.9389	3.9110	91.331%	0.9779	3.8367	91.358%
	7	θ	0.6144	0.8761	94.942%	0.6305	0.8556	94.831%	0.6526	0.8417	94.960%
		λ	1.3068	3.1111	92.608%	1.3523	3.0568	92.429%	1.4061	2.9267	92.868%
	10	θ	0.6301	0.8622	94.952%	0.6711	0.8247	94.967%	0.6780	0.8158	94.909%
		λ	1.3663	2.9843	92.910%	1.4358	2.7722	93.928%	1.5742	2.6326	93.304%
	15	θ	0.6567	0.8487	94.965%	0.6955	0.8016	94.967%	0.7046	0.7943	94.994%
		λ	1.5130	2.6832	93.755%	1.5667	2.5102	94.707%	1.6804	2.3802	94.670%
	20	θ	0.6610	0.8278	94.800%	0.7060	0.7888	94.825%	0.7138	0.7842	94.966%
		λ	1.5624	2.5615	94.321%	1.6933	2.3813	94.482%	1.7513	2.2935	94.700%
5	4	θ	0.6665	0.8204	94.682%	0.6708	0.8113	94.282%	0.6825	0.8055	94.577%
		λ	2.2760	9.8743	91.408%	2.5915	9.3797	91.214%	2.7421	9.1744	91.011%
	7	θ	0.6948	0.7999	94.890%	0.7014	0.7920	94.761%	0.7106	0.7863	94.929%
		λ	3.2671	7.7778	92.607%	3.5102	7.5021	92.126%	3.5152	7.3169	92.867%
	10	θ	0.7013	0.7945	94.910%	0.7181	0.7797	94.945%	0.7208	0.7762	94.879%
		λ	3.4157	7.4610	92.909%	3.5893	6.9307	93.928%	3.9355	6.5816	93.304%
	15	θ	0.7123	0.7891	94.985%	0.7281	0.7705	94.953%	0.7318	0.7676	94.988%
		λ	3.7825	6.7080	93.755%	3.9169	6.2755	94.707%	4.2009	5.9507	94.670%
	20	θ	0.7139	0.7810	94.746%	0.7323	0.7654	94.800%	0.7355	0.7636	94.956%
		λ	3.9061	6.4036	94.321%	4.2332	5.9532	94.482%	4.3783	5.7337	94.700%

Table 8. Lower, upper bounds of CI and CP for the GB(θ, λ) distribution under the SRS and RSS schemes when $\theta = 2$.

$\boldsymbol{\theta = 2}$			**SRS**			**RSS**			**RSS $c = 2$**		
$\boldsymbol{\lambda}$	$\boldsymbol{n}$		**Lower**	**Upper**	**CP**	**Lower**	**Upper**	**CP**	**Lower**	**Upper**	**CP**
0.25	4	θ	3.9161	9.5319	27.418%	2.5059	7.2680	40.229%	2.4984	7.3594	40.491%
		λ	0.0929	0.5038	92.529%	0.0988	0.4949	92.499%	0.0992	0.4913	92.625%
	7	θ	1.3871	6.1330	60.972%	0.6849	5.1814	79.350%	0.1731	4.6105	91.661%
		λ	0.1348	0.4011	94.194%	0.1365	0.3942	94.376%	0.1581	0.3714	94.155%
	10	θ	0.8919	5.4634	73.651%	0.3332	4.0091	94.618%	0.5120	3.6964	94.812%
		λ	0.1417	0.3836	94.516%	0.1570	0.3559	94.817%	0.1797	0.3329	94.703%
	15	θ	0.1881	4.8157	90.841%	0.8636	3.2960	94.811%	1.0577	3.0869	94.777%
		λ	0.1601	0.3439	94.980%	0.1834	0.3172	94.999%	0.2001	0.2985	94.991%
	20	θ	0.1462	3.9684	94.961%	1.0875	2.9498	94.983%	1.2490	2.8210	94.913%
		λ	0.1694	0.3231	94.895%	0.2011	0.3003	94.992%	0.2120	0.2866	94.984%
0.75	4	θ	0.5889	3.3874	94.997%	0.7194	3.1363	94.843%	0.8644	3.0550	94.941%
		λ	0.3414	1.4811	91.408%	0.3475	1.4580	91.607%	0.3660	1.4494	91.193%
	7	θ	1.0581	2.9448	95.000%	1.1774	2.7759	94.963%	1.3203	2.6759	95.000%
		λ	0.4901	1.1667	92.607%	0.4933	1.1451	92.993%	0.5273	1.0975	92.867%
	10	θ	1.1658	2.8349	95.000%	1.4526	2.5415	94.999%	1.5015	2.4735	94.971%
		λ	0.5123	1.1191	92.910%	0.5384	1.0396	93.928%	0.5903	0.9872	93.303%
	15	θ	1.3444	2.7326	94.865%	1.6180	2.3729	94.994%	1.6805	2.3200	95.000%
		λ	0.5674	1.0062	93.755%	0.5875	0.9413	94.707%	0.6301	0.8926	94.670%
	20	θ	1.3858	2.5631	94.918%	1.6909	2.2791	94.886%	1.7443	2.2467	94.986%
		λ	0.5859	0.9606	94.321%	0.6350	0.8930	94.482%	0.6567	0.8601	94.700%
2	4	θ	1.4568	2.4770	94.815%	1.4900	2.4087	94.463%	1.5601	2.3732	94.703%
		λ	0.9104	3.9497	91.408%	1.9630	3.5178	53.708%	0.8638	4.0822	91.120%
	7	θ	1.6383	2.3363	94.942%	1.6812	2.2815	94.831%	1.7403	2.2444	94.960%
		λ	1.3068	3.1111	92.608%	1.3464	3.0565	92.524%	1.4061	2.9268	92.867%
	10	θ	1.6802	2.2992	94.952%	1.7895	2.1992	94.967%	1.8079	2.1753	94.909%
		λ	1.3662	2.9844	92.910%	1.4358	2.7722	93.928%	1.5742	2.6326	93.303%
	15	θ	1.7512	2.2633	94.965%	1.8547	2.1376	94.967%	1.8790	2.1181	94.994%
		λ	1.5130	2.6832	93.755%	1.5668	2.5102	94.707%	1.6804	2.3802	94.670%
	20	θ	1.7626	2.2074	94.800%	1.8828	2.1033	94.825%	1.9035	2.0913	94.966%
		λ	1.5624	2.5615	94.321%	1.6933	2.3813	94.482%	1.7513	2.2935	94.700%
5	4	θ	1.7773	2.1878	94.682%	1.7888	2.1634	94.282%	1.8199	2.1479	94.577%
		λ	2.2759	9.8745	91.408%	4.0259	7.9454	83.342%	4.1616	7.4033	84.284%
	7	θ	1.8528	2.1331	94.890%	1.8703	2.1119	94.762%	1.8950	2.0969	94.929%
		λ	3.2671	7.7778	92.607%	3.4242	7.4985	92.715%	3.5152	7.3168	92.867%
	10	θ	1.8701	2.1186	94.910%	1.9149	2.0793	94.945%	1.9223	2.0700	94.879%
		λ	3.4156	7.4609	92.910%	3.5895	6.9305	93.928%	3.9355	6.5816	93.304%
	15	θ	1.8996	2.1042	94.985%	1.9415	2.0548	94.953%	1.9514	2.0470	94.988%
		λ	3.7825	6.7079	93.755%	3.9169	6.2755	94.707%	4.2009	5.9506	94.670%
	20	θ	1.9037	2.0827	94.746%	1.9529	2.0412	94.800%	1.9613	2.0363	94.956%
		λ	3.9061	6.4037	94.321%	4.2332	5.9532	94.482%	4.3782	5.7337	94.700%

4. Application to Real Data

In this section, a performance comparison between the proposed estimates is carried out under the SRS and RSS schemes using the real data set. The source of the data set, is a single fibre data set of 10 mm in gauge lengths with sample size 63, which is first considered by Badar and Priest [34]. Abd-Elrahman [2] utilized several estimation methods to estimate the parameters of the GB distribution for this data set and demonstrated that the GB distribution fits the data set quite well. The Kolmogorov-Smirnov (K-S) test was used to compare the different samples for fitting. For this data, Abd-Elrahman [2] reported the MLEs of the parameters θ and λ of the GB distribution, which are $\hat{\theta} = 3.3869$ and $\hat{\lambda} = 3.6186$, with the standard errors (SEs) being $\mathrm{SE}(\hat{\theta}) = 0.08511$ and $\mathrm{SE}(\hat{\lambda}) = 0.32474$, respectively. The K-S statistic (K-S(stat)) is 0.08781, and its associated p-value (K-S(stat)) is 0.7163. Figure 2 corroborated these findings by displaying the empirical cdf versus the estimated cdf $F(x;\hat{\theta},\hat{\lambda})$, the estimated pdf $f(x;\hat{\theta},\hat{\lambda})$ over the histogram of the data, and the probability-probability (PP) plot of the estimated model.

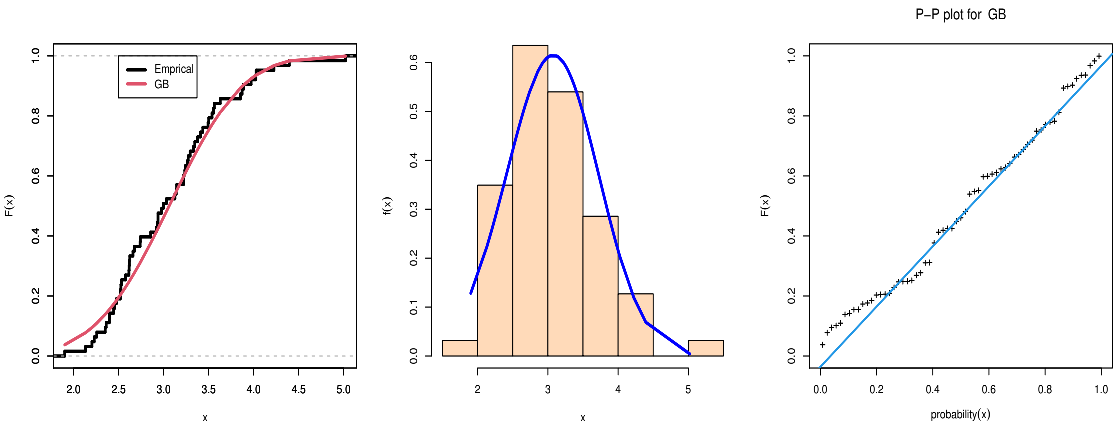

Figure 2. Estimated cdf, pdf and PP-plot of the GB(θ,λ) distribution when $\hat{\theta}$=3.3869 and $\hat{\lambda}$=3.6186.

For analysis, we observed random samples of different sizes under the SRS and RSS schemes with different set sizes and number of cycles, computed MLEs of the parameters, K-S statistic, and p-value for the observed SRS and RSS schemes in each case, and compared the performance of the estimates.

When $n = 5$, the observed SRS schemes are 2.618, 3.408, 3.628, 4.024, and 4.027. When $c = 1$ and $n = 5$, the observed RSS scheme is reported in Table 9. According to Kaur et al. [35], Latpate et al. [36] and Bhoj and Chandra [37], the performance of the RSS scheme further improves when appropriate unequal allocation is implemented instead of equal allocation. In unequal cycles ranked, when $c = 2$ and $n = 5$, the observed RSS scheme is reported in Table 10. The first cycle has $r = 3$ observations in the diagonal matrix and the second cycle has $r = 2$ observations in the diagonal matrix. The MLEs of the parameters θ and λ of the GB distribution and their corresponding SEs, and the K-S statistic with its associated p-value for the observed SRS scheme, RSS scheme with $c = 1$ and RSS scheme with $c = 2$ are presented in Table 11.

Table 9. RSS scheme from the GB(θ,λ) distribution when $c = 1$ and $n = 5$.

		One Cycle		
2.518	2.659	2.856	3.22	4.395
2.132	**2.257**	3.332	3.628	4.027
2.396	2.397	**3.971**	4.024	5.02
2.228	2.937	3.223	**3.562**	3.871
2.532	2.74	2.937	3.493	**3.886**

Table 10. RSS scheme from the GB(θ, λ) distribution when $c = 2$ and $n = 5$.

	Cycle 1			
2.856	2.996	3.537		
2.977	**3.125**	3.377		
2.396	2.616	**4.395**		
			Cycle 2	
			2.203	2.616
			3.139	**5.02**

Table 11. MLEs of the parameters θ and λ of the GB(θ, λ) distribution with lower and upper bounds of CI, SEs, Kolmogorov-Smirnov (K-S) (stat), and (K-S) p-value under the SRS and RSS schemes when $n = 5$.

	SRS		**RSS $c = 1$**		**RSS $c = 2$**	
	$\boldsymbol{\theta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\theta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\theta}$	$\boldsymbol{\lambda}$
MLEs	3.6561	3.6561	3.6591	3.1871	4.1497	1.8640
SEs	0.7766	0.7766	0.2211	0.8621	0.4384	0.5353
Lower	2.1340	2.1340	3.2257	1.4974	3.2905	0.8148
Upper	5.1783	5.1783	4.0925	4.8768	5.0090	2.9132
K-S (Stat)	0.3577		0.2489		0.1817	
K-S (p-value)	0.4417		0.8479		0.9861	

Similarly, when $n = 6$, the observed SRS schemes are 2.618, 3.220, 3.408, 3.628, 4.024, and 4.027. When $c = 1$ and $n = 6$, the observed RSS scheme is reported in Table 12. In equal cycles ranked, when $c = 2$ and $n = 6$, the observed RSS scheme is reported in Table 13, where the first and second cycles have $r = 3$ observations in the diagonal matrix. The MLEs of the parameters θ and λ of the GB distribution and their corresponding SEs, and the K-S statistic with its associated p-value for the observed SRS scheme, RSS scheme with $c = 1$, and RSS scheme with $c = 2$ are presented in Table 14.

Table 12. RSS scheme from the GB(θ, λ) distribution when $c = 1$ and $n = 6$.

		One Cycle			
2.856	2.977	2.996	3.125	3.377	3.537
2.203	**2.396**	2.616	3.139	3.852	4.395
2.624	3.408	**3.501**	3.554	3.628	3.971
2.35	2.474	2.518	**2.738**	3.294	3.493
2.132	2.257	2.659	3.264	**3.346**	5.02
2.397	2.618	3.03	3.145	3.223	**4.024**

Table 13. RSS scheme from the GB(θ, λ) distribution when $c = 2$ and $n = 6$.

	Cycle 1				
2.917	2.977	3.294			
3.125	**3.223**	3.886			
2.522	2.74	**3.408**			
				Cycle 2	
			2.624	2.937	3.886
			2.522	**2.977**	3.22
			2.532	3.125	**3.852**

Table 14. MLEs of the parameters θ and λ of the GB(θ, λ) distribution with lower and upper bounds of CI, SEs, Kolmogorov-Smirnov (K-S) (stat), and (K-S) p-value under the SRS and RSS schemes when $n = 6$.

	SRS		RSS $c = 1$		RSS $c = 2$	
	θ	λ	θ	λ	θ	λ
MLEs	2.3347	2.3347	3.4455	3.8604	3.4364	4.2626
SEs	0.5644	0.5644	0.1500	0.5498	0.1372	1.1060
Lower	1.2285	1.2285	3.1516	2.7829	3.1675	2.0948
Upper	3.4409	3.4409	3.7394	4.9380	3.7053	6.4305
K-S (Stat)	0.8198		0.1708		0.1811	
K-S (p-value)	0.0001		0.9805		0.9674	

Finally, when $n = 7$, the observed SRS schemes are 2.618, 2.856, 3.220, 3.408, 3.628, 4.024, and 4.027. When $c = 1$ and $n = 7$, the observed RSS scheme is given in Table 15. In unequal cycles ranked, when $c = 2$ and $n = 7$, the observed RSS scheme is reported in Table 16. The first cycle has $r = 3$ observations in the diagonal matrix, and the second cycle has $r = 4$ observations in the diagonal matrix. The MLEs of the parameters of the GB distribution and their corresponding SEs, and the K-S statistic with its associated p-value for the observed SRS scheme, RSS scheme with $c = 1$, and RSS scheme with $c = 2$ are presented in Table 17.

Table 15. RSS scheme from the GB(θ, λ) distribution when $c = 1$ and $n = 7$.

			One Cycle			
2.856	2.977	2.996	3.125	3.377	3.537	4.395
2.203	**2.396**	2.616	3.139	3.554	3.628	3.852
2.474	2.518	**2.624**	2.738	3.408	3.501	3.971
2.35	2.659	3.264	**3.294**	3.346	3.493	5.02
2.132	2.257	2.397	3.03	**3.145**	3.223	4.024
2.228	2.575	2.618	3.235	3.332	**3.871**	4.225
2.454	2.522	2.532	2.917	3.22	3.272	**4.027**

Table 16. RSS scheme from the GB(θ, λ) distribution when $c = 2$ and $n = 7$.

	Cycle 1					
2.257	2.35	3.272				
2.397	**2.618**	2.659				
3.125	3.294	**3.562**				
					Cycle 2	
			2.618	3.332	4.027	4.225
			2.614	**3.125**	3.272	3.554
			2.474	3.145	**3.346**	3.628
			2.454	2.532	2.977	**3.852**

The RSS scheme has a small K-S statistic (K-S (stat)) and a large K-S (p-value), while the SRS scheme has a large K-S statistic (K-S (stat)) and a small K-S (p-value). The SEs of the estimates are calculated by taking the square root of the diagonals of variance of the estimated matrix of variances- covariances. The SE of the RSS scheme is smaller than that of the SRS scheme.

Table 17. MLEs of the parameters θ and λ of the GB(θ, λ) distribution with lower and upper bounds of CI, SEs, Kolmogorov-Smirnov (K-S) (stat), and (K-S) p-value under the SRS and RSS schemes when $n = 7$.

	SRS		**RSS $c = 1$**		**RSS $c = 2$**	
	$\boldsymbol{\theta}$	λ	$\boldsymbol{\theta}$	λ	$\boldsymbol{\theta}$	$\boldsymbol{\lambda}$
MLEs	2.2814	2.2814	3.4602	4.0823	3.3762	3.8009
SEs	0.6290	0.6290	0.1252	0.5895	0.1175	0.4895
Lower	1.0485	1.0485	3.2147	2.9269	3.1458	2.8415
Upper	3.5143	3.5143	3.7056	5.2377	3.6065	4.7603
K-S (Stat)	0.8388		0.1761		0.1716	
K-S (p-value)	0.0000		0.9556		0.9593	

The application results conclude that the RSS scheme is better than the SRS scheme.

5. Conclusions

The generalized Bilal (GB) distribution was presented by Abd-Elrahman [2] as a new flexible lifetime distribution. This study examined the order statistics from this novel distribution and derived explicit expressions for single and product moments of order statistics. We also computed the means, variances, and covariances of the order statistics for selected cases. The best linear unbiased (BLU) and best linear invariant (BLI) estimators of the scale and location-scale parameters of the GB distribution, as well as BLU and BLI predictors of future unobserved order statistics, might be developed using these findings; see, for example Ahsanullah and Alzaatreh [38], Akhter et al. [39] and Akhter et al. [40]. Work on these problems is currently in progress and hope to report these findings in a future paper. The maximum likelihood (ML) estimates of the unknown parameters of the GB distribution are obtained under both simple random sampling (SRS) and ranked set sampling (RSS) schemes. The 95% confidence intervals are also obtained. A simulation study as well as a real data example are given. The theoretical results and the numerical findings emphasize that the ML estimates under the RSS scheme are more efficient than the ML estimates under the SRS scheme.

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