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Bounds for the Neuman–Sándor Mean in Terms of the Arithmetic and Contra-Harmonic Means

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Abstract: In this paper, the authors provide several sharp upper and lower bounds for the Neuman–Sándor mean in terms of the arithmetic and contra-harmonic means, and present some new sharp inequalities involving hyperbolic sine function and hyperbolic cosine function.

Keywords: Neuman–Sándor mean; arithmetic mean; contra-harmonic mean; bound; inequality; hyperbolic sine function; hyperbolic cosine function

MSC: Primary 26E60; Secondary 26D07; 33B10; 41A30

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1. Introduction

In the literature, the quantities

$$A(s,t)=\frac{s+t}{2},\quad G(s,t)=\sqrt{st}\,,\quad H(s,t)=\frac{2st}{s+t},$$
$$\overline{C}(s,t)=\frac{2(s^2+st+t^2)}{3(s+t)},\qquad C(s,t)=\frac{s^2+t^2}{s+t},$$
$$S(s,t)=\sqrt{\frac{s^2+t^2}{2}}\,,\qquad M_p(s,t)=\begin{cases}\left(\dfrac{s^p+t^p}{2}\right)^{1/p}, & p\ne 0;\\ \sqrt{st}\,, & p=0\end{cases}$$

are called in [1–3], for example, the arithmetic mean, geometric mean, harmonic mean, centroidal mean, contra-harmonic mean, root-square mean, and the power mean of order p of two positive numbers s and t, respectively.

For $s,t>0$ with $s\ne t$, the first Seiffert means $P(s,t)$, the second Seiffert means $T(s,t)$, and Neuman–Sándor mean $M(s,t)$ are, respectively, defined [4–6] by

$$P(s,t)=\frac{s-t}{4\arctan\left(\sqrt{\frac{s}{t}}\right)-\pi},\qquad T(s,t)=\frac{s-t}{2\arctan\frac{s-t}{s+t}},\qquad M(s,t)=\frac{s-t}{2\operatorname{arsinh}\frac{s-t}{s+t}},$$

where $\operatorname{arsinh} v=\ln\left(v+\sqrt{v^2+1}\,\right)$ is the inverse hyperbolic sine function.

The first Seiffert mean $P(s,t)$ can be rewritten [6] (Equation (2.4)) as

$$P(s,t)=\frac{s-t}{2\arcsin\frac{s-t}{s+t}}.$$

A chain of inequalities

$$G(s,t)<L_{-1}(s,t)<P(s,t)<A(s,t)<M(s,t)<T(s,t)<Q(s,t)$$

were given in [6], where

$$L_p(s,t)=\begin{cases}\left[\dfrac{t^{p+1}-s^{p+1}}{(p+1)(t-s)}\right]^{1/p}, & p\ne -1,0;\\ \dfrac{1}{\mathrm{e}}\left(\dfrac{t^t}{s^s}\right)^{1/(t-s)}, & p=0;\\ \dfrac{t-s}{\ln t-\ln s}, & p=-1\end{cases}$$

is the p-th generalized logarithmic mean of s and t with $s\ne t$.

In [6,7], three double inequalities

$$A(s,t)<M(s,t)<T(s,t),\quad P(s,t)<M(s,t)<T^2(s,t),$$

and

$$A(s,t)T(s,t)<M^2(s,t)<\frac{A^2(s,t)+T^2(s,t)}{2}$$

were established for $s,t>0$ with $s\ne t$.

For $0<s,t<\frac{1}{2}$ with $s\ne t$, the inequalities

$$\frac{G(s,t)}{G(1-s,1-t)}<\frac{L_{-1}(s,t)}{L_{-1}(1-s,1-t)}<\frac{P(s,t)}{P(1-s,1-t)}<\frac{A(s,t)}{A(1-s,1-t)}<\frac{M(s,t)}{M(1-s,1-t)}<\frac{T(s,t)}{T(1-s,1-t)}$$

of Ky Fan type were presented in [6] (Proposition 2.2).

In [8], Li and their two coauthors showed that the double inequality

$$L_{p_0}(s,t)<M(s,t)<L_2(s,t)$$

holds for all $s,t>0$ with $s\ne t$ and for $p_0=1.843\dots$, where p_0 is the unique solution of the equation $(p+1)^{1/p}=2\ln\left(1+\sqrt{2}\right)$.

In [9], Neuman proved that the double inequalities

$$\alpha Q(s,t)+(1-\alpha)A(s,t)<M(s,t)<\beta Q(s,t)+(1-\beta)A(s,t)$$

and

$$\lambda C(s,t)+(1-\lambda)A(s,t)<M(s,t)<\mu C(s,t)+(1-\mu)A(s,t)$$

hold for all $s,t>0$ with $s\ne t$ if and only if

$$\alpha\le\frac{1-\ln\left(1+\sqrt{2}\right)}{\left(\sqrt{2}-1\right)\ln\left(1+\sqrt{2}\right)}=0.3249\dots,\quad \beta\ge\frac{1}{3}$$

and

$$\lambda\le\frac{1-\ln\left(1+\sqrt{2}\right)}{\ln\left(1+\sqrt{2}\right)}=0.1345\dots,\quad \mu\ge\frac{1}{6}.$$

In [10], (Theorems 1.1 to 1.3), it was found that the double inequalities

$$\alpha_1H(s,t)+(1-\alpha_1)Q(s,t)<M(s,t)<\beta_1H(s,t)+(1-\beta_1)Q(s,t),$$

$$\alpha_2G(s,t)+(1-\alpha_2)Q(s,t)<M(s,t)<\beta_2G(s,t)+(1-\beta_2)Q(s,t),$$

and

$$\alpha_3H(s,t)+(1-\alpha_3)C(s,t)<M(s,t)<\beta_3H(s,t)+(1-\beta_3)C(s,t)$$

hold for all $s,t>0$ with $s\neq t$ if and only if

$$\alpha_1 \geq \frac{2}{9} = 0.2222\ldots, \quad \beta_1 \leq 1 - \frac{1}{\sqrt{2}\,\ln(1+\sqrt{2})} = 0.1977\ldots,$$
$$\alpha_2 \geq \frac{1}{3} = 0.3333\ldots, \quad \beta_2 \leq 1 - \frac{1}{\sqrt{2}\,\ln(1+\sqrt{2})} = 0.1977\ldots,$$

and

$$\alpha_3 \geq 1 - \frac{1}{2\ln(1+\sqrt{2})} = 0.4327\ldots, \quad \beta_3 \leq \frac{5}{12} = 0.4166\ldots$$

In 2017, Chen and their two coauthors [11] established bounds for Neuman–Sándor mean $M(s,t)$ in terms of the convex combination of the logarithmic mean and the second Seiffert mean $T(s,t)$. In 2022, Wang and Yin [12] obtained bounds for the reciprocals of the Neuman–Sándor mean $M(s,t)$.

In [13], it was showed that the double inequality

$$\frac{\alpha}{A(s,t)} + \frac{1-\alpha}{\overline{C}(s,t)} < \frac{1}{TD(s,t)} < \frac{\beta}{A(s,t)} + \frac{1-\beta}{\overline{C}(s,t)} \tag{1}$$

holds for all $s,t>0$ with $s\neq t$ if and only if $\alpha \leq \pi - 3$ and $\beta \geq \frac{1}{4}$, where $TD(s,t)$ is the Toader mean introduced in [14] by

$$TD(s,t) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{s^2\cos^2\phi + t^2\sin^2\phi}\,\mathrm{d}\,\phi.$$

In this paper, motivated by the double inequality (1), we will aim to find out the largest values α_1, α_2, and α_3 and the smallest values β_1, β_2, and β_3 such that the double inequalities

$$\frac{\alpha_1}{C(s,t)} + \frac{1-\alpha_1}{A(s,t)} < \frac{1}{M(s,t)} < \frac{\beta_1}{C(s,t)} + \frac{1-\beta_1}{A(s,t)}, \tag{2}$$
$$\frac{\alpha_2}{C^2(s,t)} + \frac{1-\alpha_2}{A^2(s,t)} < \frac{1}{M^2(s,t)} < \frac{\beta_2}{C^2(s,t)} + \frac{1-\beta_2}{A^2(s,t)}, \tag{3}$$

and

$$\alpha_3 C^2(s,t) + (1-\alpha_3)A^2(s,t) < M^2(s,t) < \beta_3 C^2(s,t) + (1-\beta_3)A^2(s,t) \tag{4}$$

hold for all positive real numbers s and t with $s\neq t$.

2. Lemmas

To attain our main purposes, we need the following lemmas.

Lemma 1 ([15] (Theorem 1.25))**.** *For $-\infty < s < t < \infty$, let $f,g:[s,t]\to\mathbb{R}$ be continuous on $[s,t]$, differentiable on (s,t), and $g'(v)\neq 0$ on (s,t). If $\frac{f'(v)}{g'(v)}$ is (strictly) increasing (or (strictly) decreasing, respectively) on (s,t), so are the functions*

$$\frac{f(v)-f(s)}{g(v)-g(s)} \quad \text{and} \quad \frac{f(v)-f(t)}{g(v)-g(t)}.$$

Lemma 2 ([16] (Lemma 1.1))**.** *Suppose that the power series $f(v)=\sum_{\ell=0}^{\infty} u_\ell v^\ell$ and $g(v)=\sum_{\ell=0}^{\infty} w_\ell v^\ell$ have the convergent radius $r>0$ and $w_\ell>0$ for all $\ell\in\mathbb{N}=\{0,1,2,\ldots\}$. Let $h(v)=\frac{f(v)}{g(v)}$. Then the following statements are true.*

1. *If the sequence $\{\frac{u_\ell}{w_\ell}\}_{\ell=0}^{\infty}$ is (strictly) increasing (or decreasing, respectively), then $h(v)$ is also (strictly) increasing (or decreasing, respectively) on $(0,r)$.*
2. *If the sequence $\{\frac{u_\ell}{w_\ell}\}_{\ell=0}^{\infty}$ is (strictly) increasing (or decreasing resepctively) for $0<\ell\leq\ell_0$ and (strictly) decreasing (or increasing resepctively) for $\ell>\ell_0$, then there exists $x_0\in(0,r)$*

such that $h(v)$ is (strictly) increasing (decreasing) on $(0, x_0)$ and (strictly) decreasing (or increasing resepctively) on (x_0, r).

Lemma 3. *Let*

$$h_1(v) = \frac{2v\sinh v + \cosh v - 1}{3\sinh^2 v}.$$

Then $h_1(v)$ is strictly decreasing on $(0, \infty)$ with $\lim_{v\to 0^+} h_1(v) = \frac{5}{6}$ and $\lim_{v\to\infty} h_1(v) = 0$.

Proof. Let

$$f_1(v) = 2v\sinh v + \cosh v - 1 \quad \text{and} \quad f_2(v) = 3\sinh^2 v = \frac{3}{2}[\cosh(2v) - 1].$$

Using the power series

$$\sinh v = \sum_{\ell=0}^{\infty} \frac{v^{2\ell+1}}{(2\ell+1)!} \quad \text{and} \quad \cosh v = \sum_{\ell=0}^{\infty} \frac{v^{2\ell}}{(2\ell)!}, \tag{5}$$

we can express the functions $f_1(v)$ and $f_2(v)$ as

$$f_1(v) = \sum_{\ell=0}^{\infty} \frac{2(2\ell+2)! + (2\ell+1)!}{(2\ell+1)!(2\ell+2)!} v^{2\ell+2} \quad \text{and} \quad f_2(v) = \frac{3}{2}\sum_{\ell=0}^{\infty} \frac{2^{2\ell+2}v^{2\ell+2}}{(2\ell+2)!}.$$

Hence, we have

$$h_1(v) = \frac{\sum_{\ell=0}^{\infty} u_\ell v^{2\ell+2}}{\sum_{\ell=0}^{\infty} w_\ell v^{2\ell+2}}, \tag{6}$$

where $u_\ell = \frac{2(2\ell+2)!+(2\ell+1)!}{(2\ell+1)!(2\ell+2)!}$ and $w_\ell = \frac{3\times 2^{2\ell+1}}{(2\ell+2)!}$.

Let $c_\ell = \frac{u_\ell}{w_\ell}$. Then

$$c_\ell = \frac{2(2\ell+2)! + (2\ell+1)!}{3(2\ell+1)!2^{2\ell+1}} \quad \text{and} \quad c_{\ell+1} - c_\ell = -\frac{4(3\ell+4)(2\ell+2)! + 3(2\ell+3)!}{3(2\ell+3)!2^{2\ell+3}} < 0.$$

As a result, by Lemma 2, it follows that the function $h_1(v)$ is strictly decreasing on $(0, \infty)$. From (6), it is easy to see that $\lim_{v\to 0^+} h_1(v) = \frac{u_0}{w_0} = \frac{5}{6}$.

Using the L'Hospital rule leads to $\lim_{v\to\infty} h_1(v) = 0$ immediately. The proof of Lemma 3 is complete. □

Lemma 4. *Let*

$$h_2(v) = \frac{\left(\sinh^2 v - v^2\right)\cosh^4 v}{\left(\cosh^2 v + 1\right)\sinh^4 v}.$$

Then $h_2(v)$ is strictly increasing on $v \in (0, \infty)$ and has the limit $\lim_{v\to 0^+} h_2(v) = \frac{1}{6}$ and $\lim_{v\to\infty} h_2(v) = 1$.

Proof. Let

$$f_3(v) = \left(\sinh^2 v - v^2\right)\cosh^4 v \quad \text{and} \quad f_4(v) = \left(\cosh^2 v + 1\right)\sinh^4 v.$$

Since

$$f_3'(v) = 2\left(\sinh v + 3\sinh^3 v - v\cosh v - 2v^2\sinh v\right)\cosh^3 v$$

and

$$f_4'(v) = 2\left(3\cosh^2 v + 1\right)\sinh^3 v\cosh v,$$

we obtain

$$\begin{aligned}
\frac{f_3'(v)}{f_4'(v)} &= \frac{\left(\sinh v + 3\sinh^3 v - v\cosh v - 2v^2 \sinh v\right)\cosh^2 v}{\left(3\cosh^2 v + 1\right)\sinh^3 v} \\
&= \frac{\cosh^2 v}{3\cosh^2 v + 1}\frac{\sinh v + 3\sinh^3 v - v\cosh v - 2v^2 \sinh v}{\sinh^3 v} \\
&= \frac{1}{3 + \frac{1}{\cosh^2 v}}\left(3 + \frac{\sinh v - v\cosh v - 2v^2 \sinh v}{\sinh^3 v}\right) \\
&= \frac{1}{3 + \frac{1}{\cosh^2 v}}[3 + g(v)],
\end{aligned}$$

where

$$g(v) = \frac{\sinh v - v\cosh v - 2v^2 \sinh v}{\sinh^3 v}.$$

By using the identity that $\sinh(3v) = 3\sinh v + 4\sinh^3 v$, we arrive at

$$g(v) = 4\frac{\sinh v - v\cosh v - 2v^2 \sinh v}{\sinh(3v) - 3\sinh v} \triangleq 4\frac{g_1(v)}{g_2(v)},$$

where $g_1(v) = \sinh v - v\cosh v - 2v^2 \sinh v$ and $g_2(v) = \sinh(3v) - 3\sinh v$.

Straightforward computation gives

$$\begin{aligned}
g_1'(v) &= -\left(5v\sinh v + 2v^2\cosh v\right), & g_2'(v) &= 3[\cosh(3v) - \cosh v], \\
g_1''(v) &= -\left(5\sinh v + 9v\cosh v + 2v^2 \sinh v\right), & g_2''(v) &= 3[3\sinh(3v) - \sinh v],
\end{aligned}$$

and

$$g_1(0^+) = g_2(0^+) = g_1'(0^+) = g_2'(0^+) = g_1''(0^+) = g_2''(0^+) = 0.$$

Consequently, we obtain

$$\frac{g_1''(v)}{g_2''(v)} = -\frac{5\sinh v + 9v\cosh v + 2v^2 \sinh v}{3[3\sinh(3v) - \sinh v]} \triangleq -\frac{1}{3}\frac{g_3(v)}{g_4(v)}$$

Using the power series of $\sinh v$ and $\cosh v$, we deduce

$$\begin{aligned}
g_3(v) &= 5\sum_{\ell=0}^{\infty}\frac{v^{2\ell+1}}{(2\ell+1)!} + 9\sum_{\ell=0}^{\infty}\frac{v^{2\ell+1}}{(2\ell)!} + 2\sum_{\ell=0}^{\infty}\frac{v^{2\ell+3}}{(2\ell+1)!} \\
&= 14v + \sum_{\ell=1}^{\infty}\left[\frac{5}{(2\ell+1)!} + \frac{9}{(2\ell)!} + \frac{2}{(2\ell-1)!}\right]v^{2\ell+1} \\
&= 14v + \sum_{\ell=1}^{\infty}\left[\frac{(4\ell+7)(2\ell)! + 9(2\ell+1)!}{(2\ell)!(2\ell+1)!}\right]v^{2\ell+1}
\end{aligned}$$

and

$$g_4(v) = 3\sum_{\ell=0}^{\infty}\frac{(3v)^{2\ell+1}}{(2\ell+1)!} - \sum_{\ell=0}^{\infty}\frac{v^{2\ell+1}}{(2\ell+1)!} = \sum_{\ell=0}^{\infty}\left[\frac{3^{2\ell+2} - 1}{(2\ell+1)!}\right]v^{2\ell+1}.$$

Therefore, we find

$$\frac{g_3(v)}{g_4(v)} = \frac{\sum_{\ell=0}^{\infty} u_\ell v^{2\ell+1}}{\sum_{\ell=0}^{\infty} w_\ell v^{2\ell+1}},$$

where

$$u_\ell = \begin{cases} 14, & \ell = 0; \\ \dfrac{(4\ell+7)(2\ell)! + 9(2\ell+1)!}{(2\ell)!(2\ell+1)!}, & \ell \ge 1 \end{cases} \quad \text{and} \quad w_\ell = \begin{cases} 8 > 0, & \ell = 0; \\ \dfrac{3^{2\ell+2}-1}{(2\ell+1)!}, & \ell \ge 1. \end{cases}$$

Let $c_\ell = \frac{u_\ell}{w_\ell}$. Then

$$c_\ell = \begin{cases} \dfrac{7}{4}, & \ell = 0; \\ \dfrac{(4\ell+7)(2\ell)! + 9(2\ell+1)!}{\left(3^{2\ell+2}-1\right)(2\ell)!}, & \ell \ge 1. \end{cases}$$

When $\ell = 0$, we have $c_1 - c_0 = -\frac{51}{40} < 0$. When $\ell \ge 1$, it follows that

$$\begin{aligned} c_{\ell+1} - c_\ell &= \frac{(4\ell+11)(2\ell+2)! + 9(2\ell+3)!}{\left(3^{2\ell+4}-1\right)(2\ell+2)!} - \frac{(4\ell+7)(2\ell)! + 9(2\ell+1)!}{\left(3^{2\ell+2}-1\right)(2\ell)!} \\ &= \frac{1}{\left(3^{2\ell+2}-1\right)\left(3^{2\ell+4}-1\right)(2\ell+2)!}\big\{[(4\ell+11)(2\ell+2)! \\ &\quad + 9(2\ell+3)!]\left(3^{2\ell+2}-1\right) - [(4\ell+7)(2\ell)! + 9(2\ell+1)!](2\ell+2)\left(3^{2\ell+4}-1\right)\big\} \\ &= \frac{1}{\left(3^{2\ell+2}-1\right)\left(3^{2\ell+4}-1\right)(2\ell+2)!}\big\{[(4\ell+11)(2\ell+2)! \\ &\quad + 9(2\ell+3)!]3^{2\ell+2} - [(4\ell+7)(2\ell)! + 9(2\ell+1)!](2\ell+2)3^{2\ell+4} \\ &\quad + (2\ell+2)[(4\ell+7)(2\ell)! + 9(2\ell+1)!] - [(4\ell+11)(2\ell+2)! + 9(2\ell+3)!]\big\} \\ &= -\frac{1}{\left(3^{2\ell+2}-1\right)\left(3^{2\ell+4}-1\right)(2\ell+2)!}\big\{[(8\ell+13)(2\ell+2)! \\ &\quad + 9(16\ell+15)(2\ell+1)!]3^{2\ell+2} + 9(2\ell+1)! + 4(2\ell+2)!\big\} \\ &< 0. \end{aligned}$$

By Lemma 2, it follows that the function $\frac{g_3(v)}{g_4(v)}$ is strictly decreasing on $(0,\infty)$, so the function $\frac{g_1''(v)}{g_2''(v)}$ is strictly increasing on $(0,\infty)$. Applying Lemma 1, it follows that the function $g(v)$ is strictly increasing on $(0,\infty)$. By the L'Hospital rule, we have

$$\lim_{v\to 0^+} g(v) = -\frac{7}{3} \quad \text{and} \quad \lim_{v\to\infty} g(v) = 0.$$

It is common knowledge that the function $\cosh v$ is strictly increasing on $(0,\infty)$. Hence, the function $\frac{1}{3+\frac{1}{\cosh^2 v}}$ is strictly increasing on $(0,\infty)$. Therefore, the function $h_2(v)$ is strictly increasing on $(0,\infty)$ with the limits

$$\lim_{v\to 0} h_2(v) = \frac{1}{6} \quad \text{and} \quad \lim_{v\to\infty} h_2(v) = 1.$$

The proof of Lemma 3 is complete. □

Lemma 5. *Let*

$$h_3(v) = \frac{2v\cosh^2 v}{\sinh v}.$$

Then $h_3(v)$ is strictly increasing on $(0,\infty)$ and has the limit $\lim_{v\to 0^+} h_3(v) = 2$.

Proof. Let $k_1(v) = 2v\cosh^2 v = v\cosh(2v) + v$ and $k_2(v) = \sinh v$. By Equation (5), we have

$$k_1(v) = 2v + \sum_{\ell=1}^{\infty} \frac{2^{2\ell}}{(2\ell)!} v^{2\ell+1} \quad \text{and} \quad k_2(v) = \sum_{\ell=0}^{\infty} \frac{v^{2\ell+1}}{(2\ell+1)!}.$$

Hence,

$$h_3(v) = \frac{2v + \sum_{\ell=1}^{\infty} u_\ell v^{2\ell+1}}{\sum_{\ell=0}^{\infty} w_\ell v^{2\ell+1}}, \tag{7}$$

where

$$u_\ell = \begin{cases} 2, & \ell = 0; \\ \dfrac{2^{2\ell}}{(2\ell)!}, & \ell \geq 1 \end{cases} \quad \text{and} \quad w_\ell = \frac{1}{(2\ell+1)!}.$$

Let $c_\ell = \frac{u_\ell}{w_\ell}$. Then

$$c_\ell = \begin{cases} 2, & \ell = 0; \\ \dfrac{(2\ell+1)!2^{2\ell}}{(2\ell)!}, & \ell \geq 1 \end{cases} \quad \text{and} \quad c_{\ell+1} - c_\ell = \begin{cases} 10, & \ell = 0; \\ \dfrac{(3\ell+5)(2\ell+1)!2^{2\ell+1}}{(2\ell+2)!} > 0, & \ell \geq 1. \end{cases}$$

Thus, by Lemma 2, it follows that the function $h_3(v)$ is strictly increasing on $(0, \infty)$. From (7), it is easy to see that $\lim_{v \to 0^+} h_3(v) = \frac{u_0}{w_0} = 2$. The proof of Lemma 5 is complete. □

3. Bounds for Neuman–Sándor Mean

Now we are in a position to state and prove our main results.

Theorem 1. *For $s, t > 0$ with $s \neq t$, the double inequality (2) holds if and only if*

$$\alpha_1 \geq 2\left[1 - \ln\left(1 + \sqrt{2}\right)\right] = 0.237253\ldots \quad \textit{and} \quad \beta_1 \leq \frac{1}{6}.$$

Proof. Without loss of generality, we assume that $s > t > 0$. Let $q = \frac{s-t}{s+t}$. Then $q \in (0,1)$ and

$$\frac{\frac{1}{M(s,t)} - \frac{1}{A(s,t)}}{\frac{1}{C(s,t)} - \frac{1}{A(s,t)}} = \frac{\frac{\operatorname{arsinh} q}{q} - 1}{\frac{1}{1+q^2} - 1}.$$

Let $q = \sinh\phi$. Then $\phi \in \left(0, \ln\left(1 + \sqrt{2}\right)\right)$ and

$$\frac{\frac{1}{M(s,t)} - \frac{1}{A(s,t)}}{\frac{1}{C(s,t)} - \frac{1}{A(s,t)}} = \frac{\frac{\phi}{\sinh\phi} - 1}{\frac{1}{\cosh^2\phi} - 1} = \frac{(\sinh\phi - \phi)\cosh^2\phi}{\sinh^3\phi} \triangleq F(\phi) = \frac{k_1(\phi)}{k_2(\phi)}.$$

Let

$$k_1(\phi) = (\sinh\phi - \phi)\cosh^2\phi \quad \text{and} \quad k_2(\phi) = \sinh^3\phi.$$

Then elaborated computations lead to $k_1(0^+) = k_2(0^+) = 0$ and

$$\frac{k_1'(\phi)}{k_2'(\phi)} = \frac{2(\sinh\phi - \phi)\sinh\phi + (\cosh\phi - 1)\cosh\phi}{3\sinh^2\phi} = 1 - \frac{2\phi\sinh\phi + \cosh\phi - 1}{3\sinh^2\phi}.$$

Combining this with Lemmas 1 and 3 reveals that the function $F(\phi)$ is strictly increasing on $\left(0, \ln\left(1 + \sqrt{2}\right)\right)$. Moreover, it is easy to compute the limits

$$\lim_{\phi \to 0^+} F(\phi) = \frac{1}{6} \quad \text{and} \quad \lim_{\phi \to \ln(1+\sqrt{2})^-} F(\phi) = 2 - 2\ln\left(1 + \sqrt{2}\right).$$

The proof of Theorem 1 is thus complete. □

Corollary 1. *For all $\phi \in \left(0, \ln\left(1+\sqrt{2}\right)\right)$, the double inequality*

$$1-\beta_1\left(1-\frac{1}{\cosh^2\phi}\right)<\frac{\phi}{\sinh\phi}<1-\alpha_1\left(1-\frac{1}{\cosh^2\phi}\right) \tag{8}$$

holds if and only if

$$\alpha_1 \le \frac{1}{6} \quad \text{and} \quad \beta_1 \ge 2\left[1-\ln\left(1+\sqrt{2}\right)\right]=0.237253\dots.$$

Theorem 2. *For $s,t>0$ with $s\ne t$, the double inequality (3) holds if and only if*

$$\alpha_2 \ge \frac{4}{3}[1-\ln^2(1+\sqrt{2})]=0.297574\dots \quad \text{and} \quad \beta_2 \le \frac{1}{6}.$$

Proof. Without loss of generality, we assume that $s>t>0$. Let $q=\frac{s-t}{s+t}$. Then $q\in(0,1)$ and

$$\frac{\frac{1}{M^2(s,t)}-\frac{1}{A^2(s,t)}}{\frac{1}{C^2(s,t)}-\frac{1}{A^2(s,t)}}=\frac{\frac{\operatorname{arsinh}^2 q}{q^2}-1}{\frac{1}{(1+q^2)^2}-1}.$$

Let $q=\sinh\phi$. Then $\phi\in\left(0,\ln\left(1+\sqrt{2}\right)\right)$ and

$$\frac{\frac{1}{M^2(s,t)}-\frac{1}{A^2(s,t)}}{\frac{1}{C^2(s,t)}-\frac{1}{A^2(s,t)}}=\frac{\frac{\phi^2}{\sinh^2\phi}-1}{\frac{1}{\cosh^4\phi}-1}=\frac{(\sinh^2\phi-\phi^2)\cosh^4\phi}{(\cosh^2\phi+1)\sinh^4\phi}\triangleq H(\phi).$$

By Lemma 4, it is easy to show that $H(\phi)$ is strictly increasing on $\left(0,\ln\left(1+\sqrt{2}\right)\right)$. Moreover, the limits

$$\lim_{\phi\to0^+}H(\phi)=\frac{1}{6} \quad \text{and} \quad \lim_{\phi\to\ln(1+\sqrt{2})^-}H(\phi)=\frac{4}{3}\left[1-\ln^2\left(1+\sqrt{2}\right)\right]$$

can be computed readily. The double inequality (3) is thus proved. □

Corollary 2. *For all $\phi \in \left(0, \ln\left(1+\sqrt{2}\right)\right)$, the double inequality*

$$1-\beta_2\left(1-\frac{1}{\cosh^4\phi}\right)<\left(\frac{\phi}{\sinh\phi}\right)^2<1-\alpha_2\left(1-\frac{1}{\cosh^4\phi}\right) \tag{9}$$

holds if and only if

$$\alpha_2 \le \frac{1}{6} \quad \text{and} \quad \beta_2 \ge \frac{4}{3}\left[1-\ln^2\left(1+\sqrt{2}\right)\right]=0.297574\dots.$$

Theorem 3. *For $s,t>0$ with $s\ne t$, the double inequality (4) holds if and only if*

$$\alpha_3 \le \frac{1-\ln^2\left(1+\sqrt{2}\right)}{3\ln^2\left(1+\sqrt{2}\right)}=0.095767\dots \quad \text{and} \quad \beta_3 \ge \frac{1}{6}.$$

Proof. Without loss of generality, we assume that $s>t>0$. Let $q=\frac{s-t}{s+t}$. Then $q\in(0,1)$ and

$$\frac{M^2(s,t)-A^2(s,t)}{C^2(s,t)-A^2(s,t)}=\frac{\frac{q^2}{\operatorname{arsinh}^2 q}-1}{(1+q^2)^2-1}.$$

Let $q = \sinh\phi$. Then $\phi \in \big(0, \ln(1+\sqrt{2})\big)$ and

$$\frac{M^2(s,t) - A^2(s,t)}{C^2(s,t) - A^2(s,t)} = \frac{\frac{\sinh^2\phi}{\phi^2} - 1}{\cosh^4\phi - 1} \triangleq G(\phi) = \frac{k_1(\phi)}{k_2(\phi)},$$

where

$$k_1(\phi) = \frac{\sinh^2\phi}{\phi^2} - 1 \quad \text{and} \quad k_2(\phi) = \cosh^4\phi - 1.$$

Then $k_1(0^+) = k_2(0^+) = 0$ and

$$\frac{k_1'(\phi)}{k_2'(\phi)} = \frac{\phi\cosh\phi - \sinh\phi}{2\phi^3\cosh^3\phi}.$$

Denote

$$k_3(\phi) = \phi\cosh\phi - \sinh\phi \quad \text{and} \quad k_4(\phi) = 2\phi^3\cosh^3\phi,$$

it is easy to obtain $k_3(0^+) = k_4(0^+) = 0$ and

$$\frac{k_4'(\phi)}{k_3'(\phi)} = \frac{6\phi\cosh^2\phi}{\sinh\phi} + 6\phi^2\cosh^2\phi. \tag{10}$$

Since the function $v^2\cosh^2 v$ is strictly increasing on $(0,\infty)$, by Lemma 5, we see that the ratio in (10) is strictly increasing and $\frac{k_3'(\phi)}{k_4'(\phi)}$ is strictly decreasing on $\big(0, \ln(1+\sqrt{2})\big)$. Consequently, from Lemma 1, it follows that $G(\phi)$ is strictly decreasing on $\big(0, \ln(1+\sqrt{2})\big)$.

The limits

$$\lim_{\phi\to 0^+} G(\phi) = \frac{1}{6} \quad \text{and} \quad \lim_{\phi\to \ln(1+\sqrt{2})^-} G(\phi) = \frac{1-\ln^2(1+\sqrt{2})}{3\ln^2(1+\sqrt{2})}$$

can be computed easily. The proof of Theorem 3 is thus complete. □

Corollary 3. *For all $\phi \in \big(0, \ln(1+\sqrt{2})\big)$, the double inequality*

$$1 + \alpha_3\big(\cosh^4\phi - 1\big) < \left(\frac{\sinh\phi}{\phi}\right)^2 < 1 + \beta_3\big(\cosh^4\phi - 1\big) \tag{11}$$

holds if and only if

$$\alpha_3 \le \frac{1-\ln^2(1+\sqrt{2})}{3\ln^2(1+\sqrt{2})} = 0.095767\ldots \quad \textit{and} \quad \beta_3 \ge \frac{1}{6}.$$

4. A Double Inequality

From Lemma 5, we can deduce

$$\frac{\sinh v}{v} < \cosh^2 v \quad \text{and} \quad \frac{\sinh v}{v} > \frac{\tanh^2 x}{v^2} \tag{12}$$

for $v \in (0,\infty)$. The inequality

$$\left(\frac{\sinh v}{v}\right)^3 > \cosh v \tag{13}$$

for $v \in (0, \infty)$ can be found and has been applied in [17] (p. 65), [18] (p. 300), [19] (pp. 279, 3.6.9), and [20] (p. 260). In [21], (Lemma 3), Zhu recovered the fact stated in [19] (pp. 279, 3.6.9) that the exponent 3 in the inequality (13) is the least possible, that is, the inequality

$$\left(\frac{\sinh v}{v}\right)^p > \cosh v \tag{14}$$

for $x > 0$ holds if and only if $p \leq 3$.

Inspired by (12) and (14), we find out the following double inequality.

Theorem 4. *The inequality*

$$\cosh^\alpha v < \frac{\sinh v}{v} < \cosh^\beta v \tag{15}$$

for $v \neq 0$ holds if and only if $\alpha \leq \frac{1}{3}$ and $\beta \geq 1$.

Proof. Let

$$h(v) = \frac{\ln \sinh v - \ln v}{\ln \cosh v} \triangleq \frac{f_1(v)}{f_2(v)}.$$

Direct calculation yields

$$\frac{f_1'(v)}{f_2'(v)} = \frac{v\cosh^2 v - \sinh v \cosh v}{v \sinh^2 v} = \frac{v\cosh(2v) + v - \sinh(2v)}{v\cosh(2v) - v} \triangleq \frac{f_3(v)}{f_4(v)}.$$

Using the power series of $\sinh v$ and $\cosh v$, we obtain

$$\begin{aligned} f_3(v) &= v + v\sum_{\ell=0}^{\infty}\frac{(2v)^{2\ell}}{(2\ell)!} - \sum_{\ell=0}^{\infty}\frac{(2v)^{2\ell+1}}{(2\ell+1)!} = \sum_{\ell=1}^{\infty}\left[\frac{2^{2\ell}}{(2\ell)!} - \frac{2^{2\ell+1}}{(2\ell+1)!}\right]v^{2\ell+1} \\ &= \sum_{\ell=0}^{\infty}\frac{(2\ell+1)2^{2\ell+2}}{(2\ell+3)!}v^{2\ell+3} \triangleq \sum_{\ell=0}^{\infty} u_\ell v^{2\ell+3} \end{aligned}$$

and

$$f_4(v) = v\sum_{\ell=0}^{\infty}\frac{(2v)^{2\ell}}{(2\ell)!} - v = \sum_{\ell=1}^{\infty}\frac{2^{2\ell}}{(2\ell)!}v^{2\ell+1} = \sum_{\ell=0}^{\infty}\frac{2^{2\ell+2}}{(2\ell+2)!}v^{2\ell+3} \triangleq \sum_{\ell=0}^{\infty} w_\ell v^{2\ell+3},$$

where

$$u_\ell = \frac{(2\ell+1)2^{2\ell+2}}{(2\ell+3)!} \quad \text{and} \quad w_\ell = \frac{2^{2\ell+2}}{(2\ell+2)!}.$$

When setting $c_\ell = \frac{u_\ell}{w_\ell}$, we obtain

$$c_\ell = \frac{2\ell+1}{2\ell+3} = 1 - \frac{2}{2\ell+3}$$

is increasing on $\ell \in \mathbb{N}$. Therefore, by Lemma 2, the ratio $\frac{f_3(v)}{f_4(v)}$ is increasing on $(0, \infty)$. Using Lemma 1, we obtain that

$$h(v) = \frac{f_1(v)}{f_2(v)} = \frac{f_1(v) - f_1(0^+)}{f_2(v) - f_2(0^+)}$$

is increasing on $(0, \infty)$.

Moreover, the limits $\lim_{v\to 0^+} h_1 = \frac{1}{3}$ and $\lim_{v\to\infty} h_1 = 1$ are obvious. The proof of Lemma 4 is thus complete. □

5. A Remark

For $v, r \in \mathbb{R}$, we have

$$\left(\frac{\sinh v}{v}\right)^r = 1 + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{2m}\frac{(-r)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}}\right]\frac{(2v)^{2m}}{(2m)!}, \tag{16}$$

where the rising factorial $(r)_k$ is defined by

$$(r)_k = \prod_{\ell=0}^{k-1}(r+\ell) = \begin{cases} r(r+1)\cdots(r+k-1), & k \geq 1 \\ 1, & k = 0 \end{cases}$$

and $T(2m+j, j)$ is called central factorial numbers of the second kind and can be computed by

$$T(n,\ell) = \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\left(\frac{\ell}{2}-j\right)^n.$$

for $n \geq \ell \geq 0$.

The series expansion (16) was recently derived in [22] (Corollary 4.1).

Can one find bounds of the function $\left(\frac{\sinh v}{v}\right)^r$ for $v, r \in \mathbb{R} \setminus \{0\}$?

6. Conclusions

In this paper, we found out the largest values α_1, α_2, α_3 and the smallest values β_1, β_2, β_3 such that the double inequalities (2), (3), and (4) hold for all positive real number $s, t > 0$ with $s \neq t$. Moreover, we presented some new sharp inequalities (8), (9), (11), and (15) involving the hyperbolic sine function $\sinh \phi$ and the hyperbolic cosine function $\cosh \phi$.

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References

1. Jiang, W.-D.; Qi, F. Sharp bounds for Neuman–Sándor's mean in terms of the root-mean-square. *Period. Math. Hung.* **2014**, *69*, 134–138. [CrossRef]
2. Jiang, W.-D.; Qi, F. Sharp bounds in terms of the power of the contra-harmonic mean for Neuman–Sándor mean. *Cogent Math.* **2015**, *2*, 995951. [CrossRef]
3. Qi, F.; Li, W.-H. A unified proof of inequalities and some new inequalities involving Neuman–Sándor mean. *Miskolc Math. Notes* **2014**, *15*, 665–675. [CrossRef]
4. Seiffert, H.-J. Problem 887. *Nieuw Arch. Wiskd.* **1993**, *11*, 176.
5. Seiffert, H.-J. Aufgabe β 16. *Wurzel* **1995**, *29*, 221–222.
6. Neuman, E.; Sándor, J. On the Schwab–Borchardt mean. *Math. Pannon.* **2003**, *14*, 253–266.
7. Neuman, E.; Sándor, J. On the Schwab–Borchardt mean II. *Math. Pannon.* **2006**, *17*, 49–59.
8. Li, Y.-M.; Long, B.-Y.; Chu, Y.-M. Sharp bounds for the Neuman–Sándor mean in terms of generalized logarithmic mean. *J. Math. Inequal.* **2012**, *6*, 567–577. [CrossRef]
9. Neuman, E. A note on a certain bivariate mean. *J. Math. Inequal.* **2012**, *6*, 637–643. [CrossRef]
10. Zhao, T.-H.; Chu, Y.-M.; Liu, B.-Y. Optimal bounds for Neuman–Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic, and contraharmonic means. *Abstr. Appl. Anal.* **2012**, *2012*, 302635. [CrossRef]

11. Chen, J.-J.; Lei, J.-J.; Long, B.-Y. Optimal bounds for Neuman–Sándor mean in terms of the convex combination of the logarithmic and the second Seiffert means. *J. Inequal. Appl.* **2017**, *2017*, 251. [CrossRef] [PubMed]
12. Wang, X.-L.; Yin, L. Sharp bounds for the reciprocals of the Neuman–Sándor mean. *J. Interdiscip. Math.* **2022**.
13. Hua, Y.; Qi, F. The best bounds for Toader mean in terms of the centroidal and arithmetic means. *Filomat* **2014**, *28*, 775–59780. [CrossRef]
14. Toader, G. Some mean values related to the arithmetic-geometric mean. *J. Math. Anal. Appl.* **1998**, *218*, 358–368. [CrossRef]
15. Anderson, G.D.; Vamanamurthy, M.K.; Vuorinen, M. *Conformal Invariants, Inequalities, and Quasiconformal Maps*; John Wiley & Sons: New York, NY, USA, 1997,
16. Simić, S.; Vuorinen, M. Landen inequalities for zero-balanced hypergeometric functions. *Abstr. Appl. Anal.* **2012**, *2012*, 932061. [CrossRef]
17. Guo, B.-N.; Qi, F. The function $(b^x - a^x)/x$: Logarithmic convexity and applications to extended mean values. *Filomat* **2011**, *25*, 63–73. [CrossRef]
18. Kuang, J.C. *Applied Inequalities*, 3rd ed.; Shangdong Science and Technology Press: Jinan, China, 2004; pp. 300–301.
19. Mitrinović, D.S. *Analytic Inequalities*; Springer: Berlin/Heidelberg, Germany, 1970.
20. Qi, F. On bounds for norms of sine and cosine along a circle on the complex plane. *Kragujevac J. Math.* **2024**, *48*, 255–266. [CrossRef]
21. Zhu, L. On Wilker-type inequalities. *Math. Inequal. Appl.* **2007**, *10*, 727–731. [CrossRef]
22. Qi, F.; Taylor, P. Several series expansions for real powers and several formulas for partial Bell polynomials of sinc and sinhc functions in terms of central factorial and Stirling numbers of second kind. *arXiv* **2022**, arxiv:2204.05612v4.