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Edge Neighbor Toughness of Graphs

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Abstract: A new graph parameter, edge neighbor toughness is introduced to measure how difficult it is for a graph to be broken into many components through the deletion of the closed neighborhoods of a few edges. Let $G = (V, E)$ be a graph. An edge e is said to be subverted when its neighborhood and the two endvertices are deleted from G. An edge set $S \subseteq E(G)$ is called an edge cut-strategy if all the edges in S has been subverted from G and the survival subgraph, denoted by G/S, is disconnected, or is a single vertex, or is . The edge neighbor toughness of a graph G is defined to be $t_{EN}(G) = \min\limits_{S \subseteq E(G)} \{\frac{|S|}{c(G/S)}\}$, where S is any edge cut strategy of G, $c(G/S)$ is the number of the components of G/S. In this paper, the properties of this parameter are investigated, and the proof of the computation problem of edge neighbor toughness is NP-complete; finally, a polynomial algorithm for computing the edge neighbor toughness of trees is given.

Keywords: graph; edge neighbor toughness; NP-complete; polynomial algorithm; tree

MSC: 05C85; 68R10

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1. Introduction

Gunther and Hartnell [1] introduced the idea of modeling a spy network by a graph whose vertices represent the spies and whose edges represent their connection. If a spy is arrested, the spies who are in direct contact with them are unreliable; therefore, some new graphical parameters such as vertex neighbor connectivity [1] and edge neighbor connectivity [2] were introduced to measure the invulnerability of networks under the "neighbor" case.

Observing that the behavior of spies in a spy network is similar to the spread of biological viruses in social networks, we introduced edge neighbor scattering number (ENS) [3] and vertex neighbor scattering number (VNS) [4]. It is shown that they are alternative invulnerability measures of the above networks. Since removing some vertices (or edges) from a graph, all of their adjacent vertices (or edges) are removed simultaneously, we call ENS and VNS neighbor invulnerability parameters.

Let $G = (V, E)$ be a graph and $e \in V(G)$. The *open-edge neighborhood* of e is defined $N(e) = \{f \in E(G) | f \neq e, e$ and f are adjacent$\}$. The *closed-edge neighborhood* of e is $N[e] = N(e) \cup \{e\}$. An edge e is said to be *subverted* when $N[e]$ and the two endvertices of e are deleted from G. We call S an *edge subversion strategy* of G if $S \subseteq E(G)$ and each of the edges in S is subverted from G. The survival subgraph is denoted by G/S. An edge subversion strategy S is called an *edge cut strategy* of G if G/S is disconnected, or is a single vertex, or is .

The *edge neighbor scattering number* of a connected graph G is defined as [3] $ENS(G) = \max\limits_{S \subseteq E(G)} \{c(G/S) - |S|\}$, where S is any edge cut strategy of G, and $c(G/S)$ is the number of the components of G/S. We call $S^* \subseteq E(G)$ a ENS-*set* of G if $ENS(G) = c(G/S^*) - |S^*|$.

Inspired by the definitions of vertex neighbor connectivity and toughness, we define *edge neighbor toughness*(ENT) of a connected graph G as $t_{EN}(G) = \min_{S \subseteq E(G)} \{\frac{|S|}{c(G/S)}\}$, where S is any edge cut strategy of G, and $c(G/X)$ is the number of the components of G/S. We call $S^*(\subseteq E(G))$ an t_{EN}*-set* of G if $t_{EN}(G) = \frac{|S^*|}{c(G/S^*)}$.

In this paper, we prove that the computation problem of edge neighbor toughness of a graph is NP-complete. We also give a polynomial algorithm of the ENT of trees, which is a class of special and important graphs. Throughout this paper, we consider the simple, undirected graphs, and use Bondy and Murty [5] for terminologies and notations not defined here.

2. Preliminaries

Clearly, it is of prime importance to determine the edge neighbor toughness of a graph when this parameter is used to measure the neighbor invulnerability of a network. In this section, we give the edge neighbor toughness of several basic graphs.

Theorem 1. *Let P_n be a path of order $n(\geq 3)$. Then*

$$t_{EN}(P_n) = \begin{cases} 1, & \text{if } n = 3; \\ \frac{1}{2}, & \text{if } n \geq 4. \end{cases}$$

Proof. The case $n = 3$ is trivial. When $n \geq 4$, for any edge cut strategy S, $c(P_n/S) \leq |S| + 1$. We thus have $t_{EN}(P_n) \geq \frac{|S|}{|S|+1}$. Let $f(x) = \frac{x}{x+1}$ be a function of variate x, where $x \in Z_+$. When $x = 1$, $f(x)$ reaches its minimum value $\frac{1}{2}$. So, we have $t_{EN}(P_n) \geq \frac{1}{2}$.

On the other hand, let $e = uv$ be an edge of P_n such that $d(u) = d(v) = 2$. Then $\{e\}$ is an edge cut strategy of P_n and $c(P_n/\{e\}) = 2$. By the definition of ENT, we have $t_{EN}(P_n) \leq \frac{|\{e\}|}{c(P_n/\{e\})} = \frac{1}{2}$.

Therefore, $t_{EN}(P_n) = \frac{1}{2}$. □

Theorem 2. *Let C_n be a cycle of order $n(\geq 3)$. Then*

$$t_{EN}(C_n) = \begin{cases} 2, & \text{if } n = 4, 5; \\ 1, & \text{if } n = 3 \text{ or } n \geq 6. \end{cases}$$

Proof. $n = 3, 4, 5$ is trivial. When $n \geq 6$, for any edge cut strategy S of C_n, $c(C_n/S) \leq |S|$, we have $t_{EN}(C_n) \geq 1$.

On the other hand, there must exist two edges $e, f \in V(C_n)$ such that $\{e, f\}$ is an edge cut strategy of C_n and $c(C_n/\{e, f\}) = 2$. By the definition of ENT, we have $t_{EN}(C_n) \leq \frac{|\{e,f\}|}{c(C_n/\{e,f\})} = 1$.

Therefore, $t_{EN}(C_n) = 1$. □

Theorem 3. *Let K_n be a complete graph of order $n(\geq 3)$. Then $t_{EN}(K_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. Observe that for any $e \in E(K_n)$, $K_n/\{e\} = K_{n-2}$. Cozzens [6] proved that the edge neighbor connectivity of K_n is $\lfloor \frac{n}{2} \rfloor$; therefore, if S is an edge cut strategy of K_n, then $|S| \geq \lfloor \frac{n}{2} \rfloor$, and $c(K_n/S) \leq 1$. By the definition of ENT, we have $t_{EN}(K_n) \geq \frac{|S|}{c(K_n/S)} \geq \lfloor \frac{n}{2} \rfloor$.

On the other hand, if n is even, let $M = \{u_1v_1, u_2v_2, \ldots, u_{\frac{n}{2}}v_{\frac{n}{2}}\}$ be a maximum matching of K_n. Replace u_1v_1 by u_1v_2, denote $M' = (M - \{u_1v_1\}) \cup \{u_1v_2\}$. Then $K_n/M' = v_1$. By the definition of ENT, we have $t_{EN}(K_n) \leq \frac{|M'|}{c(K_n/M')} = \lfloor \frac{n}{2} \rfloor$. If n is odd, then there exists a maximum matching M in K_n such that $|M| = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$ and K_n/M is an isolated vertex. So we have $t_{EN}(K_n) \leq \lfloor \frac{n}{2} \rfloor$, too.

Therefore, $t_{EN}(K_n) = \lfloor \frac{n}{2} \rfloor$. □

Theorem 4. *Let $K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with vertex partition $(X_1, X_2, \ldots, X_p)$. Assume $|X_i| = n_i, i = 1, 2, \ldots, p$, $p \geq 2$ and $n_p \geq n_{p-1} \geq \ldots \geq n_1$. Then*

$$t_{VN}(K_{n_1,n_2,\ldots,n_p}) = \frac{1}{n_p}(\sum_{i=1}^{p-1} n_i - \alpha').$$

where α' is the matching number of $K_{n_1,n_2,\ldots,n_{p-1}}$.

Proof. For convenience, denote $K_{n_1,n_2,\ldots,n_p}$ as G. Obviously, for any edge cut strategy, S, of G, if $G/S \neq$, then there exists some i such that all the vertices of G/S are included in N_i. Let S^* be a t_{EN}-set of G. It is not difficult to know that $c(G/S^*) = n_p$ and $S^* \subset E(G - X_p)$ such that $(G - X_p)/S^* =$.

Let M^* be a maximum matching of $G - X_p$(that is, $K_{n_1,n_2,\ldots,n_{p-1}}$), and $U = \{u_1, u_2, \ldots, u_k\}$ be the unsaturated vertex set under M^*. Then $\{u_1v_1, u_2v_2, \ldots, u_kv_k\} \cup M^*$ is a t_{EN}-set of G, where $v_1, v_2, \ldots, v_k$ are arbitrary k vertices in $V(G) - X_p - U$. Since the unsaturated vertex number in $G - X_p$ is $\sum_{i=1}^{p-1} n_i - 2|M^*|$, we have $t_{VN}(K_{n_1,n_2,\ldots,n_p}) = \frac{1}{n_p}(\sum_{i=1}^{p-1} n_i - \alpha')$. □

Remark 1. *If $K_{n_1,n_2,\ldots,n_{p-1}}$ has a perfect matching, then $t_{VN}(K_{n_1,n_2,\ldots,n_p}) = \frac{n}{2n_p} - \frac{1}{2}$, where $n = \sum_{i=1}^{p} n_i$ and $n_p \geq n_{p-1} \geq \ldots \geq n_1$.*

Example 1. *Let $S_{1,n-1}$ be a star of order n. Then $t_{EN}(S_{1,n-1}) = \frac{1}{n-2}$.*

A comet, denoted by $C_{n,k}$, is a graph by coinciding an end vertex of path P_{n-k} with the center vertex of a star $S_{1,k}$, where $1 \leq k \leq n - 2$ and $n \geq 4$. The order of comet $C_{n,k}$ is n, and the center of $S_{1,k}$ is called the center of $C_{n,k}$.

Theorem 5. *Let $C_{n,k}$ be a comet, where $1 \leq k \leq n - 3$ and $n \geq 4$. Then $t_{EN}(C_{n,k}) = \frac{1}{k+1}$.*

Proof. Let $V(P_{n-k}) = \{v_1, v_2, \ldots, v_{n-k}\}$, and v_1 be the center of $C_{n,k}$. Obviously, if S is an edge cut strategy of $C_{n,k}$, then $|S| \geq 1$. Moreover, for any edge cut strategy S, $c(C_{n,k}/S) \leq k + |S|$ and the function $f(x) = \frac{x}{x+k}(x \in Z_+)$ is decreased with x, we then have $t_{EN}(C_{n,k}) \geq \frac{1}{k+1}$.

On the other hand, since $\{v_1v_2\}$ is an edge cut strategy of $C_{n,k}$, and $c(C_{n,k}/\{v_1v_2\}) = k + 1$, by the definition of ENT, we have $t_{EN}(C_{n,k}) \leq \frac{1}{k+1}$.

Therefore, $t_{EN}(C_{n,k}) = \frac{1}{k+1}$, and we complete the proof. □

3. The Main Result

In this section, we consider the computational problems of ENT. We prove that the problem of computing the edge neighbor toughness is NP-complete and give a polynomial algorithm for computing the edge neighbor toughness of trees.

Problem 1. **EDGE NEIGHBOR TOUGHNESS**
Instance: An undirected graph G; a positive rational number t.
Question: Does there exist an edge cut strategy S of G such that $\frac{|S|}{c(G/S)} \leq t$?

We solve this problem by considering the following

Problem 2. **EDGE DOMINATION SET**
Instance: A bipartite graph G; a positive integer d.
Question: Does there exist an edge dominating set D of G such that $|D| \leq d$?

It was proved by Yannakakis and Gavril [7] that Problem 2 is **NP**-complete. Based on this conclusion, we prove that Problem 1 is **NP**-complete by reducing Problem 2 to a special case of Problem 1.

Theorem 6. *EDGE NEIGHBOR TOUGHNESS is* **NP***-complete.*

Proof. Clearly, EDGE NEIGHBOR TOUGHNESS is in the class **NP**, since a nondeterministic algorithm need only guess an edge cut strategy $S \subseteq E(G)$ and check in polynomial time that $\frac{|S|}{c(G/S)} \leq t$.

Let $G = (V, E)$ be a bipartite graph of order n. Denote $V = \{v_1, v_2, \ldots, v_n\}$. Replace each vertex $v_i \in V$ by a copy of the complete graph K_n, denote this copy by G_i. Choose a vertex from G_i, v_i^*, add edges $v_i^* v_j^*$ if $v_i v_j \in E$, $i = 1, 2, \ldots, n$. Denote the resulting graph by G^*.

Denote the subgraph induced by $\{v_1^*, v_2^*, \ldots, v_n^*\}$ in G^* as G'. Obviously, $G' \cong G$. Let D be a smallest edge dominating set of G and S^* be an ENS-set of G^*. In [8], we proved the following claims.

Fact 1. *If e is an edge in G_i, which is not incident with v_i^*, then $e \notin S^*$, $i = 1, 2, \ldots, n$.*

Fact 2. *Let $E_i^* = \{e : e \in E(G_i)$ and e be incident with $v_i^*\}$. Then $|E_i^* \cap S^*| \leq 1$, $i = 1, 2, \ldots, n$.*

Fact 3. *There exists a t_{EN}-set S of G^* such that $E(G^*/S) \cap E(G') =$ and $S \subseteq E(G')$.*

From the above claims, we conclude that $t_{EN}(G^*) = \frac{|D|}{n}$. By the construction of G^* and the NP-completeness of Problem 2, this is sufficient for the conclusion. □

Theorem 7. *Let T be a tree with order* n*($\geq$4). Then $t_{EN}(T) = \frac{1}{d-2}$, where $d = \max_{(u,v) \in E(T)} \{d(u) + d(v)\}$.*

Proof. Let $e^* = u^* v^* \in E(T)$ such that $d = d(u^*) + d(v^*) = \max_{(u,v) \in E(T)} \{d(u) + d(v)\}$. Since $n \geq 4$, $\{e^*\}$ is an edge cut strategy of T and $\frac{|\{e^*\}|}{c(T/\{e^*\})} = \frac{1}{d-2}$. Assume that $S \subset E(T)$ is any nonempty edge cut strategy of T. By the selection of e^*, we have

$$\frac{|S|}{c(T/S)} \geq \frac{|S|}{\sum\limits_{e \in S} c(T/\{e\}} \geq \frac{|S|}{|S|\omega(T/\{e^*\})} = \frac{1}{d-2}.$$

Therefore, by the definition of ENT, $\{e^*\}$ is a t_{EN}-set of T, and $t_{EN}(T) = \frac{1}{d-2}$. The proof is completed. □

Let T be a tree of order $n(\geq 4)$, $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $M(T) = (m_{ij})_{n \times (n-1)}$ be its incident matrix. Then $d(v_i) = \sum\limits_{j=1}^{n-1} m_{ij}$, $i = 1, 2, \ldots, n$.

Based on the above results, we give an algorithm for the problem of computing the ENT of trees.

Input A tree $T = (V; E)$.
Output $t_{EN}(T)$ and the corresponding t_{EN}-set.
Step 1. Compute the degree $d(v)$ for each vertex $v \in V(T)$;
Step 2. Compute $d_e = d(u) + d(v)$ for each edge $e = uv \in E(T)$;
Step 3. Search an edge $e^* = u^* v^*$ satisfying $d = d(u^*) + d(v^*) = \max_{e \in E(T)} d_e$, output $t_{EN}(T) = \frac{1}{d-2}$ and a t_{EN}-set $\{u^* v^*\}$.

A graph-theoretic algorithm is good if the number of computational steps for its implementation on any graph is bounded above by a polynomial in the order and the size of the graph [5]. We show that the above algorithm is good by the following complexity analysis.

Computing the degree for each vertex requires $n-2$ additions, so the computations involved in step 1 require $n(n-2)$ additions. Computing $d_e = d(u) + d(v)$ for an edge

$uv \in E(T)$ requires 1 additions only. Since T is a tree, $m = n - 1$, step 2 requires $(n-1)$ additions. To find the edge $e^* = u^*v^*$, it is sufficient to sort the $n-1$ number d_e; therefore, step 3, in the worst case, requires $(n-1)\log(n-1)$ comparisons (Quick Sort Algorithm).

It is well known that a comparison or an addition is a basic computational unit. So, the total number of computations of this algorithm is approximately $n(n-2) + (n-1) + (n-1)\log(n-1)$, and thus is of order n^2.

Remark 2. *If the adjacency list is used instead of the incidence matrix, the complexity can be lowered to* $O(n)$.

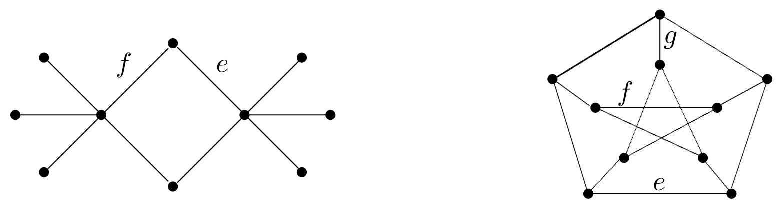
4. Invulnerability Analysis Based on ENT

Observe the definition of the neighbor invulnerability parameters such as ENS, VNS, and VNT, they all measure the state of a network after being most severely damaged. That is, the subversion strategy must be a cut strategy. Based on this common characteristic, we replace the subtraction in ENS by division to define the concept of ENT. By definition, the larger the ENT is, the more resilient the network is.

The following examples illustrate that the parameters ENT and ENS are independent, so they are well defined.

Example 2. *Consider the comet* $C_{10,3}$ *and* G_1*(see Figure 1a). They have the equal order* 10*, vertex neighbor connectivity* 1*, edge neighbor connectivity* 1*, and ENT* $\frac{1}{4}$*; however,* $ENS(C_{10,3}) = 3$*,* $ENS(G_1) = 5$*. Furthermore,* $\{e\}$ *is a* t_{EN}*-set of* G_1 *but is not an ENS-set;* $\{e, f\}$ *is an ENS-set of* G_1 *but is not a* t_{EN}*-set of* G_1*.*

Example 3. *Consider the Petersen graph* $P(5,2)$*(see Figure 1b) and the path* P_{10}*. They have the equal order* 10 *and ENS* 1*, but* $t_{EN}(P(5,2)) = \frac{3}{4}$*,* $t_{EN}(P_{10}) = \frac{1}{2}$*. The edge set* $\{e, f, g\}$ *is both of a* t_{EN}*-set and an ENS-set of* $P(5,2)$*.*



(a) G_1 (b) $P(5,2)$

Figure 1. Two graphs with order 10.

5. Conclusions

The above two examples also show that the parameters ENT and ENS have their own characteristics and advantages when measuring the neighbor invulnerability of networks. Network invulnerability is an important problem in graph theory. Many parameters have been introduced to measure the relationship of invulnerability and structure of networks; however, there exist a lot of unresolved problems about the parameters computing [3,9,10]. Since trees have special structure and wide range of applications [11,12], the polynomial algorithm of ENT of trees has a certain theoretical and practical significance.

Observing the conclusion of Theorem 4, it is easy to know that the matching number of the complete p-partite graph $K_{n_1,n_2,\ldots,n_p}$ is determined by the numbers $n_1, n_2, \ldots, n_p$; therefore, to find a maximum matching of $K_{n_1,n_2,\ldots,n_p}$ is equivalent to divide $\{1, 2, \ldots, p\}$ into two nonempty subset N_1 and N_2 such that $|\sum\limits_{i \in N_1} n_i - \sum\limits_{j \in N_2} n_j|$ is as small as possible. Later, we will consider this integer programming and related problems. Furthermore, the algorithm and the bound of ENT for graphs of general structure are also worth considering.

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