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Article

General Opial Type Inequality and New Green Functions

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Abstract: In this paper we provide many new results involving Opial type inequalities. We consider two functions—one is convex and the other is concave—and prove a new general inequality on a measure space (Ω, Σ, μ). We give an new result involving four new Green functions. Our results include Grüss and Ostrowski type inequalities related to the generalized Opial type inequality. The obtained inequalities are of Opial type because the integrals contain the function and its integral representation. They are not a direct generalization of the Opial inequality.

Keywords: Opial inequality; Green function; Jensen inequality

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1. Introduction

First, we start with the Opial inequality. Opial [1] proved in 1960 the following inequality: If $f \in C^1[0, h]$ is such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0, h)$, then:

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4}\int_0^h [f'(x)]^2 dx, \tag{1}$$

where $h/4$ is the best possible.

This inequality has been generalized and extended in many different directions (for more details see e.g., [2–9]).

Now we continue with the following result. In 2009, Krulić, K. et al., in [10] , observed two measure spaces $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ and the general integral operator A_k defined by:

$$A_k f(x) = \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\, d\mu_2(y),\ x \in \Omega_1, \tag{2}$$

where $f : \Omega_2 \longrightarrow \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative, and

$$K(x) = \int_{\Omega_2} k(x,y)\, d\mu_2(y) > 0,\ x \in \Omega_1. \tag{3}$$

The authors proved the weighted inequality by using Jensen's inequality and Fubini's theorem. Their result is:

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x) \leq \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y), \tag{4}$$

where $u : \Omega_1 \to \mathbb{R}$ is a non-negative measurable function, $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, v is defined on Ω_2 by

$$v(y) = \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\, d\mu_1(x), \tag{5}$$

Φ is a convex function on an interval $I \subseteq \mathbb{R}$, and $f : \Omega_2 \to \mathbb{R}$ is such that $f(y) \in I$ for all $y \in \Omega_2$. We mention that inequality (4) unifies and generalizes many of the results of this type (including the classical ones by Hardy, Hilbert and Godunova).

In the sequel, let (Ω, Σ, μ) be a measure space and let $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric non-negative or nonpositive function such that $K(x)$ is defined by:

$$K(x) := \int_{\Omega} k(x,y)d\mu(y),\ K(x) \neq 0,\ a.e.\, x \in \Omega, \tag{6}$$

and $|K(x)| < \infty$. In the rest of the paper we assume that all integrals are well defined. We continue with the following result that is given in [11].

Theorem 1. *Let $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric nonpositive or non-negative function. If f is a positive convex function, and g a positive concave function on an interval $I \subseteq \mathbb{R}$, $v : \Omega \to \mathbb{R}$ is either nonpositive or non-negative, such that $Im|v| \subseteq I$ and u is defined by:*

$$u(x) := \int_{\Omega} k(x,y)v(y)d\mu(y) < \infty. \tag{7}$$

The following inequality:

$$\int_{\Omega} |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|)d\mu(x) \leq \int_{\Omega} |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \tag{8}$$

holds, where K is defined by (6).

In our main results, we will use the following generalized Montgomery identity:

Theorem 2 ([12])**.** *Let $n \in \mathbb{N}$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$ and $\alpha < \beta$. Then the following identity holds:*

$$\phi(x) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(t)dt + \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\alpha)}{k!(k+2)} \frac{(x-\alpha)^{k+2}}{\beta - \alpha} - \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\beta)}{k!(k+2)} \frac{(x-\beta)^{k+2}}{\beta - \alpha} + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} T_n(x,s)\phi^{(n)}(s)ds, \tag{9}$$

where

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha}(x-s)^{n-1}, & \alpha \leq s \leq x, \\ -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha}(x-s)^{n-1}, & x < s \leq \beta. \end{cases} \tag{10}$$

In case $n = 1$ the sum $\sum_{k=0}^{n-2} \ldots$ is empty, so the identity (9) reduces to the well-known: Montgomery identity

$$\phi(x) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(t)dt + \int_{\alpha}^{\beta} P(x,s)\phi'(s)ds,$$

where $P(x,s)$ is the Peano kernel, defined by:

$$P(x,s) = \begin{cases} \frac{s-\alpha}{\beta-\alpha}, & \alpha \leq s \leq x, \\ \frac{s-\beta}{\beta-\alpha}, & x < s \leq \beta. \end{cases}$$

Now we recall the definition of new Green functions. For any function $\phi : [\alpha, \beta] \to \mathbb{R}, \phi \in C^2([\alpha, \beta])$, we can easily show by integrating by parts that the following is valid:

$$\phi(u) = \phi(\alpha) + (u - \alpha)\phi'(\beta) + \int_\alpha^\beta G_1(u,s)\phi''(s)ds, \tag{11}$$

where the function $G_1 : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is Green's function of the boundary value problem

$$z'' = 0, z(\alpha) = z(\beta)$$

and is defined by:

$$G_1(u,s) = \begin{cases} \alpha - s, & s \le u; \\ \alpha - u, & u \le s. \end{cases} \tag{12}$$

The function G_1 is convex under u and s, it is a symmetric nonpositive function and it is continuous under s and continuous under u.

Here we give three new types of Green's functions defined on $[\alpha, \beta] \times [\alpha, \beta]$ as follows:

$$G_2(u,s) = \begin{cases} u - \beta, & s \le u; \\ s - \beta, & u \le s. \end{cases} \tag{13}$$

$$G_3(u,s) = \begin{cases} u - \alpha, & s \le u; \\ s - \alpha, & u \le s. \end{cases} \tag{14}$$

$$G_4(u,s) = \begin{cases} \beta - s, & s \le u; \\ \beta - u, & u \le s. \end{cases} \tag{15}$$

All three functions are continuous, symmetric and convex with respect to both variables u and s.

Lemma 1. *Let $G_k(\cdot, s), s \in [\alpha, \beta], k = 2, 3, 4$ be defined by (13)–(15). Then for every function $\phi \in C^2([\alpha, \beta])$, it holds that:*

$$\phi(u) = \phi(\beta) + (u - \beta)\phi'(\alpha) + \int_\alpha^\beta G_2(u,s)\phi''(s)ds, \tag{16}$$

$$\phi(u) = \phi(\beta) + (\beta - \alpha)\phi'(\beta) + (u - \alpha)\phi'(\alpha) + \int_\alpha^\beta G_3(u,s)\phi''(s)ds, \tag{17}$$

$$\phi(u) = \phi(\alpha) + (\beta - \alpha)\phi'(\alpha) - (\beta - u)\phi'(\beta) + \int_\alpha^\beta G_4(u,s)\phi''(s)ds. \tag{18}$$

In paper [11], you can see results involving the Green function defined by:

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \le s \le t; \\ \frac{(t-\alpha)(s-\beta)}{\beta-\alpha}, & t \le s \le \beta. \end{cases} \tag{19}$$

Motivated by those results we give general Opial type inequalities. The new inequalities are not direct generalizations of the Opial inequality. They are of Opial type because the integrals contain function and its integral representation. There are many papers involving Green functions; here we mention only a few of them. In [13] you can find results involving Sherman's inequality and new Green's functions. Here we also mention new results about Hilbert-type inequalities; see [14–16]. This paper is organized in the following way: after the Introduction, where we recall the original Opial inequality from 1960 and also provide newer results involving two measure spaces, Section 2 follows. There we give our main results. They contain two functions—convex and concave—and four new Green's functions. In this section there are many new results, six new Theorems and many new Corollaries. In

Section 3, titled Grüss and Ostrowski type inequalities related to the generalized Opial type inequality, we also provide many new results. We conclude our paper with the Discussion.

2. The Main Results

We give our first result, which involves two functions, one positive convex and the other a positive concave function.

Theorem 3. *Let f be a positive convex function and g a positive concave function on an interval $I \subseteq \mathbb{R}$. Then the inequality:*

$$\int_a^b (-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab) f\left(\frac{|\phi(x) - \phi(a) - (x-a)\phi'(b)|}{-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab} \right) g(|(\phi)''(x)|) d\mu(x)$$
$$\leq \int_a^b (-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab) f(|\phi''(x)|) g\left(\frac{|\phi(x) - \phi(a) - (x-a)\phi'(b)|}{-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab} \right) d\mu(x) \tag{20}$$

holds for all nonpositive or non-negative functions $\phi : [a,b] \to \mathbb{R}, \phi \in C^2([a,b])$.

Proof. Function G_1 defined by (12) is a nonpositive symmetric function so we can apply Theorem 1. Let $\Omega = [a,b], k(x,s) = G_1(x,s), v(s) = \varphi''(s)$. Then

$$u(x) = \int_a^b G_1(x,s)\phi''(s) ds = \phi(x) - \phi(a) - (x-a)\phi'(b),$$

$$|K(x)| = \int_a^b |G_1(x,s)| ds = \int_a^x (s-a) ds + \int_x^b (x-a) ds = -\frac{x^2}{2} + bx + \frac{a^2}{2} - ab$$

and inequality (8) becomes (20) so the proof is complete. □

Now we give a special case of Theorem 3 for $\phi(a) = \phi(b) = 0$.

Remark 1. *If $\phi(a) = \phi(b) = 0$ then inequality (20) becomes:*

$$\int_a^b (-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab) f\left(\frac{|\phi(x)|}{-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab} \right) g(|(\phi)''(x)|) d\mu(x)$$
$$\leq \int_a^b (-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab) f(|\phi''(x)|) g\left(\frac{|\phi(x)|}{-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab} \right) d\mu(x). \tag{21}$$

We continue with the other three new Green's functions. We will give the result without the proof since the proof is similar to the proof of Theorem 3.

Corollary 1. *If f is a positive convex function and g is a positive concave function on an interval $I \subseteq \mathbb{R}$, then the following inequalities:*

$$\int_a^b (-\frac{x^2}{2} + bx + \frac{a^2}{2} - ab) f\left(\frac{|\phi(x) - \phi(b) - (x-b)\phi'(a)|}{-\frac{x^2}{2} + ax + \frac{b^2}{2} - ab} \right) g(|(\phi)''(x)|) d\mu(x)$$
$$\leq \int_a^b (-\frac{x^2}{2} + ax + \frac{b^2}{2} - ab) f(|\phi''(x)|) g\left(\frac{|\phi(x) - \phi(b) - (x-b)\phi'(a)|}{-\frac{x^2}{2} + ax + \frac{b^2}{2} - ab} \right) d\mu(x) \tag{22}$$

$$\int_a^b (\frac{x^2}{2} - ax + a^2 + \frac{b^2}{2} - ab) f\left(\frac{|\phi(x) - \phi(b) - (b-a)\phi'(b) - (x-a)\phi'(a)|}{\frac{x^2}{2} - ax + a^2 + \frac{b^2}{2} - ab}\right)$$
$$\cdot g(|(\phi)''(x)|) d\mu(x)$$
$$\leq \int_a^b (\frac{x^2}{2} - ax + a^2 + \frac{b^2}{2} - ab) f(|\phi''(x)|)$$
$$\cdot g\left(\frac{|\phi(x) - \phi(b) - (b-a)\phi'(b) - (x-a)\phi'(a)|}{\frac{x^2}{2} - ax + a^2 + \frac{b^2}{2} - ab}\right) d\mu(x) \tag{23}$$

$$\int_a^b (\frac{x^2}{2} - bx + \frac{a^2}{2} + b^2 - ab) f\left(\frac{|\phi(x) - \phi(a) - (b-a)\phi'(a) - (b-x)\phi'(b)|}{\frac{x^2}{2} - bx + \frac{a^2}{2} + b^2 - ab}\right)$$
$$\cdot g(|(\phi)''(x)|) d\mu(x)$$
$$\leq \int_a^b (\frac{x^2}{2} - bx + \frac{a^2}{2} + b^2 - ab) f(|\phi''(x)|)$$
$$\cdot g\left(\frac{|\phi(x) - \phi(a) - (b-a)\phi'(a) - (b-x)\phi'(b)|}{\frac{x^2}{2} - bx + \frac{a^2}{2} + b^2 - ab}\right) d\mu(x) \tag{24}$$

hold for all nonpositive or non-negative functions $\phi : [a,b] \to \mathbb{R}, \phi \in C^2([a,b])$.

The results given in Theorem 3 and Corollary 1 are new. Similar results can be found in paper [11].

We continue with the following result.

Theorem 4. *Let $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric nonpositive or non-negative function. If f is a positive convex function $f \in C^2([a,b])$, g a positive concave function on an interval $[a,b] \subseteq \mathbb{R}$, $v : \Omega \to \mathbb{R}$ is either nonpositive or non-negative, such that $Im|v| \subseteq [a,b]$, u defined by (7), $K(x)$ is defined by (6) and $G_1(.,s)$ is defined by (12). Then the following result follows:*

$$\int_\Omega |K(x)| \int_a^b G_1(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) f''(s) ds d\mu(x)$$
$$- \int_\Omega |K(x)| \int_a^b G_1\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) f''(s) ds d\mu(x)$$
$$\geq \int_\Omega |K(x)| f(a) \left[g(|v(x)| - g\left(\left|\frac{u(x)}{K(x)}\right|\right)\right] d\mu(x)$$
$$- \int_\Omega |K(x)| f'(b) \left[(|v(x)| - a) g\left(\left|\frac{u(x)}{K(x)}\right|\right) - \left(\left|\frac{u(x)}{K(x)}\right| - a\right) g(|v(x)|\right] d\mu(x). \tag{25}$$

Proof. For every function $f \in C^2([a,b])$, the following is valid:

$$f(u) = f(a) + (u-a) f'(b) + \int_a^b G_1(u,s) f''(s) ds, \tag{26}$$

where G_1 is Green's function defined by (12). Now we insert (26) to (8) and we get:

$$\begin{aligned}
&\int_{\Omega}|K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)-\int_{\Omega}|K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x) \\
&=\int_{\Omega}|K(x)|\left[f(a)+(|v(x)|-a)f'(b)+\int_a^b G_1(|v(x)|,s)f''(s)ds\right]g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x) \\
&-\int_{\Omega}|K(x)|\left[f(a)+\left(\left|\frac{u(x)}{K(x)}\right|-a\right)f'(b)+\int_a^b G_1\left(\left|\frac{u(x)}{K(x)}\right|,s\right)f''(s)ds\right]g(|v(x)|)d\mu(x) \\
&\geq 0.
\end{aligned} \tag{27}$$

Now we rearrange the integrals and get (25). □

Remark 2. *If $f(a)=f(b)=0$ the inequality (25) becomes:*

$$\begin{aligned}
&\int_{\Omega}|K(x)|\int_a^b G_1(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)f''(s)dsd\mu(x) \\
&-\int_{\Omega}|K(x)|\int_a^b G_1\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)f''(s)dsd\mu(x)\geq 0.
\end{aligned} \tag{28}$$

We continue with analogue results with three other Green functions.

Corollary 2. *If $k:\Omega\times\Omega\to\mathbb{R}$ is a symmetric nonpositive or non-negative function, f a positive convex function $f\in C^2([a,b])$, g a positive concave function on an interval $[a,b]\subseteq\mathbb{R}$, $v:\Omega\to\mathbb{R}$ is either nonpositive or non-negative, such that $Im|v|\subseteq[a,b]$, u is defined by (7), $K(x)$ is defined by (6) and $G_i(.,s), i=2,3,4$ are defined by (13)–(15), then the following results follow:*

(i)

$$\begin{aligned}
&\int_{\Omega}|K(x)|\int_a^b G_2(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)f''(s)dsd\mu(x) \\
&-\int_{\Omega}|K(x)|\int_a^b G_2\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)f''(s)dsd\mu(x) \\
&\geq\int_{\Omega}|K(x)|f(b)\left[g(|v(x)|-g\left(\left|\frac{u(x)}{K(x)}\right|\right)\right]d\mu(x) \\
&-\int_{\Omega}|K(x)|f'(a)\left[(|v(x)|-b)g\left(\left|\frac{u(x)}{K(x)}\right|\right)-\left(\left|\frac{u(x)}{K(x)}\right|-b\right)g(|v(x)|\right]d\mu(x);
\end{aligned} \tag{29}$$

(ii)

$$\begin{aligned}
&\int_{\Omega}|K(x)|\int_a^b G_3(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)f''(s)dsd\mu(x) \\
&-\int_{\Omega}|K(x)|\int_a^b G_3\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)f''(s)dsd\mu(x) \\
&\geq\int_{\Omega}|K(x)|f(b)\left[g(|v(x)|-g\left(\left|\frac{u(x)}{K(x)}\right|\right)\right]d\mu(x) \\
&-\int_{\Omega}|K(x)|f'(a)\left[(|v(x)|-a)g\left(\left|\frac{u(x)}{K(x)}\right|\right)-\left(\left|\frac{u(x)}{K(x)}\right|-a\right)g(|v(x)|\right]d\mu(x) \\
&-\int_{\Omega}|K(x)|(b-a)f'(b)\left[g(|v(x)|-g\left(\left|\frac{u(x)}{K(x)}\right|\right)\right]d\mu(x);
\end{aligned} \tag{30}$$

(iii)

$$\begin{aligned}
&\int_\Omega |K(x)| \int_a^b G_4(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) f''(s) ds d\mu(x) \\
&- \int_\Omega |K(x)| \int_a^b G_4\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) f''(s) ds d\mu(x) \\
&\geq \int_\Omega |K(x)| f(a) \left[g(|v(x)| - g\left(\left|\frac{u(x)}{K(x)}\right|\right) \right] d\mu(x) \\
&- \int_\Omega |K(x)| f'(b) \left[(b - |v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) - \left(b - \left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)| \right] d\mu(x) \\
&- \int_\Omega |K(x)| (b-a) f'(a) \left[g(|v(x)| - g\left(\left|\frac{u(x)}{K(x)}\right|\right) \right]. \qquad (31)
\end{aligned}$$

Proof. Similar to the proof of Theorem 4. □

Remark 3. *If $f(a) = f(b) = 0$ the inequalities (29)–(31) reduce to:*

$$\begin{aligned}
&\int_\Omega |K(x)| \int_a^b G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) f''(s) ds d\mu(x) \\
&- \int_\Omega |K(x)| \int_a^b G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) f''(s) ds d\mu(x) \geq 0, i = 2, 3, 4. \qquad (32)
\end{aligned}$$

We continue with the following result. It holds only for G_3 and G_4 since they are non-negative functions.

Theorem 5. *If f is a positive convex function $f \in C^2([a, b])$ such that $f(a) = f(b) = 0$, g a positive concave function on an interval $[a, b] \subseteq \mathbb{R}$, $v : \Omega \to \mathbb{R}$ is either non-negative or nonpositive such that $Im|v| \subseteq [a, b]$, u is defined by (7), $K(x)$ is defined by (6) and $G_i(., s), i = 3, 4$ are defined by (14) and (15) then the following statements are equivalent:*

(i)

$$\int_\Omega |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \leq \int_\Omega |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x); \qquad (33)$$

(ii)

$$\int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x) \leq \int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x), \quad (34)$$

where

$$i = 3, 4. \qquad (35)$$

Proof. We only give the proof for $i = 3$; the proof for $i = 4$ is similar.

(i) ⇒ (ii): Let (i) hold. We consider the Green function G_3 defined by (14). We know that function $G_3(., s),\ s \in [a, b]$ is positive and convex on $[a, b]$, so (34) holds for $G_3(., s)$;

(ii) ⇒ (i): Let (ii) holds. Every function $f : [a, b] \to \mathbb{R}$, $f \in C^2([a, b])$, can be written in the form (16). Therefore by some simple calculations, we deduce:

$$
\begin{aligned}
&\int_{\Omega}|K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)-\int_{\Omega}|K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x)\\
&=\int_{\Omega}|K(x)|\int_a^b G_3(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)f''(s)dsd\mu(x)\\
&-\int_{\Omega}|K(x)|\int_a^b G_3\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)f''(s)dsd\mu(x)\\
&+\int_{\Omega}|K(x)|f(b)\left[g\left(\left|\frac{u(x)}{K(x)}\right|\right)-g(|v(x)|\right]d\mu(x)\\
&+\int_{\Omega}|K(x)|f'(a)\left[(|v(x)|-a)g\left(\left|\frac{u(x)}{K(x)}\right|\right)-\left(\left|\frac{u(x)}{K(x)}\right|-a\right)g(|v(x)|\right]d\mu(x)\\
&+\int_{\Omega}|K(x)|(b-a)f'(b)\left[g(|v(x)|-g\left(\left|\frac{u(x)}{K(x)}\right|\right)\right]d\mu(x). \qquad (36)
\end{aligned}
$$

Since $f(a)=f(b)=0$ (36) reduces to:

$$
\begin{aligned}
&\int_{\Omega}|K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)-\int_{\Omega}|K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x)\\
&=\int_{\Omega}|K(x)|\int_a^b G_3(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)f''(s)dsd\mu(x)\\
&-\int_{\Omega}|K(x)|\int_a^b G_3\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)f''(s)dsd\mu(x). \qquad (37)
\end{aligned}
$$

Since f is convex, therefore $f''(s)\geq 0$ for $s\in[a,b]$. Furthermore, if for every $s\in[a,b]$ the inequality (34) holds, then the right hand side of (37) is non-negative and hence (36) holds. □

Now we continue with the following result.

Theorem 6. *Let $n\in\mathbb{N}, n\geq 4$, $\phi: I\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I\subset\mathbb{R}$ is an open interval, $a,b\in I$, $a<b$. Let f be a positive convex function $f\in C^2([a,b])$ such that $f(a)=f(b)=0$, g is a positive concave function on an interval $[a,b]\subseteq\mathbb{R}$, $v:\Omega\to\mathbb{R}$ is either non-negative or nonpositive, such that $Im|v|\subseteq[a,b]$, u defined by (7), $K(x)$ is defined by (6), $G_i(.,s), i=1,2,3,4$ are defined by (12)–(15) and T_n is defined by (10). Then the following results follow:*

(i)

$$
\begin{aligned}
&\int_{\Omega}|K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)-\int_{\Omega}|K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x) \qquad (38)\\
&=\int_a^b\left[\int_{\Omega}|K(x)|G_i(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)\right.\\
&\qquad\left.-\int_{\Omega}|K(x)|G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)d\mu(x)\right]\\
&\times\sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}\right)ds\\
&+\frac{1}{(n-3)!}\int_a^b\int_a^b\left[\int_{\Omega}|K(x)|G_i(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)\right.\\
&\qquad\left.-\int_{\Omega}|K(x)|G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)d\mu(x)\right]\cdot\\
&\cdot\tilde{T}_{n-2}(s,t)\phi^{(n)}(t)dtds,
\end{aligned}
$$

where

$$\tilde{T}_{n-2}(s,t) = \begin{cases} \frac{1}{b-a}\left[\frac{(s-t)^{n-2}}{n-2} + (s-a)(s-t)^{n-3}\right], & a \le t \le s, \\ \\ \frac{1}{b-a}\left[\frac{(s-t)^{n-2}}{n-2} + (s-b)(s-t)^{n-3}\right], & s < t \le b. \end{cases}; \tag{39}$$

(ii)

$$\begin{aligned} &\int_{\Omega} |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_{\Omega} |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \\ =& \frac{f'(b)-f'(a)}{b-a} \int_a^b \left[\int_{\Omega} |K(x)| G_i(|v(x)|,s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \\ &\quad \left. - \int_{\Omega} |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right) g(|v(x)|) d\mu(x)\right] ds \end{aligned} \tag{40}$$

$$\begin{aligned} &+ \int_a^b \left[\int_{\Omega} |K(x)| G_i(|v(x)|,s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \\ &\quad \left. - \int_{\Omega} |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right) g(|v(x)|) d\mu(x)\right] \\ &\times \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right) ds \\ &+ \frac{1}{(n-3)!} \int_a^b \int_a^b \left[\int_{\Omega} |K(x)| G_i(|v(x)|,s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \\ &\quad \left. - \int_{\Omega} |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right) g(|v(x)|) d\mu(x)\right] T_{n-2}(s,t) f^{(n)}(t)\, dt\, ds. \end{aligned} \tag{41}$$

Proof. Fix $i = 1,2,3,4$. Using (11), (16)–(18) in

$$\int_{\Omega} |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_{\Omega} |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x)$$

we obtain:

$$\begin{aligned} &\int_{\Omega} |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_{\Omega} |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \\ =& \int_a^b \left[|K(x)| G_i(|v(x)|,s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \\ &\quad \left. - \int_{\Omega} |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right) g(|v(x)|) d\mu(x)\right] f''(s)\, ds, \end{aligned} \tag{42}$$

(i) Differentiating (9) twice with respect to s and rearranging the terms, we get:

$$\begin{aligned} f''(s) = \sum_{k=1}^{n-1} \frac{k}{(k-1)!} &\left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right) \\ &+ \frac{1}{(n-3)!} \int_a^b \tilde{T}_{n-2}(s,t) f^{(n)}(t)\, dt. \end{aligned} \tag{43}$$

Substituting (43) in (42) we obtain (38);

(ii) Replacing f with f'' and then n with $n-2$ in (9), we have:

$$f''(s) = \frac{1}{b-a}\int_a^b f''(s)ds + \sum_{k=0}^{n-4}\frac{f^{(k+3)}(a)}{k!(k+2)}\frac{(s-a)^{k+2}}{b-a} - \sum_{k=0}^{n-4}\frac{f^{(k+3)}(b)}{k!(k+2)}\frac{(s-b)^{k+2}}{b-a}$$
$$+ \frac{1}{(n-3)!}\int_a^b T_{n-2}(s,t)f^{(n)}(t)dt;$$

this implies that:

$$f''(s) = \frac{f'(b)-f'(a)}{b-a} + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right)$$
$$+\frac{1}{(n-3)!}\int_a^b T_{n-2}(s,t)f^{(n)}(t)dt. \quad (44)$$

Combining (44) with (42), we get (40). □

Notice that in Theorem 6 we calculated, under some conditions, the difference between the right-hand and left-hand side of inequality 8.

Theorem 7. *Suppose that all assumptions of Theorem 6 hold. Let for even n the function $f : I \to \mathbb{R}$ be n-convex and*

$$\int_\Omega |K(x)|G_i(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x) - \int_\Omega |K(x)|G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)d\mu(x)$$
$$\geq 0, \text{ for } i = 1,2,3,4. \quad (45)$$

Then the following inequalities hold:

(i)

$$\int_\Omega |K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x) - \int_\Omega |K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x)$$
$$\geq \int_a^b \left[\int_\Omega |K(x)|G_i(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)\right.$$
$$\left. - \int_\Omega |K(x)|G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)d\mu(x)\right]$$
$$\times \sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right)ds; \quad (46)$$

(ii)

$$\int_\Omega |K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x) - \int_\Omega |K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x)$$
$$\geq \frac{f'(b)-f'(a)}{b-a}\int_a^b \left[\int_\Omega |K(x)|G_i(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)\right.$$
$$\left. - \int_\Omega |K(x)|G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)d\mu(x)\right]ds$$
$$+ \int_a^b \left[\int_\Omega |K(x)|G_i(|v(x)|,s)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x)\right. \quad (47)$$
$$\left. - \int_\Omega |K(x)|G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right)g(|v(x)|)d\mu(x)\right]$$
$$\times \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right)ds. \quad (48)$$

Proof. (i) Since the function f is n-convex, we have $f^{(n)} \geq 0$. It is also obvious that if n is even then $\tilde{T}_{n-2} \geq 0$ because:

Case I : If $a \leq t \leq s$, then $s - t \geq 0$ and hence $\frac{(s-t)^{n-2}}{n-2} \geq 0$. Also $(s-a) \geq 0$ and $(s-t)^{n-3} \geq 0$. So in this case from (39) we have $\tilde{T}_{n-2} \geq 0$.

Case II : If $s < t \leq b$, then $(s-t)^{n-3}$ and $(t-b)$ are non positive. As n is even so we have $(t-b)(s-t)^{n-3} \geq 0$; also $\frac{(s-t)^{n-2}}{n-2} \geq 0$. So in this case from (39) we have $\tilde{T}_{n-2} \geq 0$.

Now using (45) and the positivity of $\tilde{T}_{n-2}$ and $f^{(n)}$ in (38) we get (46);

(ii) The proof is similar to the proof of part (i).

□

We continue with the last result in this section.

Theorem 8. *Let $n \in \mathbb{N}, n \geq 4$, $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$ and $f(a) = f(b) = 0$. Let g be a positive concave function on an interval $[a, b] \subseteq \mathbb{R}$, $v : \Omega \to \mathbb{R}$ is either non-negative or nonpositive, such that $Im|v| \subseteq [a, b]$, u is defined by (7), $K(x)$ is defined by (6) and $G_i(., s), i = 1, 2, 3, 4$ are defined by (12)–(15). If n is even and f is an n-convex function, then (46) and (47) hold. Moreover, if (46) and (47) hold and the functions defined by:*

$$L_1(., s) = \int_a^b G_w(., s) \times \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} \right) ds, \tag{49}$$

$$L_2(., s) = \frac{f'(b) - f'(a)}{b-a} \int_a^b G_w(., s) ds + \int_a^b G_w(., s) \times$$
$$\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} \right) ds, \tag{50}$$

where $\omega = 1, 2, 3, 4$ are convex on $[a, b]$, then

$$\int_\Omega |K(x)| f\left(\left| \frac{u(x)}{K(x)} \right| \right) g(|v(x)|) d\mu(x) \leq \int_\Omega |K(x)| f(|v(x)|) g\left(\left| \frac{u(x)}{K(x)} \right| \right) d\mu(x). \tag{51}$$

Proof. Since the functions $G_w(., t)$, $w \in \{1, 2, 3, 4\}$, $t \in [a, b]$, are convex, so it holds that

$$\int_\Omega |K(x)| G_\omega\left(\left| \frac{u(x)}{K(x)} \right|, s \right) g(|v(x)|) d\mu(x) \leq \int_\Omega |K(x)| G_\omega(|v(x)|, s) g\left(\left| \frac{u(x)}{K(x)} \right| \right) d\mu(x),$$

$s \in [a, b]$. Applying Theorem 6, we obtain (46) and (47).

Since (46) holds, the right hand side of (46) can be rewritten in the form:

$$\int_\Omega |K(x)| L_1\left(\left| \frac{u(x)}{K(x)} \right|, s \right) g(|v(x)|) d\mu(x) \leq \int_\Omega |K(x)| L_1(|v(x)|, s) g\left(\left| \frac{u(x)}{K(x)} \right| \right) d\mu(x),$$

where L_1 is defined by (49). Since L_1 is convex, therefore by Theorem 1 we have:

$$\int_\Omega |K(x)| L_1(|v(x)|, s) g\left(\left| \frac{u(x)}{K(x)} \right| \right) d\mu(x) - \int_\Omega |K(x)| L_1\left(\left| \frac{u(x)}{K(x)} \right|, s \right) g(|v(x)|) \geq 0,$$

i.e., the right hand side of (46) is non-negative, so the inequality (8) immediately follows. Similarly, we may get (8) by using the convexity of L_2. □

3. Grüss and Ostrowski Type Inequalities Related to the Generalized Opial Type Inequality

Cerone et al. [17] considered Čebyšev functional

$$T(f,g) := \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt$$

for Lebesgue integrable functions $f, g : [\alpha,\beta] \to \mathbb{R}$, proving the following two results which contain the Grüss and Ostrowski type inequalities.

Theorem 9. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be a Lebesgue integrable function and $g : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(g')^2 \in L[\alpha,\beta]$. Then*

$$|T(f,g)| \leq \frac{1}{\sqrt{2}}|T(f,f)|^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (x-\alpha)(\beta-x)[g'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{52}$$

The constant $\frac{1}{\sqrt{2}}$ in (52) is the best possible.

Theorem 10. *Let $g : [\alpha,\beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha,\beta]$. Then*

$$|T(f,g)| \leq \frac{1}{2(\beta-\alpha)}\|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dg(x). \tag{53}$$

The constant $\frac{1}{2}$ in (53) is the best possible.

Using the previous two theorems we obtain upper bounds for the identities related to generalizations of the Opial type inequality.

To avoid many notations, under the assumptions of Theorem 6, we define functions P_1 and P_2 from $[\alpha,\beta]$ to $\mathbb{R}$ by:

$$\begin{aligned} P_{1,w}(t) = \int_a^b \Bigg[&\int_\Omega |K(x)| G_i(|v(x)|,s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \\ &- \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right) g(|v(x)|) d\mu(x) \Bigg] \cdot \tilde{T}_{n-2}(s,t)ds \\ &\text{for,} \quad w = 1,2,3,4. \end{aligned} \tag{54}$$

$$\begin{aligned} P_{2,w}(t) = \int_a^b \Bigg[&\int_\Omega |K(x)| G_i(|v(x)|,s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \\ &- \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|,s\right) g(|v(x)|) d\mu(x) \Bigg] \cdot T_{n-2}(s,t)ds \\ &\text{for,} \quad w = 1,2,3,4. \end{aligned} \tag{55}$$

Theorem 11. *Let $n \in \mathbb{N}, n \geq 4$, $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(\cdot-a)(b-\cdot)(f^{(n+1)})^2 \in L[a,b]$, $P_{1,w}$, $P_{2,w}$, $w = 1,2,3,4$, be defined as in (54) and (55) respectively. Then,*

(i) *the remainder $\kappa^1(f;a,b)$, defined by*

$$\begin{aligned}\kappa^1(f;a,b) = &\int_\Omega |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_\Omega |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \\ &- \int_a^b \left[\int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \qquad (56)\\ &\left.- \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x)\right] \\ &\times \sum_{k=1}^{n-1} \frac{k}{(k-1)!}\left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right) ds \\ &- \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b P_{1,w}(t) dt \qquad (57)\end{aligned}$$

satisfies the estimation

$$\left|\kappa^1(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}(n-3)!} |T(P_{1,w}, P_{1,w})|^{\frac{1}{2}} \left(\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right)^{\frac{1}{2}}; \qquad (58)$$

(ii) *The remainder $\kappa^2(f;a,b)$, defined by*

$$\begin{aligned}\kappa^2(f;a,b) = &\int_\Omega |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_\Omega |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \\ &- \frac{f'(b) - f'(a)}{b-a} \int_a^b \left[\int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \\ &\left.- \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x)\right] ds \\ &- \int_a^b \left[\int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x)\right. \qquad (59)\\ &\left.- \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x)\right] \\ &\times \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!}\left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\right) ds \\ &- \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b P_{2,w}(t) dt, \qquad (60)\end{aligned}$$

satisfies the estimation

$$\left|\kappa^2(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}(n-3)!} |T(P_{2,w}, P_{2,w})|^{\frac{1}{2}} \left(\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right)^{\frac{1}{2}}. \qquad (61)$$

Proof. (i) Comparing (38) and (57) we get

$$\kappa^1(f;a,b) = \frac{1}{(n-3)!} \int_a^b P_{1,w}(s) f^{(n)}(s) ds - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b P_{1,w}(s) ds. \qquad (62)$$

Applying Theorem 9 for $f \to P_{1,w}$, $g \to f^{(n)}$ and using the Čebyšev functional, we get:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b P_{1,w}(s) f^{(n)}(s) ds - \frac{1}{b-a}\int_a^b P_{1,w}(s) ds \cdot \frac{1}{b-a}\int_a^b f^{(n)}(s) ds\right| \\ &\le \frac{1}{\sqrt{2}} |T(P_{1,w}, P_{1,w})|^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}. \qquad (63)\end{aligned}$$

Therefore from (62) and (63) we get (58);

(ii) Similarly to part (i), we obtain (61).
□

We recall that the symbol $L_p[a,b]$ $(1 \leq p < \infty)$ denotes the space of $p-$power integrable functions defined on the interval $[a,b]$, equipped with the norm

$$\| \phi \|_p = \left(\int_a^b |\phi(t)|^p dt \right)^{\frac{1}{p}} \text{ for all } \phi \in L_p[a,b],$$

and the space of essentially bounded functions on $[a,b]$, denoted by $L_\infty[a,b]$, with the norm

$$\| \phi \|_\infty = ess \sup_{t \in [a,b]} |\phi(t)|.$$

Theorem 12. *Let $n \in \mathbb{N}, n \geq 4$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is monotonic nondecreasing on $[a,b]$ and let $P_{1,w}$, $P_{2,w}$, $w = 1,2,3,4$, be defined as in (54) and (55) respectively. Then:*

(i) The remainder $\kappa^1(f;a,b)$ defined by (57) satisfies the estimation

$$\left|\kappa^1(f;a,b)\right| \leq \frac{\|P'_{1,w}\|_\infty}{(n-3)!} \left[\frac{(b-a)\left(f^{(n-1)}(b) + f^{(n-1)}(a)\right)}{2} - \left\{ f^{(n-2)}b) - f^{(n-2)}(a) \right\} \right]; \quad (64)$$

(ii) The remainder $\kappa^2(f;a,b)$ defined by (60) satisfies the estimation

$$\left|\kappa^2(\phi;\alpha,\beta)\right| \leq \frac{\|P'_{2,w}\|_\infty}{(n-3)!} \left[\frac{(\beta-\alpha)\left(\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha)\right)}{2} - \left\{ \phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha) \right\} \right]. \quad (65)$$

Proof. (i) Since

$$\kappa^1(f;a,b) = \frac{1}{(n-3)!} \int_a^b P_{1,w}(s) f^{(n)}(s) ds - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b P_{1,w}(s) ds. \quad (66)$$

Applying Theorem 10 for $f \to P_{1,w}$, $g \to f^{(n)}$ and using Čebyšev functional, we get:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b P_{1,w}(s) f^{(n)}(s) ds - \frac{1}{b-a} \int_a^b P_{1,w}(s) ds \cdot \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right| \\ &\leq \frac{1}{2(b-a)} \|P'_{1,w}\|_\infty \int_a^b (s-a)(b-s) f^{(n+1)}(s) ds. \end{aligned} \quad (67)$$

We calculated

$$\begin{aligned} \int_a^b (s-a)(b-s) f^{(n+1)}(s) ds &= \int_a^b [2s - (a+b)] f^{(n)}(s) ds \\ &= (b-a)\left[f^{(n-1)}(b) + f^{(n-1)}(a) \right] - 2\left[f^{(n-2)}(b) - f^{(n-2)}(a) \right]. \end{aligned}$$

Therefore, from (66) and (67), we get (64).

(i) Similarly, we can prove (65).
□

In the following theorem we present Ostrowski type inequality related to generalizations of Opial's inequality.

Theorem 13. *Let $n \in \mathbb{N}, n \geq 4$, (p,q) be a pair of conjugate exponents, i.e., $1 \leq p,q \leq \infty$ and $1/p + 1/q = 1$. Let $f : [a,b] \to \mathbb{R}$ be such that $\left|f^{(n)}\right|^p \in L[a,b]$. Let $P_{1,w}$ and $P_{2,w}$, $w = 1,2,3,4$, be defined as in (54), (55), respectively. Then the following inequalities hold.*

(i)

$$
\begin{aligned}
&\Bigg| \int_\Omega |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_\Omega |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \\
&- \int_a^b \left[\int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \right. \\
&\qquad \left. - \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x) \right] \\
&\times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} \right) ds \Bigg| \\
&\leq \frac{1}{(n-3)!} \left\| f^{(n)} \right\|_p \|P_{1,w}\|_q.
\end{aligned}
$$

The constant $\|P_{1,w}\|_q$ is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$;

(ii)

$$
\begin{aligned}
&\Bigg| \int_\Omega |K(x)| f(|v(x)|) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) - \int_\Omega |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|) d\mu(x) \\
&- \frac{f'(b) - f'(a)}{b-a} \int_a^b \left[\int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \right. \\
&\qquad \left. - \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x) \right] ds \\
&- \int_a^b \left[\int_\Omega |K(x)| G_i(|v(x)|, s) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \right. \\
&\qquad \left. - \int_\Omega |K(x)| G_i\left(\left|\frac{u(x)}{K(x)}\right|, s\right) g(|v(x)|) d\mu(x) \right] \\
&\times \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} \right) ds \Bigg| \\
&\leq \frac{1}{(n-3)!} \left\| f^{(n)} \right\|_p \|P_{2,w}\|_q.
\end{aligned}
$$

The constant $\|P_{2,w}\|_q$ is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The proof is similar to the proof of Theorem 12 in [13]. □

More results of this type, but involving Sherman's inequality, can be found in paper [13].

4. Discussion

In this paper we provide many new results involving Opial type inequalities. In paper [11] we considered the Green function defined by (19). Here we gave an analogous result involving four new Green functions. In Section 3 we present Grüss and Ostrowski type inequalities related to the generalized Opial type inequality. There are many papers with those types of result but involving different inequalities, see e.g., [13]. There you can find results involving Sherman's inequality. We are motivated by those results and in this paper our goal was to provide those types of results for Opial type inequalities. We consider this paper to be a natural sequel of the paper [11]. In our next papers we plan to present new results involving Green functions and Hardy's inequality. Results proved in this paper are theoretical but we are open to all suggestions involving applications and further investigation.

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