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Up and Down h-Pre-Invex Fuzzy-Number Valued Mappings and Some Certain Fuzzy Integral Inequalities

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Abstract: The objective of the current paper is to incorporate the new class and concepts of convexity and Hermite–Hadamard inequality with the fuzzy Riemann integral operators because almost all classical single-valued and interval-valued convex functions are special cases of fuzzy-number valued convex mappings. Therefore, a new class of nonconvex mapping in the fuzzy environment has been defined; up and down h-pre-invex fuzzy-number valued mappings ($U.D$ h-pre-invex $F\text{-}N\cdot V\cdot M$s). With the help of this newly defined class, some new versions of Hermite–Hadamard (HH) type inequalities have been also presented. Moreover, some related inequalities such as HH Fejér- and Pachpatte-type inequalities for $U\cdot D$ h-pre-invex $F\text{-}N\cdot V\cdot M$s are also introduced. Some exceptional cases have been discussed, which can be seen as applications of the main results. We have provided some nontrivial examples. Finally, we also discuss some future scopes.

Keywords: up and down h-pre-invex fuzzy-number valued mappings; fuzzy Riemann integral operators; Hermite–Hadamard Fejér type inequalities; Hermite–Hadamard Pachpatte type inequalities

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1. Introduction

The area of mathematics known as convex analysis is where we explore the characteristics of convex sets and convex functions. These traditional ideas have numerous uses in both the pure and applied sciences. Everyone is aware of, for instance, how convexity is used in mathematical economics, operations research, optimization theory, and the theory of means, among other fields. The traditional notions of convexity have recently been expanded upon and developed in many ways using fresh and original concepts. For instance, Dragomir [1] proposed the class of coordinated convex functions and expanded the idea of classical convex functions on the coordinates. The concept of harmonically convex functions was first suggested by Iscan [2], who also noted that this class benefits from several good features shared by convex functions. The class of interval-valued convex functions was introduced by Nikodem [3], and its characteristics were covered. Interval-valued harmonically convex functions were first described by Zhao et al. in their publication [4]. Readers who are interested in more information are advised to read the book [5]. Mohan and Neogy [6] introduced the well-established class of nonconvex functions which is known as preinvex functions. Moreover, they defined a condition to handle a bi-function that is used in invex sets.

The idea of convexity's relationship to the theory of inequalities is another endearing feature. Numerous inequalities that are well-known to us are a direct result of using the convexity condition of functions. The Hermite–Hadamard inequality is among one of the findings in this area that have received the most research.

The HH inequality [7,8] for convex mapping $\mathfrak{U} : K \to \mathbb{R}$ on an interval $K = [a,\ z]$

$$\mathfrak{U}\left(\frac{a+z}{2}\right) \leq \frac{1}{z-a}\int_a^z \mathfrak{U}(v)dv \leq \frac{\mathfrak{U}(a)+\mathfrak{U}(z)}{2} \tag{1}$$

for all $a,\ z \in K$, where K is a convex set. If the mapping is concave, then inequality (1) is reversed.

Fejér considered the major generalizations of HH inequality in [9] which is known as HH–Fejér inequality.

Let $\mathfrak{U} : K = [a,\ z] \to \mathbb{R}$ be a convex mapping on a convex set K and $a,\ z \in K$. Then,

$$\mathfrak{U}\left(\frac{a+z}{2}\right) \leq \frac{1}{\int_a^z \mathcal{C}(v)dv}\ \int_a^z \mathfrak{U}(v)\mathcal{C}(v)dv \leq \frac{\mathfrak{U}(a)+\mathfrak{U}(z)}{2} \tag{2}$$

If $\mathcal{C}(v) = 1$, then we obtain (1) from (2). For concave mapping, the above inequality (2) is reversed. Many inequalities may be found using special symmetric mapping $\mathcal{C}(v)$ for convex mappings with the help of inequality (2).

With the use of fractional calculus, Sarikaya et al. [10] were able to derive fractional analogs of the Hermite–Hadamard inequality. See [11] for some more current research on Hermite–Hadamard's inequality and its uses.

On the other hand, interval analysis is a crucial component of mathematics and is employed in computer models as one method for addressing interval uncertainty. Although Archimedes' calculation of a circle's circumference is where this theory first appeared, significant research on the subject was not published until the 1950s. The first book [12] on interval analysis was published in 1966 by Moore, the inventor of interval calculus. After that, other academics studied the theory and uses of interval analysis. Integral inequalities resulting from interval-valued functions have recently attracted the attention of numerous authors. The Hermite–Hadamard inequality for set-valued functions, a more extensive kind of interval-valued mapping, was discovered by Sadowska [13]:

Let $\mathfrak{U} : K = [a,\ z] \to \mathcal{K}_C^+$ be a convex interval-valued mapping such that $\mathfrak{U}(v) = [\mathfrak{U}_*(v),\ \mathfrak{U}^*(v)]$ for all $\omega \in [a,\ z]$. Then

$$\mathfrak{U}\left(\frac{a+z}{2}\right) \supseteq \frac{1}{z-a}\int_a^z \mathfrak{U}(v)dv \supseteq \frac{\mathfrak{U}(a)+\mathfrak{U}(z)}{2} \tag{3}$$

If $\mathfrak{U}$ is concave interval-valued mapping, then the above double inclusion relation (3) is reversed.

Many publications have focused on generalizing the inclusions (1)–(3). For instance, Budak et al. [14] used Riemann–Liouville fractional integrals of interval-valued functions to demonstrate the Hermite–Hadamard inclusion. Several works [15–17] examined the generalization of (3) using various general convexities. The analogous Hermite–Hadamard inclusions for interval-valued functions with two variables were also demonstrated by numerous writers [18–21]. We recommend the following articles [21–24] for readers interested.

Khan and his colleagues recently extended the concept of convex interval-valued mappings (convex $I{\cdot}V{\cdot}M$s) and the fuzzy interval-valued mappings (convex F-$I{\cdot}V{\cdot}M$s) term of fuzzy interval-valued convex mappings by using fuzzy-order relation such that the convex F-$I{\cdot}V{\cdot}M$s (apparently new) concept includes (h1, h2)-convex F-$I{\cdot}V{\cdot}M$s, see [25] and harmonic convex F-$I{\cdot}V{\cdot}M$s, see [26]. To illustrate inequalities of the Hermite–Hadamard, Hermite–Hadamard–Fejér, and Pachpatte types, his team utilized h-preinvex F-$I{\cdot}V{\cdot}M$s, see [27], (h1, h2)-preinvex F-$I{\cdot}V{\cdot}M$s, see [28], and higher-order preinvex F-$I{\cdot}V{\cdot}M$s, see [29], Recently Khan et al. [30] introduced new versions of Hermite–Hadamard and Hermite–

Hadamard–Fejér type inequalities by using the introduced concept of fuzzy Riemann–Liouville fractional integrals via *U·D F-N·V·Ms*. For various recent achievements related to the notion of fuzzy interval-valued analysis of some well-known integral inequalities, we refer interested readers to study some basic concepts related to fuzzy calculus, see [31–55] and the references therein.

Motivated and inspired by existing research, we have presented a new extension of *HH* inequalities for the newly introduced class of *U·D h*-pre-invex *F-N·V·Ms* using fuzzy inclusion relation. With the aid of this class, we have created new versions of the *HH* inequalities that take advantage of the fuzzy Riemann integral operators. We also looked at the applicability of our findings in exceptional circumstances.

2. Preliminaries

Let $\mathcal{X}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $₦ \in \mathcal{X}_C$ be defined by

$$₦ = \left[₦_*,\ ₦^*\right] = \left\{v \in \mathbb{R}\middle|\ ₦_* \leq v \leq ₦^*\right\}, \left(₦_*,\ ₦^* \in \mathbb{R}\right). \tag{4}$$

If $₦_* = ₦^*$, then $₦$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $₦_* \geq 0$, then $\left[₦_*,\ ₦^*\right]$ is called the positive interval. The set of all positive intervals is denoted by $\mathcal{X}_C^+$ and defined as $\mathcal{X}_C^+ = \left\{\left[₦_*,\ ₦^*\right] : \left[₦_*,\ ₦^*\right] \in \mathcal{X}_C \text{ and } ₦_* \geq 0\right\}$.

Let $\sigma \in \mathbb{R}$ and $\sigma \cdot ₦$ be defined by

$$\sigma \cdot ₦ = \begin{cases} \left[\sigma ₦_*,\ \sigma ₦^*\right] & \text{if } \sigma > 0, \\ \{0\} & \text{if } \sigma = 0, \\ \left[\sigma ₦^*, \sigma ₦_*\right] & \text{if } \sigma < 0. \end{cases} \tag{5}$$

Then, the Minkowski difference $₩ - ₦$, addition $₦ + ₩$ and $₦ \times ₩$ for $₦, ₩ \in \mathcal{X}_C$ are defined by

$$\left[₩_*,\ ₩^*\right] + \left[₦_*,\ ₦^*\right] = \left[₩_* + ₦_*,\ ₩^* + ₦^*\right], \tag{6}$$

$$\left[₩_*,\ ₩^*\right] \times \left[₦_*,\ ₦^*\right] = \left[\min\left\{₩_*₦_*,\ ₩^*₦_*,\ ₩_*₦^*,\ ₩^*₦^*\right\}, \max\left\{₩_*₦_*,\ ₩^*₦_*,\ ₩_*₦^*,\ ₩^*₦^*\right\}\right] \tag{7}$$

$$\left[₩_*,\ ₩^*\right] - \left[₦_*,\ ₦^*\right] = \left[₩_* - ₦^*,\ ₩^* - ₦_*\right], \tag{8}$$

Remark 1. (i) For given $\left[₩_*,\ ₩^*\right], \left[₦_*,\ ₦^*\right] \in \mathbb{R}_I$, the relation " $\supseteq_I$ " defined on $\mathbb{R}_I$ by

$$\left[₦_*,\ ₦^*\right] \supseteq_I \left[₩_*,\ ₩^*\right] \text{ if, and only if, } ₦_* \leq ₩_*,\ \ ₩^* \leq ₦^*, \tag{9}$$

for all $\left[₩_*,\ ₩^*\right], \left[₦_*,\ ₦^*\right] \in \mathbb{R}_I$, it is a partial interval inclusion relation. The relation $\left[₦_*,\ ₦^*\right] \supseteq_I \left[₩_*,\ ₩^*\right]$ coincident to $\left[₦_*,\ ₦^*\right] \supseteq \left[₩_*,\ ₩^*\right]$ on $\mathbb{R}_I$. It can be easily seen that "$\supseteq_I$" looks like "up and down" on the real line $\mathbb{R}$, so we determine that " $\supseteq_I$ " is "up and down" (or "*U·D*" order, in short) [40].

(ii) For each given $\left[₩_*,\ ₩^*\right], \left[₦_*,\ ₦^*\right] \in \mathbb{R}_I$, we say that $\left[₩_*,\ ₩^*\right] \leq_I \left[₦_*,\ ₦^*\right]$ if and only if $₩_* \leq ₦_*,\ ₩^* \leq ₦^*$ or $₩_* \leq ₦_*,\ ₩^* < ₦^*$, it is a partial interval order relation. The relation $\left[₩_*,\ ₩^*\right] \leq_I \left[₦_*,\ ₦^*\right]$ coincident to $\left[₩_*,\ ₩^*\right] \leq \left[₦_*,\ ₦^*\right]$ on $\mathbb{R}_I$. It can be easily seen that " $\leq_I$ " looks like "left and right" on the real line $\mathbb{R}$, so we determine that " $\leq_I$ " is "left and right" (or "LR" order, in short) [39,40].

For $\left[\text{₩}_*,\ \text{₩}^*\right],\ \left[\text{₦}_*, \text{₦}^*\right] \in \mathcal{X}_C$, the Hausdorff–Pompeiu distance between intervals $\left[\text{₩}_*,\ \text{₩}^*\right]$, and $\left[\text{₦}_*,\ \text{₦}^*\right]$ is defined by

$$d_H\left(\left[\text{₩}_*,\ \text{₩}^*\right],\ \left[\text{₦}_*,\ \text{₦}^*\right]\right) = \max\left\{\left|\text{₩}_* - \text{₦}_*\right|,\ \left|\text{₩}^* - \text{₦}^*\right|\right\}. \tag{10}$$

It is familiar fact that $(\mathcal{X}_C, d_H)$ is a complete metric space, see [33,37,38].

Definition 1 ([32]). *A fuzzy subset L of $\mathbb{R}$ is distinguished by a mapping $\widetilde{\text{₦}} : \mathbb{R} \to [0,1]$ called the membership mapping of L. That is, a fuzzy subset L of $\mathbb{R}$ is a mapping $\widetilde{\text{₦}} : \mathbb{R} \to [0,1]$. So for further study, we have chosen this notation. We appoint ₤ to denote the set of all fuzzy subsets of $\mathbb{R}$.*

Let $\widetilde{\text{₦}} \in$ ₤. Then, $\widetilde{\text{₦}}$ is known as a fuzzy number or fuzzy number if the following properties are satisfied by $\widetilde{\text{₦}}$:

(1) $\widetilde{\text{₦}}$ should be normal if there exists $v \in \mathbb{R}$ and $\widetilde{\text{₦}}(v) = 1$;

(2) $\widetilde{\text{₦}}$ should be upper semi-continuous on $\mathbb{R}$ if for given $v \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\widetilde{\text{₦}}(v) - \widetilde{\text{₦}}(s) < \varepsilon$ for all $s \in \mathbb{R}$ with $|v - s| < \delta$;

(3) $\widetilde{\text{₦}}$ should be fuzzy convex that is $\widetilde{\text{₦}}((1-\sigma)v + \sigma s) \geq min\left(\widetilde{\text{₦}}(v),\ \widetilde{\text{₦}}(s)\right)$, for all $v, s \in \mathbb{R}$, and $\sigma \in [0,\ 1]$

(4) $\widetilde{\text{₦}}$ should be compactly supported that is $cl\left\{v \in \mathbb{R}\middle|\ \widetilde{\text{₦}}(v)\right\rangle 0\Big\}$ is compact.

We appoint ₤_C to denote the set of all fuzzy numbers of $\mathbb{R}$.

Definition 2 ([32,33]). *Given $\widetilde{\text{₦}} \in \text{₤}_C$, the level sets or cut sets are given by $\left[\widetilde{\text{₦}}\right]^{\mathfrak{o}} = \left\{v \in \mathbb{R}\middle|\ \widetilde{\text{₦}}(v)\right\rangle \mathfrak{o}\Big\}$ for all $\mathfrak{o} \in [0,\ 1]$ and by $\left[\widetilde{\text{₦}}\right]^{0} = \left\{v \in \mathbb{R}\middle|\ \widetilde{\text{₦}}(v)\right\rangle 0\Big\}$. These sets are known as $\mathfrak{o}$-level sets or $\mathfrak{o}$-cut sets of $\widetilde{\text{₦}}$.*

Proposition 1 ([34]). *Let $\widetilde{\text{₦}}, \widetilde{\text{₩}} \in \text{₤}_C$. Then relation " $\leq_{\mathbb{F}}$ " given on ₤_C by*

$$\widetilde{\text{₦}} \leq_{\mathbb{F}} \widetilde{\text{₩}} \text{ when, and only when, } \left[\widetilde{\text{₦}}\right]^{\mathfrak{o}} \leq_I \left[\widetilde{\text{₩}}\right]^{\mathfrak{o}}, \text{ for every } \mathfrak{o} \in [0,\ 1], \tag{11}$$

it is left and right order relation.

Proposition 2 ([30]). *Let $\widetilde{\text{₦}}, \widetilde{\text{₩}} \in \text{₤}_C$. Then relation " $\supseteq_{\mathbb{F}}$ " given on ₤_C by*

$$\widetilde{\text{₦}} \supseteq_{\mathbb{F}} \widetilde{\text{₩}} \text{ when, and only when, } \left[\widetilde{\text{₦}}\right]^{\mathfrak{o}} \supseteq_I \left[\widetilde{\text{₩}}\right]^{\mathfrak{o}}, \text{ for every } \mathfrak{o} \in [0,\ 1], \tag{12}$$

it is up and down order relation on ₤_C.

Proof. The proof follows directly from the up and down relation $\supseteq_I$ defined on $\mathcal{X}_C$. □

Remember the approaching notions, which are offered in the literature. If $\widetilde{\text{₦}}, \widetilde{\text{₩}} \in \text{₤}_C$ and $\mathfrak{o} \in \mathbb{R}$, then, for every $\mathfrak{o} \in [0,\ 1]$, the arithmetic operations are defined by

$$\left[\widetilde{\text{₦}} \oplus \widetilde{\text{₩}}\right]^{\mathfrak{o}} = \left[\widetilde{\text{₦}}\right]^{\mathfrak{o}} + \left[\widetilde{\text{₩}}\right]^{\mathfrak{o}}, \tag{13}$$

$$\left[\widetilde{₦} \otimes \widetilde{₩}\right]^{\mathfrak{o}} = \left[\widetilde{₦}\right]^{\mathfrak{o}} \times \left[\widetilde{₩}\right]^{\mathfrak{o}}, \tag{14}$$

$$\left[\sigma \odot \widetilde{₦}\right]^{\mathfrak{o}} = \sigma.\left[\widetilde{₦}\right]^{\mathfrak{o}} \tag{15}$$

These operations follow directly from the Equations (5)–(7), respectively.

Theorem 1 ([33]). *The space* $£_C$ *dealing with a supremum metric i.e., for* $\widetilde{₦},\ \widetilde{₩} \in £_C$

$$d_\infty\left(\widetilde{₦},\ \widetilde{₩}\right) = \sup_{0 \le \mathfrak{o} \le 1} d_H\left(\left[\widetilde{₦}\right]^{\mathfrak{o}},\ \left[\widetilde{₩}\right]^{\mathfrak{o}}\right), \tag{16}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on space of intervals.

3. Riemann Integral Operators for the Interval- and Fuzzy-Number Valued Mappings

Now we define and discuss some properties of fractional integral operators of interval- and fuzzy-number valued mappings.

Theorem 2 ([33,34]). *If* $\mathfrak{U} : [a, z] \subset \mathbb{R} \to \mathcal{X}_C$ *is an interval-valued mapping (I-V·M) satisfying that* $\mathfrak{U}(v) = [\mathfrak{U}_*(v),\ \mathfrak{U}^*(v)]$, *then* $\mathfrak{U}$ *is Aumann integrable (IA-integrable) over* $[a, z]$ *when and only when* $\mathfrak{U}_*(v)$ *and* $\mathfrak{U}^*(v)$ *both are integrable over* $[a, z]$ *such that*

$$(IA)\int_a^z \mathfrak{U}(v)dv = \left[\int_a^z \mathfrak{U}_*(v)dv,\ \int_a^z \mathfrak{U}^*(v)dv\right] \tag{17}$$

Definition 3 ([39]). *Let* $\widetilde{\mathfrak{U}} : \mathbb{I} \subset \mathbb{R} \to £_C$ *is called fuzzy-number valued mapping. Then, for every* $\mathfrak{o} \in [0,\ 1]$, *as well as* $\mathfrak{o}$*-levels define the family of I-V·Ms* $\mathfrak{U}_{\mathfrak{o}} : \mathbb{I} \subset \mathbb{R} \to \mathcal{X}_C$ *satisfying that* $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})]$ *for every* $v \in \mathbb{I}$. *Here, for every* $\mathfrak{o} \in [0,\ 1]$, *the endpoint real-valued mappings* $\mathfrak{U}_*(\bullet, \mathfrak{o}),\ \mathfrak{U}^*(\bullet, \mathfrak{o}) : \mathbb{I} \to \mathbb{R}$ *are called lower and upper mappings of* $\mathfrak{U}_{\mathfrak{o}}$.

Definition 4 ([39]). *Let* $\widetilde{\mathfrak{U}} : \mathbb{I} \subset \mathbb{R} \to £_C$ *be a F-N·V·M. Then* $\widetilde{\mathfrak{U}}(v)$ *is said to be continuous at* $v \in \mathbb{I}$, *if for every* $\mathfrak{o} \in [0,\ 1]$,$\mathfrak{U}_{\mathfrak{o}}(v)$ *is continuous when and only when, both endpoint mappings* $\mathfrak{U}_*(v, \mathfrak{o})$, *and* $\mathfrak{U}^*(v, \mathfrak{o})$ *are continuous at* $v \in \mathbb{I}$.

Definition 5 ([33]). *Let* $\widetilde{\mathfrak{U}} : [a,\ z] \subset \mathbb{R} \to £_C$ *be F-N·V·M. The fuzzy Aumann integral ((FA)-integral) of* $\mathfrak{U}$ *over* $[a,\ z]$, *denoted by* $(FA)\int_a^z \widetilde{\mathfrak{U}}(v)dv$, *is defined level-wise by*

$$\left[(FA)\int_a^z \widetilde{\mathfrak{U}}(v)dv\right]^{\mathfrak{o}} = (IA)\int_a^z \mathfrak{U}_{\mathfrak{o}}(v)dv = \left\{\int_a^z \mathfrak{U}(v, \mathfrak{o})dv : \mathfrak{U}(v, \mathfrak{o}) \in S(\mathfrak{U}_{\mathfrak{o}})\right\}, \tag{18}$$

where $S(\mathfrak{U}_{\mathfrak{o}}) = \{\mathfrak{U}(., \mathfrak{o}) \to \mathbb{R} : \mathfrak{U}(., \mathfrak{o})$ *is integrable and* $\mathfrak{U}(v, \mathfrak{o}) \in \mathfrak{U}_{\mathfrak{o}}(v)\}$, *for every* $\mathfrak{o} \in [0,\ 1]$. $\mathfrak{U}$ *is (FA)-integrable over* $[a,\ z]$ *if* $(FA)\int_a^z \widetilde{\mathfrak{U}}(v)dv \in £_C$.

Theorem 3 ([34]). *Let* $\widetilde{\mathfrak{U}} : [a,\ z] \subset \mathbb{R} \to £_C$ *be a F-N·V·M as well as* $\mathfrak{o}$*-levels define the family of I-V·Ms* $\mathfrak{U}_{\mathfrak{o}} : [a,\ z] \subset \mathbb{R} \to \mathcal{X}_C$ *satisfying that* $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})]$ *for every* $v \in [a,\ z]$

and for every $\mathfrak{o} \in [0,\ 1]$. *Then* $\widetilde{\mathfrak{U}}$ *is* (FA)*-integrable over* $[a,\ z]$ *when, and only when,* $\mathfrak{U}_*(v, \mathfrak{o})$ *and* $\mathfrak{U}^*(v, \mathfrak{o})$ *both are integrable over* $[a,\ z]$. *Moreover, if* $\mathfrak{U}$ *is* (FA)*-integrable over* $[a,\ z]$*, then*

$$\left[(FA)\int_a^z \widetilde{\mathfrak{U}}(v)dv\right]^o = \left[\int_a^z \mathfrak{U}_*(v, \mathfrak{o})dv,\ \int_a^z \mathfrak{U}^*(v, \mathfrak{o})dv\right] = (IA)\int_a^z \mathfrak{U}_{\mathfrak{o}}(v)dv \tag{19}$$

for every $\mathfrak{o} \in [0,\ 1]$.

Breckner discussed the emerging idea of interval-valued convexity in [35].

An interval valued mapping $\mathfrak{U} : \mathbb{I} = [a,\ z] \to \mathcal{X}_C$ is called convex inteval valued mapping if

$$\mathfrak{U}(\sigma v + (1-\sigma)s) \supseteq \sigma\mathfrak{U}(v) + (1-\sigma)\mathfrak{U}(s), \tag{20}$$

for all $v,\ s \in [a,\ z]$, $\sigma \in [0,\ 1]$, where $\mathcal{X}_C$ is the collection of all real valued intervals. If (20) is reversed, then $\mathfrak{U}$ is called concave.

Definition 6 ([31]). *The F-N·V·M* $\widetilde{\mathfrak{U}} : [a,\ z] \to \text{£}_C$ *is called convex F-N·V·M on* $[a,\ z]$ *if*

$$\widetilde{\mathfrak{U}}(\sigma v + (1-\sigma)s) \leq_{\mathbb{F}} \sigma \odot \widetilde{\mathfrak{U}}(v) \oplus (1-\sigma) \odot \widetilde{\mathfrak{U}}(s), \tag{21}$$

for all $v,\ s \in [a,\ z]$, $\sigma \in [0,\ 1]$, *where* $\widetilde{\mathfrak{U}}(v) \geq_{\mathbb{F}} \widetilde{0}$ *for all* $v \in [a,\ z]$. *If (21) is reversed then,* $\widetilde{\mathfrak{U}}$ *is called concave F-N·V·M on* $[a,\ z]$. $\widetilde{\mathfrak{U}}$ *is affine if and only if it is both convex and concave F-N·V·M.*

Definition 7 ([40]). *The F-N·V·M* $\widetilde{\mathfrak{U}} : [a,\ z] \to \text{£}_C$ *is called U·D convex F-N·V·M on* $[a,\ z]$ *if*

$$\widetilde{\mathfrak{U}}(\sigma v + (1-\sigma)s) \supseteq_{\mathbb{F}} \sigma \odot \widetilde{\mathfrak{U}}(v) \oplus (1-\sigma) \odot \widetilde{\mathfrak{U}}(s), \tag{22}$$

for all $v,\ s \in [a,\ z]$, $\sigma \in [0,\ 1]$, *where* $\widetilde{\mathfrak{U}}(v) \geq_{\mathbb{F}} \widetilde{0}$ *for all* $v \in [a,\ z]$. *If (22) is reversed then,* $\widetilde{\mathfrak{U}}$ *is called U·D concave F-N·V·M on* $[a,\ z]$. $\widetilde{\mathfrak{U}}$ *is U·D affine F-N·V·M if and only if it is both U·D convex and U·D concave F-N·V·M.*

Definition 8 ([44]). *Let K be an invex set and* $h : [0,\ 1] \to \mathbb{R}$ *such that* $h(v) > 0$. *Then F-N·V·M* $\widetilde{\mathfrak{U}} : K \to \text{£}_C$ *is said to be U·D h-pre-invex on K with respect to* ϖ *if*

$$\widetilde{\mathfrak{U}}(v + (1-\sigma)\varpi(y, v)) \supseteq_{\mathbb{F}} \sigma \odot \widetilde{\mathfrak{U}}(v) \oplus (1-\sigma) \odot \widetilde{\mathfrak{U}}(y), \tag{23}$$

for all $v,\ y \in K$, $\sigma \in [0,\ 1]$, *where* $\widetilde{\mathfrak{U}}(v) \geq_{\mathbb{F}} \widetilde{0}$, $\varpi : K \times K \to \mathbb{R}$. *The mapping* $\widetilde{\mathfrak{U}}$ *is said to be U·D h-pre-incave on K with respect to* ϖ *if inequality (23) is reversed.*

Theorem 4 ([44]). *Let* $\widetilde{\mathfrak{U}} : [a, z] \to \text{£}_C$ *be an F-N·V·M, whose* $\mathfrak{o}$*-levels define the family of I-V·Ms* $\mathfrak{U}_{\mathfrak{o}} : [a, z] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ *are given by*

$$\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})], \tag{24}$$

for all $v \in [a, z]$ *and for all* $\mathfrak{o} \in [0,\ 1]$. *Then,* $\widetilde{\mathfrak{U}}$ *is U·D h-pre-invex F-N·V·M on* $[a, z]$*, if and only if, for all* $\mathfrak{o} \in [0,\ 1]$, $\mathfrak{U}_*(v,\ \mathfrak{o})$ *is a h-pre-invex mapping and* $\mathfrak{U}^*(v,\ \mathfrak{o})$ *is a h-pre-incave mapping.*

The following assumption is required to prove the next result regarding the bi-function $\varpi : K \times K \to \mathbb{R}$ which is known as:

Condition C. See [6]. Let K be an invex set with respect to ϖ. For any $a,\ z \in K$ and $\sigma \in [0,\ 1]$,

$$\varpi(z, a + \sigma\varpi(z, a)) = (1-\sigma)\varpi(z, a),$$
$$\varpi(a, a + \sigma\varpi(z, a)) = -\sigma\varpi(z, a).$$

Clearly, for $\sigma = 0$, we have $\varpi(z, a) = 0$ if, and only if, $z = a$, for all $a,\ z \in K$. For the applications of Condition C, see [6,27–29,41,44,45].

4. Up and Down Fuzzy-Number Valued Mappings and Related Fuzzy Integral Inequalities

In this section, we discuss our key findings. We begin by introducing the category of $U{\cdot}D$ h-pre-invex mappings with fuzzy number values.

Definition 9. *Let K be an invex set and $h : [0,\ 1] \to \mathbb{R}$ such that $h(v) > 0$. Then F-N·V·M $\widetilde{\mathfrak{U}} : K \to \mathbb{E}_C$ is said to be U·D h-pre-invex on K with respect to ϖ if*

$$\widetilde{\mathfrak{U}}(v + (1-\sigma)\varpi(y, v)) \supseteq_{\mathbb{F}} h(\sigma) \odot \widetilde{\mathfrak{U}}(v) \oplus h(1-\sigma) \odot \widetilde{\mathfrak{U}}(y), \quad (25)$$

for all $v,\ y \in K$, $\sigma \in [0,\ 1]$, where $\widetilde{\mathfrak{U}}(v) \geq_{\mathbb{F}} \widetilde{0}$, $\varpi : K \times K \to \mathbb{R}$. The mapping $\widetilde{\mathfrak{U}}$ is said to be U·D h-pre-incave on K with respect to ϖ if inequality (25) is reversed.

Remark 2. *The U·D h-pre-invex F-N·V·Ms have some very nice properties similar to pre-invex F-N·V·M,*

(1) if $\widetilde{\mathfrak{U}}$ is U·D h-pre-invex F-N·V·M, then $\Upsilon\widetilde{\mathfrak{U}}$ is also U·D h-pre-invex for $\Upsilon \geq 0$.

(2) if $\widetilde{\mathfrak{U}}$ and $\widetilde{\mathcal{J}}$ both are U·D h-pre-invex F-N·V·Ms, then max$\left(\widetilde{\mathfrak{U}}(v), \widetilde{\mathcal{J}}(v)\right)$ is also U·D h-pre-invex F-N·V·M.

Now we discuss some new special cases of $U{\cdot}D$ h-pre-invex F-$N{\cdot}V{\cdot}Ms$:

If $h(\sigma) = \sigma^s$, then $U{\cdot}D$ h-pre-invex F-$N{\cdot}V{\cdot}M$ becomes $U{\cdot}D$ s-pre-invex F-$N{\cdot}V{\cdot}M$, that is

$$\widetilde{\mathfrak{U}}(v + (1-\sigma)\varpi(y, v)) \supseteq_{\mathbb{F}} \sigma^s \odot \widetilde{\mathfrak{U}}(v) \oplus (1-\sigma)^s \odot \widetilde{\mathfrak{U}}(y),\ \forall\ v,\ y \in K,\ \sigma \in [0,\ 1]. \quad (26)$$

If $\varpi(y, v) = y - v$, then $\widetilde{\mathfrak{U}}$ is called U·D s-convex F-$N{\cdot}V{\cdot}M$.

If $h(\sigma) = \sigma$, then $U{\cdot}D$ h-pre-invex F-$N{\cdot}V{\cdot}M$ becomes $U{\cdot}D$ pre-invex F-$N{\cdot}V{\cdot}M$, see [44].

If $\varpi(y, v) = y - v$, then $\widetilde{\mathfrak{U}}$ is called $U{\cdot}D$ convex F-$N{\cdot}V{\cdot}M$, this is the resulting new one:

$$\widetilde{\mathfrak{U}}(\sigma v + (1-\sigma)y) \supseteq_{\mathbb{F}} h(\sigma) \odot \widetilde{\mathfrak{U}}(v) \oplus h(1-\sigma) \odot \widetilde{\mathfrak{U}}(y), \quad (27)$$

If $h(\sigma) \equiv 1$, and $\varpi(y, v) = y - v$, then $\widetilde{\mathfrak{U}}$ is called $U{\cdot}D$ convex F-$N{\cdot}V{\cdot}M$, this is the resulting new one:

$$\widetilde{\mathfrak{U}}(\sigma v + (1-\sigma)y) \supseteq_{\mathbb{F}} \sigma \odot \widetilde{\mathfrak{U}}(v) \oplus (1-\sigma) \odot \widetilde{\mathfrak{U}}(y), \quad (28)$$

If $h(\sigma) \equiv 1$, then $U{\cdot}D$ h-pre-invex F-$N{\cdot}V{\cdot}M$ becomes $U{\cdot}D$ P-pre-invex F-$N{\cdot}V{\cdot}M$, this is the resulting new one:

$$\widetilde{\mathfrak{U}}(v + (1-\sigma)\varpi(y, v)) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{U}}(v) \oplus \widetilde{\mathfrak{U}}(y),\ \forall\ v,\ y \in K,\ \sigma \in [0,\ 1]. \quad (29)$$

If $\varpi(y, v) = y - v$, then $\widetilde{\mathfrak{U}}$ is called P-F-$N{\cdot}V{\cdot}M$.

Theorem 5. *Let K be an invex set and $h : [0,\ 1] \subseteq K \to \mathbb{R}^+$, and let $\widetilde{\mathfrak{U}} : K \to \mathbb{E}_C$ be a F-N·V·M with $\widetilde{\mathfrak{U}}(v) \geq_{\mathbb{F}} \widetilde{0}$, whose $\mathfrak{o}$-levels define the family of I·V·Ms $\mathfrak{U}_{\mathfrak{o}} : K \subset \mathbb{R} \to \mathcal{K}_C{}^+ \subset \mathcal{K}_C$ is given by*

$$\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})],\ \forall\ v \in K. \quad (30)$$

for all $v \in K$ and for all $\mathfrak{o} \in [0,\ 1]$. Then, $\widetilde{\mathfrak{U}}$ is U·D h-pre-invex F-N·V·M on K, if, and only if, for all $\mathfrak{o} \in [0,\ 1]$, $\mathfrak{U}_*(v,\ \mathfrak{o})$, and $\mathfrak{U}^*(v,\ \mathfrak{o})$ are h-pre-invex and h-pre-incave functions, respectively.

Proof. Assume that for each $\mathfrak{o} \in [0,\ 1]$, $\mathfrak{U}_*(v,\ \mathfrak{o})$ and $\mathfrak{U}^*(v,\ \mathfrak{o})$ are h-pre-invex and h-pre-incave functions on K, respectively. Then from (25), we have

$$\mathfrak{U}_*(v + (1-\sigma)\varpi(y,v),\ \mathfrak{o}) \le h(\sigma)\mathfrak{U}_*(v,\ \mathfrak{o}) + h(1-\sigma)\mathfrak{U}_*(y,\ \mathfrak{o}),\ \forall\ v,y \in K,\ \sigma \in [0,\ 1],$$

and

$$\mathfrak{U}^*(v + (1-\sigma)\varpi(y,v),\ \mathfrak{o}) \ge h(\sigma)\mathfrak{U}^*(v,\ \mathfrak{o}) + h(1-\sigma)\mathfrak{U}^*(y,\ \mathfrak{o}), \forall\ v,y \in K,\ \sigma \in [0,\ 1].$$

Then by (30), (13), and (15), we obtain

$$\mathfrak{U}_{\mathfrak{o}}(v + (1-\sigma)\varpi(y,v)) = [\mathfrak{U}_*(v + (1-\sigma)\varpi(y,v),\ \mathfrak{o}),\ \mathfrak{U}^*(v + (1-\sigma)\varpi(y,v),\ \mathfrak{o})]$$
$$\supseteq_I [h(\sigma)\mathfrak{U}_*(v,\ \mathfrak{o}),\ h(\sigma)\mathfrak{U}^*(v,\ \mathfrak{o})] + [h(1-\sigma)\mathfrak{U}_*(y,\ \mathfrak{o}),\ h(1-\sigma)\mathfrak{U}^*(y,\ \mathfrak{o})],$$

that is

$$\widetilde{\mathfrak{U}}(v + (1-\sigma)\varpi(y,v)) \supseteq_{\mathbb{F}} h(\sigma) \odot \widetilde{\mathfrak{U}}(v) \oplus h(1-\sigma) \odot \widetilde{\mathfrak{U}}(y), \forall\ v,y \in K,\ \sigma \in [0,\ 1].$$

Hence, $\widetilde{\mathfrak{U}}$ is $U{\cdot}D$ h-pre-invex F-$N{\cdot}V{\cdot}M$ on K.

Conversely, let $\widetilde{\mathfrak{U}}$ be an U·D h-pre-invex F-$N{\cdot}V{\cdot}M$ on K. Then, for all $v, y \in K$ and $\sigma \in [0,\ 1]$, we have $\widetilde{\mathfrak{U}}(v + (1-\sigma)\varpi(y,v)) \supseteq_{\mathbb{F}} h(\sigma) \odot \widetilde{\mathfrak{U}}(v) \oplus h(1-\sigma) \odot \widetilde{\mathfrak{U}}(y)$. Therefore, from (13), we have

$$\mathfrak{U}_{\mathfrak{o}}(v + (1-\sigma)\varpi(y,v)) = [\mathfrak{U}_*(v + (1-\sigma)\varpi(y,v),\ \mathfrak{o}),\ \mathfrak{U}^*(v + (1-\sigma)\varpi(y,v),\ \mathfrak{o})].$$

Again, from (30), (13), and (15), we obtain

$$\begin{aligned}&h(\sigma)\mathfrak{U}_{\mathfrak{o}}(v) + h(1-\sigma)\mathfrak{U}_{\mathfrak{o}}(v)\\ &= [h(\sigma)\mathfrak{U}_*(v,\ \mathfrak{o}),\ h(\sigma)\mathfrak{U}^*(v,\ \mathfrak{o})] + [h(1-\sigma)\mathfrak{U}_*(y,\ \mathfrak{o}),\ h(1-\sigma)\mathfrak{U}^*(y,\ \mathfrak{o})],\end{aligned}$$

for all $v, y \in K$ and $\sigma \in [0,\ 1]$. Then by $U{\cdot}D$ h-pre-invexity of $\widetilde{\mathfrak{U}}$, we have for all $v, y \in K$ and $\sigma \in [0,\ 1]$ such that

$$\mathfrak{U}_*(v + (1-\sigma)\varpi(y,v), \mathfrak{o}) \le h(\sigma)\mathfrak{U}_*(v,\ \mathfrak{o}) + h(1-\sigma)\mathfrak{U}_*(y,\ \mathfrak{o}),$$

and

$$\mathfrak{U}^*(v + (1-\sigma)\varpi(y,v), \mathfrak{o}) \ge h(\sigma)\mathfrak{U}^*(v,\ \mathfrak{o}) + h(1-\sigma)\mathfrak{U}^*(y,\ \mathfrak{o}),$$

for each $\mathfrak{o} \in [0,\ 1]$. Hence, the result follows. □

Example 1. *We consider $h(\sigma) = \sigma$, for $\sigma \in [2,\ 3]$ and the F-N·V·M $\widetilde{\mathfrak{U}} : \mathbb{R}^+ \to \mathbb{E}_C$ defined by,*

$$\widetilde{\mathfrak{U}}(v)(\varrho) = \begin{cases} \frac{\varrho - 2 + v^{\frac{1}{2}}}{1 - v^{\frac{1}{2}}} & \varrho \in \left[2 - v^{\frac{1}{2}},\ 3\right] \\ \frac{4 + v^{\frac{1}{2}} - \varrho}{1 + v^{\frac{1}{2}}} & \varrho \in \left(3, 4 + v^{\frac{1}{2}}\right] \\ 0 & otherwise, \end{cases}$$

then, for each $\mathfrak{o} \in [0,\ 1]$, we have $\mathfrak{U}_{\mathfrak{o}}(v) = \left[(1-\mathfrak{o})\left(2 - v^{\frac{1}{2}}\right) + 3\mathfrak{o}, (1-\mathfrak{o})\left(4 + v^{\frac{1}{2}}\right) + 3\mathfrak{o}\right]$. Since $\mathfrak{U}_*(v, \mathfrak{o}), \mathfrak{U}^*(v, \mathfrak{o})$ are h-pre-invex functions $\varpi(y,v) = y - v$ for each $\mathfrak{o} \in [0,\ 1]$. Hence $\widetilde{\mathfrak{U}}(v)$ is U·D h-pre-invex F-N·V·M.

Now we have obtained some new definitions from the literature which will be helpful to investigate some classical and new results as special cases of the main results.

Definition 10. *Let $\widetilde{\mathfrak{U}} : [a, z] \to \mathbb{E}_C$ be a F-N·V·M, whose $\mathfrak{o}$-levels define the family of I-V·Ms $\mathfrak{U}_{\mathfrak{o}} : [a, z] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ are given by*

$$\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})], \tag{31}$$

for all $v \in [a, z]$ and for all $\mathfrak{o} \in [0,\ 1]$. Then,$\widetilde{\mathfrak{U}}$ is lower U·D h-pre-invex (h-pre-incave) F-N·V·M on $[a, z]$, if, and only if, for all $\mathfrak{o} \in [0,\ 1]$,

$$\mathfrak{U}_*(v + (1-\sigma)\varpi(y,\ v),\ \mathfrak{o}) \le (\ge) h(\sigma)\mathfrak{U}_*(v, \mathfrak{o}) + h(1-\sigma)\mathfrak{U}_*(y, \mathfrak{o}), \tag{32}$$

and

$$\mathfrak{U}^*(v + (1-\sigma)\varpi(y,\ v),\ \mathfrak{o}) = h(\sigma)\mathfrak{U}^*(v, \mathfrak{o}) + h(1-\sigma)\mathfrak{U}^*(y, \mathfrak{o}). \tag{33}$$

Definition 11. *Let $\widetilde{\mathfrak{U}} : [a, z] \to \mathbb{E}_C$ be a F-N·V·M, whose $\mathfrak{o}$-levels define the family of I-V·Ms $\mathfrak{U}_{\mathfrak{o}} : [a, z] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ are given by*

$$\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})], \tag{34}$$

for all $v \in [a, z]$ and for all $\mathfrak{o} \in [0,\ 1]$. Then, $\widetilde{\mathfrak{U}}$ is upper U·D h-pre-invex (h-pre-incave) F-N·V·M on $[a, z]$, if, and only if, for all $\mathfrak{o} \in [0,\ 1]$,

$$\mathfrak{U}_*(v + (1-\sigma)\varpi(y,\ v),\ \mathfrak{o}) = h(\sigma)\mathfrak{U}_*(v, \mathfrak{o}) + h(1-\sigma)\mathfrak{U}_*(y, \mathfrak{o}), \tag{35}$$

and

$$\mathfrak{U}^*(v + (1-\sigma)\varpi(y,\ v),\ \mathfrak{o}) \le (\ge) h(\sigma)\mathfrak{U}^*(v, \mathfrak{o}) + h(1-\sigma)\mathfrak{U}^*(y, \mathfrak{o}). \tag{36}$$

Remark 3. *Both concepts "U·D h-pre-invex F-N·V·M" and "h-pre-invex F-N·V·M, see [28]" behave alike when $\widetilde{\mathfrak{U}}$ is lower U·D h-pre-invex F-N·V·M.*

If we take $\varpi(y,\ v) = y - v$, then we acquire classical and new results from Definitions 7–9, Remarks 1 and 2, and Theorem 5, see [16,25,27,30,41,42,44,45].

The up and down h-pre-invex fuzzy-number valued mappings version of a Hermite–Hadamard type inequality can be represented as follows.

Theorem 6 . *Let $\widetilde{\mathfrak{U}} : [a,\ a + \varpi(z,\ a)] \to \mathbb{E}_C$ be an U·D h-pre-invex F-N·V·M with $h : [0,\ 1] \to \mathbb{R}^+$ and $h\left(\frac{1}{2}\right) \ne 0$, whose $\mathfrak{o}$-levels define the family of I·V·Ms $\mathfrak{U}_{\mathfrak{o}} : [a,\ a + \varpi(z,\ a)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})]$ for all $v \in [a,\ a + \varpi(z,\ a)]$ and for all $\mathfrak{o} \in [0,\ 1]$. If $\widetilde{\mathfrak{U}} \in \mathcal{FR}_{([a,\ a+\varpi(z,\ a)],\ \mathfrak{o})}$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a + \varpi(z,\ a)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)dv \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{U}}(a) \oplus \widetilde{\mathfrak{U}}(z)\right] \odot \int_0^1 h(\sigma)d\sigma. \tag{37}$$

If $\widetilde{\mathfrak{U}}$ is U·Dh-pre-incave F-N·V·M, then (37) is reversed such that

$$\frac{1}{2h\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a + \varpi(z,\ a)}{2}\right) \subseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)dv \subseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{U}}(a) \oplus \widetilde{\mathfrak{U}}(z)\right] \odot \int_0^1 h(\sigma)d\sigma. \tag{38}$$

Proof. Let $\widetilde{\mathfrak{U}} : [a,\ a + \varpi(z,\ a)] \to \mathbb{E}_C$ be an U·D h-pre-invex F-N·V·M. Then, by hypothesis, we have

$$\frac{1}{h\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{U}}(a+(1-\sigma)\varpi(z,\ a)) \oplus \widetilde{\mathfrak{U}}(a+\sigma\varpi(z,\ a)).$$

Therefore, for every $\mathfrak{o} \in [0,\ 1]$, we have

$$\begin{aligned}
\tfrac{1}{h\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\tfrac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) &\leq \mathfrak{U}_*(a+(1-\sigma)\varpi(z,\ a),\ \mathfrak{o}) + \mathfrak{U}_*(a+\sigma\varpi(z,\ a),\ \mathfrak{o}),\\
\tfrac{1}{h\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\tfrac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) &\geq \mathfrak{U}^*(a+(1-\sigma)\varpi(z,\ a),\mathfrak{o}) + \mathfrak{U}^*(a+\sigma\varpi(z,\ a),\ \mathfrak{o}).
\end{aligned}$$

Then

$$\begin{aligned}
\tfrac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \mathfrak{U}_*\left(\tfrac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right)d\sigma &\leq \int_0^1 \mathfrak{U}_*(a+(1-\sigma)\varpi(z,\ a),\ \mathfrak{o})d\sigma + \int_0^1 \mathfrak{U}_*(a+\sigma\varpi(z,\ a),\mathfrak{o})d\sigma,\\
\tfrac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \mathfrak{U}^*\left(\tfrac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right)d\sigma &\geq \int_0^1 \mathfrak{U}^*(a+(1-\sigma)\varpi(z,\ a),\mathfrak{o})d\sigma + \int_0^1 \mathfrak{U}^*(a+\sigma\varpi(z,\ a),\ \mathfrak{o})d\sigma.
\end{aligned}$$

It follows that

$$\begin{aligned}
\tfrac{1}{h\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\tfrac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) &\leq \tfrac{2}{\varpi(z,\ a)}\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})dv,\\
\tfrac{1}{h\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\tfrac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) &\geq \tfrac{2}{\varpi(z,\ a)}\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\ \mathfrak{o})dv.
\end{aligned}$$

That is

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[\mathfrak{U}_*\left(\frac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right),\ \mathfrak{U}^*\left(\frac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right)\right] \supseteq_I \frac{2}{\varpi(z,\ a)}\left[\int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})dv,\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\ \mathfrak{o})dv\right].$$

Thus,

$$\frac{1}{2h\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)dv. \tag{39}$$

In a similar way as above, we have

$$\frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)dv \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{U}}(a) \oplus \widetilde{\mathfrak{U}}(z)\right] \odot \int_0^1 h(\sigma)d\sigma. \tag{40}$$

Combining (39) and (40), we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)dv \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{U}}(a) \oplus \widetilde{\mathfrak{U}}(z)\right] \odot \int_0^1 h(\sigma)d\sigma,$$

which complete the proof. □

Note that, inequality (14) is known as fuzzy *HH* inequality for *U·D* *h*-pre-invex *F-N·V·M*.

Remark 4. *If* $h(\sigma) = \sigma^s$, *then Theorem 7 reduces to the result for U·D U·D s-pre-invex F-N·V·M:*

$$2^{s-1}\odot\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\supseteq_{\mathbb{F}}\frac{1}{s+1}\odot\left[\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z)\right]. \tag{41}$$

If $h(\sigma)=\sigma$, then Theorem 6 reduces to the result for U·D pre-invex F-N·V·M, see [44]*:*

$$\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\supseteq_{\mathbb{F}}\frac{\widetilde{\mathfrak{U}}(a)\ \oplus\ \widetilde{\mathfrak{U}}(z)}{2}. \tag{42}$$

If $h(\sigma)\equiv 1$, then Theorem 6 reduces to the result for U·DP-pre-invex F-N·V·M:

$$\frac{1}{2}\odot\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\supseteq_{\mathbb{F}}\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z). \tag{43}$$

If $\widetilde{\mathfrak{U}}$ is lower U·D h-pre-invex F-N·V·M, then we can get the following coming inequality, see [28]*:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\leq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\leq_{\mathbb{F}}\left[\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z)\right]\odot\int_{0}^{1}h(\sigma)d\sigma \tag{44}$$

If $h(\sigma)=\sigma^{s}$, then Theorem 6 reduces to the result for lower U·D s-pre-invex F-N·V·M, see [28]*:*

$$2^{s-1}\odot\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\leq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\leq_{\mathbb{F}}\frac{1}{s+1}\odot\left[\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z)\right]. \tag{45}$$

If $h(\sigma)=\sigma$, then Theorem 6 reduces to the result for lower U·D pre-invex F-N·V·M, see [28]*:*

$$\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\leq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\leq_{\mathbb{F}}\frac{\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z)}{2}. \tag{46}$$

If $h(\sigma)\equiv 1$, then Theorem 6 reduces to the result for lower U·D P-pre-invex F-N·V·M, see [28]*:*

$$\frac{1}{2}\odot\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\leq_{\mathbb{F}}\frac{1}{\varpi(z,\ a)}\odot(FR)\int_{a}^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)dv\leq_{\mathbb{F}}\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z). \tag{47}$$

If $\mathfrak{U}_{*}(v,\ \mathfrak{o})=\mathfrak{U}^{*}(v,\ \mathfrak{o})$ and $\mathfrak{o}=1$, then Theorem 6 reduces to the result for h-pre-invex function, see [41]*:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{U}\left(\frac{2a+\varpi(z,\ a)}{2}\right)\leq\frac{1}{\varpi(z,\ a)}(IR)\int_{a}^{a+\varpi(z,\ a)}\mathfrak{U}(v)dv\leq[\mathfrak{U}(a)+\mathfrak{U}(z)]\int_{0}^{1}h(\sigma)d\sigma. \tag{48}$$

Note that, if $\varpi(y,\ v)=y-v$, then integral inequalities (18)–(21) reduce to new ones.

Example 2. *We consider $h(\sigma)=\sigma$, for $\sigma\in[0,\ 1]$, and the F-N·V·M $\widetilde{\mathfrak{U}}:[a,\ a+\varpi(z,\ a)]=[2,\ 3+\varpi(3,\ 2)]\rightarrow \text{₤}_{C}$ defined by,*

$$\widetilde{\mathfrak{U}}(v)(\varrho)=\begin{cases}\frac{\varrho-2+v^{\frac{1}{2}}}{1-v^{\frac{1}{2}}} & \varrho\in\left[2-v^{\frac{1}{2}},\ 3\right]\\ \frac{2+v^{\frac{1}{2}}-\varrho}{v^{\frac{1}{2}}-1} & \varrho\in\left(3,\ 2+v^{\frac{1}{2}}\right]\\ 0 & \text{otherwise,}\end{cases} \tag{49}$$

Then, for each $\mathfrak{o} \in [0,\ 1]$*, we have* $\mathfrak{U}_{\mathfrak{o}}(v) = \left[(1-\mathfrak{o})\left(2 - v^{\frac{1}{2}}\right) + 3\mathfrak{o}, (1+\mathfrak{o})\left(2 + v^{\frac{1}{2}}\right) + 3\mathfrak{o}\right]$. *Since left and right end point mappings* $\mathfrak{U}_*(v, \mathfrak{o}) = (1-\mathfrak{o})\left(2 - v^{\frac{1}{2}}\right) + 3\mathfrak{o}$, *and* $\mathfrak{U}^*(v,\ \mathfrak{o}) = (1+\mathfrak{o})\left(2 + v^{\frac{1}{2}}\right) + 3\mathfrak{o}$, *are pre-invex and pre-incave mappings with* $\varpi(y,\ v) = y - v$ *for each* $\mathfrak{o} \in [0,\ 1]$, *respectively, then* $\widetilde{\mathfrak{U}}(v)$ *is U·D pre-invex F-N·V·M with* $\varpi(y,\ v) = y - v$. *We clearly see that* $\widetilde{\mathfrak{U}} \in L([a, z], \varpi_C)$ *and*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\,\mathfrak{U}_*\left(\frac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) \leq \frac{1}{\varpi(z,\ a)}\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})dv \leq [\mathfrak{U}_*(a,\ \mathfrak{o}) + \mathfrak{U}_*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)d\sigma.$$

$$\frac{1}{2h\left(\frac{1}{2}\right)}\,\mathfrak{U}_*\left(\frac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) = \mathfrak{U}_*\left(\frac{5}{2},\ \mathfrak{o}\right) = (1-\mathfrak{o})\frac{4-\sqrt{10}}{2} + 3\mathfrak{o},$$

$$\frac{1}{\varpi(z,\ a)}\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})dv = \int_2^3\left((1-\mathfrak{o})\left(2 - v^{\frac{1}{2}}\right) + 3\mathfrak{o}\right)dv = \frac{843}{2000}(1-\mathfrak{o}) + 3\mathfrak{o},$$

$$[\mathfrak{U}_*(a,\ \mathfrak{o}) + \mathfrak{U}_*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)d\sigma = (1-\mathfrak{o})\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right) + 3\mathfrak{o},$$

for all $\mathfrak{o} \in [0,\ 1]$.

Similarly, it can be easily shown that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\,\mathfrak{U}^*\left(\frac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) \geq \frac{1}{\varpi(z,\ a)}\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\ \mathfrak{o})dv \geq [\mathfrak{U}^*(a,\ \mathfrak{o}) + \mathfrak{U}^*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)d\sigma.$$

for all $\mathfrak{o} \in [0,\ 1]$, *such that*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\,\mathfrak{U}^*\left(\frac{2a+\varpi(z,\ a)}{2},\ \mathfrak{o}\right) = \mathfrak{U}^*\left(\frac{5}{2},\ \mathfrak{o}\right) = (1-\mathfrak{o})\frac{4+\sqrt{10}}{2} + 3\mathfrak{o},$$

$$\frac{1}{\varpi(z,\ a)}\ \int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\ \mathfrak{o})dv = \frac{1}{2}\ \int_0^2 (4-2\mathfrak{o})v^2dv = \frac{179}{50}(1-\mathfrak{o}) + 3\mathfrak{o},$$

$$[\mathfrak{U}^*(a,\ \mathfrak{o}) + \mathfrak{U}^*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)d\sigma = (1-\mathfrak{o})\left(\frac{4+\sqrt{2}+\sqrt{3}}{2}\right) + 3\mathfrak{o}.$$

that is

$$\begin{aligned}\left[(1-\mathfrak{o})\tfrac{4-\sqrt{10}}{2} + 3\mathfrak{o},\ (1-\mathfrak{o})\tfrac{4+\sqrt{10}}{2} + 3\mathfrak{o}\right] &\supseteq_I \left[\tfrac{843}{2000}(1-\mathfrak{o}) + 3\mathfrak{o},\ \tfrac{179}{50}(1-\mathfrak{o}) + 3\mathfrak{o}\right] \\ &\supseteq_I \left[(1-\mathfrak{o})\left(\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\right) + 3\mathfrak{o},\ (1-\mathfrak{o})\left(\tfrac{4+\sqrt{2}+\sqrt{3}}{2}\right) + 3\mathfrak{o}\right]\end{aligned}$$

for all $\mathfrak{o} \in [0,\ 1]$.
Hence,

$$\frac{1}{2h\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)dv \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{U}}(a) \oplus \widetilde{\mathfrak{U}}(z)\right] \odot \int_0^1 h(\sigma)d\sigma,$$

and the Theorem 6 is verified.

The product of two up and down h-pre-invex fuzzy-number valued mapping versions of a Hermite–Hadamard type inequality can be represented as follows.

Theorem 7. *Let* $\widetilde{\mathfrak{U}}, \widetilde{\mathcal{J}} : [a,\ a+\varpi(z,\ a)] \to \text{₤}_C$ *be two U·D* h_1 *and* h_2*-pre-invex F-N·V·Ms with* $h_1,\ h_2 : [0,\ 1] \to \mathbb{R}^+$ *and* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$*, whose* $\mathfrak{o}$*-levels define the family of I·V·Ms* $\mathfrak{U}_{\mathfrak{o}},\ \mathcal{J}_{\mathfrak{o}} : [a,\ a+\varpi(z,\ a)] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ *are given by* $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v,\mathfrak{o}),\ \mathfrak{U}^*(v,\mathfrak{o})]$ *and* $\mathcal{J}_{\mathfrak{o}}(v) = [\mathcal{J}_*(v,\mathfrak{o}),\ \mathcal{J}^*(v,\mathfrak{o})]$ *for all* $v \in [a,\ a+\varpi(z,\ a)]$ *and for all* $\mathfrak{o} \in [0,\ 1]$*. If* $\widetilde{\mathfrak{U}} \otimes \widetilde{\mathcal{J}} \in \mathcal{FR}_{([a,\ a+\varpi(z,\ a)],\ \mathfrak{o})}$*, then*

$$\begin{aligned} &\frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v) \otimes \widetilde{\mathcal{J}}(v)dv \\ &\supseteq_{\mathbb{F}} \widetilde{\mathcal{M}}(a,z) \odot \int_0^1 h_1(\sigma)h_2(\sigma)d\sigma \oplus \widetilde{\mathcal{N}}(a,z) \odot \int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma, \end{aligned} \tag{50}$$

where $\widetilde{\mathcal{M}}(a,z) = \widetilde{\mathfrak{U}}(a) \otimes \widetilde{\mathcal{J}}(a) \oplus \widetilde{\mathfrak{U}}(z) \otimes \widetilde{\mathcal{J}}(z)$, $\widetilde{\mathcal{N}}(a,z) = \widetilde{\mathfrak{U}}(a) \otimes \widetilde{\mathcal{J}}(z) \oplus \widetilde{\mathfrak{U}}(z) \otimes \widetilde{\mathcal{J}}(a)$ *with* $\mathcal{M}_{\mathfrak{o}}(a,z) = [\mathcal{M}_*((a,z),\ \mathfrak{o}),\ \mathcal{M}^*((a,z),\ \mathfrak{o})]$ *and* $\mathcal{N}_{\mathfrak{o}}(a,z) = [\mathcal{N}_*((a,z),\ \mathfrak{o}),\ \mathcal{N}^*((a,z),\ \mathfrak{o})]$.

Example 3. *We consider* $h_1(\sigma) = \sigma = h_2(\sigma)$*, for* $\sigma \in [0,\ 1]$*, and the F-N·V·Ms* $\widetilde{\mathfrak{U}},\ \widetilde{\mathcal{J}} : [a,\ a+\varpi(z,\ a)] = [0,\ \varpi(2,\ 0)] \to \text{₤}_C$ *defined by,*

$$\widetilde{\mathfrak{U}}(v)(\varrho) = \begin{cases} \frac{\varrho}{v} & \varrho \in [0,\ v] \\ \frac{2v-\varrho}{v} & \varrho \in (v,\ 2v] \\ 0 & otherwise, \end{cases} \tag{51}$$

$$\widetilde{\mathcal{J}}(v)(\varrho) = \begin{cases} \frac{\varrho-v}{2-v} & \varrho \in [v,\ 2] \\ \frac{8-e^v-\varrho}{8-e^v-2} & \varrho \in (2,\ 8-e^v] \\ 0 & otherwise. \end{cases} \tag{52}$$

Then, for each $\mathfrak{o} \in [0,\ 1]$*, we have* $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{o}v, (2-\mathfrak{o})v]$ *and* $\mathcal{J}_{\mathfrak{o}}(v) = [(1-\mathfrak{o})v + 2\mathfrak{o}, (1-\mathfrak{o})(8-e^v) + 2\mathfrak{o}]$*. Since* $\mathfrak{U}_*(v,\mathfrak{o}) = \mathfrak{o}v$ *and* $\mathfrak{U}^*(v,\ \mathfrak{o}) = 2-\mathfrak{o})v$ *both are* h_1*-pre-invex functions, and* $\mathcal{J}_*(v,\mathfrak{o}) = (1-\mathfrak{o})v + 2\mathfrak{o}$*, and* $\mathcal{J}^*(v,\ \mathfrak{o}) = (1-\mathfrak{o})(8-e^v) + 2\mathfrak{o}$ *both are also* h_2*-pre-invex functions with respect to same* $\varpi(z,\ a) = z - a$*, for each* $\mathfrak{o} \in [0,\ 1]$ *then,* $\widetilde{\mathfrak{U}}$ *and* $\widetilde{\mathcal{J}}$ *both are* h_1 *and* h_2*-pre-invex F-N·V·Ms, respectively. Now we compute the following:*

$$\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\mathfrak{o}) \times \mathcal{J}_*(v,\mathfrak{o})dv = \frac{1}{2}\int_0^2 \left(\mathfrak{o}(1-\mathfrak{o})v^2 + 2\mathfrak{o}^2 v\right)dv = \frac{2}{3}o(2+\mathfrak{o}),$$

$$\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\mathfrak{o}) \times \mathcal{J}^*(v,\mathfrak{o})dv = \frac{1}{2}\int_0^2 ((1-\mathfrak{o})(2-\mathfrak{o})v(8-e^v) + 2\mathfrak{o}(2-\mathfrak{o})v)dv \approx \frac{(2-\mathfrak{o})}{2}\left(\frac{1903}{250} - \frac{903}{250}\mathfrak{o}\right),$$

$$\begin{gathered} \mathcal{M}_*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma = \frac{4\mathfrak{o}}{3}, \\ \mathcal{M}^*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma = \frac{2(2-\mathfrak{o})\left[(1-\mathfrak{o})\left(8-e^2\right)+2\mathfrak{o}\right]}{3}, \\ \mathcal{N}_*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma = \frac{2\mathfrak{o}^2}{3} \\ \mathcal{N}^*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma = \frac{(2-\mathfrak{o})(7-5\mathfrak{o})}{3}, \end{gathered}$$

for each $\mathfrak{o} \in [0,\ 1]$*, that means*

$$\left[\frac{2}{3}\mathfrak{o}(1+2\mathfrak{o}), \frac{(2-\mathfrak{o})}{2}\left(\frac{1903}{250} - \frac{903}{250}\mathfrak{o}\right)\right] \supseteq_I \frac{1}{3}\left[2\mathfrak{o}(2+\mathfrak{o}), (2-\mathfrak{o})\left[2(1-\mathfrak{o})\left(8-e^2\right) - \mathfrak{o} + 7\right]\right]$$

Hence, Theorem 7 is verified.

Theorem 8. *Let* $\widetilde{\mathfrak{U}}, \widetilde{\mathcal{J}} : [a,\ a+\varpi(z,\ a)] \to \text{₤}_C$ *be two U·D* h_1*- and* h_2*-pre-invex F-N·V·Ms with* $h_1,\ h_2 : [0,\ 1] \to \mathbb{R}^+$ *and* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$*, respectively, whose* $\mathfrak{o}$*-levels define the family of I·V·Ms* $\mathfrak{U}_{\mathfrak{o}},\ \mathcal{J}_{\mathfrak{o}} : [a,\ a+\varpi(z,\ a)] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ *are given by* $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v,\mathfrak{o}),\ \mathfrak{U}^*(v,\mathfrak{o})]$ *and*

$\mathcal{J}_{\mathfrak{o}}(v) = [\mathcal{J}_*(v, \mathfrak{o}),\ \mathcal{J}^*(v, \mathfrak{o})]$ for all $v \in [a,\ a + \varpi(z,\ a)]$ and for all $\mathfrak{o} \in [0,\ 1]$. If $\widetilde{\mathfrak{U}}$, $\widetilde{\mathcal{J}}$ and $\widetilde{\mathfrak{U}} \otimes \widetilde{\mathcal{J}} \in \mathcal{FR}_{([a,\ a+\varpi(z,\ a)],\ \mathfrak{o})}$ and condition C hold for ϖ, then

$$\begin{aligned}\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \otimes \widetilde{\mathcal{J}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) &\supseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v) \otimes \widetilde{\mathcal{J}}(v)dv \\ &\oplus \widetilde{\mathcal{M}}(a,z) \odot \int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma \oplus \widetilde{\mathcal{N}}(a,z) \odot \int_0^1 h_1(\sigma)h_2(\sigma)d\sigma,\end{aligned} \tag{53}$$

where $\widetilde{\mathcal{M}}(a,z) = \widetilde{\mathfrak{U}}(a) \otimes \widetilde{\mathcal{J}}(a) \oplus \widetilde{\mathfrak{U}}(z) \otimes \widetilde{\mathcal{J}}(z)$, $\widetilde{\mathcal{N}}(a,z) = \widetilde{\mathfrak{U}}(a) \otimes \widetilde{\mathcal{J}}(z) \oplus \widetilde{\mathfrak{U}}(z) \otimes \widetilde{\mathcal{J}}(a)$, and $\mathcal{M}_{\mathfrak{o}}(a,z) = [\mathcal{M}_*((a,z),\ \mathfrak{o}),\ \mathcal{M}^*((a,z),\ \mathfrak{o})]$ and $\mathcal{N}_{\mathfrak{o}}(a,z) = [\mathcal{M}_*((a,z),\ \mathfrak{o}),\ \mathcal{N}^*((a,z),\ \mathfrak{o})]$

Proof. Using condition C, we can write

$$a + \frac{1}{2}\varpi(z,\ a) = a + \sigma\varpi(z,\ a) + \frac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a),\ a + \sigma\varpi(z,a)).$$

By hypothesis, for each $\mathfrak{o} \in [0,\ 1]$, we have

$$\begin{gathered}\mathfrak{U}_*\left(\frac{2a+\varpi(z,\ a)}{2}, \mathfrak{o}\right) \times \mathcal{J}_*\left(\frac{2a+\varpi(z,\ a)}{2}, \mathfrak{o}\right) \\ \mathfrak{U}^*\left(\frac{2a+\varpi(z,\ a)}{2}, \mathfrak{o}\right) \times \mathcal{J}^*\left(\frac{2a+\varpi(z,\ a)}{2}, \mathfrak{o}\right)\end{gathered}$$

$$\begin{aligned}&= \mathfrak{U}_*\left(a + \sigma\varpi(z,\ a) + \tfrac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a),\ a + \sigma\varpi(z,a)), \mathfrak{o}\right) \\ &\times \mathcal{J}_*\left(a + \sigma\varpi(z,\ a) + \tfrac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a),\ a + \sigma\varpi(z,a)), \mathfrak{o}\right), \\ &= \mathfrak{U}^*\left(a + \sigma\varpi(z,\ a) + \tfrac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a),\ a + \sigma\varpi(z,a)), \mathfrak{o}\right) \\ &\times \mathcal{J}^*\left(a + \sigma\varpi(z,\ a) + \tfrac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a),\ a + \sigma\varpi(z,a)), \mathfrak{o}\right),\end{aligned}$$

$$\begin{aligned}&\leq h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \times \mathcal{J}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \\ +\mathfrak{U}_*(a + (1?\sigma)\varpi(z,a), \mathfrak{o}) \times \mathcal{J}_*(a + \sigma\varpi(z,a), \mathfrak{o})\end{bmatrix} \\ &+h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}_*(a + \sigma\varpi(z,a), \mathfrak{o}) \times \mathcal{J}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \\ +\mathcal{U}_*(a + \sigma\varpi(z,a), \mathfrak{o}) \times \mathcal{J}_*(a + \sigma\varpi(z,a), \mathfrak{o})\end{bmatrix}, \\ &\geq h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \times \mathcal{J}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \\ +\mathfrak{U}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \times \mathcal{J}^*(a + \sigma\varpi(z,a), \mathfrak{o})\end{bmatrix} \\ &+h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}^*(a + \sigma\varpi(z,a), \mathfrak{o}) \times \mathcal{J}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \\ +\mathfrak{U}^*(a + \sigma\varpi(z,a), \mathfrak{o}) \times \mathcal{J}^*(a + \sigma\varpi(z,a), \mathfrak{o})\end{bmatrix},\end{aligned}$$

$$\begin{aligned}&\leq h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \times \mathcal{J}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \\ +\mathfrak{U}_*(a + \sigma\varpi(z,a), \mathfrak{o}) \times \mathcal{J}_*(a + \sigma\varpi(z,a), \mathfrak{o})\end{bmatrix} \\ &+h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}(h_1(\sigma)\mathfrak{U}_*(a, \mathfrak{o}) + h_1(1-\sigma)\mathfrak{U}_*(z, \mathfrak{o})) \\ \times(h_2(1-\sigma)\mathcal{J}_*(a, \mathfrak{o}) + h_2(\sigma)\mathcal{J}_*(z, \mathfrak{o})) \\ +(h_1(1-\sigma)\mathfrak{U}_-(a, \mathfrak{o}) + h_1(\sigma)\mathfrak{U}_*(z, \mathfrak{o})) \\ \times(h_2(\sigma)\mathcal{J}_*(a, \mathfrak{o}) + h_2(1-\sigma)\mathcal{J}_*(z, \mathfrak{o}))\end{bmatrix}, \\ &\leq h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \times \mathcal{J}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) \\ +\mathfrak{U}^*(a + \sigma\varpi(z,a), \mathfrak{o}) \times \mathcal{J}^*(a + \sigma\varpi(z,a), \mathfrak{o})\end{bmatrix} \\ &+h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}(h_1(\sigma)\mathfrak{U}^*(a, \mathfrak{o}) + h_1(1-\sigma)\mathfrak{U}^*(z, \mathfrak{o})) \\ \times(h_2(1-\sigma)\mathcal{J}^*(a, \mathfrak{o}) + h_2(\sigma)\mathcal{J}^*(z, \mathfrak{o})) \\ +(h_1(1-\sigma)\mathfrak{U}^*(a, \mathfrak{o}) + h_1(\sigma)\mathfrak{U}^*(z, \mathfrak{o})) \\ \times(h_2(\sigma)\mathcal{J}^*(a, \mathfrak{o}) + h_2(1-\sigma)\mathcal{J}^*(z, \mathfrak{o}))\end{bmatrix},\end{aligned}$$

$$
\begin{aligned}
&= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}_*(a+(1-\sigma)\varpi(z,a),\mathfrak{o}) \times \mathcal{J}_*(a+(1-\sigma)\varpi(z,a),\mathfrak{o}) \\ +\mathfrak{U}_*(a+\sigma\varpi(z,a),\mathfrak{o}) \times \mathcal{J}_*(a+\sigma\varpi(z,a),\mathfrak{o})\end{bmatrix} \\
&+2h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\{h_1(\sigma)h_2(\sigma)+h_1(1-\sigma)h_2(1-\sigma)\}\mathcal{N}_*((a,z),\mathfrak{o}) \\ +\{h_1(\sigma)h_2(1-\sigma)+h_1(1-\sigma)h_2(\sigma)\}\mathcal{M}_*((a,z),\mathfrak{o})\end{bmatrix}, \\
&= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{U}^*(a+(1-\sigma)\varpi(z,a),\mathfrak{o}) \times \mathcal{J}^*(a+(1-\sigma)\varpi(z,a),\mathfrak{o}) \\ +\mathfrak{U}^*(a+\sigma\varpi(z,a),\mathfrak{o}) \times \mathcal{J}^*(a+\sigma\varpi(z,a),\mathfrak{o})\end{bmatrix} \\
&+2h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\{h_1(\sigma)h_2(\sigma)+h_1(1-\sigma)h_2(1-\sigma)\}\mathcal{N}^*((a,z),\mathfrak{o}) \\ +\{h_1(\sigma)h_2(1-\sigma)+h_1(1-\sigma)h_2(\sigma)\}\mathcal{M}^*((a,z),\mathfrak{o})\end{bmatrix}.
\end{aligned}
$$

Integrating over $[0,\ 1]$, we have

$$
\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\frac{2a+\varpi(z,a)}{2},\mathfrak{o}\right) \times \mathcal{J}_*\left(\frac{2a+\varpi(z,a)}{2},\mathfrak{o}\right) \le \frac{1}{\varpi(z,a)}\int_a^{a+\varpi(z,a)}\mathfrak{U}_*(v,\mathfrak{o}) \times \mathcal{J}_*(v,\mathfrak{o})dv \\
&\quad +\mathcal{M}_*((a,z),\mathfrak{o})\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma+\mathcal{N}_*((a,z),\mathfrak{o})\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma, \\
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\frac{2a+\varpi(z,a)}{2},\mathfrak{o}\right) \times \mathcal{J}^*\left(\frac{2a+\varpi(z,a)}{2},\mathfrak{o}\right) \ge \frac{1}{\varpi(z,a)}\int_a^{a+\varpi(z,a)}\mathfrak{U}^?(v,\mathfrak{o}) \times \mathcal{J}^*(v,\mathfrak{o})dv \\
&\quad +\mathcal{M}^*((a,z),\mathfrak{o})\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma+\mathcal{N}^*((a,z),\mathfrak{o})\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma,
\end{aligned}
$$

from which, we have

$$
\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\mathfrak{U}_*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right) \times \mathcal{J}_*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right),\ \mathfrak{U}^*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right) \times \mathcal{J}^*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right)\right] \\
&\supseteq\ _I\frac{1}{\varpi(z,\ a)}\left[\int_a^{a+\varpi(z,\ a)}\mathfrak{U}_*(v,\mathfrak{o}) \times \mathcal{J}_*(v,\mathfrak{o})dv\ ,\ \int_a^{a+\varpi(z,\ a)}\mathfrak{U}^*(v,\mathfrak{o}) \times \mathcal{J}^*(v,\mathfrak{o})dv\right] \\
&\quad + \int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma[\mathcal{M}_*((a,z),\ \mathfrak{o}),\ \mathcal{M}^*((a,z),\ \mathfrak{o})] \\
&\quad + [\mathcal{N}_*((a,z),\ \mathfrak{o}),\ \mathcal{N}^*((a,z),\ \mathfrak{o})]\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \otimes \widetilde{\mathcal{J}}\left(\frac{2a+\varpi(z,\ a)}{2}\right) \\
&\supseteq_{\mathbb{F}} \frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v) \otimes \widetilde{\mathcal{J}}(v)dv \\
&\oplus \widetilde{\mathcal{M}}(a,z) \odot \int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma \oplus \widetilde{\mathcal{N}}(a,z) \odot \int_0^1 h_1(\sigma)h_2(\sigma)d\sigma,
\end{aligned}
$$

this completes the result. □

Example 4. *We consider $h_1(\sigma) = \sigma$, $h_2(\sigma) = \sigma$, for $\sigma \in [0,\ 1]$, and the F-N·V·Ms $\widetilde{\mathfrak{U}}$, $\widetilde{\mathcal{J}} : [a,\ a+\varpi(z,\ a)] = [0,\ \varpi(2,\ 0)] \to \mathbb{E}_C$ defined by, for each $\mathfrak{o} \in [0,\ 1]$, we have $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{o}v, (2-\mathfrak{o})v]$ and $\mathcal{J}_{\mathfrak{o}}(v) = [(1-\mathfrak{o})v+2\mathfrak{o}, (1-\mathfrak{o})(8-e^v)+2\mathfrak{o}]$, as in Example 3, and $\widetilde{\mathfrak{U}}(v)$, $\widetilde{\mathcal{J}}(v)$ both are U·D h_1- and h_2-pre-invex F-N·V·Ms with respect to $\varpi(z,\ a) = z-a$, respectively. Since $\mathfrak{U}_*(v,\mathfrak{o}) = \mathfrak{o}v$,$\mathfrak{U}^*(v,\ \mathfrak{o}) = (2-\mathfrak{o})v$ and $\mathcal{J}_*(v,\mathfrak{o}) = (1-\mathfrak{o})v+2\mathfrak{o}$, $\mathcal{J}^*(v,\ \mathfrak{o}) = (1-\mathfrak{o})(8-e^v)+2\mathfrak{o}$ then, we have*

$$
\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\ \mathfrak{U}_*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right) \times \mathcal{J}_*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right) = 2o(1+\mathfrak{o}), \\
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\ \mathfrak{U}^*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right) \times \mathcal{J}^*\left(\frac{2a+\varpi(z,\ a)}{2},\mathfrak{o}\right) = 2\left[16-20\mathfrak{o}+6\mathfrak{o}^2+\left(2-3\mathfrak{o}+\mathfrak{o}^2\right)e\right],
\end{aligned}
$$

$$\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)}\mathfrak{U}_*(v,\mathfrak{o})\times\mathcal{J}_*(v,\mathfrak{o})dv=\frac{1}{2}\int_0^2\big(\mathfrak{o}(1-\mathfrak{o})v^2+2\mathfrak{o}^2v\big)dv=\frac{4}{3}o(3-\mathfrak{o})$$
$$\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)}\mathfrak{U}^*(v,\mathfrak{o})\times\mathcal{J}^*(v,\mathfrak{o})dv=\frac{1}{2}\int_0^2((1-\mathfrak{o})(2-\mathfrak{o})v(8-e^v)+2\mathfrak{o}(2-\mathfrak{o})v)dv\approx\frac{(2-\mathfrak{o})}{2}\left(\frac{1903}{250}-\frac{903}{250}\mathfrak{o}\right),$$

$$\mathcal{M}_*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma=\frac{2\mathfrak{o}}{3},$$
$$\mathcal{M}^*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(1-\sigma)d\sigma=\frac{(2-\mathfrak{o})\big[(1-\mathfrak{o})\big(8-e^2\big)+2\mathfrak{o}\big]}{3},$$
$$\mathcal{N}_*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma=\frac{4\mathfrak{o}^2}{3},$$
$$\mathcal{N}^*((a,z),\ \mathfrak{o})\int_0^1 h_1(\sigma)h_2(\sigma)d\sigma=\frac{2(2-\mathfrak{o})(7-5\mathfrak{o})}{3},$$

for each $\mathfrak{o}\in[0,\ 1]$,*that means*

$$2\big[\mathfrak{o}(1+\mathfrak{o}),\big[16-20\mathfrak{o}+6\mathfrak{o}^2+\big(2-3\mathfrak{o}+\mathfrak{o}^2\big)e\big]\big]\supseteq_I\left[\frac{2}{3}\mathfrak{o}(2+\mathfrak{o}),\frac{(2-\mathfrak{o})}{2}\left(\frac{1903}{250}-\frac{903}{250}\mathfrak{o}\right)\right]$$
$$+\frac{1}{3}\big[2\mathfrak{o}(1+2\mathfrak{o}),(2-\mathfrak{o})\big[(1-\mathfrak{o})\big(8-e^2\big)-8\mathfrak{o}+14\big]\big]$$

hence, Theorem 8 is demonstrated.

The *HH* Fejér inequalities for *U·D h*-pre-invex FNVMs are now provided. The second *HH* Fejér inequality is first found for both *U·D* h-pre-invex FNVM.

Theorem 9. *Let* $\widetilde{\mathfrak{U}}:[a,\ a+\varpi(z,\ a)]\to\mathbb{E}_C$ *be an U·D h-pre-invex F-N·V·M with* $a<a+\varpi(z,\ a)$ *and* $h:[0,\ 1]\to\mathbb{R}^+$, *whose* $\mathfrak{o}$*-levels define the family of I·V·Ms* $\mathfrak{U}_\mathfrak{o}:[a,\ a+\varpi(z,\ a)]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ *are given by* $\mathfrak{U}_\mathfrak{o}(v)=[\mathfrak{U}_*(v,\mathfrak{o}),\ \mathfrak{U}^*(v,\mathfrak{o})]$ *for all* $v\in[a,\ a+\varpi(z,\ a)]$ *and for all* $\mathfrak{o}\in[0,\ 1]$. *If* $\widetilde{\mathfrak{U}}\in\mathcal{FR}_{([a,\ a+\varpi(z,\ a)],\ \mathfrak{o})}$ *and* $\mathcal{C}:[a,\ a+\varpi(z,\ a)]\to\mathbb{R}$, $\mathcal{C}(v)\geq 0$, *symmetric with respect to* $a+\frac{1}{2}\varpi(z,\ a)$, *then*

$$\frac{1}{\varpi(z,\ a)}\odot(FR)\int_a^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)\odot\mathcal{C}(v)dv\supseteq_{\mathbb{F}}\left[\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z)\right]\odot\int_0^1 h(\sigma)\mathcal{C}(a+\sigma\varpi(z,\ a))d\sigma. \tag{54}$$

Proof. Let $\widetilde{\mathfrak{U}}$ be an U·D *h*-pre-invex F-N·V·M. Then, for each $\mathfrak{o}\in[0,\ 1]$, we have

$$\begin{array}{c}\mathfrak{U}_*(a+(1-\sigma)\varpi(z,a),v)C(a+(1-\sigma)\varpi(z,a))\\ \leq(h(\sigma)\mathfrak{U}_*(a,\mathfrak{o})+h(1-\sigma)\mathfrak{U}_*(z,\mathfrak{o}))C(a+(1-\sigma)\varpi(z,a))\\ \mathfrak{U}^*(a+(1-\sigma)\varpi(z,a),v)C(a+(1-\sigma)\varpi(z,a))\\ \geq(h(\sigma)\mathfrak{U}^*(a,\mathfrak{o})+h(1-\sigma)\mathfrak{U}^*(z,\mathfrak{o}))C(a+(1-\sigma)\varpi(z,a))\end{array} \tag{55}$$

and

$$\begin{array}{l}\mathfrak{U}_*(a+\sigma\varpi(z,\ a),\ \mathfrak{o})C(a+\sigma\varpi(z,\ a))\leq(h(1-\sigma)\mathfrak{U}_*(a,\ \mathfrak{o})+h(\sigma)\mathfrak{U}_*(z,\ \mathfrak{o}))C(a+\sigma\varpi(z,\ a)),\\ \mathfrak{U}^*(a+\sigma\varpi(z,\ a),\ \mathfrak{o})C(a+\sigma\varpi(z,\ a))\geq(h(1-\sigma)\mathfrak{U}^*(a,\ \mathfrak{o})+h(\sigma)\mathfrak{U}^*(z,\ \mathfrak{o}))C(a+\sigma\varpi(z,\ a)).\end{array} \tag{56}$$

After adding (55) and (56), and integrating over $[0,\ 1]$, we get

$$\begin{gathered}\int_0^1 \mathfrak{U}_*(a+(1-\sigma)\varpi(z,a),\mathfrak{o})\mathcal{C}(a+(1-\sigma)\varpi(z,a))d\sigma \\ +\int_0^1 \mathfrak{U}_*(a+\sigma\varpi(z,a),\mathfrak{o})\mathcal{C}(a+\sigma\varpi(z,a))d\sigma \\ \leq \int_0^1 \begin{bmatrix} \mathfrak{U}_*(a,\mathfrak{o})\{h(\sigma)\mathcal{C}(a+(1-\sigma)\varpi(z,a))+h(1-\sigma)\mathcal{C}(a+\sigma\varpi(z,a))\} \\ +\mathfrak{U}_*(z,\mathfrak{o})\{h(1-\sigma)\mathcal{C}(a+(1-\sigma)\varpi(z,a))+h(\sigma)\mathcal{C}(a+\sigma\varpi(z,a))\}\end{bmatrix} d\sigma, \\ \int_0^1 \mathfrak{U}^*(a+\sigma\varpi(z,a),\mathfrak{o})\mathcal{C}(a+\sigma\varpi(z,a))d\sigma \\ +\int_0^1 \mathfrak{U}^*(a+(1-\sigma)\varpi(z,a),\mathfrak{b})\mathcal{C}(a+(1-\sigma)\varpi(z,a))d\sigma \\ \geq \int_0^1 \begin{bmatrix} \mathfrak{U}^*(a,\mathfrak{o})\{h(\sigma)\mathcal{C}(a+(1-\sigma)\varpi(z,a))+h(1-\sigma)\mathcal{C}(a+\sigma\varpi(z,a))\} \\ +\mathfrak{U}^*(z,\mathfrak{o})\{h(1-\sigma)\mathcal{C}(a+(1-\sigma)\varpi(z,a))+h(\sigma)\mathcal{C}(a+\sigma\varpi(z,a))\}\end{bmatrix} d\sigma\end{gathered}$$

$$\begin{aligned}&= 2\mathfrak{U}_*(a,\ \mathfrak{o})\int_0^1 h(\sigma)\mathcal{C}(a+(1-\sigma)\varpi(z,a))d\sigma + 2\mathfrak{U}_*(z,\ \mathfrak{o})\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma, \\ &= 2\mathfrak{U}^*(a,\ \mathfrak{o})\int_0^1 h(\sigma)\mathcal{C}(a+(1-\sigma)\varpi(z,a))d\sigma + 2\mathfrak{U}^*(z,\ \mathfrak{o})\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma.\end{aligned}$$

Since $\mathcal{C}$ is symmetric, then

$$\begin{aligned}&= 2[\mathfrak{U}_*(a,\ \mathfrak{o})+\mathfrak{U}_*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma, \\ &= 2[\mathfrak{U}^*(a,\ \mathfrak{o})+\mathfrak{U}^*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma.\end{aligned} \tag{57}$$

Since

$$\begin{gathered}\int_0^1 \mathfrak{U}_*(a+(1-\sigma)\varpi(z,a),\mathfrak{o})\mathcal{C}(a+(1-\sigma)\varpi(z,a))d\sigma \\ = \int_0^1 \mathfrak{U}_*(a+\sigma\varpi(z,\ a),\mathfrak{o})\mathcal{C}(a+\sigma\varpi(z,\ a))d\sigma, \\ = \frac{1}{\sigma\varpi(z,\ a)}\int_a^{a+\sigma\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})\mathcal{C}(v)dv, \\ \int_0^1 \mathfrak{U}^*(a+\sigma\varpi(z,\ a),\mathfrak{o})\mathcal{C}(a+\sigma\varpi(z,\ a))d\sigma \\ = \int_0^1 \mathfrak{U}^*(a+(1-\sigma)\varpi(z,a),\mathfrak{o})\mathcal{C}(a+(1-\sigma)\varpi(z,a))d\sigma, \\ = \frac{1}{\sigma\varpi(z,\ a)}\int_a^{a+\sigma\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})\mathcal{C}(v)dv,\end{gathered} \tag{58}$$

From (54) and (55), we have

$$\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv \leq [\mathfrak{U}_*(a,\ \mathfrak{o})+\mathfrak{U}_*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma,$$

$$\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv \geq [\mathfrak{U}^*(a,\ \mathfrak{o})+\mathfrak{U}^*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma,$$

that is

$$\left[\frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv,\ \frac{1}{\varpi(z,\ a)}\int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv\right]$$

$$\supseteq_I [\mathfrak{U}_*(a,\ \mathfrak{o})+\mathfrak{U}_*(z,\ \mathfrak{o}),\ \mathfrak{U}^*(a,\ \mathfrak{o})+\mathfrak{U}^*(z,\ \mathfrak{o})]\int_0^1 h(\sigma)C(a+\sigma\varpi(z,\ a))d\sigma$$

hence

$$\frac{1}{\varpi(z,\ a)} \odot (FR)\int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v)\odot\mathcal{C}(v)dv \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{U}}(a)\oplus\widetilde{\mathfrak{U}}(z)\right]\odot\int_0^1 h(\sigma)\mathcal{C}(a+\sigma\varpi(z,\ a))d\sigma.$$

this completes the proof. □

Next, we construct the first *HH* Fejér inequality for the *U·D h*-pre-invex *F-N·V·M*, which generalizes the first *HH* Fejér inequality for the *U·D h*-pre-invex function, see [4].

Theorem 10. *Let* $\widetilde{\mathfrak{U}} : [a,\ a+\varpi(z,\ a)] \to \mathbb{E}_C$ *be an* $U{\cdot}D$ *h-pre-invex F-N·V·M with* $a < a + \varpi(z,\ a)$ *and* $h : [0,\ 1] \to \mathbb{R}^+$, *whose* $\mathfrak{o}$*-levels define the family of I·V·Ms* $\mathfrak{U}_{\mathfrak{o}} : [a,\ a+\varpi(z,\ a)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are given by* $\mathfrak{U}_{\mathfrak{o}}(v) = [\mathfrak{U}_*(v, \mathfrak{o}),\ \mathfrak{U}^*(v, \mathfrak{o})]$ *for all* $v \in [a,\ a+\varpi(z,\ a)]$ *and for all* $\mathfrak{o} \in [0,\ 1]$. *If* $\widetilde{\mathfrak{U}} \in \mathcal{FR}_{([a,\ a+\varpi(z,\ a)],\ \mathfrak{o})}$ *and* $\mathcal{C} : [a,\ a+\varpi(z,\ a)] \to \mathbb{R}$, $\mathcal{C}(v) \geq 0$, *symmetric with respect to* $a + \frac{1}{2}\varpi(z,\ a)$, *and* $\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv > 0$, *and Condition C for* ϖ, *then*

$$\widetilde{\mathfrak{U}}\left(a + \frac{1}{2}\varpi(z,\ a)\right) \supseteq_{\mathbb{F}} \frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv} \odot (FR) \int_a^{a+\varpi(z,\ a)} \widetilde{\mathfrak{U}}(v) \odot \mathcal{C}(v)dv. \tag{59}$$

Proof. Using condition C, we can write

$$a + \frac{1}{2}\varpi(z,\ a) = a + \sigma\varpi(z,\ a) + \frac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a),\ a + \sigma\varpi(z,a)).$$

Since $\widetilde{\mathfrak{U}}$ is an $U{\cdot}D$ h-pre-invex, then for $\mathfrak{o} \in [0,\ 1]$, we have

$$\begin{aligned} \mathfrak{U}_*\left(a + \tfrac{1}{2}\varpi(z,a), \mathrm{o}\right) &= \mathfrak{U}_*\left(a + \sigma\varpi(z,a) + \tfrac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a), a + \sigma\varpi(z,a)),\ \mathfrak{o}\right) \\ &\leq h\left(\tfrac{1}{2}\right)(\mathfrak{U}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) + \mathfrak{U}_*(a + \sigma\varpi(z,a), \mathfrak{o})), \\ \mathfrak{U}^*\left(a + \tfrac{1}{2}\varpi(z,a), 0\right) &= \mathfrak{U}^*\left(a + \sigma\varpi(z,a) + \tfrac{1}{2}\varpi(a + (1-\sigma)\varpi(z,a), a + \sigma\varpi(z,a)), \mathrm{o}\right) \\ &\geq h\left(\tfrac{1}{2}\right)(\mathfrak{U}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o}) + \mathfrak{U}^*(a + \sigma\varpi(z,a), \mathfrak{o})), \end{aligned} \tag{60}$$

By multiplying (57) by $\mathcal{C}(a + (1-\sigma)\varpi(z,a)) = \mathcal{C}(a + \sigma\varpi(z,a))$ and integrate it by σ over $[0,\ 1]$, we obtain

$$\begin{aligned} &\mathfrak{U}_*\left(a + \tfrac{1}{2}\sigma\varpi(z,a), \mathfrak{o}\right) \int_0^1 \mathcal{C}(a + \sigma\varpi(z,a))d\sigma \\ &\leq h\left(\tfrac{1}{2}\right)\Bigg(\int_0^1 \mathfrak{U}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o})\mathcal{C}(a + (1-\sigma)\varpi(z,a))d\sigma \\ &\qquad + \int_0^1 \mathfrak{U}_*(a + \sigma\varpi(z,\ a), \mathfrak{o})\mathcal{C}(a + \sigma\varpi(z,\ a))d\sigma, \\ &\mathfrak{U}^*\left(a + \tfrac{1}{2}\sigma\varpi(z,a), \mathfrak{o}\right) \int_0^1 \mathcal{C}(a + \sigma\varpi(z,a))d\sigma \\ &\geq h\left(\tfrac{1}{2}\right)\Bigg(\int_0^1 \mathfrak{U}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o})\mathcal{C}(a + (1-\sigma)\varpi(z,a))d\sigma \\ &\qquad + \int_0^1 \mathfrak{U}^*(a + \sigma\varpi(z,\ a), \mathfrak{o})\mathcal{C}(a + \sigma\varpi(z,\ a))d\sigma, \end{aligned} \tag{61}$$

Since

$$\begin{aligned} &\int_0^1 \mathfrak{U}_*(a + (1-\sigma)\varpi(z,a), \mathfrak{o})\mathcal{C}(a + (1-\sigma)\varpi(z,a))d\sigma \\ &= \int_0^1 \mathfrak{U}_*(a + \sigma\varpi(z,\ a), \mathfrak{o})\mathcal{C}(a + \sigma\varpi(z,\ a))d\sigma, \\ &= \tfrac{1}{\sigma\varpi(z,\ a)} \int_a^{a+\sigma\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})\mathcal{C}(v)dv, \\ &\int_0^1 \mathfrak{U}^*(a + \sigma\varpi(z,\ a), \mathfrak{o})\mathcal{C}(a + \sigma\varpi(z,\ a))d\sigma \\ &= \int_0^1 \mathfrak{U}^*(a + (1-\sigma)\varpi(z,a), \mathfrak{o})\mathcal{C}(a + (1-\sigma)\varpi(z,a))d\sigma, \\ &= \tfrac{1}{\sigma\varpi(z,\ a)} \int_a^{a+\sigma\varpi(z,\ a)} \mathfrak{U}_*(v,\ \mathfrak{o})\mathcal{C}(v)dv, \end{aligned} \tag{62}$$

From (58) and (59), we have

$$\begin{aligned} \mathfrak{U}_*\left(a + \tfrac{1}{2}\varpi(z,\ a),\ \mathfrak{o}\right) &\leq \frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv} \int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v, \mathfrak{o})\mathcal{C}(v)dv, \\ \mathfrak{U}^*\left(a + \tfrac{1}{2}\varpi(z,\ a),\ \mathfrak{o}\right) &\geq \frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv} \int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v, \mathfrak{o})\mathcal{C}(v)dv. \end{aligned}$$

From which, we have

$$\left[\mathfrak{U}_*\left(a+\tfrac{1}{2}\varpi(z,a),\mathfrak{o}\right),\mathfrak{U}^*\left(a+\tfrac{1}{2}\varpi(z,a),\mathfrak{o}\right)\right]$$
$$\supseteq_I \frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,a)}\mathcal{C}(v)dv}\left[\int_a^{a+\varpi(z,a)}\mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv,\int_a^{a+\varpi(z,a)}\mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv\right],$$

that is

$$\widetilde{\mathfrak{U}}\left(a+\frac{1}{2}\varpi(z,\ a)\right)\supseteq_{\mathbb{F}}\frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,\ a)}\mathcal{C}(v)dv}\odot(FR)\int_a^{a+\varpi(z,\ a)}\widetilde{\mathfrak{U}}(v)\odot\mathcal{C}(v)dv,$$

Then we complete the proof. □

Remark 5. *If $h(\sigma)=\sigma$ then inequalities in Theorems 9 and 10 reduces for U·D pre-invex F-N·V·Ms, see [44].*

If $h(\sigma)=\sigma$ and $\varpi(y,\ v)=y-v$, then inequalities in Theorems 9 and 10 reduce for U·D convex F-N·V·Ms, see [44].

If $\widetilde{\mathfrak{U}}$ is lower U·D h-pre-invex F-N·V·M, then inequalities in Theorems 9 and 10 reduce for h-pre-invex F-N·V·Ms, see [28].

If $h(\sigma)=\sigma$ and $\widetilde{\mathfrak{U}}$ is lower U·D h-pre-invex F-N·V·M, then inequalities in Theorems 9 and 10 reduce for pre-invex F-N·V·Ms, see [28].

If $\mathfrak{U}_*(v,\mathfrak{o})=\mathfrak{U}^*(v,\ \mathfrak{o})$ with $\mathfrak{o}=1$, then Theorems 9 and 10 reduce to classical first and second HH Fejér inequality for h*-pre-invex function, see [41].*

If $\mathfrak{U}_*(v,\mathfrak{o})=\mathfrak{U}^*(v,\ \mathfrak{o})$ with $\mathfrak{o}=1$ and $\varpi(y,\ v)=y-v$, then Theorems 9 and 10 reduce to classical first and second HH Fejér inequality for h*-convex function, see [9].*

Example 5. *We consider $h(\sigma)=\sigma$, for $\sigma\in[0,\ 1]$ and the F-N·V·M $\widetilde{\mathfrak{U}}:[0,\ \partial(2,\ 0)]\to\mathbb{E}_C$ defined by,*

$$\widetilde{\mathfrak{U}}(v)(\varrho)=\begin{cases}\frac{\varrho-2+v^{\frac{1}{2}}}{\frac{3}{2}-2-v^{\frac{1}{2}}} & \varrho\in\left[2-v^{\frac{1}{2}},\ \frac{3}{2}\right]\\ \frac{2+v^{\frac{1}{2}}-\varrho}{2+v^{\frac{1}{2}}-\frac{3}{2}} & \varrho\in\left(\frac{3}{2},\ 2+v^{\frac{1}{2}}\right]\\ 0 & otherwise,\end{cases}\tag{63}$$

Then, for each$\mathfrak{o}\in[0,\ 1]$,we have$\mathfrak{U}_{\mathfrak{o}}(v)=\left[(1-\mathfrak{o})\left(2-v^{\frac{1}{2}}\right)+\frac{3}{2}\mathfrak{o},(1+\mathfrak{o})\left(2+v^{\frac{1}{2}}\right)+\frac{3}{2}\mathfrak{o}\right]$. Since$\mathfrak{U}_*(v,\mathfrak{o})and\mathfrak{U}^*(v,\mathfrak{o})$areh-pre-invex functions$\varpi(y,v)=y-v$for each$\mathfrak{o}\in[0,\ 1]$, then$\widetilde{\mathfrak{U}}(v)$ish-pre-invex F-N·V·M. If

$$\mathcal{C}(v)=\begin{cases}\sqrt{v}, & \sigma\in[0,1],\\ \sqrt{2-v}, & \sigma\in(1,\ 2],\end{cases}\tag{64}$$

then, we have

$$\frac{1}{\varpi(2,0)}\int_0^{\varpi(2,0)}\mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv=\frac{1}{2}\int_0^2\mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv$$
$$=\frac{1}{2}\int_0^1\mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv+=\frac{1}{2}\int_1^2\mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv,$$
$$\frac{1}{\varpi(2,0)}\int_0^{\varpi(2,0)}\mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv=\frac{1}{2}\int_0^2\mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv$$
$$=\frac{1}{2}\int_0^1\mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv+=\frac{1}{2}\int_1^2\mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv,$$

$$\begin{aligned}&=\frac{1}{2}\int_0^1\left[(1-\mathfrak{o})\left(2-v^{\frac{1}{2}}\right)+\frac{3}{2}\mathfrak{o}\right](\sqrt{v})dv+\frac{1}{2}\int_1^2\left[(1-\mathfrak{o})\left(2-v^{\frac{1}{2}}\right)+\frac{3}{2}\mathfrak{o}\right](\sqrt{2-v})dv\\ &=\frac{1}{4}\left[\frac{13}{3}-\frac{\pi}{2}\right]+v\left[\frac{\pi}{8}-\frac{1}{12}\right],\\ &=\frac{1}{2}\int_0^1\left[(1+\mathfrak{o})\left(2+v^{\frac{1}{2}}\right)+\frac{3}{2}\mathfrak{o}\right](\sqrt{v})dv+\frac{1}{2}\int_1^2\left[(1+\mathfrak{o})\left(2+v^{\frac{1}{2}}\right)+\frac{3}{2}\mathfrak{o}\right](\sqrt{2-v})dv\\ &=\frac{1}{4}\left[\frac{19}{3}+\frac{\pi}{2}\right]+v\left[\frac{\pi}{8}+\frac{31}{12}\right].\end{aligned}\tag{65}$$

and

$$\begin{array}{c} [\mathfrak{U}_*(a,\ \mathfrak{o}) + \mathfrak{U}_*(z,\ \mathfrak{o})] \int_0^1 h(\sigma)C(a + \sigma\varpi(z,\ a))d\sigma \\ [\mathfrak{U}^*(a,\ \mathfrak{o}) + \mathfrak{U}^*(z,\ \mathfrak{o})] \int_0^1 h(\sigma)C(a + \sigma\varpi(z,\ a))d\sigma \\ = \left[4(1-\mathfrak{o}) - \sqrt{2}(1-\mathfrak{o}) + 3\mathfrak{o}\right]\left[\int_0^{\frac{1}{2}} \sigma\sqrt{2\sigma}d\sigma + \int_{\frac{1}{2}}^1 \sigma\sqrt{2(1-\sigma)}d\sigma\right] \\ = \frac{1}{3}\left(4(1-\mathfrak{o}) - \sqrt{2}(1-\mathfrak{o}) + 3\mathfrak{o}\right), \\ = \left[4(1+\mathfrak{o}) + \sqrt{2}(1+\mathfrak{o}) + 3\mathfrak{o}\right]\left[\int_0^{\frac{1}{2}} \sigma\sqrt{2\sigma}d\sigma + \int_{\frac{1}{2}}^1 \sigma\sqrt{2(1-\sigma)}d\sigma\right] \\ = \frac{1}{3}\left(4(1+\mathfrak{o}) + \sqrt{2}(1+\mathfrak{o}) + 3\mathfrak{o}\right) \end{array} \tag{66}$$

From (62) and (63), we have

$$\left[\frac{1}{4}\left[\frac{13}{3} - \frac{\pi}{2}\right] + \mathfrak{o}\left[\frac{\pi}{4} - \frac{7}{6}\right],\ \frac{1}{4}\left[\frac{19}{3} + \frac{\pi}{2}\right] + \mathfrak{o}\left[\frac{\pi}{4} + \frac{25}{6}\right]\right],$$

$$\supseteq_I \left[\frac{1}{3}\left(4(1-\mathfrak{o}) - \sqrt{2}(1-\mathfrak{o}) + 3\mathfrak{o}\right),\ \frac{1}{3}\left(4(1+\mathfrak{o}) + \sqrt{2}(1+\mathfrak{o}) + 3\mathfrak{o}\right)\right]$$

for each$\mathfrak{o} \in [0,\ 1]$.*Hence, Theorem 9 is verified.*

For Theorem 10, we have

$$\begin{array}{c} \mathfrak{U}_*\left(a + \frac{1}{2}\varpi(z,\ a),\ \mathfrak{o}\right) = \frac{2+\mathfrak{o}}{2}\ , \\ \mathfrak{U}^*\left(a + \frac{1}{2}\varpi(z,\ a),\ \mathfrak{o}\right) = \frac{3(2+3\mathfrak{o})}{2}\ , \end{array} \tag{67}$$

$$\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv = \int_0^1 \sqrt{v}dv + \int_1^2 \sqrt{2-v}dv = \frac{4}{3},$$

$$\begin{array}{c} \frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv} \int_a^{a+\varpi(z,\ a)} \mathfrak{U}_*(v,\mathfrak{o})\mathcal{C}(v)dv = \frac{3}{8}\left[\frac{13}{3} - \frac{\pi}{2}\right] + \frac{3\mathfrak{o}}{2}\left[\frac{\pi}{8} - \frac{1}{12}\right] \\ \frac{2h\left(\frac{1}{2}\right)}{\int_a^{a+\varpi(z,\ a)} \mathcal{C}(v)dv} \int_a^{a+\varpi(z,\ a)} \mathfrak{U}^*(v,\mathfrak{o})\mathcal{C}(v)dv = \frac{3}{8}\left[\frac{19}{3} + \frac{\pi}{2}\right] + \frac{3\mathfrak{o}}{2}\left[\frac{\pi}{8} + \frac{31}{12}\right]. \end{array} \tag{68}$$

From (64) and (65), we have

$$\left[\frac{2+\mathfrak{o}}{2},\ \frac{3(2+3\mathfrak{o})}{2}\right] \supseteq_I \left[\frac{3}{8}\left[\frac{13}{3} - \frac{\pi}{2}\right] + \frac{3\mathfrak{o}}{2}\left[\frac{\pi}{8} - \frac{1}{12}\right],\ \frac{3}{8}\left[\frac{19}{3} + \frac{\pi}{2}\right] + \frac{3\mathfrak{o}}{2}\left[\frac{\pi}{8} + \frac{31}{12}\right]\right].$$

Hence, Theorem 10 is verified.

5. Conclusions

The Hermite and Hadamard's Fejér-type containments have been examined in the most recent study in relation to fuzzy analysis. We define the new class of nonconvex mappings which are known as U·D h-pre-invex fuzzy-number valued mappings, and this is illustrated by an example, in order to examine our results. We first established some generalized Hermite–Hadamard–Fejér-type fuzzy inclusions in one dimension involving U·D h-pre-invex fuzzy-number valued mappings, after obtaining fuzzy integral inclusions in association with U·D h-pre-invex fuzzy-number valued mappings, and their numerical verifications. Convexity and fuzzy-number analysis theory have several uses in both optimization and error analysis. We hope that the style of this paper will pique readers' curiosity and encourage more research in the related field.

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