axioms

MDPI

Article

Some New Families of Finite Orthogonal Polynomials in Two Variables

Esra Güldoğan Lekesiz [1,†] and Iván Area [2,*,†]

[1] Department of Mathematics, Faculty of Science, Gazi University, 06500 Ankara, Turkey; esragldgn@gmail.com
[2] CITMAga, Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo, Campus As Lagoas-Ourense, Universidade de Vigo, 32004 Ourense, Spain
* Correspondence: area@uvigo.gal
† These authors contributed equally to this work.

Abstract: In this paper, we generalize the study of finite sequences of orthogonal polynomials from one to two variables. In doing so, twenty three new classes of bivariate finite orthogonal polynomials are presented, obtained from the product of a finite and an infinite family of univariate orthogonal polynomials. For these new classes of bivariate finite orthogonal polynomials, we present a bivariate weight function, the domain of orthogonality, the orthogonality relation, the recurrence relations, the second-order partial differential equations, the generating functions, as well as the parameter derivatives. The limit relations among these families are also presented in Labelle's flavor.

Keywords: orthogonal polynomial; weight function; differential equation; recurrence relation; generating function

MSC: 33C50

Citation: Güldoğan Lekesiz, E.; Area, I. Some New Families of Finite Orthogonal Polynomials in Two Variables. *Axioms* **2023**, *12*, 932. https://doi.org/10.3390/axioms12100932

Academic Editor: Gradimir V. Milovanović

Received: 10 August 2023
Revised: 20 September 2023
Accepted: 25 September 2023
Published: 29 September 2023

1. Introduction

Let us consider a second-order linear differential equation of the form:

$$\left(ax^2+bx+c\right)y_n''+(dx+e)y_n'=n(d+(n-1)a)y_n, \tag{1}$$

where a, b, c, d, and e are all real parameters and n is a nonnegative integer. The problem of finding all linear second-order differential equations of the Sturm–Liouville type with polynomial coefficients having orthogonal polynomial solutions goes back to Bochner [1] in 1929. Under some assumptions about the parameters a, b, c, d, and e, for each n, the differential equation can have orthogonal polynomial solutions [2,3]. If we also impose that the weight function for the orthogonality is positive, then we obtain the three very classical orthogonal polynomial sequences of Jacobi, orthogonal with respect to the beta weight function, Laguerre, orthogonal with respect to the gamma weight function, and Hermite, orthogonal with respect to the normal weight function. These three families are infinite sequences in the sense that, for each nonnegative integer n, there exists one element of the family that is a polynomial of degree n. Recently, Masjed-Jamei [4] studied three other finite classes of hypergeometric orthogonal polynomials, which are special solutions to (1). These three families, denoted by $M_n^{(\lambda,\gamma)}(x)$, $N_n^{(\lambda)}(x)$, and $I_n^{(\lambda)}(x)$, are finitely orthogonal with respect to the F sampling distribution, inverse Gamma distribution, and T sampling distribution.

Very classical families of univariate orthogonal polynomials have been enlarged to classical univariate orthogonal polynomials [2,3]. Following [5] (p. 189), we recall the following definition of univariate classical orthogonal polynomials [6]: "An orthogonal polynomial sequence is classical if it is a special case or limiting case of the ${}_4\phi_3$ polynomials given by the q-Racah or the Askey-Wilson polynomials". We refer to [2] for the required notations, as well as the definitions of q-Racah and Askey–Wilson polynomials. The limit transitions among all univariate families known as the "Askey scheme"—or Tableau d'Askey—was nicely designed by J. Labelle [7], and it appears, e.g., in [2].

In [8], Güldoğan et al. introduced some new finite classes of two-variable orthogonal polynomials derived from two finite orthogonal polynomials in one variable by means of Koornwinder's method [9], giving some properties of these families. In [10], Fourier transforms of the finite sets defined in [8] were studied, and some new orthogonal functions were obtained via Parseval's identity, presenting some limit relationships between finite and infinite sequences of orthogonal polynomials in two variables.

In this paper, we constructed some new classes of two-variable finite orthogonal polynomials obtained from the product of a finite and an infinite family of univariate orthogonal polynomials. Furthermore, some transitions are given, and some new families are defined by taking the limit of the bivariate orthogonal polynomials. For the 23 new families introduced, recurrence relations, generating functions, and second-order partial differential equations are presented. Since we obtained a large number of finite families of bivariate orthogonal polynomials, we shall just give the details of the proofs for the first family in Section 3.1. The other results can be obtained *mutatis mutandis*, and they deserve also to be presented in Sections 3.2–3.23. The results were checked with the help of Mathematica [11].

The main aims of this study were to increase and extend the number of families of finite orthogonal polynomials and to provide tools that can serve as inspiration for future studies.

The work is organized as follows. In Section 2, we recall the basic properties of univariate orthogonal polynomials, both infinite and finite situations. In Section 3, we introduce 23 new families of finite bivariate orthogonal polynomials. For each family, we present the polynomials as the product of a finite and an infinite family of univariate orthogonal polynomials described in Section 2. The orthogonality weight function, as well as the domain and orthogonality relation are explicitly given. Next, recurrence relations, second-order partial differential equations, and generating functions are derived. Furthermore, parameter derivatives are also investigated. Finally, the limit relations in Labelle's flavor [7] are given. As already mentioned, the sketch of the proofs will just be given for the first family.

2. Infinite and Finite Univariate Families of Orthogonal Polynomials

Let us recall the general properties of the infinite sequences of the Jacobi, Laguerre, and Hermite polynomials and the sets of finite orthogonal polynomials $M_n^{(\lambda,\gamma)}(x)$, $N_n^{(\lambda)}(x)$, and $I_n^{(\lambda)}(x)$.

2.1. The Jacobi Polynomials

The Jacobi polynomial $P_n^{(\lambda,\gamma)}(x)$ is defined by the explicit series [2] (p. 216):

$$P_n^{(\lambda,\gamma)}(x) = 2^{-n}\sum_{k=0}^{n}\frac{\Gamma(n+\lambda+1)\Gamma(n+\gamma+1)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(n+\lambda-k+1)\Gamma(\gamma+k+1)} \times (x+1)^k(x-1)^{n-k}, \quad (2)$$

where $\Gamma(z)$ denotes the Gamma function. These polynomials are orthogonal on the interval $[-1,1]$ with respect to the (beta) weight function $w(x) = (1-x)^\lambda(1+x)^\gamma$. Jacobi polynomials satisfy the orthogonality relation:

$$\int_{-1}^{1}(1-x)^\lambda(1+x)^\gamma P_n^{(\lambda,\gamma)}(x)P_m^{(\lambda,\gamma)}(x)dx = \frac{2^{\lambda+\gamma+1}\Gamma(\lambda+n+1)\Gamma(\gamma+n+1)\delta_{m,n}}{n!(\lambda+\gamma+2n+1)\Gamma(\lambda+\gamma+n+1)},$$

where $\min\{\Re(\lambda), \Re(\gamma)\} > -1$, $m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\delta_{m,n}$ is the Kronecker delta [12]. The set of polynomials has the generating functions (see, e.g., [13] (p. 82, Equations (1) and (3)), as well as [14] for the use of generating functions in discrete mathematics):

$$\sum_{n=0}^{\infty} P_n^{(\lambda,\gamma)}(x) t^n = \frac{2^{\lambda+\gamma}}{R(1-t+R)^{\lambda}(1+t+R)^{\gamma}}, \tag{3}$$

where $R = \left(1 - 2xt + t^2\right)^{1/2}$ and

$$\sum_{n=0}^{\infty} P_n^{(\lambda-\beta n,\gamma-\theta n)}(x) t^n = \frac{(1+\xi)^{\lambda+1}(1+\eta)^{\gamma+1}}{1+\beta\xi+\theta\eta-(1-\beta-\theta)\xi\eta}, \tag{4}$$

where

$$\xi = \frac{1}{2}(x+1)t(1+\xi)^{1-\beta}(1+\eta)^{1-\theta}, \quad \eta = \frac{1}{2}(x-1)t(1+\xi)^{1-\beta}(1+\eta)^{1-\theta}.$$

Jacobi polynomials $P_n^{(\lambda,\gamma)}(x)$ satisfy the following four recurrence relations [15,16]:

$$\begin{aligned} 2n(n+\lambda+\gamma)(2n+\lambda+\gamma-2)P_n^{(\lambda,\gamma)}(x) \\ = \Big((2n+\lambda+\gamma-2)_3 x + (2n+\lambda+\gamma-1)\left(\lambda^2-\gamma^2\right)\Big) P_{n-1}^{(\lambda,\gamma)}(x) \\ - 2(n+\lambda-1)(n+\gamma-1)(2n+\lambda+\gamma)P_{n-2}^{(\lambda,\gamma)}(x), \end{aligned} \tag{5}$$

$$(2n+\lambda+\gamma+1)P_n^{(\lambda,\gamma)}(x) = (n+\lambda+\gamma+1)P_n^{(\lambda,\gamma+1)}(x) + (n+\lambda)P_{n-1}^{(\lambda,\gamma+1)}(x), \tag{6}$$

$$(x-1)P_{n-1}^{(\lambda+1,\gamma+1)}(x) = 2\left(P_n^{(\lambda,\gamma+1)}(x) - P_n^{(\lambda,\gamma)}(x)\right), \tag{7}$$

$$\frac{n+\lambda+\gamma+1}{2}\left(x^2-1\right)P_{n-1}^{(\lambda+1,\gamma+1)}(x) = (2\gamma+n+nx)P_n^{(\lambda,\gamma)}(x) - 2(\gamma+n)P_n^{(\lambda,\gamma-1)}(x), \tag{8}$$

where $(\mu)_n$ stands for the Pochhammer symbol defined by $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$ for $n = 1, 2, \dots$ and $(\mu)_0 = 1$.

Furthermore, we have the following limit relations between the Jacobi and Laguerre polynomials introduced in the next section (see, e.g., [2] (Equation (9.8.16)) or [16,17]):

$$\lim_{\lambda\to\infty} P_n^{(\lambda,\gamma)}\left(\frac{2x}{\lambda} - 1\right) = (-1)^n L_n^{(\gamma)}(x)$$

and

$$\lim_{\gamma\to\infty} P_n^{(\lambda,\gamma)}\left(1 - \frac{2x}{\gamma}\right) = L_n^{(\lambda)}(x). \tag{9}$$

Furthermore, for $\lambda, \gamma > -1$, we can compute the parameter derivatives of Jacobi polynomials with respect to λ or γ, giving rise to (see [18] (p. 9, Equation (4.7)) or [19])

$$\begin{aligned} \frac{\partial P_n^{(\lambda,\gamma)}(x)}{\partial\lambda} &= \sum_{k=0}^{n-1} \frac{1}{n+k+\lambda+\gamma+1} P_n^{(\lambda,\gamma)}(x) \\ &+ \frac{(\gamma+1)_n}{(\lambda+\gamma+1)_n} \sum_{k=0}^{n-1} \frac{(2k+\lambda+\gamma+1)(\lambda+\gamma+1)_k}{(n-k)(n+k+\lambda+\gamma+1)(\gamma+1)_k} P_k^{(\lambda,\gamma)}(x) \end{aligned}$$

and

$$\frac{\partial P_n^{(\lambda,\gamma)}(x)}{\partial\gamma} = \sum_{k=0}^{n-1} \frac{1}{n+k+\lambda+\gamma+1} P_n^{(\lambda,\gamma)}(x) + \frac{(\lambda+1)_n}{(\lambda+\gamma+1)_n} \sum_{k=0}^{n-1} \frac{(-1)^{n+k}(2k+\lambda+\gamma+1)(\lambda+\gamma+1)_k}{(n-k)(n+k+\lambda+\gamma+1)(\lambda+1)_k} P_k^{(\lambda,\gamma)}(x). \quad (10)$$

2.2. The Generalized Laguerre Polynomials

The generalized Laguerre polynomials $L_n^{(\lambda)}(x)$ can be introduced by the series representation [2] (p. 241):

$$L_n^{(\lambda)}(x) = \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+\lambda+1)}{\Gamma(n-k+1)\Gamma(\lambda+k+1)} \frac{x^k}{k!} \quad (11)$$

and the corresponding orthogonality relation with respect to the gamma weight function is explicitly given as [12]

$$\int_0^\infty x^\lambda e^{-x} L_n^{(\lambda)}(x) L_m^{(\lambda)}(x) dx = \frac{\Gamma(\lambda+n+1)}{n!} \delta_{m,n},$$

where $\Re(\lambda) > -1$, $m,n \in \mathbb{N}_0$. These polynomials have the generating functions (see, e.g., [12] (p. 202, Equation (4)), [13] (p. 84, Equation (16)) or [20]):

$$\sum_{n=0}^{\infty} L_n^{(\lambda)}(x) t^n = \frac{1}{(1-t)^{\lambda+1}} \exp\left(\frac{-tx}{1-t}\right), \quad (12)$$

and

$$\sum_{n=0}^{\infty} L_n^{(\lambda+\beta n)}(x) t^n = \frac{(1+v)^{\lambda+1} e^{-vx}}{1-\beta v} \; ; \; \begin{cases} v = t(1+v)^{\beta+1}, \\ v(0) = 0. \end{cases} \quad (13)$$

Furthermore, for $\lambda > -1$, we have the following derivative with respect to the unique parameter λ (see [19] or [21] (p. 80)):

$$\frac{\partial L_n^{(\lambda)}(x)}{\partial\lambda} = \sum_{k=0}^{n-1} \frac{1}{n-k} L_k^{(\lambda)}(x).$$

2.3. The Hermite Polynomials

The Hermite polynomials $H_n(x)$ defined by [12] (p. 187, Equation (2)):

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}, \quad (14)$$

have the orthogonality relation in the form [13] (p. 73, Equation (13)):

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \delta_{m,n}.$$

The set of the polynomials is generated by [2] (Equation (9.15.10))

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}. \quad (15)$$

2.4. First Class of Finite Classical Orthogonal Polynomials $M_n^{(\lambda,\gamma)}(x)$

The polynomials $M_n^{(\lambda,\gamma)}(x)$, which are defined by the Rodrigues formula:

$$M_n^{(\lambda,\gamma)}(x) = (-1)^n \frac{(1+x)^{\lambda+\gamma}}{x^\gamma} \frac{d^n\left(x^{n+\gamma}(1+x)^{n-\lambda-\gamma}\right)}{dx^n}, \quad n = 0,1,\dots, \tag{16}$$

are polynomial solutions of the equation:

$$x(x+1)y_n''(x) + ((2-\lambda)x+\gamma+1)y_n'(x) - n(n+1-\lambda)y_n(x) = 0.$$

The finite set $\left\{M_n^{(\lambda,\gamma)}(x)\right\}_{n=0}^N$ is orthogonal with respect to the weight function $W_1(x,\lambda,\gamma) = x^\gamma(1+x)^{-(\lambda+\gamma)}$ on $[0,\infty)$ if and only if $\lambda > 2N+1$, $\gamma > -1$. That is,

$$\int_0^\infty \frac{x^\gamma}{(1+x)^{\lambda+\gamma}} M_n^{(\lambda,\gamma)}(x) M_m^{(\lambda,\gamma)}(x)dx = \frac{\Gamma(n+1)\Gamma(\lambda-n)\Gamma(\gamma+n+1)}{(\lambda-2n-1)\Gamma(\lambda+\gamma-n)}\delta_{n,m}$$

for $m,n = 0,1,2,\dots,N < \dfrac{\lambda-1}{2}$, $\gamma > -1$, $N = \max\{m,n\}$. The polynomials $M_n^{(\lambda,\gamma)}(x)$ satisfy the following recurrence relations:

$$\begin{aligned}(\lambda-n-1)(\lambda-2n)M_{n+1}^{(\lambda,\gamma)}(x) &= [(\lambda-2n-2)_3 x \\ &\quad +(\lambda-2n-1)(2n(n+1)-\lambda(\gamma+2n+1))]M_n^{(\lambda,\gamma)}(x) \\ &\quad - n(\lambda-2n-2)(\lambda+\gamma-n)(\gamma+n)M_{n-1}^{(\lambda,\gamma)}(x),\end{aligned}$$

$$M_{n+1}^{(\lambda,\gamma)}(x) = ((\lambda-2)x-(\gamma+1))M_n^{(\lambda-2,\gamma+1)}(x) - n(\lambda-n-3)x(x+1)M_{n-1}^{(\lambda-4,\gamma+2)}(x),$$

as well as

$$\frac{d}{dx}\left(M_n^{(\lambda,\gamma)}(x)\right) = n(\lambda-n-1)M_{n-1}^{(\lambda-2,\gamma+1)}(x).$$

for $n = 1,2,\dots$.

Furthermore, the set of the polynomials has a generating function of the form:

$$\sum_{n=0}^\infty M_n^{(\lambda,\gamma)}(x)\frac{t^n}{n!} = \frac{2^{-\lambda}\left(1-t+\sqrt{(1+t)^2+4xt}\right)^{\lambda+\gamma}}{\sqrt{(1+t)^2+4xt}\left(1+t+\sqrt{(1+t)^2+4xt}\right)^\gamma}.$$

Formally, the polynomials $M_n^{(\lambda,\gamma)}(x)$ are related to the Jacobi polynomials defined in Equation (2) by

$$M_n^{(\lambda,\gamma)}(x) = (-1)^n n! P_n^{(\gamma,-\lambda-\gamma)}(2x+1) \Leftrightarrow P_n^{(\lambda,\gamma)}(x) = \frac{(-1)^n}{n!} M_n^{(-\lambda-\gamma,\lambda)}\left(\frac{x-1}{2}\right). \tag{17}$$

Furthermore [22],

$$\lim_{\lambda\to\infty} M_n^{(\lambda,\gamma)}\left(\frac{x}{\lambda}\right) = (-1)^n n! L_n^{(\gamma)}(x).$$

2.5. Second Class of Finite Classical Orthogonal Polynomials $N_n^{(\lambda)}(x)$

The polynomials $N_n^{(\lambda)}(x)$ are also defined through a Rodrigues formula:

$$N_n^{(\lambda)}(x) = (-1)^n x^\lambda e^{1/x} \frac{d^n\left(x^{-\lambda+2n}e^{-1/x}\right)}{dx^n}, \quad n = 0,1,\dots \tag{18}$$

and they satisfy the second-order differential equation:

$$x^2 y_n''(x) + ((2-\lambda)x+1)y_n'(x) - n(n+1-\lambda)y_n(x) = 0.$$

The finite set $\left\{N_n^{(\lambda)}(x)\right\}_{n=0}^{N}$ is orthogonal with respect to the weight function $W_2(x,\lambda) = x^{-\lambda}e^{-1/x}$ on $[0,\infty)$ if and only if $\lambda > 2N+1$, and

$$\int_0^\infty x^{-\lambda}e^{-1/x} N_n^{(\lambda)}(x) N_m^{(\lambda)}(x) dx = \frac{\Gamma(n+1)\Gamma(\lambda-n)}{(\lambda-2n-1)}\delta_{n,m},$$

is satisfied for $m,n = 0,1,2,\ldots,N < \frac{\lambda-1}{2}$, $N = \max\{m,n\}$. The polynomials $N_n^{(\lambda)}(x)$ satisfy the following recurrence relations:

$$\begin{aligned}((\lambda-2n-2)_3 x - \lambda(\lambda-2n-1))N_n^{(\lambda)}(x) - n(\lambda-2n-2)N_{n-1}^{(\lambda)}(x)& \\ = (\lambda-n-1)(\lambda-2n)N_{n+1}^{(\lambda)}(x),&\end{aligned}$$

$$(\lambda x - 1)N_n^{(\lambda)}(x) - n(\lambda-n-1)x^2 N_{n-1}^{(\lambda-2)}(x) = N_{n+1}^{(\lambda+2)}(x), \tag{19}$$

and

$$\frac{d}{dx}\left(N_n^{(\lambda)}(x)\right) = n(\lambda-n-1)N_{n-1}^{(\lambda-2)}(x) \tag{20}$$

for $n = 1,2,\ldots$. Moreover, the set of the polynomials is generated by:

$$\sum_{n=0}^{\infty} N_n^{(\lambda+2n)}(x)\frac{t^n}{n!} = (1-tx)^{-\lambda}\exp\left(\frac{-t}{1-tx}\right).$$

The relationship:

$$N_n^{(\lambda)}(x) = n!x^n L_n^{(\lambda-2n-1)}\left(\frac{1}{x}\right) \Leftrightarrow L_n^{(\lambda)}(x) = \frac{x^n}{n!}N_n^{(\lambda+2n+1)}\left(\frac{1}{x}\right) \tag{21}$$

between the polynomials $N_n^{(\lambda)}(x)$ and the Laguerre polynomials $L_n^{(\lambda)}(x)$ holds true.

2.6. Third Class of Finite Classical Orthogonal Polynomials $I_n^{(\lambda)}(x)$

The polynomials $I_n^{(\lambda)}(x)$ are defined as follows:

$$I_n^{(\lambda)}(x) = \frac{(-2)^n(\lambda-n)_n}{(2\lambda-2n-1)_n}\left(1+x^2\right)^{\lambda-1/2}\frac{d^n\left(\left(1+x^2\right)^{n-(\lambda-1/2)}\right)}{dx^n}, \quad n = 0,1,\ldots. \tag{22}$$

These polynomials are solutions to the equation:

$$\left(1+x^2\right)y_n''(x) + (3-2\lambda)xy_n'(x) - n(n+2-2\lambda)y_n(x) = 0,$$

and are orthogonal with respect to the weight function $W_3(x,\lambda) = \left(1+x^2\right)^{-(\lambda-1/2)}$ in the interval $(-\infty,\infty)$ if and only if $\lambda > N+1$. The orthogonality relation corresponding to these polynomials is given by

$$\int_{-\infty}^{\infty} \left(1+x^2\right)^{-(\lambda-1/2)} I_n^{(\lambda)}(x) I_m^{(\lambda)}(x) dx$$
$$= \frac{2^{2n-1}\sqrt{\pi}\Gamma(n+1)\Gamma^2(\lambda)\Gamma(2\lambda-2n)}{(\lambda-n-1)\Gamma(\lambda-n)\Gamma(\lambda-n+1/2)\Gamma(2\lambda-n-1)}\delta_{n,m}$$

for $m, n = 0, 1, 2, \ldots, N < \lambda - 1$, $N = \max\{m, n\}$.

For $n \geq 1$, the following recurrence relations hold:

$$I_{n+1}^{(\lambda)}(x) = 2(\lambda-n-1)x I_n^{(\lambda)}(x) - n(2\lambda-n-1) I_{n-1}^{(\lambda)}(x),$$

$$4n\lambda(\lambda-1)\left(1+x^2\right) I_{n-1}^{(\lambda-1)}(x) - 2\lambda(2\lambda-1)x I_n^{(\lambda)}(x) = (n+1-2\lambda) I_{n+1}^{(\lambda+1)}(x)$$

as well as

$$\frac{d}{dx}\left(I_n^{(\lambda)}(x)\right) = 2n(\lambda-1) I_{n-1}^{(\lambda-1)}(x).$$

Furthermore, the set of polynomials has a generating function of the form:

$$\sum_{n=0}^{\infty} I_n^{(\lambda)}(x)\frac{t^n}{n!} = \left(1+2tx-t^2\right)^{\lambda-1},$$

and the relation:

$$I_n^{(\lambda)}(x) = n! i^n C_n^{(1-\lambda)}(ix) \Leftrightarrow C_n^{(\lambda)}(x) = \frac{1}{n! i^n} I_n^{(1-\lambda)}(-ix) \tag{23}$$

holds true between the polynomials $I_n^{(\lambda)}(x)$ and the ultraspherical polynomials $C_n^{(\lambda)}(x)$ defined in terms of Jacobi polynomials by [2] (p. 222):

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x).$$

Furthermore, we have the relation [13] (p. 125, Equation (6)):

$$C_n^{(\lambda)}(x) = (-2)^n\left(\sqrt{x^2-1}\right)^n P_n^{(-\lambda-n,-\lambda-n)}\left(\frac{x}{\sqrt{x^2-1}}\right) \tag{24}$$

between the ultraspherical polynomials $C_n^{(\lambda)}(x)$ and the Jacobi polynomials. In [22], the following limit relation:

$$\lim_{\lambda\to\infty}\left(\lambda^{-\frac{n}{2}} I_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right)\right) = H_n(x)$$

is given.

Now, we recall Koornwinder's method to derive orthogonal polynomials in two variables from two orthogonal polynomials in one variable [9,23].

Theorem 1 ([9]). *Assume that $v(x)$ and $w(y)$ are positive weight functions on the interval (c_1, d_1) and (c_2, d_2), respectively. Let $\tau(x)$ be a positive function on (c_1, d_1) and satisfy one of the following assumptions:*

Case 1: $\tau(x)$ is a polynomial of degree ≤ 1.

Case 2: $\tau^2(x)$ is a polynomial of degree ≤ 2; (c_2, d_2) is a symmetric interval; $w(y)$ is an even function.

For each integer $s \geq 0$, let $\kappa_m(x; s)$, $(m = 0, 1, \ldots)$ denote a sequence of orthogonal polynomials in one variable with respect to the weight function $\tau^{2s+1}(x)v(x)$. Let $\eta_m(y)$ be a sequence of orthogonal polynomials with respect to $w(y)$. Then, polynomials $\phi_{n,k}$ in two variables can be defined by

$$\phi_{m,s}(x,y)=\kappa_{m-s}(x;s)\tau^s(x)\eta_s\left(\frac{y}{\tau(x)}\right),\qquad 0\le s\le m.$$

These polynomials are orthogonal with respect to the weight function:

$$\rho(x,y)=v(x)w(y/\tau(x)),$$

over the domain $\Omega=\{(x,y):c_1<x<d_1,\ c_2\tau(x)<y<d_2\tau(x)\}$.

In the present paper, we define 23 sets of finite orthogonal polynomials in two variables in terms of the finite univariate orthogonal polynomials $M_n^{(\lambda,\gamma)}(x)$, $N_n^{(\lambda)}(x)$, $I_n^{(\lambda)}(x)$ and very classical orthogonal polynomials $P_n^{(\lambda,\gamma)}(x)$, $L_n^{(\lambda)}(x)$ and $H_n(x)$ by using Koornwinder's method. We present a number of properties for each family such as the orthogonality relation, the recurrence relations, the partial differential equation, the generating function, as well as the parameter derivatives of these polynomials.

3. The Finite Sets of the Bivariate Orthogonal Polynomials Obtained by the Product of a Finite and an Infinite Univariate Orthogonal Polynomials

By means of the polynomials given by (2), (11), (14), (16), (18), and (22), we define the following 23 sets of bivariate finite orthogonal polynomials in the next subsections.

3.1. The Set of Polynomials ${}_1E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 1. *Let us define*

$$ {}_1E_{n,k}^{(\lambda,\gamma)}(x,y)=P_{n-k}^{(\lambda+2k+1,\gamma)}(x)(1-x)^kN_k^{(\lambda)}\left(\frac{y}{1-x}\right),\quad k=0,1,\dots,n. \tag{25}$$

The set $\left\{{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_1(x,y)=(1-x)^{2\lambda}(1+x)^{\gamma}y^{-\lambda}\exp\left(-\frac{1-x}{y}\right)$$

over the domain $\Omega_1=\{(x,y):-1<x<1,0<y<\infty\}$ for $\lambda>2N+1$ and $\gamma>-1$. Indeed, the following relation holds:

$$\int_{-1}^{1}\int_{0}^{\infty}(1-x)^{2\lambda}(1+x)^{\gamma}y^{-\lambda}\exp(-(1-x)/y)\,{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)\,{}_1E_{r,s}^{(\lambda,\gamma)}(x,y)dxdy$$
$$=\frac{2^{2k+\lambda+\gamma+2}k!\Gamma(n+k+\lambda+2)\Gamma(n-k+\gamma+1)\Gamma(\lambda-k)}{(\lambda-2k-1)\,(n-k)!(2n+\lambda+\gamma+2)\Gamma(n+k+\lambda+\gamma+2)}\delta_{n,r}\delta_{k,s},$$

for $n,r,=0,1,\dots,N<\dfrac{\lambda-1}{2},\gamma>-1,N=\max\{n,r\}$.

Theorem 2. *The polynomials* ${}_1E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the recurrence relations:*

$$(2n+\lambda+\gamma+2)\,{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)=(n+k+\lambda+\gamma+2)\,{}_1E_{n,k}^{(\lambda,\gamma+1)}(x,y)$$
$$+(n+k+\lambda+1)\,{}_1E_{n-1,k}^{(\lambda,\gamma+1)}(x,y)\quad,\quad n\ge1, \tag{26}$$

$$(n+k+\lambda+\gamma+2)(1+x)\,{}_1E_{n,k}^{(\lambda,\gamma+1)}(x,y)$$
$$=((2n+\lambda+\gamma+2)(1+x)+2\gamma)\,{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)-2(n-k+\gamma)\,{}_1E_{n,k}^{(\lambda,\gamma-1)}(x,y) \tag{27}$$

and the three-term recurrence relation of the form:

$$A_{n,k}\widetilde{E}_{n+1,k}(x,y) = (B_{n,k}x + C_{n,k})\widetilde{E}_{n,k}(x,y) - D_{n,k}\widetilde{E}_{n-1,k}(x,y) \quad , \quad n \geq 1 \tag{28}$$

where $\widetilde{E}_{n,k}(x,y) = {}_1E^{(\lambda,\gamma)}_{n,k}(x,y)$ *and the coefficients are*

$$\begin{aligned}
A_{n,k} &= 2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1), \\
B_{n,k} &= (2n+\lambda+\gamma+1)_3, \\
C_{n,k} &= (2n+\lambda+\gamma+2)(\lambda+2k+1-\gamma)(\lambda+2k+1+\gamma), \\
D_{n,k} &= 2(n+k+\lambda+1)(n-k+\gamma)(2n+\lambda+\gamma+3).
\end{aligned}$$

Proof. To obtain (26)–(28), we substitute $n \to n-k$, $\lambda \to \lambda+2k+1$, $\gamma \to \gamma$ and, then, multiply the equality by $(1-x)^k N_k^{(\lambda)}\left(\frac{y}{1-x}\right)$ in (6) and in the equality of (7) and (8) in (5), respectively. □

Theorem 3. *The polynomials* (25) *satisfy the partial differential equation:*

$$y^2E_{yy} - ((\lambda-2)y - (1-x))E_y + k(\lambda-k-1)E = 0. \tag{29}$$

Proof. Let us consider $n \to k-1$, $\lambda \to \lambda-2$, and $x \to \frac{y}{1-x}$ in (19) and multiply the equation by $P^{(\lambda+2k+1,\gamma)}_{n-k}(x)(1-x)^k$. Let us use the first partial derivative with respect to the y variable:

$$\frac{\partial}{\partial y}\, {}_1E^{(\lambda,\gamma)}_{n,k}(x,y) = k(\lambda-k-1)P^{(\lambda+2k+1,\gamma)}_{n-k}(x)(1-x)^{k-1}N^{(\lambda-2)}_{k-1}\left(\frac{y}{1-x}\right)$$

and the second partial derivative with respect to the y variable:

$$\frac{\partial^2}{\partial y^2}\, {}_1E^{(\lambda,\gamma)}_{n,k}(x,y) = (k-1)_2(\lambda-k-2)_2P^{(\lambda+2k+1,\gamma)}_{n-k}(x)(1-x)^{k-2}N^{(\lambda-4)}_{k-2}\left(\frac{y}{1-x}\right).$$

The partial differential Equation (29) is obtained by using (20). □

Theorem 4. *The set of polynomials* ${}_1E^{(\lambda-2k,\gamma)}_{n+k,k}(x,y)$ *is generated by*

$$\sum_{n,k=0}^{\infty} {}_1E^{(\lambda-2k,\gamma)}_{n+k,k}(x,y)\frac{t^{n+k}}{k!} = \frac{2^{\lambda+\gamma+1}(1+v)^{\lambda}\exp\left(-\frac{v(1-x)}{y}\right)}{(1+4v)R(1-t+R)^{\lambda+1}(1+t+R)^{\gamma}}$$

where $R = \sqrt{1-2xt+t^2}$ *and* $v = ty/(1+v)^3$ *with* $v(0) = 0$.

Proof. The result follows from Relations (3), (13), and (21). □

Theorem 5. *The polynomials* (25) *have the following parameter derivative:*

$$\begin{aligned}
\frac{\partial}{\partial \gamma}\, {}_1E^{(\lambda,\gamma)}_{n,k}(x,y) = &\sum_{l=0}^{n-k-1} \frac{1}{n+k+\lambda+\gamma+l+2}\, {}_1E^{(\lambda,\gamma)}_{n,k}(x,y) + \sum_{l=0}^{n-k-1}\frac{(-1)^{l+1}}{(l+1)} \\
&\times \frac{(2(n-l)+\lambda+\gamma)(n+k+\lambda-l+1)_{l+1}}{(2n-l+\lambda+\gamma+1)(n+k+\lambda+\gamma-l+1)_{l+1}}\, {}_1E^{(\lambda,\gamma)}_{n-l-1,k}(x,y),
\end{aligned}$$

for $n \geq k+1$, $k \geq 0$, *and* $\frac{\partial}{\partial \gamma}\, {}_1E^{(\lambda,\gamma)}_{n,n}(x,y) = 0$.

Proof. From (10), we have

$$
\begin{aligned}
&\frac{\partial}{\partial\gamma}\,{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)=\frac{\partial}{\partial\gamma}\left(P_{n-k}^{(\lambda+2k+1,\gamma)}(x)\right)(1-x)^k N_k^{(\lambda)}\left(\frac{y}{1-x}\right)\\
&=\sum_{l=0}^{n-k-1}\frac{{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)}{n+k+\lambda+\gamma+l+2}+(1-x)^k N_k^{(\lambda)}\left(\frac{y}{1-x}\right)\frac{(\lambda+2k+2)_{n-k}}{(\lambda+\gamma+2k+2)_{n-k}}\\
&\times\sum_{l=0}^{n-k-1}\frac{(-1)^{l+1}(2(n-l)+\lambda+\gamma)(\lambda+\gamma+2k+2)_{n-k-l-1}}{(l+1)(2n-l+\lambda+\gamma+1)(\lambda+2k+2)_{n-k-l-1}}P_{n-k-l-1}^{(\lambda+2k+1,\gamma)}(x)\\
&=\sum_{l=0}^{n-k-1}\frac{{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)}{n+k+\lambda+\gamma+l+2}+\sum_{l=0}^{n-k-1}\frac{(-1)^{l+1}(2(n-l)+\lambda+\gamma)}{(l+1)(2n-l+\lambda+\gamma+1)}\\
&\times\frac{(\lambda+n+k-l+1)\cdots(\lambda+n+k+1)}{(\lambda+\gamma+n+k-l+1)\cdots(\lambda+\gamma+n+k+1)}\,{}_1E_{n-1-l,k}^{(\lambda,\gamma)}(x,y)\\
&=\sum_{l=0}^{n-k-1}\frac{1}{n+k+\lambda+\gamma+l+2}\,{}_1E_{n,k}^{(\lambda,\gamma)}(x,y)+\sum_{l=0}^{n-k-1}\frac{(-1)^{l+1}}{(l+1)}\\
&\times\frac{(2(n-l)+\lambda+\gamma)(n+k+\lambda-l+1)_{l+1}}{(2n-l+\lambda+\gamma+1)(n+k+\lambda+\gamma-l+1)_{l+1}}\,{}_1E_{n-l-1,k}^{(\lambda,\gamma)}(x,y).
\end{aligned}
$$

□

Lemma 1. *If we substitute $x\to 1-\frac{2x}{\gamma}$ and $y\to\frac{2y}{\gamma}$ and take the limit as $\gamma\to\infty$ in Definition (25), from Relation (9), we obtain:*

$$
\begin{aligned}
\lim_{\gamma\to\infty}\left[\left(\frac{\gamma}{2}\right)^k{}_1E_{n,k}^{(\lambda,\gamma)}\left(1-\frac{2x}{\gamma},\frac{2y}{\gamma}\right)\right]&=\lim_{\gamma\to\infty}\left[P_{n-k}^{(\lambda+2k+1,\gamma)}\left(1-\frac{2x}{\gamma}\right)x^k N_k^{(\lambda)}\left(\frac{y}{x}\right)\right]\\
&=L_{n-k}^{(\lambda+2k+1)}(x)x^k N_k^{(\lambda)}\left(\frac{y}{x}\right)={}_2E_{n,k}^{(\lambda)}(x,y)
\end{aligned}
$$

which is a new bivariate orthogonal polynomial set expressed as the product of a finite set and an infinite sequence of univariate orthogonal polynomials.

3.2. The Set of Polynomials ${}_2E_{n,k}^{(\lambda)}(x,y)$

Definition 2. *Let us define*

$$
{}_2E_{n,k}^{(\lambda)}(x,y)=L_{n-k}^{(\lambda+2k+1)}(x)x^k N_k^{(\lambda)}\left(\frac{y}{x}\right),\quad k=0,1,\dots,n. \tag{30}
$$

The set $\left\{{}_2E_{n,k}^{(\lambda)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$
w_2(x,y)=x^{2\lambda}y^{-\lambda}e^{-\left(x+\frac{x}{y}\right)}
$$

over the domain $\Omega_2=\{(x,y):0<x,y<\infty\}$ for $\lambda>2N+1$. In fact, the corresponding orthogonality relation is

$$
\int_0^\infty\int_0^\infty\frac{x^{2\lambda}}{y^\lambda e^{x+\frac{x}{y}}}\,{}_2E_{n,k}^{(\lambda)}(x,y)\,{}_2E_{r,s}^{(\lambda)}(x,y)dxdy=\frac{k!\Gamma(n+k+\lambda+2)\Gamma(\lambda-k)}{(n-k)!(\lambda-2k-1)}\delta_{n,r}\delta_{k,s}
$$

for $n,r=0,1,\dots,N<\dfrac{\lambda-1}{2}$, $N=\max\{n,r\}$.

Theorem 6. *The polynomials defined in (30) satisfy the recurrence relation (28) for $\widetilde{E}_{n,k}(x,y)={}_2E_{n,k}^{(\lambda)}(x,y)$, $A_{n,k}=n+k+1$, $B_{n,k}=-1$, $C_{n,k}=2n+\lambda+2$, and $D_{n,k}=n+k+\lambda+1$.*

Theorem 7. *The polynomials* ${}_2E_{n,k}^{(\lambda)}(x,y)$ *satisfy the partial differential equations:*

$$x^2E_{xx} + 2xyE_{xy} + y^2E_{yy} + x(\lambda + 2 - x)E_x + y(\lambda + 2 - x)E_y + (nx - k(\lambda + k + 1))E = 0$$

and

$$y^2E_{yy} - ((\lambda - 2)y - x)E_y + k(\lambda - k - 1)E = 0.$$

Theorem 8. *The set of the polynomials* (30) *has the generating function:*

$$\sum_{n,k=0}^{\infty} {}_2E_{n+k,k}^{(\lambda-2k)}(x,y)\frac{t^{n+k}}{k!} = \frac{(1+v)^{\lambda}}{(1+4v)(1-t)^{\lambda+2}} \exp\left(-\frac{tx}{1-t} - \frac{vx}{y}\right),$$

with the conditions $v = ty(1+v)^{-3}$ *and* $v(0) = 0$.

3.3. The Set of Polynomials ${}_3E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 3. *Let us define*

$${}_3E_{n,k}^{(\lambda,\gamma)}(x,y) = P_{n-k}^{(\lambda,\gamma+2k+1)}(x)(1+x)^k N_k^{(\gamma)}\left(\frac{y}{1+x}\right), \quad k = 0,1,\ldots,n. \tag{31}$$

The set $\left\{ {}_3E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_3(x,y) = (1-x)^{\lambda}(1+x)^{2\gamma}y^{-\gamma}\exp(-(1+x)/y)$$

over the domain $\Omega_3 = \{(x,y) : -1 < x < 1, 0 < y < \infty\}$ for $\lambda > -1, \gamma > 2N+1$. Indeed, the orthogonality relation corresponding to these polynomials is:

$$\begin{aligned}&\int_{-1}^{1}\int_{0}^{\infty}(1-x)^{\lambda}(1+x)^{2\gamma}y^{-\gamma}\exp\left(-\frac{1+x}{y}\right) {}_3E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_3E_{r,s}^{(\lambda,\gamma)}(x,y)dxdy \\ &\qquad = \frac{2^{2k+\lambda+\gamma+2}k!\Gamma(n-k+\lambda+1)\Gamma(n+k+\gamma+2)\Gamma(\gamma-k)}{(n-k)!(\gamma-2k-1)(2n+\lambda+\gamma+2)\Gamma(n+k+\lambda+\gamma+2)}\delta_{n,r}\delta_{k,s}\end{aligned}$$

for $n,r = 0,1,\ldots,N < \dfrac{\gamma-1}{2}$, $\lambda > -1$, and $N = \max\{n,r\}$.

Theorem 9. *The polynomials given by* (31) *satisfy the recurrence relations:*

$$\begin{aligned}&(1-x)(2n+\lambda+\gamma+3)\, {}_3E_{n,k}^{(\lambda+1,\gamma)}(x,y) \\ &\qquad = 2(n-k+\lambda+1)\, {}_3E_{n,k}^{(\lambda,\gamma)}(x,y) - 2(n-k+1)\, {}_3E_{n+1,k}^{(\lambda,\gamma)}(x,y).\end{aligned}$$

and the relation (28) *for* $A_{n,k} = 2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1)$, $B_{n,k} = (2n+\lambda+\gamma+1)_3$, $C_{n,k} = (2n+\lambda+\gamma+2)\left(\lambda^2 - (\gamma+2k+1)^2\right)$, $\widetilde{E}_{n,k}(x,y) = {}_3E_{n,k}^{(\lambda,\gamma)}(x,y)$, *and* $D_{n,k} = 2(n-k+\lambda)(n+k+\gamma+1)(2n+\lambda+\gamma+3)$.

Theorem 10. *The polynomials* ${}_3E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equation:*

$$y^2E_{yy} - ((\gamma-2)y - (1+x))E_y + k(\gamma - k - 1)E = 0.$$

Theorem 11. *The set of the polynomials* (31) *is generated by:*

$$\sum_{n,k=0}^{\infty} {}_3E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)\frac{t^{n+k}}{k!} = \frac{2^{\lambda+\gamma+1}(1+v)^{\gamma}(1+t+R)^{-(\gamma+1)}}{(1+4v)R(1-t+R)^{\lambda}} \exp\left(-\frac{v(1+x)}{y}\right)$$

where $R = \sqrt{1-2xt+t^2}$ *and* $v = ty(1+v)^{-3}$, $v(0) = 0$.

Lemma 2. *If we substitute* $x \to \frac{2x}{\lambda} - 1$ *and* $y \to \frac{2y}{\lambda}$ *and take the limit as* $\lambda \to \infty$ *in Definition* (31), *we obtain:*

$$\begin{aligned}\lim_{\lambda\to\infty}\left[\left(\frac{\lambda}{2}\right)^{k} {}_3E_{n,k}^{(\lambda,\gamma)}\left(\frac{2x}{\lambda}-1,\frac{2y}{\lambda}\right)\right] &= \lim_{\lambda\to\infty}\left[P_{n-k}^{(\lambda,\gamma+2k+1)}\left(\frac{2x}{\lambda}-1\right)x^k N_k^{(\gamma)}\left(\frac{y}{x}\right)\right] \\ &= (-1)^{n-k}L_{n-k}^{(\gamma+2k+1)}(x)x^k N_k^{(\gamma)}\left(\frac{y}{x}\right) = (-1)^{n-k}\, {}_2E_{n,k}^{(\gamma)}(x,y),\end{aligned}$$

where ${}_2E_{n,k}^{(\gamma)}(x,y)$ *is defined in* (30).

Theorem 12. *The parameter derivative of the polynomials* ${}_3E_{n,k}^{(\lambda,\gamma)}(x,y)$ *is given by:*

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\, {}_3E_{n,k}^{(\lambda,\gamma)}(x,y) \\ &= (1+x)^k N_k^{(\gamma)}\left(\frac{y}{1+x}\right)\left\{\sum_{l=0}^{n-k-1}\frac{P_{n-k}^{(\lambda,\gamma+2k+1)}(x)}{n+k+\lambda+\gamma+l+2} + \frac{(\gamma+2k+2)_{n-k}}{(\lambda+\gamma+2k+2)_{n-k}}\right. \\ &\quad\left.\times\sum_{l=0}^{n-k-1}\frac{(2(k+l)+\lambda+\gamma+2)(\lambda+\gamma+2k+2)_l P_l^{(\lambda,\gamma+2k+1)}(x)}{(n-k-l)(n+k+\lambda+\gamma+l+2)(\gamma+2k+2)_l}\right\} \\ &= \sum_{l=0}^{n-k-1}\frac{1}{n+k+\lambda+\gamma+l+2}\, {}_3E_{n,k}^{(\lambda,\gamma)}(x,y) \\ &+ \sum_{l=0}^{n-k-1}\frac{(2(n-l)+\lambda+\gamma)(n+k+\gamma-l+1)_{l+1}}{(l+1)(2n-l+\lambda+\gamma+1)(n+k+\lambda+\gamma-l+1)_{l+1}}\, {}_3E_{n-l-1,k}^{(\lambda,\gamma)}(x,y),\end{aligned}$$

for $n \geq k+1$, $k \geq 0$ *and* $\frac{\partial}{\partial\lambda}\, {}_3E_{n,n}^{(\lambda,\gamma)}(x,y) = 0$.

3.4. The Set of Polynomials ${}_4E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 4. *Let us define*

$$ {}_4E_{n,k}^{(\lambda,\gamma)}(x,y) = N_{n-k}^{(\lambda-2k-1)}(x)x^k P_k^{(\lambda,\gamma)}\left(\frac{y}{x}\right), \quad k = 0,1,\ldots,n. \tag{32}$$

The set $\left\{ {}_4E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_4(x,y) = (x-y)^{\lambda}(x+y)^{\gamma}x^{-(2\lambda+\gamma)}\exp(-1/x)$$

over the domain $\Omega_4 = \{(x,y) : 0 < x < \infty, -x < y < x\}$ for $\lambda > 2N+2$, $\gamma > -1$. The orthogonality relation corresponding to these polynomials is given by:

$$\begin{aligned}&\int_0^{\infty}\int_{-x}^{x} x^{-(2\lambda+\gamma)}e^{-1/x}(x-y)^{\lambda}(x+y)^{\gamma}\, {}_4E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_4E_{r,s}^{(\lambda,\gamma)}(x,y)dydx \\ &= \frac{2^{\lambda+\gamma+1}(n-k)!\Gamma(\lambda-n-k-1)\Gamma(k+\lambda+1)\Gamma(k+\gamma+1)}{k!(\lambda-2n-2)(2k+\lambda+\gamma+1)\Gamma(k+\lambda+\gamma+1)}\delta_{n,r}\delta_{k,s},\end{aligned}$$

for $n, r = 0, 1, \ldots, N < \dfrac{\lambda - 2}{2}, \gamma > -1, N = \max\{n, r\}$.

Theorem 13. *The polynomials ${}_4E_{n,k}^{(\lambda,\gamma)}(x, y)$ satisfy the recurrence relation (28), where $A_{n,k} = (\lambda - 2n - 1)(\lambda - n - k - 2)$, $B_{n,k} = (\lambda - 2n - 3)_3$, $C_{n,k} = (2n - \lambda + 2)(\lambda - 2k - 1)$, $D_{n,k} = (n - k)(\lambda - 2n - 3)$, as well as $\widetilde{E}_{n,k}(x, y) = {}_4E_{n,k}^{(\lambda,\gamma)}(x, y)$.*

Theorem 14. *The polynomials ${}_4E_{n,k}^{(\lambda,\gamma)}(x, y)$ satisfy the second-order partial differential equation:*

$$x^3 E_{xx} + 2x^2 y E_{xy} + xy^2 E_{yy} - x((\lambda - 3)x - 1)E_x \\ - y((\lambda - 3)x - 1)E_y + (n(\lambda - n - 2)x - k)E = 0.$$

Theorem 15. *For the polynomials ${}_4E_{n+k,k}^{(\lambda+2k,\gamma)}(x, y)$, we have the generating function:*

$$\sum_{n,k=0}^{\infty} {}_4E_{n+k,k}^{(\lambda+2k,\gamma)}(x, y) \frac{t^{n+k}}{n!} = \frac{(1 + v)^{\lambda-1} e^{-v/x}}{1 + 2v} \frac{(1 + \xi)^{\lambda+1}(1 + \eta)^{\gamma+1}}{1 - 2\xi - 3\xi\eta}$$

where we denoted $v = \dfrac{tx}{1 + v}$, with $v(0) = 0$, $\xi = \dfrac{t(y + x)}{2}(1 + \eta)(1 + \xi)^3$, as well as $\eta = \dfrac{t(y - x)}{2}(1 + \eta)(1 + \xi)^3$.

Lemma 3. *If we substitute $y \to x - \frac{2y}{\gamma}$ and take the limit as $\gamma \to \infty$ in Definition (32), then*

$$\lim_{\gamma \to \infty} \left[{}_4E_{n,k}^{(\lambda,\gamma)}\left(x, x - \frac{2y}{\gamma}\right) \right] = \lim_{\gamma \to \infty} \left[N_{n-k}^{(\lambda-2k-1)}(x) x^k P_k^{(\lambda,\gamma)}\left(1 - \frac{2y}{\gamma x}\right) \right] \\ = N_{n-k}^{(\lambda-2k-1)}(x) x^k L_k^{(\lambda)}\left(\frac{y}{x}\right) = {}_5E_{n,k}^{(\lambda)}(x, y).$$

3.5. The Set of Polynomials ${}_5E_{n,k}^{(\lambda)}(x, y)$

Definition 5. *Let us define*

$$ {}_5E_{n,k}^{(\lambda)}(x, y) = N_{n-k}^{(\lambda-2k-1)}(x) x^k L_k^{(\lambda)}\left(\frac{y}{x}\right), \quad k = 0, 1, \ldots, n. \tag{33}$$

The set $\left\{ {}_5E_{n,k}^{(\lambda)}(x, y) \right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_5(x, y) = x^{-2\lambda} y^{\lambda} \exp(-(1 + y)/x)$$

over the domain $\Omega_5 = \{(x, y) : 0 < x, y < \infty\}$ for $\lambda > 2N + 2$. The orthogonality relation reads as

$$\int_0^\infty \int_0^\infty x^{-2\lambda} y^{\lambda} \exp\left(-\frac{1 + y}{x}\right) {}_5E_{n,k}^{(\lambda)}(x, y)\, {}_5E_{r,s}^{(\lambda)}(x, y) dy dx \\ = \frac{(n - k)!\Gamma(\lambda - n - k - 1)\Gamma(k + \lambda + 1)}{k!(\lambda - 2n - 2)} \delta_{n,r}\delta_{k,s}$$

for $n, r = 0, 1, \ldots, N < \dfrac{\lambda - 2}{2}$, $N = \max\{n, r\}$.

Theorem 16. *For the polynomials ${}_5E_{n,k}^{(\lambda)}(x, y)$ defined by (33), the recurrence relation (28) holds true for $\widetilde{E}_{n,k}(x, y) = {}_5E_{n,k}^{(\lambda)}(x, y)$, with the coefficients:*

$$A_{n,k} = (\lambda - 2n - 1)(\lambda - n - k - 2), \quad B_{n,k} = (\lambda - 2n - 3)_3,$$
$$C_{n,k} = (2n - \lambda + 2)(\lambda - 2k - 1), \quad D_{n,k} = (n - k)(\lambda - 2n - 3).$$

Theorem 17. *The polynomials* ${}_5E_{n,k}^{(\lambda)}(x,y)$ *satisfy the second-order partial differential equations:*

$$x^3 E_{xx} + 2x^2 y E_{xy} + xy^2 E_{yy} - x((\lambda - 3)x - 1)E_x - y((\lambda - 3)x - 1)E_y + (n(\lambda - n - 2)x - k)E = 0$$

and

$$xyE_{yy} + ((\lambda + 1)x - y)E_y + kE = 0.$$

Theorem 18. *For the polynomials* ${}_5E_{n+k,k}^{(\lambda+2k)}(x,y)$*, the generating function:*

$$\sum_{n,k=0}^{\infty} {}_5E_{n+k,k}^{(\lambda+2k)}(x,y) \frac{t^{n+k}}{n!k!} = \frac{(1+v)^{\lambda-1}(1+w)^{\lambda+1} \exp\left(-\frac{v+wy}{x}\right)}{(1+2v)(1-2w)}$$

holds where $v = xt/(1+v)$*,* $v(0) = 0$*, and* $w = ty(1+w)^3$*,* $w(0) = 0$*.*

3.6. The Set of Polynomials ${}_6E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 6. *Let us define*

$${}_6E_{n,k}^{(\lambda,\gamma)}(x,y) = M_{n-k}^{(\lambda-2k-1,\gamma+2k+1)}(x)x^k H_k\left(\frac{y}{x}\right), \quad k = 0,1,\ldots,n. \tag{34}$$

The set $\left\{ {}_6E_{n,k}^{(\lambda,\gamma)}(x,y) \right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_6(x,y) = x^\gamma (1+x)^{-(\lambda+\gamma)} \exp\left(-y^2/x^2\right)$$

over the domain $\Omega_6 = \{(x,y) : 0 < x < \infty, -\infty < y < \infty\}$ for $\lambda > 2N+2$, $\gamma > -2$. The corresponding orthogonality relation takes the form:

$$\int_0^\infty \int_{-\infty}^\infty x^\gamma (1+x)^{-(\lambda+\gamma)} \exp\left(-\frac{y^2}{x^2}\right) {}_6E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_6E_{r,s}^{(\lambda,\gamma)}(x,y) dxdy$$
$$= \frac{2^k (n-k)!k!\sqrt{\pi}\,\Gamma(\lambda - n - k - 1)\Gamma(\gamma + n + k + 2)}{(\lambda - 2n - 2)\Gamma(\lambda + \gamma - n + k)} \delta_{n,r}\delta_{k,s},$$

for $n,r = 0,1,\ldots,N < \dfrac{\lambda - 2}{2}$, $\gamma > -2$, $N = max\{n,r\}$.

Theorem 19. *The polynomials* ${}_6E_{n,k}^{(\lambda,\gamma)}(x,y)$ *defined in* (34) *satisfy the following recurrence relations:*

$${}_6E_{n,k}^{(\lambda,\gamma)}(x,y) = ((\lambda - 2k - 3)x - (\gamma + 2k + 2))\, {}_6E_{n-1,k}^{(\lambda-2,\gamma+1)}(x,y) - (n - k - 1)(\lambda - n - k - 3)x(x+1)\, {}_6E_{n-2,k}^{(\lambda-4,\gamma+2)}(x,y), \qquad n \geq 2,$$

$${}_6E_{n,k}^{(\lambda,\gamma)}(x,y) = 2y\, {}_6E_{n-1,k-1}^{(\lambda-2,\gamma+2)}(x,y) - 2(k-1)x^2\, {}_6E_{n-2,k-2}^{(\lambda-4,\gamma+4)}(x,y),$$

for $n \geq 2$*,* $0 \leq k \leq n-2$*, the differential relation:*

$$\frac{\partial^j}{\partial y^j}\, {}_6E_{n,k}^{(\lambda,\gamma)}(x,y) = 2^j (k - j + 1)_j\, {}_6E_{n-j,k-j}^{(\lambda-2j,\gamma+2j)}(x,y), \qquad 0 \leq j \leq k \leq n$$

as well as the relation (28) *by substituting* $\widetilde{E}_{n,k}(x,y) = {}_6E_{n,k}^{(\lambda,\gamma)}(x,y)$, *and the coefficients are given by* $A_{n,k} = (\lambda-n-k-2)(\lambda-2n-1)$, $B_{n,k} = (\lambda-2n-3)_3$, $C_{n,k} = (\lambda-2n-2)(2(n-k)_2 - (\lambda-2k-1)(\gamma+2n+2))$, *and* $D_{n,k} = (n-k)(\lambda-2n-3)(\lambda+\gamma-n+k)(n+k+\gamma+1)$.

Theorem 20. *The polynomials* ${}_6E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equations:*

$$x^2(x+1)E_{xx} + 2xy(x+1)E_{xy} + y^2(x+1)E_{yy} - x[(\lambda-3)x-(\gamma+2)]E_x \\ - y[(\lambda-3)x-(\gamma+2)]E_y + [n(\lambda-n-2)x - k(\gamma+k+1)]E = 0,$$

and

$$x^2E_{yy} - 2yE_y + 2kE = 0.$$

Theorem 21. *The polynomials* ${}_6E_{n+k,k}^{(\lambda+2k,\gamma-2k)}(x,y)$ *are generated by*

$$\sum_{n,k=0}^{\infty} {}_6E_{n+k,k}^{(\lambda+2k,\gamma-2k)}(x,y)\frac{t^{n+k}}{n!k!} = \frac{2^{1-\lambda}\left(1-t+\sqrt{(1+t)^2+4xt}\right)^{\lambda+\gamma} e^{2yt-(xt)^2}}{\sqrt{(1+t)^2+4xt}\left(1+t+\sqrt{(1+t)^2+4xt}\right)^{\gamma+1}}.$$

Lemma 4. *If we substitute* $x \to \dfrac{x}{\lambda}$ *and* $y \to \dfrac{y}{\lambda}$ *and take the limit as* $\lambda \to \infty$ *in Definition* (34), *we obtain*

$$\lim_{\lambda\to\infty}\left[\lambda^k\, {}_6E_{n,k}^{(\lambda,\gamma)}\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)\right] = \lim_{\lambda\to\infty}\left[M_{n-k}^{(\lambda-2k-1,\gamma+2k+1)}\left(\frac{x}{\lambda}\right)x^kH_k\left(\frac{y}{x}\right)\right] \\ = (-1)^{n-k}(n-k)!L_{n-k}^{(\gamma+2k+1)}(x)x^kH_k\left(\frac{y}{x}\right) = (-1)^{n-k}(n-k)!Z_{n,k}^{(\gamma)}(x,y)$$

where we used the notation $Z_{n,k}^{(\gamma)}(x,y)$ *introduced in* [22].

Remark 1. *The bivariate orthogonal polynomials* $Z_{n,k}^{(\gamma)}(x,y)$ *are the product of two infinite families of univariate orthogonal polynomials.*

3.7. The Set of Polynomials ${}_7E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 7. *Let us define*

$${}_7E_{n,k}^{(\lambda,\gamma)}(x,y) = M_{n-k}^{(\lambda-2k-1,\gamma)}(x)(1+x)^kH_k\left(\frac{y}{1+x}\right), \quad k=0,1,\dots,n. \tag{35}$$

The set $\left\{{}_7E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_7(x,y) = x^\gamma(1+x)^{-(\lambda+\gamma)}\exp\left(-y^2/(1+x)^2\right)$$

over the domain $\Omega_7 = \{(x,y) : 0 < x < \infty,\ -\infty < y < \infty\}$ for $\lambda > 2N+2$, $\gamma > -1$. It follows from the orthogonality relation:

$$\int_0^\infty\int_{-\infty}^\infty x^\gamma(1+x)^{-(\lambda+\gamma)}e^{-\frac{y^2}{(1+x)^2}}\, {}_7E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_7E_{r,s}^{(\lambda,\gamma)}(x,y)dydx \\ = \frac{2^k(n-k)!k!\sqrt{\pi}\Gamma(\lambda-n-k-1)\Gamma(\gamma+n-k+1)}{(\lambda-2n-2)\Gamma(\lambda+\gamma-n-k-1)}\delta_{n,r}\delta_{k,s}$$

for $n,r = 0,1,\dots,N < \frac{\lambda-2}{2}$, $\gamma > -1$, $N = max\{n,r\}$.

Theorem 22. *The polynomials* ${}_7E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the following recurrence relations:*

$$ {}_7E_{n,k}^{(\lambda,\gamma)}(x,y) = 2y\, {}_7E_{n-1,k-1}^{(\lambda-2,\gamma)}(x,y) - 2(k-1)(1+x)^2\, {}_7E_{n-2,k-2}^{(\lambda-4,\gamma)}(x,y), \quad n \geq 2, $$

$$ \begin{aligned} {}_7E_{n,k}^{(\lambda,\gamma)}(x,y) = {} & ((\lambda-2k-3)x-(\gamma+1))\, {}_7E_{n-1,k}^{(\lambda-2,\gamma+1)}(x,y) \\ & - (n-k-1)(\lambda-n-k-3)x(x+1)\, {}_7E_{n-2,k}^{(\lambda-4,\gamma+2)}(x,y), \end{aligned} $$

for $n \geq 2, 0 \leq k \leq n-2$, *the differential relation:*

$$ \frac{\partial^j}{\partial y^j}\, {}_7E_{n,k}^{(\lambda,\gamma)}(x,y) = 2^j(k-j+1)_j\, {}_7E_{n-j,k-j}^{(\lambda-2j,\gamma)}(x,y)\ ,\ 0 \leq j \leq k \leq n $$

as well as the relation (28) *by substituting* $\widetilde{E}_{n,k}(x,y) = {}_7E_{n,k}^{(\lambda,\gamma)}(x,y)$, *and the coefficients are explicitly given by* $A_{n,k} = (\lambda-n-k-2)(\lambda-2n-1)$, $B_{n,k} = (\lambda-2n-3)_3$, $C_{n,k} = (\lambda-2n-2)(2(n-k)_2-(\lambda-2k-1)(2(n-k)+\gamma+1))$, *and* $D_{n,k} = (n-k)(\lambda-2n-3) \times(\lambda+\gamma-n-k-1)(\gamma+n-k)$.

Theorem 23. *The polynomials* ${}_7E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equations:*

$$ \begin{aligned} & x(1+x)^2E_{xx} + 2xy(1+x)E_{xy} + xy^2E_{yy} - (1+x)((\lambda-3)x-(\gamma+1))E_x \\ & \qquad - y((\lambda-3)x-(\gamma+1))E_y + (n(\lambda-n-2)(1+x) - k(\lambda+\gamma-k-1))E = 0 \end{aligned} $$

and

$$ (1+x)^2E_{yy} - 2yE_y + 2kE = 0. $$

Theorem 24. *The set* ${}_7E_{n+k,k}^{(\lambda+2k,\gamma)}(x,y)$ *is generated by*

$$ \sum_{n,k=0}^{\infty} {}_7E_{n+k,k}^{(\lambda+2k,\gamma)}(x,y)\frac{t^{n+k}}{n!k!} = \frac{e^{2ty-t^2(1+x)^2}\left(1-t+\sqrt{(1+t)^2+4xt}\right)^{\lambda+\gamma-1}}{2^{\lambda-1}\sqrt{(1+t)^2+4xt}\left(1+t+\sqrt{(1+t)^2+4xt}\right)^{\gamma}}. $$

Lemma 5. *If we substitute* $x \to \frac{x}{\lambda}$ *and take the limit as* $\lambda \to \infty$ *in Definition* (35), *we obtain*

$$ \lim_{\lambda\to\infty}\left[{}_7E_{n,k}^{(\lambda,\gamma)}\left(\frac{x}{\lambda},y\right)\right] = (-1)^{n-k}(n-k)!L_{n-k}^{(\gamma)}(x)H_k(y) $$

which gives the product of the Laguerre and Hermite polynomials [23–25].

3.8. The Set of Polynomials ${}_8E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 8. *Let us define*

$$ {}_8E_{n,k}^{(\lambda,\gamma)}(x,y) = P_{n-k}^{(\lambda+2k+1,\gamma)}(x)(1-x)^kI_k^{(\lambda)}\left(\frac{y}{1-x}\right), \quad k = 0,1,\ldots,n. \tag{36} $$

The set $\left\{ {}_8E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$ w_8(x,y) = (1-x)^{3\lambda-1}(1+x)^{\gamma}\left((1-x)^2+y^2\right)^{-(\lambda-1/2)} $$

over the domain $\Omega_8 = \{(x,y) : -1 < x < 1, -\infty < y < \infty\}$ for $\lambda > N+1, \gamma > -1$. The orthogonality relation corresponding to these polynomials is given by

$$\int_{-1}^{1}\int_{-\infty}^{\infty}\frac{(1-x)^{3\lambda-1}(1+x)^{\gamma}}{\left((1-x)^2+y^2\right)^{(\lambda-1/2)}}\,{}_8E_{n,k}^{(\lambda,\gamma)}(x,y)\,{}_8E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$=\frac{2^{2k+3\lambda+\gamma}k!\Gamma^2(\lambda)\Gamma(n+k+\lambda+2)\Gamma(n-k+\gamma+1)\delta_{n,r}\delta_{k,s}}{(n-k)!(2n+\lambda+\gamma+2)(\lambda-k-1)\Gamma(2\lambda-k-1)\Gamma(n+k+\lambda+\gamma+2)}$$

for $n,r=0,1,\dots,N<\lambda-1,\gamma>-1, N=max\{n,r\}$.

Theorem 25. *The polynomials* ${}_8E_{n,k}^{(\lambda,\gamma)}(x,y)$ *defined in* (36) *satisfy the recurrence relation* (28) *for* $\widetilde{E}_{n,k}(x,y)={}_8E_{n,k}^{(\lambda,\gamma)}(x,y)$, *and the coefficients defined by*

$$A_{n,k}=2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1),$$
$$B_{n,k}=(2n+\lambda+\gamma+1)_3,\quad C_{n,k}=(2n+\lambda+\gamma+2)\left((\lambda+2k+1)^2-\gamma^2\right),$$
$$D_{n,k}=2(n+k+\lambda+1)(n-k+\gamma)(2n+\lambda+\gamma+3).$$

Theorem 26. *The polynomials* ${}_8E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equation:*

$$\left(y^2+(1-x)^2\right)E_{yy}-(2\lambda-3)yE_y-k(k+2-2\lambda)E=0.$$

Theorem 27. *For the polynomials given by* (36), *we have the generating function:*

$$\sum_{n,k=0}^{\infty}{}_8E_{n+k,k}^{(\lambda-2k,\gamma)}(x,y)\frac{t^{n+k}}{k!}=\frac{2^{\lambda+\gamma+1}(1+\xi)^{\lambda}(1+\eta)^{\lambda}}{R(1-t+R)^{\lambda+1}(1+t+R)^{\gamma}(1+3\xi+3\eta+5\xi\eta)}$$

where $R=\sqrt{1-2xt+t^2}$, $\xi=\frac{t\left(y+\sqrt{y^2+(1-x)^2}\right)}{(1+\xi)^2(1+\eta)^2}$ *and* $\eta=\frac{t\left(y-\sqrt{y^2+(1-x)^2}\right)}{(1+\xi)^2(1+\eta)^2}$.

Theorem 28. *The parameter derivative of the polynomials* (36) *is given by*

$$\frac{\partial}{\partial\gamma}\left({}_8E_{n,k}^{(\lambda,\gamma)}(x,y)\right)=\sum_{l=0}^{n-k-1}\frac{1}{n+k+\lambda+\gamma+2+l}\,{}_8E_{n,k}^{(\lambda,\gamma)}(x,y)$$
$$+\sum_{l=0}^{n-k-1}\frac{(-1)^{l+1}(2(n-l)+\lambda+\gamma)(n+k+\lambda+1-l)_{l+1}}{(l+1)(2n+\lambda+\gamma+1-l)(n+k+\lambda+\gamma+1-l)_{l+1}}\,{}_8E_{n-1-l,k}^{(\lambda,\gamma)}(x,y)$$

for $n\geq k+1$, $k\geq 0$, *and* $\frac{\partial}{\partial\gamma}\,{}_8E_{n,n}^{(\lambda,\gamma)}(x,y)=0$.

Lemma 6. *If we substitute* $x\to\frac{2x}{\lambda}-1$ *and* $y\to\frac{2y}{\sqrt{\lambda}}\left(1-\frac{x}{\lambda}\right)$ *and take the limit as* $\lambda\to\infty$ *in Definition* (36), *we obtain*

$$\lim_{\lambda\to\infty}\left[2^{-k}\lambda^{-\frac{k}{2}}\,{}_8E_{n,k}^{(\lambda,\gamma)}\left(\frac{2x}{\lambda}-1,\frac{2y}{\sqrt{\lambda}}\left(1-\frac{x}{\lambda}\right)\right)\right]=(-1)^{n-k}L_{n-k}^{(\gamma)}(x)H_k(y)$$

where $L_{n-k}^{(\gamma)}(x)H_k(y)$ *defined in* [23–25] *is the product of two infinite sequences of univariate orthogonal polynomials.*

From a different viewpoint, we give the following limit case.

Lemma 7. *If we substitute* $x\to 1-\frac{2x}{\gamma}$ *and* $y\to\frac{2y}{\gamma}$ *and take the limit as* $\gamma\to\infty$ *in Definition* (36), *we obtain*

$$\lim_{\gamma\to\infty}\left[\left(\frac{\gamma}{2}\right)^k {}_8E_{n,k}^{(\lambda,\gamma)}\left(1-\frac{2x}{\gamma},\frac{2y}{\gamma}\right)\right] = L_{n-k}^{(\lambda+2k+1)}(x)x^k I_k^{(\lambda)}\left(\frac{y}{x}\right) = {}_9E_{n,k}^{(\lambda)}(x,y).$$

3.9. The Set of Polynomials ${}_9E_{n,k}^{(\lambda)}(x,y)$

Definition 9. *Let us introduce*

$$ {}_9E_{n,k}^{(\lambda)}(x,y) = L_{n-k}^{(\lambda+2k+1)}(x)x^k I_k^{(\lambda)}\left(\frac{y}{x}\right), \quad k=0,1,\dots,n. \tag{37}$$

The set $\left\{ {}_9E_{n,k}^{(\lambda)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_9(x,y) = x^\lambda e^{-x}\left(1+y^2/x^2\right)^{-(\lambda-1/2)}$$

over the domain $\Omega_9 = \{(x,y) : 0 < x < \infty, -\infty < y < \infty\}$ for $\lambda > N+1$. That is, the corresponding orthogonality relation:

$$\int_0^\infty\int_{-\infty}^\infty x^\lambda e^{-x}\left(1+y^2/x^2\right)^{-(\lambda-1/2)} {}_9E_{n,k}^{(\lambda)}(x,y)\,{}_9E_{r,s}^{(\lambda)}(x,y)dydx$$
$$= \frac{2^{2\lambda-2}k!\Gamma^2(\lambda)\Gamma(n+k+\lambda+2)\delta_{n,r}\delta_{k,s}}{(n-k)!(\lambda-k-1)\Gamma(2\lambda-k-1)}$$

is satisfied for $n,r = 0,1,\dots,N < \lambda-1$, $N = \max\{n,r\}$.

Theorem 29. *For the polynomials (37), we have the recurrence relation (28) for $\widetilde{E}_{n,k}(x,y) = {}_9E_{n,k}^{(\lambda)}(x,y)$, $A_{n,k} = n-k+1$, $B_{n,k} = -1$, $C_{n,k} = 2n+2+\lambda$, and $D_{n,k} = n+k+\lambda+1$.*

Theorem 30. *The polynomials ${}_9E_{n,k}^{(\lambda)}(x,y)$ satisfy the partial differential equations:*

$$x^2E_{xx} + 2xyE_{xy} + y^2E_{yy} + x(\lambda+2-x)E_x + y(\lambda+2-x)E_y + [nx - k(\lambda+k+1)]E = 0$$

and

$$\left(x^2+y^2\right)E_{yy} - (2\lambda-3)yE_y - k(k+2-2\lambda)E = 0.$$

Theorem 31. *The set of the polynomials ${}_9E_{n+k,k}^{(\lambda-2k)}(x,y)$ is generated by*

$$\sum_{n,k=0}^{\infty} {}_9E_{n+k,k}^{(\lambda-2k)}(x,y)\frac{t^{n+k}}{k!} = \frac{\exp\left(\frac{-tx}{1-t}\right)(1+\xi)^\lambda(1+\eta)^\lambda}{(1-t)^{\lambda+2}(1+3\xi+3\eta+5\xi\eta)}$$

where $\xi = \frac{t\left(y+\sqrt{x^2+y^2}\right)}{(1+\xi)^2(1+\eta)^2}$ *and* $\eta = \frac{t\left(y-\sqrt{x^2+y^2}\right)}{(1+\xi)^2(1+\eta)^2}$.

3.10. The Set of Polynomials ${}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 10. *Let us define*

$$ {}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y) = P_{n-k}^{(\lambda,\gamma+2k+1)}(x)(1+x)^k I_k^{(\gamma)}\left(\frac{y}{1+x}\right), \quad k=0,1,\dots,n. \tag{38}$$

The set $\left\{ {}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{10}(x,y) = (1-x)^{\lambda}(1+x)^{3\gamma-1}\left((1+x)^2+y^2\right)^{-\left(\gamma-\frac{1}{2}\right)}$$

over the domain $\Omega_{10} = \{(x,y) : -1 < x < 1, -\infty < y < \infty\}$ for $\lambda > -1$, $\gamma > N+1$. That is,

$$\int_{-1}^{1}\int_{-\infty}^{\infty} \frac{(1-x)^{\lambda}(1+x)^{3\gamma-1}}{\left((1+x)^2+y^2\right)^{\gamma-1/2}}\, {}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_{10}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$= \frac{2^{2k+\lambda+3\gamma}k!\Gamma^2(\gamma)\Gamma(n-k+\lambda+1)\Gamma(n+k+\gamma+2)\delta_{n,r}\delta_{k,s}}{(n-k)!(2n+\lambda+\gamma+2)(\gamma-k-1)\Gamma(n+k+\lambda+\gamma+2)\Gamma(2\gamma-k-1)}$$

for $n,r = 0,1,\ldots,N < \gamma-1$, $\lambda > -1$, $N = max\{n,r\}$.

Theorem 32. *The polynomials defined by (38) satisfy the recurrence relation:*

$$(2n+\lambda+\gamma+3)(1-x)\, {}_{10}E_{n,k}^{(\lambda+1,\gamma)}(x,y)$$
$$= 2(n-k+\lambda+1)\, {}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y) - 2(n-k+1)\, {}_{10}E_{n+1,k}^{(\lambda,\gamma)}(x,y)$$

and the relation (28) if we consider $\widetilde{E}_{n,k}(x,y) = {}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *and the coefficients*

$$\begin{aligned}
A_{n,k} &= 2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1),\\
B_{n,k} &= (2n+\lambda+\gamma+1)_3,\\
C_{n,k} &= (2n+\lambda+\gamma+2)(\lambda-\gamma-2k-1)(\lambda+\gamma+2k+1),\\
D_{n,k} &= 2(n-k+\lambda)(n+k+\gamma+1)(2n+\lambda+\gamma+3).
\end{aligned}$$

Theorem 33. *The polynomials* ${}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equation:*

$$\left((1+x)^2+y^2\right)E_{yy} - (2\gamma-3)yE_y - k(k+2-2\gamma)E = 0.$$

Theorem 34. *The polynomials* ${}_{10}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)$ *have the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{10}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)\frac{t^{n+k}}{k!} = \frac{2^{\lambda+\gamma+1}(1+\xi)^{\gamma}(1+\eta)^{\gamma}}{R(1-t+R)^{\lambda}(1+t+R)^{\gamma+1}(1+3\xi+3\eta+5\xi\eta)},$$

where $R = \sqrt{1-2xt+t^2}$ *and* $\xi = \frac{t\left(y+\sqrt{y^2+(1+x)^2}\right)}{(1+\xi)^2(1+\eta)^2}$, $\eta = \frac{t\left(y-\sqrt{y^2+(1+x)^2}\right)}{(1+\xi)^2(1+\eta)^2}$.

Lemma 8. *If we substitute* $x \to \frac{2x}{\lambda}-1$ *and* $y \to \frac{2y}{\lambda}$ *and take the limit as* $\lambda \to \infty$ *in Definition (38), we obtain*

$$\lim_{\lambda\to\infty}\left[\left(\frac{\lambda}{2}\right)^k {}_{10}E_{n,k}^{(\lambda,\gamma)}\left(\frac{2x}{\lambda}-1, \frac{2y}{\lambda}\right)\right]$$
$$= (-1)^{n-k}L_{n-k}^{(\gamma+2k+1)}(x)x^k I_k^{(\gamma)}\left(\frac{y}{x}\right) = (-1)^{n-k}\, {}_9E_{n,k}^{(\gamma)}(x,y).$$

From a different viewpoint, we give the following limit case.

Lemma 9. *If we substitute* $x \to 1-\frac{2x}{\gamma}$ *and* $y \to \frac{2y}{\sqrt{\gamma}}\left(1-\frac{x}{\gamma}\right)$ *and take the limit as* $\gamma \to \infty$ *in Definition (38), we obtain*

$$\lim_{\gamma\to\infty}\left[2^{-k}\gamma^{-\frac{k}{2}}\ {}_{10}E_{n,k}^{(\lambda,\gamma)}\left(1-\frac{2x}{\gamma},\frac{2y}{\sqrt{\gamma}}\left(1-\frac{x}{\gamma}\right)\right)\right]=L_{n-k}^{(\lambda)}(x)H_k(y).$$

Theorem 35. *The parameter derivative of the polynomials* ${}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *is given by*

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left({}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)\right)&=\sum_{l=0}^{n-k-1}\frac{1}{n+k+\lambda+\gamma+2+l}\ {}_{10}E_{n,k}^{(\lambda,\gamma)}(x,y)\\&+\sum_{l=0}^{n-k-1}\frac{(2(n-l)+\lambda+\gamma)(n+k+\gamma+1-l)_{l+1}}{(l+1)(2n+\lambda+\gamma+1-l)(n+k+\lambda+\gamma+1-l)_{l+1}}\ {}_{10}E_{n-1-l,k}^{(\lambda,\gamma)}(x,y)\end{aligned}$$

for $n\geq k+1$, $k\geq 0$ *and* $\frac{\partial}{\partial\lambda}\ {}_{10}E_{n,n}^{(\lambda,\gamma)}(x,y)=0$.

3.11. The Set of Polynomials ${}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 11. *Let us define*

$$ {}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)=P_{n-k}^{(\lambda,\gamma+2k+1)}(x)(1+x)^kI_k^{(\lambda)}\left(\frac{y}{1+x}\right),\quad k=0,1,\dots,n. \tag{39}$$

The set $\left\{{}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{11}(x,y)=(1-x)^{\lambda}(1+x)^{2\lambda+\gamma-1}\left((1+x)^2+y^2\right)^{-(\lambda-1/2)}$$

over the domain $\Omega_{11}=\{(x,y):-1<x<1,-\infty<y<\infty\}$ for $\lambda>N+1,\gamma>-2$. In fact,

$$\begin{aligned}&\int_{-1}^{1}\int_{-\infty}^{\infty}\frac{(1-x)^{\lambda}(1+x)^{2\lambda+\gamma-1}}{\left((1+x)^2+y^2\right)^{\lambda-\frac{1}{2}}}\ {}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)\ {}_{11}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx\\&\quad=\frac{2^{2k+3\lambda+\gamma}k!\Gamma^2(\lambda)\Gamma(n-k+\lambda+1)\Gamma(\gamma+n+k+2)\delta_{n,r}\delta_{k,s}}{(n-k)!(2n+\lambda+\gamma+2)(\lambda-k-1)\Gamma(n+k+\lambda+\gamma+2)\Gamma(2\lambda-k-1)}\end{aligned}$$

for $n,r=0,1,\dots,N<\lambda-1,\gamma>-2$, $N=max\{n,r\}$.

Theorem 36. *The polynomials* (39) *satisfy the following recurrence relations:*

$$\begin{aligned}&{}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)-2(\lambda-k)y\ {}_{11}E_{n-1,k-1}^{(\lambda,\gamma+2)}(x,y)\\&\qquad+(k-1)(2\lambda-k)(1+x)^2\ {}_{11}E_{n-2,k-2}^{(\lambda,\gamma+4)}(x,y)=0,\qquad n\geq k\geq 2,\end{aligned}$$

$$\begin{aligned}(2n+\lambda+\gamma+2)\ {}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)&=(n+k+\lambda+\gamma+2)\ {}_{11}E_{n,k}^{(\lambda,\gamma+1)}(x,y)\\&\quad+(n-k+\lambda)\ {}_{11}E_{n-1,k}^{(\lambda,\gamma+1)}(x,y),\qquad n\geq 1,\end{aligned}$$

$$\begin{aligned}&(n+k+\lambda+\gamma+2)(1+x)\ {}_{11}E_{n,k}^{(\lambda,\gamma+1)}(x,y)+2(n+k+\gamma+1)\ {}_{11}E_{n,k}^{(\lambda,\gamma-1)}(x,y)\\&\qquad-((2n+\lambda+\gamma+2)x+2n+4k+\lambda+3\gamma+4)\ {}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)=0\end{aligned}$$

and the relation (28) *for the coefficients* $A_{n,k}=2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1)$, $B_{n,k}=(2n+\lambda+\gamma+1)_3$,$C_{n,k}=(2n+\lambda+\gamma+2)\left(\lambda^2-(\gamma+2k+1)^2\right)$,$D_{n,k}=2(n-k+\lambda)$ $\times(n+k+\gamma+1)(2n+\lambda+\gamma+3)$ *and* $\widetilde{E}_{n,k}(x,y)=\ {}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)$.

Theorem 37. *The polynomials* ${}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *are solutions to the partial differential equation:*

$$\Big((1+x)^2+y^2\Big)E_{yy}-(2\lambda-3)yE_y-k(k+2-2\lambda)E=0.$$

Theorem 38. *The polynomials* ${}_{11}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)$ *have the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{11}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)\frac{t^{n+k}}{k!}=\frac{2^{\lambda+\gamma+1}\Big(1+2ty-t^2(1+x)^2\Big)^{\lambda-1}}{R(1-t+R)^{\lambda}(1+t+R)^{\gamma+1}},$$

where the function $R=\sqrt{1-2xt+t^2}$.

Theorem 39. *The parameter derivative of the polynomials* (39) *is given by*

$$\begin{aligned}\frac{\partial}{\partial\gamma}\,{}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)&=\sum_{l=0}^{n-k-1}\frac{1}{n+k+\lambda+\gamma+l+2}\,{}_{11}E_{n,k}^{(\lambda,\gamma)}(x,y)\\&+\sum_{l=0}^{n-k-1}\frac{(-1)^{l+1}(2(n-l)+\lambda+\gamma)(\lambda+n-k-l)_{l+1}}{(l+1)(2n-l+\lambda+\gamma+1)(n+k+\lambda+\gamma-l+1)_{l+1}}\,{}_{11}E_{n-l-1,k}^{(\lambda,\gamma)}(x,y),\end{aligned}$$

for $n\geq k+1$, $k\geq 0$, *and* $\dfrac{\partial}{\partial\gamma}\,{}_{11}E_{n,n}^{(\lambda,\gamma)}(x,y)=0$.

Lemma 10. *If we substitute* $x\to\dfrac{2x}{\lambda}-1$ *and* $y\to\dfrac{2y}{\lambda\sqrt{\lambda}}$ *and take the limit as* $\lambda\to\infty$ *in Definition* (39), *we obtain*

$$\begin{aligned}&\lim_{\lambda\to\infty}\left[\left(\frac{\sqrt{\lambda}}{2}\right)^{k}{}_{11}E_{n,k}^{(\lambda,\gamma)}\left(\frac{2x}{\lambda}-1,\frac{2y}{\lambda\sqrt{\lambda}}\right)\right]\\&\qquad=(-1)^{n-k}L_{n-k}^{(\gamma+2k+1)}(x)x^kH_k\left(\frac{y}{x}\right)=(-1)^{n-k}\,Z_{n,k}^{(\gamma)}(x,y),\end{aligned}$$

where $Z_{n,k}^{(\gamma)}(x,y)$ *is defined in* [22], *and moreover,*

$$\lim_{\gamma\to\infty}\left[2^{-k}\,{}_{11}E_{n,k}^{(\lambda,\gamma)}\left(1-\frac{2x}{\gamma},2y\left(1-\frac{x}{\gamma}\right)\right)\right]=L_{n-k}^{(\lambda)}(x)I_k^{(\lambda)}(y)=\,{}_{12}E_{n,k}^{(\lambda)}(x,y).$$

3.12. The Set of Polynomials ${}_{12}E_{n,k}^{(\lambda)}(x,y)$

Definition 12. *Let us define*

$${}_{12}E_{n,k}^{(\lambda)}(x,y)=L_{n-k}^{(\lambda)}(x)I_k^{(\lambda)}(y),\quad k=0,1,\ldots,n. \tag{40}$$

The set $\left\{{}_{12}E_{n,k}^{(\lambda)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{12}(x,y)=x^{\lambda}e^{-x}\Big(1+y^2\Big)^{-(\lambda-1/2)}$$

over the domain $\Omega_{12}=\{(x,y):0<x<\infty,-\infty<y<\infty\}$ for $\lambda>N+1$. The orthogonality relation corresponding to these polynomials is

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} x^{\lambda}e^{-x}\left(1+y^2\right)^{-\left(\lambda-\frac{1}{2}\right)} {}_{12}E_{n,k}^{(\lambda)}(x,y)\, {}_{12}E_{r,s}^{(\lambda)}(x,y)dydx = \frac{2^{2\lambda-2}k!\Gamma^2(\lambda)\Gamma(\lambda+n-k+1)}{(n-k)!(\lambda-k-1)\Gamma(2\lambda-k-1)}\delta_{n,r}\delta_{k,s}$$

for $n,r = 0,1,\ldots,N < \lambda-1,\ N = max\{n,r\}$.

Theorem 40. *The polynomials defined in (40) can also be computed by the Rodrigues representation:*

$$ {}_{12}E_{n,k}^{(\lambda)}(x,y) = \frac{(-2)^k(\lambda-k)_k e^x x^{-\lambda}\left(1+y^2\right)^{\lambda-\frac{1}{2}}}{(n-k)!(2\lambda-2k-1)_k}\frac{\partial^n\left(e^{-x}x^{n-k+\lambda}\left(1+y^2\right)^{k-\lambda+\frac{1}{2}}\right)}{\partial x^{n-k}\partial y^k}.$$

Theorem 41. *For the polynomials defined by (40), the recurrence relation:*

$$ {}_{12}E_{n,k}^{(\lambda)}(x,y) = 2(\lambda-k)y\, {}_{12}E_{n-1,k-1}^{(\lambda)}(x,y) - (k-1)(2\lambda-k)\, {}_{12}E_{n-2,k-2}^{(\lambda)}(x,y)$$

holds true, as well as Relation (28), where $\widetilde{E}_{n,k}(x,y) = {}_{12}E_{n,k}^{(\lambda)}(x,y)$ *and coefficients* $A_{n,k} = n-k+1$, $B_{n,k} = -1$, $C_{n,k} = 2(n-k)+\lambda+1$, *and* $D_{n,k} = n-k+\lambda$.

Theorem 42. *The polynomials* ${}_{12}E_{n,k}^{(\lambda)}(x,y)$ *defined in (40) satisfy the partial differential equations:*

$$xE_{xx} + (\lambda+1-x)E_x + (n-k)E = 0$$

and

$$\left(1+y^2\right)E_{yy} - y(2\lambda-3)E_y - k(k+2-2\lambda)E = 0.$$

Theorem 43. *The polynomials* ${}_{12}E_{n+k,k}^{(\lambda)}(x,y)$ *have the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{12}E_{n+k,k}^{(\lambda)}(x,y)\frac{t^{n+k}}{k!} = \frac{\left(1+2ty-t^2\right)^{\lambda-1}}{(1-t)^{\lambda+1}}\exp\left(-\frac{tx}{1-t}\right).$$

3.13. The Set of Polynomials ${}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 13. *Let us define*

$$ {}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y) = P_{n-k}^{(\lambda+k+1/2,\gamma+k+1/2)}(x)\left(\sqrt{1-x^2}\right)^k I_k^{(\lambda)}\left(\frac{y}{\sqrt{1-x^2}}\right), \tag{41}$$

for $k = 0,1,\ldots,n$.

The set $\left\{ {}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{13}(x,y) = (1-x)^{2\lambda-\frac{1}{2}}(1+x)^{\lambda+\gamma-\frac{1}{2}}\left(1-x^2+y^2\right)^{-(\lambda-1/2)}$$

over the domain $\Omega_{13} = \{(x,y) : -1 < x < 1, -\infty < y < \infty\}$ for $\lambda > N+1$, $\gamma > -3/2$. That is,

$$\int_{-1}^{1}\int_{-\infty}^{\infty} \frac{(1-x)^{2\lambda-\frac{1}{2}}(1+x)^{\lambda+\gamma-\frac{1}{2}}}{(1-x^2+y^2)^{\lambda-\frac{1}{2}}}\, {}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_{13}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$= \frac{2^{2k+3\lambda+\gamma}k!\Gamma^2(\lambda)\Gamma(n+\lambda+3/2)\Gamma(n+\gamma+3/2)\delta_{n,r}\delta_{k,s}}{(n-k)!(2n+\lambda+\gamma+2)(\lambda-k-1)\Gamma(n+k+\lambda+\gamma+2)\Gamma(2\lambda-k-1)}$$

holds true for $n, r = 0, 1, \ldots, N < \lambda - 1, \gamma > -3/2, N = max\{n, r\}$.

Theorem 44. *The polynomials ${}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)$ defined in (41) satisfy the relation (28), where in this case, the coefficients are given by*

$$A_{n,k} = 2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1),$$
$$B_{n,k} = (2n+\lambda+\gamma+1)_3, \quad C_{n,k} = (2n+\lambda+\gamma+2)(\lambda-\gamma)(\lambda+\gamma+2k+1),$$
$$D_{n,k} = 2(n+\lambda+1/2)(n+\gamma+1/2)(2n+\lambda+\gamma+3).$$

Theorem 45. *The polynomials (41) satisfy the partial differential equation:*

$$\left(1-x^2+y^2\right)E_{yy} - (2\lambda-3)yE_y - k(k+2-2\lambda)E = 0.$$

Theorem 46. *The polynomials ${}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)$ have the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{13}E_{n+k,k}^{(\lambda-k,\gamma-k)}(x,y)\frac{t^{n+k}}{k!} = \frac{2^{\lambda+\gamma+1}(1+\xi)^{\lambda}(1+\eta)^{\lambda}(1+2\xi+2\eta+3\xi\eta)^{-1}}{R(1-t+R)^{\lambda+1/2}(1+t+R)^{\gamma+1/2}},$$

where $R = \sqrt{1-2xt+t^2}$, $\xi = \dfrac{t(y+\sqrt{y^2+1-x^2})}{(1+\xi)(1+\eta)}$, *and* $\eta = \dfrac{t(y-\sqrt{y^2+1-x^2})}{(1+\xi)(1+\eta)}$.

Theorem 47. *The parameter derivative of the polynomials ${}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)$ is given by*

$$\frac{\partial}{\partial\gamma}\, {}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y) = \sum_{l=0}^{n-k-1} \frac{1}{n+k+\lambda+\gamma+l+2}\, {}_{13}E_{n,k}^{(\lambda,\gamma)}(x,y)$$
$$+ \sum_{l=0}^{n-k-1} \frac{(-1)^{l+1}(2(n-l)+\lambda+\gamma)(n+\lambda-l+1/2)_{l+1}}{(l+1)(2n-l+\lambda+\gamma+1)(n+k+\lambda+\gamma-l+1)_{l+1}}\, {}_{13}E_{n-1-l,k}^{(\lambda,\gamma)}(x,y)$$

for $n \geq k+1$, $k \geq 0$, and $\frac{\partial}{\partial\gamma}\, {}_{13}E_{n,n}^{(\lambda,\gamma)}(x,y) = 0$.

Lemma 11. *If we substitute $x \to 1 - \frac{2x}{\gamma}$ and $y \to \frac{2y}{\sqrt{\gamma}}$ and take the limit as $\gamma \to \infty$ in Definition (41), we obtain*

$$\lim_{\gamma\to\infty}\left[\left(\frac{\sqrt{\gamma}}{2}\right)^k {}_{13}E_{n,k}^{(\lambda,\gamma)}\left(1-\frac{2x}{\gamma}, \frac{2y}{\sqrt{\gamma}}\right)\right]$$
$$= \lim_{\gamma\to\infty}\left[P_{n-k}^{(\lambda+k+1/2,\gamma+k+1/2)}\left(1-\frac{2x}{\gamma}\right)\left(\sqrt{x-\frac{x^2}{\gamma}}\right)^k I_k^{(\lambda)}\left(\frac{y}{\sqrt{x-\frac{x^2}{\gamma}}}\right)\right]$$
$$= L_{n-k}^{(\lambda+k+1/2)}(x)\left(\sqrt{x}\right)^k I_k^{(\lambda)}\left(\frac{y}{\sqrt{x}}\right) = {}_{14}E_{n,k}^{(\lambda)}(x,y).$$

3.14. The Set of Polynomials ${}_{14}E_{n,k}^{(\lambda)}(x,y)$

Definition 14. *Let us define*

$$ {}_{14}E_{n,k}^{(\lambda)}(x,y) = L_{n-k}^{(\lambda+k+1/2)}(x)\left(\sqrt{x}\right)^k I_k^{(\lambda)}\left(\frac{y}{\sqrt{x}}\right), \quad k = 0,1,\ldots,n. \tag{42}$$

The set $\left\{ {}_{14}E_{n,k}^{(\lambda)}(x,y) \right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{14}(x,y) = e^{-x}x^{2\lambda-1/2}\left(x+y^2\right)^{-(\lambda-1/2)}$$

over the domain $\Omega_{14} = \{(x,y) : 0 < x < \infty, -\infty < y < \infty\}$ for $\lambda > N+1$, and

$$\int_0^\infty \int_{-\infty}^\infty x^{2\lambda-\frac{1}{2}}e^{-x}\left(x+y^2\right)^{-(\lambda-1/2)} {}_{14}E_{n,k}^{(\lambda)}(x,y)\, {}_{14}E_{r,s}^{(\lambda)}(x,y)dydx$$
$$= \frac{2^{2\lambda-2}k!\Gamma^2(\lambda)\Gamma(n+\lambda+3/2)}{(n-k)!(\lambda-k-1)\Gamma(2\lambda-k-1)}\delta_{n,r}\delta_{k,s}$$

for $n,r = 0,1,\ldots,N < \lambda-1$, $N = max\{n,r\}$.

Theorem 48. *For the polynomials* ${}_{14}E_{n,k}^{(\lambda)}(x,y)$ *given by* (42)*, the recurrence relation* (28) *holds true, where* $\widetilde{E}_{n,k}(x,y) = {}_{14}E_{n,k}^{(\lambda)}(x,y)$ *and the coefficients* $A_{n,k} = n-k+1$, $B_{n,k} = -1$, $C_{n,k} = 2n-k+\lambda+3/2$*, and* $D_{n,k} = n+\lambda+1/2$.

Theorem 49. *The polynomials* ${}_{14}E_{n,k}^{(\lambda)}(x,y)$ *are solutions of the partial differential equations:*

$$4x^2E_{xx} + 4xyE_{xy} + y^2E_{yy} + 4x(\lambda+3/2-x)E_x + 2y(\lambda+1-x)E_y$$
$$+ (2(2n-k)x - k(k+1+2\lambda))E = 0$$

and

$$\left(x+y^2\right)E_{yy} - (2\lambda-3)yE_y - k(k+2-2\lambda)E = 0.$$

Theorem 50. *The set of the polynomials* (42) *is generated by*

$$\sum_{n,k=0}^\infty {}_{14}E_{n+k,k}^{(\lambda-k)}(x,y)\frac{t^{n+k}}{k!} = \frac{(1+\xi)^\lambda(1+\eta)^\lambda \exp\left(\frac{xt}{t-1}\right)}{(1-t)^{\lambda+3/2}(1+2\xi+2\eta+3\xi\eta)},$$

where $\xi = \frac{t\left(y+\sqrt{x+y^2}\right)}{(1+\xi)(1+\eta)}$ *and* $\eta = \frac{t\left(y-\sqrt{x+y^2}\right)}{(1+\xi)(1+\eta)}$.

3.15. The Set of Polynomials ${}_{15}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 15. *Let us define*

$$ {}_{15}E_{n,k}^{(\lambda,\gamma)}(x,y) = M_{n-k}^{(\lambda-2k-1,\gamma+2k+1)}(x)x^k L_k^{(\gamma)}\left(\frac{y}{x}\right), \quad k = 0,1,\ldots,n. \tag{43}$$

The set $\left\{ {}_{15}E_{n,k}^{(\lambda,\gamma)}(x,y) \right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{15}(x,y) = y^\gamma(1+x)^{-(\lambda+\gamma)}\exp(-y/x)$$

over the domain $\Omega_{15} = \{(x,y) : 0 < x,y < \infty\}$ for $\lambda > 2N+2$, $\gamma > -1$, and

$$\int_0^\infty \int_0^\infty y^\gamma e^{-\frac{y}{x}}(1+x)^{-(\lambda+\gamma)}\; {}_{15}E_{n,k}^{(\lambda,\gamma)}(x,y)\; {}_{15}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$= \frac{(n-k)!\Gamma(\lambda-n-k-1)\Gamma(n+k+\gamma+2)\Gamma(k+\gamma+1)}{k!(\lambda-2n-2)\Gamma(\lambda+\gamma-n+k)}\delta_{n,r}\delta_{k,s}$$

for $n,r = 0,1,\ldots,N < \frac{\lambda-2}{2}, \gamma > -1$.

Theorem 51. *The polynomials* ${}_{15}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the relation* (28) *for* $\widetilde{E}_{n,k}(x,y) = {}_{15}E_{n,k}^{(\lambda,\gamma)}$ (x,y)*, with coefficients* $A_{n,k} = (\lambda-n-k-2)\ (\lambda-2n-1)$*,* $B_{n,k} = (\lambda-2n-3)_3$*,* $C_{n,k} = (\lambda-2n-2)\times(2(n-k)_2-(\lambda-2k-1)(\gamma+2n+2))$*, as well as* $D_{n,k} = (n-k)(\lambda-2n-3)$ $(\lambda+\gamma-n+k)\times(n+k+\gamma+1)$.

Theorem 52. *The set of the polynomials* ${}_{15}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfies the partial differential equations:*

$$x^2(x+1)E_{xx} + 2xy(x+1)E_{xy} + y^2(x+1)E_{yy} - x((\lambda-3)x-(\gamma+2))E_x$$
$$- y((\lambda-3)x-(\gamma+2))E_y + (n(\lambda-n-2)x - k(k+\gamma+1))E = 0$$

and

$$xyE_{yy} + ((\gamma+1)x-y)E_y + kE = 0.$$

Theorem 53. *The polynomials* ${}_{15}E_{n+k,k}^{(\lambda+2k,\gamma-2k)}(x,y)$ *are generated by*

$$\sum_{n,k=0}^{\infty} {}_{15}E_{n+k,k}^{(\lambda+2k,\gamma-2k)}(x,y)\frac{t^{n+k}}{n!} = \frac{2^{1-\lambda}(1+v)^{\gamma+1}e^{-\frac{vy}{x}}(1-t+A(x,t))^{\lambda+\gamma}}{(1+2v)(1+t+A(x,t))^{\gamma+1}A(x,t)}$$

where $A(x,t) = \sqrt{(1+t)^2+4xt}$ *and* $v = \frac{xt}{1+v}$, $v(0) = 0$.

Lemma 12. *If we substitute* $x \to \frac{x}{\lambda}$ *and* $y \to \frac{y}{\lambda}$ *and take the limit as* $\lambda \to \infty$ *in Definition* (43)*, we obtain*

$$\lim_{\lambda\to\infty}\left[\lambda^k\; {}_{15}E_{n,k}^{(\lambda,\gamma)}\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)\right] = \lim_{\lambda\to\infty}\left[M_{n-k}^{(\lambda-2k-1,\gamma+2k+1)}\left(\frac{x}{\lambda}\right)x^kL_k^{(\gamma)}\left(\frac{y}{x}\right)\right]$$
$$= (-1)^{n-k}(n-k)!L_{n-k}^{(\gamma+2k+1)}(x)x^kL_k^{(\gamma)}\left(\frac{y}{x}\right) = (-1)^{n-k}(n-k)!R_{n,k}^{(\gamma,\gamma)}(x,y)$$

where $R_{n,k}^{(\gamma,\gamma)}(x,y)$ *are actually the Laguerre–Laguerre–Koornwinder polynomials [26].*

3.16. The Set of Polynomials ${}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 16. *Let us define*

$$ {}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y) = M_{n-k}^{(\lambda-2k-1,\gamma)}(x)(1+x)^kL_k^{(\gamma)}\left(\frac{y}{1+x}\right), \quad k = 0,1,\ldots,n. \tag{44}$$

The set $\left\{ {}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{16}(x,y) = x^\gamma y^\gamma(1+x)^{-(\lambda+2\gamma)}e^{-y/(1+x)}$$

over the domain $\Omega_{16} = \{(x,y) : 0 < x,y < \infty\}$ for $\lambda > 2N+2$, $\gamma > 0$. The corresponding orthogonality relation is

$$\int_0^\infty\int_0^\infty x^\gamma y^\gamma e^{-\frac{y}{1+x}}(1+x)^{-(\lambda+2\gamma)}\,{}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y)\,{}_{16}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$=\frac{(n-k)!\Gamma(\lambda-n-k-1)\Gamma(k+\gamma+1)\Gamma(n-k+\gamma+1)}{k!(\lambda-2n-2)\Gamma(\lambda+\gamma-n-k-1)}\delta_{n,r}\delta_{k,s}$$

for $n,r=0,1,\ldots,N<\frac{\lambda-2}{2},\gamma>0,N=max\{n,r\}$.

Theorem 54. *The polynomials given by (44) satisfy the recurrence relation:*

$$k\,{}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y)-((2k+\gamma-1)(1+x)-y)\,{}_{16}E_{n-1,k-1}^{(\lambda-2,\gamma)}(x,y)$$
$$+(k+\gamma-1)(1+x)^2\,{}_{16}E_{n-2,k-2}^{(\lambda-4,\gamma)}(x,y)=0$$

as well as the relation (28) for $\widetilde{E}_{n,k}(x,y)={}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y)$, *with*

$$A_{n,k}=(\lambda-n-k-2)(\lambda-2n-1),\quad B_{n,k}=(\lambda-2n-3)_3,$$
$$C_{n,k}=(\lambda-2n-2)(2(n-k)_2-(\lambda-2k-1)(\gamma+2(n-k)+1)),$$
$$D_{n,k}=(n-k)(\lambda-2n-3)(\lambda+\gamma-n-k-1)(n-k+\gamma).$$

Theorem 55. *The polynomials* ${}_{16}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equations:*

$$x(1+x)^2E_{xx}+2xy(1+x)E_{xy}+xy^2E_{yy}-(1+x)((\lambda-3)x-(\gamma+1))E_x$$
$$-y((\lambda-3)x-(\gamma+1))E_y+(n(\lambda-n-2)(1+x)-k(\lambda+\gamma-k-1))E=0$$

and

$$y(1+x)E_{yy}+((\gamma+1)(1+x)-y)E_y+kE=0.$$

Theorem 56. *The set of polynomials* ${}_{16}E_{n+k,k}^{(\lambda+2k,\gamma)}(x,y)$ *has the generating function:*

$$\sum_{n,k=0}^{\infty}{}_{16}E_{n+k,k}^{(\lambda+2k,\gamma)}(x,y)\frac{t^{n+k}}{n!}=\frac{2^{1-\lambda}(1-t+A(x,t))^{\lambda+\gamma-1}\exp\left(\frac{yt}{t(1+x)-1}\right)}{A(x,t)(1-t(1+x))^{\gamma+1}(1+t+A(x,t))^{\gamma}}$$

where $A(x,t)=\sqrt{(1+t)^2+4xt}$.

Lemma 13. *If we substitute* $x\to\frac{x}{\lambda}$ *and take the limit as* $\lambda\to\infty$ *in Definition (44), we obtain*

$$\lim_{\lambda\to\infty}\left[{}_{16}E_{n,k}^{(\lambda,\gamma)}\left(\frac{x}{\lambda},y\right)\right]=\lim_{\lambda\to\infty}\left[M_{n-k}^{(\lambda-2k-1,\gamma)}\left(\frac{x}{\lambda}\right)\left(1+\frac{x}{\lambda}\right)^kL_k^{(\gamma)}\left(\frac{y}{1+\frac{x}{\lambda}}\right)\right]$$
$$=(-1)^{n-k}(n-k)!L_{n-k}^{(\gamma)}(x)L_k^{(\gamma)}(y)=(-1)^{n-k}(n-k)!L_{n,k}^{(\gamma,\gamma)}(x,y)$$

where the polynomials $L_{n,k}^{(\gamma,\gamma)}(x,y)$ *are defined in [23–25].*

3.17. The Set of Polynomials ${}_{17}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 17. *Let us define*

$${}_{17}E_{n,k}^{(\lambda,\gamma)}(x,y)=L_{n-k}^{(\gamma+2k+1)}(x)x^kM_k^{(\lambda,\gamma)}\left(\frac{y}{x}\right),\quad k=0,1,\ldots,n. \tag{45}$$

The set $\left\{{}_{17}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{17}(x,y) = y^{\gamma} x^{\lambda+\gamma} (x+y)^{-(\lambda+\gamma)} \exp(-x)$$

over the domain $\Omega_{17} = \{(x,y) : 0 < x, y < \infty\}$ for $\lambda > 2N+1, \gamma > -1$. The corresponding orthogonality relation takes the form:

$$\int_0^{\infty}\int_0^{\infty} y^{\gamma} x^{\lambda+\gamma} e^{-x} (x+y)^{-(\lambda+\gamma)} \ {}_{17}E_{n,k}^{(\lambda,\gamma)}(x,y) \ {}_{17}E_{r,s}^{(\lambda,\gamma)}(x,y) dxdy$$
$$= \frac{k!\Gamma(n+k+\gamma+2)\Gamma(\lambda-k)\Gamma(\gamma+k+1)}{(n-k)!(\lambda-2k-1)\Gamma(\lambda+\gamma-k)} \delta_{n,r}\delta_{k,s}$$

for $n, r = 0, 1, \ldots, N < \frac{\lambda-1}{2}, \gamma > -1, N = max\{n,r\}$.

Theorem 57. *The polynomials defined in (45) satisfy the three-term recurrence relation (28) for* $\widetilde{E}_{n,k}(x,y) = {}_{17}E_{n,k}^{(\lambda,\gamma)}(x,y)$, $A_{n,k} = n-k+1$, $B_{n,k} = -1$, $C_{n,k} = 2n+\gamma+2$, *and* $D_{n,k} = n+k+\gamma+1$.

Theorem 58. *The polynomials* ${}_{17}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equations:*

$$x^2 E_{xx} + 2xyE_{xy} + y^2 E_{yy} + x(\gamma+2-x)E_x + y(\gamma+2-x)E_y + (nx - k(\gamma+k+1))E = 0$$

and

$$y(x+y)E_{yy} + ((\gamma+1)x - (\lambda-2)y)E_y + k(\lambda-k-1)E = 0.$$

Theorem 59. *The polynomials* ${}_{17}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)$ *are generated by*

$$\sum_{n,k=0}^{\infty} {}_{17}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y) \frac{t^{n+k}}{k!} = \frac{\exp\left(\frac{xt}{t-1}\right)(1+\xi)^{\gamma+1}(1+\eta)^{1-\lambda-\gamma}}{(1-t)^{\gamma+2}(1+2\xi-2\eta-\xi\eta)}$$

where $\xi = \dfrac{-t(x+y)(1+\eta)^3}{1+\xi}$ *and* $\eta = \dfrac{-ty(1+\eta)^3}{1+\xi}$.

Lemma 14. *If we substitute* $y \to \frac{y}{\lambda}$ *and take the limit as* $\lambda \to \infty$ *in Definition (45), we obtain*

$$\lim_{\lambda\to\infty}\left[{}_{17}E_{n,k}^{(\lambda,\gamma)}\left(x, \frac{y}{\lambda}\right)\right] = \lim_{\lambda\to\infty}\left[L_{n-k}^{(\gamma+2k+1)}(x) x^k M_k^{(\lambda,\gamma)}\left(\frac{y}{\lambda x}\right)\right]$$
$$= (-1)^k k! L_{n-k}^{(\gamma+2k+1)}(x) x^k L_k^{(\gamma)}\left(\frac{y}{x}\right) = (-1)^k k! R_{n,k}^{(\gamma,\gamma)}(x,y),$$

where $R_{n,k}^{(\gamma,\gamma)}(x,y)$ *is defined in [26].*

3.18. The Set of Polynomials ${}_{18}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 18. *Let us define*

$$ {}_{18}E_{n,k}^{(\lambda,\gamma)}(x,y) = M_{n-k}^{(\lambda-2k-1,\gamma)}(x)(1+x)^k P_k^{(\lambda,\gamma)}\left(\frac{y}{1+x}\right), \quad k = 0, 1, \ldots, n. \tag{46}$$

The set $\left\{ {}_{18}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{18}(x,y) = x^{\gamma}(1+x)^{-2(\lambda+\gamma)}(1+x-y)^{\lambda}(1+x+y)^{\gamma}$$

over the domain $\Omega_{18} = \{(x,y) : 0 < x < \infty, -(1+x) < y < 1+x\}$ for $\lambda > 2N+2$, $\gamma > -1$. The corresponding orthogonality relation is

$$\int_{0}^{\infty}\int_{-(1+x)}^{1+x} \frac{x^{\gamma}(1+x-y)^{\lambda}(1+x+y)^{\gamma}}{(1+x)^{2(\lambda+\gamma)}} \, {}_{18}E_{n,k}^{(\lambda,\gamma)}(x,y) \, {}_{18}E_{r,s}^{(\lambda,\gamma)}(x,y) dydx$$
$$= \frac{(n-k)!\Gamma(\lambda-n-k-1)\Gamma(n-k+\gamma+1)}{2^{-(\lambda+\gamma+1)}k!(\lambda-2n-2)(2k+\lambda+\gamma+1)} \frac{\Gamma(k+\lambda+1)\Gamma(k+\gamma+1)}{\Gamma(\lambda+\gamma-n-k-1)\Gamma(k+\lambda+\gamma+1)}\delta_{n,r}\delta_{k,s}$$

for $n, r = 0, 1, \ldots, N < \frac{\lambda-2}{2}, \gamma > -1, N = max\{n, r\}$.

Theorem 60. *The polynomials defined by* (46) *satisfy the recurrence relation* (28)*, where* $\widetilde{E}_{n,k}(x,y) = {}_{18}E_{n,k}^{(\lambda,\gamma)}(x,y)$*, and the coefficients:*

$$A_{n,k} = (\lambda-n-k-2)(\lambda-2n-1), \quad B_{n,k} = (\lambda-2n-3)_3,$$
$$C_{n,k} = (\lambda-2n-2)(2(n-k)(n-k+1)-(\gamma+2(n-k)+1)(\lambda-2k-1)),$$
$$D_{n,k} = (n-k)(\lambda-2n-3)(\lambda+\gamma-n-k-1)(n-k+\gamma).$$

Theorem 61. *The polynomials* ${}_{18}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equation:*

$$x(1+x)^2E_{xx} + 2xy(1+x)E_{xy} + xy^2E_{yy}$$
$$-(1+x)((\lambda-3)x-(\gamma+1))E_x - y((\lambda-3)x-(\gamma+1))E_y$$
$$+(n(\lambda-n-2)(1+x)-k(\lambda+\gamma-k-1))E = 0.$$

Theorem 62. *The set of the polynomials* ${}_{18}E_{n+k,k}^{(\lambda+2k,\gamma)}(x,y)$ *has the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{18}E_{n+k,k}^{(\lambda+2k,\gamma)}(x,y)\frac{t^{n+k}}{n!} = \frac{2^{1-\lambda}(1+\xi)^{\lambda+1}(1+\eta)^{\gamma+1}(1-t+A(x,t))^{\lambda+\gamma-1}}{A(x,t)(1+t+A(x,t))^{\gamma}(1-2\xi-3\xi\eta)}$$

where we denoted $\xi = \frac{t(y+1+x)(1+\eta)(1+\xi)^3}{2}$*,* $\eta = \frac{t(y-1-x)(1+\eta)(1+\xi)^3}{2}$*, as well as* $A(x,t) = \sqrt{(1+t)^2+4xt}$*.*

Lemma 15. *If we substitute* $x \to \frac{x}{\lambda}$ *and* $y \to \left(\frac{2y}{\lambda}-1\right)\left(1+\frac{x}{\lambda}\right)$ *and take the limit as* $\lambda \to \infty$ *in Definition* (46)*, we obtain*

$$\lim_{\lambda\to\infty}\left[{}_{18}E_{n,k}^{(\lambda,\gamma)}\left(\frac{x}{\lambda}, \left(\frac{2y}{\lambda}-1\right)\left(1+\frac{x}{\lambda}\right)\right)\right]$$
$$= (-1)^n(n-k)!L_{n-k}^{(\gamma)}(x)L_k^{(\gamma)}(y) = (-1)^n(n-k)!L_{n,k}^{(\gamma,\gamma)}(x,y)$$

where $L_{n,k}^{(\gamma,\gamma)}(x,y)$ *is defined in* [23–25].

3.19. The Set of Polynomials ${}_{19}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 19. *Let us define*

$$_{19}E_{n,k}^{(\lambda,\gamma)}(x,y) = M_{n-k}^{(\lambda-2k-1,\gamma+2k+1)}(x)x^kP_k^{(\lambda,\gamma)}\left(\frac{y}{x}\right), \quad k = 0,1,\ldots,n. \tag{47}$$

The set $\left\{ {}_{19}E_{n,k}^{(\lambda,\gamma)}(x,y) \right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{19}(x,y) = x^{-\lambda}(1+x)^{-(\lambda+\gamma)}(x-y)^{\lambda}(x+y)^{\gamma}$$

over the domain $\Omega_{19} = \{(x,y) : 0 < x < \infty, -x < y < x\}$ for $\lambda > 2N+2, \gamma > -1$. That is,

$$\int_0^\infty \int_{-x}^{x} x^{-\lambda}(1+x)^{-(\lambda+\gamma)}(x-y)^{\lambda}(x+y)^{\gamma}\,{}_{19}E_{n,k}^{(\lambda,\gamma)}(x,y)\,{}_{19}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$= \frac{(n-k)!\Gamma(\lambda-n-k-1)\Gamma(n+k+\gamma+2)\Gamma(k+\lambda+1)\Gamma(k+\gamma+1)\delta_{n,r}\delta_{k,s}}{2^{-(\lambda+\gamma+1)}k!(\lambda-2n-2)(2k+\lambda+\gamma+1)\Gamma(\lambda+\gamma-n+k)\Gamma(k+\lambda+\gamma+1)}$$

for $n, r = 0, 1, \ldots, N < \frac{\lambda-2}{2}, \gamma > -1, N = max\{n, r\}$.

Theorem 63. *The polynomials (47) satisfy the relation (28) for $\widetilde{E}_{n,k}(x,y) = {}_{19}E_{n,k}^{(\lambda,\gamma)}(x,y)$, and coefficients:*

$$\begin{aligned} A_{n,k} &= (\lambda-n-k-2)(\lambda-2n-1), \quad B_{n,k} = (\lambda-2n-3)_3, \\ C_{n,k} &= (\lambda-2n-2)(2(n-k)_2-(\lambda-2k-1)(\gamma+2n+2)), \\ D_{n,k} &= (n-k)(\lambda-2n-3)(\lambda+\gamma-n+k)(n+k+\gamma+1). \end{aligned}$$

Theorem 64. *The polynomials ${}_{19}E_{n,k}^{(\lambda,\gamma)}(x,y)$ satisfy the partial differential equation:*

$$\begin{aligned} &x^2(1+x)E_{xx}+2xy(1+x)E_{xy}+y^2(1+x)E_{yy}-x((\lambda-3)x-(\gamma+2))E_x \\ &\qquad -y((\lambda-3)x-(\gamma+2))E_y+(n(\lambda-n-2)x-k(\gamma+k+1))E=0. \end{aligned}$$

Theorem 65. *The polynomials ${}_{19}E_{n+k,k}^{(\lambda+2k,\gamma-2k)}(x,y)$ have the generating function*

$$\sum_{n,k=0}^{\infty} {}_{19}E_{n+k,k}^{(\lambda+2k,\gamma-2k)}(x,y)\frac{t^{n+k}}{n!} = \frac{2^{1-\lambda}(1+\xi)^{\lambda+1}(1+\eta)^{\gamma+1}(1-t+A(x,t))^{\lambda+\gamma}}{A(x,t)(1+t+A(x,t))^{\gamma+1}(1-2\xi+2\eta-\xi\eta)}$$

where $A(x,t) = \sqrt{(1+t)^2+4xt}$ and $\xi = \frac{t(y+x)(1+\xi)^3}{2(1+\eta)}$, $\eta = \frac{t(y-x)(1+\xi)^3}{2(1+\eta)}$.

Lemma 16. *If we substitute $x \to \frac{x}{\lambda}$ and $y \to \frac{2y}{\lambda^2} - \frac{x}{\lambda}$ and take the limit as $\lambda \to \infty$ in Definition (47), we obtain*

$$\lim_{\lambda\to\infty}\left[\lambda^k\,{}_{19}E_{n,k}^{(\lambda,\gamma)}\left(\frac{x}{\lambda}, \frac{2y}{\lambda^2}-\frac{x}{\lambda}\right)\right] = (-1)^n(n-k)!L_{n-k}^{(\gamma+2k+1)}(x)x^kL_k^{(\gamma)}\left(\frac{y}{x}\right)$$
$$= (-1)^n(n-k)!R_{n,k}^{(\gamma,\gamma)}(x,y)$$

where $R_{n,k}^{(\gamma,\gamma)}(x,y)$ is defined in [26].

3.20. The Set of Polynomials ${}_{20}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 20. *Let us define*

$${}_{20}E_{n,k}^{(\lambda,\gamma)}(x,y) = P_{n-k}^{(\lambda+2k+1,\gamma)}(x)(1-x)^kM_k^{(\lambda,\gamma)}\left(\frac{y}{1-x}\right), \quad k = 0,1,\ldots,n. \tag{48}$$

The set $\left\{{}_{20}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{20}(x,y) = y^{\gamma}(1+x)^{\gamma}(1-x)^{2\lambda}(1-x+y)^{-(\lambda+\gamma)}$$

over the domain $\Omega_{20} = \{(x,y) : -1 < x < 1, 0 < y < \infty\}$ for $\lambda > 2N+1, \gamma > -1$. The corresponding orthogonality relation takes the form:

$$\int_{-1}^{1}\int_{0}^{\infty} \frac{y^{\gamma}(1+x)^{\gamma}(1-x)^{2\lambda}}{(1-x+y)^{\lambda+\gamma}}\, {}_{20}E_{n,k}^{(\lambda,\gamma)}(x,y)\, {}_{20}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$= \frac{2^{2k+\lambda+\gamma+2}k!\Gamma(n+k+\lambda+2)\Gamma(n-k+\gamma+1)}{(n-k)!(2n+\lambda+\gamma+2)(\lambda-2k-1)} \frac{\Gamma(\lambda-k)\Gamma(k+\gamma+1)}{\Gamma(n+k+\lambda+\gamma+2)\Gamma(\lambda+\gamma-k)}\delta_{n,r}\delta_{k,s}$$

for $n, r = 0, 1, \ldots, N < \frac{\lambda-1}{2}$, $\gamma > -1$, $N = max\{n, r\}$.

Theorem 66. *The polynomials given by* (48) *satisfy* (28)*, where*

$$\begin{aligned}
A_{n,k} &= 2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1),\\
B_{n,k} &= (2n+\lambda+\gamma+1)_3,\\
C_{n,k} &= (2n+\lambda+\gamma+2)\left((\lambda+2k+1)^2-\gamma^2\right),\\
D_{n,k} &= 2(n+k+\lambda+1)(n-k+\gamma)(2n+\lambda+\gamma+3)
\end{aligned}$$

and $\widetilde{E}_{n,k}(x,y) = {}_{20}E_{n,k}^{(\lambda,\gamma)}(x,y)$.

Theorem 67. *The polynomials* ${}_{20}E_{n,k}^{(\lambda,\gamma)}(x,y)$ *satisfy the partial differential equation:*

$$y(1-x+y)E_{yy} - ((\lambda-2)y - (\gamma+1)(1-x))E_y + k(\lambda-k-1)E = 0.$$

Theorem 68. *The set of the polynomials* ${}_{20}E_{n+k,k}^{(\lambda-2k,\gamma)}(x,y)$ *has the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{20}E_{n+k,k}^{(\lambda-2k,\gamma)}(x,y)\frac{t^{n+k}}{k!} = \frac{2^{\lambda+\gamma+1}(1+\xi)^{\gamma+1}(1-t+R)^{-(\lambda+1)}}{(1-2\eta-3\xi\eta)(1+\eta)^{\lambda+\gamma-1}R(1+t+R)^{\gamma}},$$

where $R = \sqrt{1-2xt+t^2}$, $\xi = t(x-y-1)(1+\xi)(1+\eta)^3$, *and* $\eta = -ty(1+\xi)(1+\eta)^3$.

Lemma 17. *If we substitute* $x \to \frac{2x}{\lambda} - 1$ *and* $y \to \frac{2y(\lambda-x)}{\lambda^2(1-x)}$ *and take the limit as* $\lambda \to \infty$ *in Definition* (48)*, we obtain*

$$\lim_{\lambda\to\infty}\left[2^{-k}\, {}_{20}E_{n,k}^{(\lambda,\gamma)}\left(\frac{2x}{\lambda}-1, \frac{2y(\lambda-x)}{\lambda^2(1-x)}\right)\right]$$
$$= (-1)^n k! L_{n-k}^{(\gamma)}(x) L_k^{(\gamma)}\left(\frac{y}{1-x}\right) = (-1)^n k! L_{n,k}^{(\gamma,\gamma)}\left(x, \frac{y}{1-x}\right)$$

where $L_{n,k}^{(\gamma,\gamma)}(x, \frac{y}{1-x})$ *are defined in* [23–25] *by replacing* $y \to \frac{y}{1-x}$.

3.21. The Set of Polynomials ${}_{21}E_{n,k}^{(\lambda,\gamma)}(x,y)$

Definition 21. *Let us define*

$$ {}_{21}E_{n,k}^{(\lambda,\gamma)}(x,y) = P_{n-k}^{(\lambda,\gamma+2k+1)}(x)(1+x)^k M_k^{(\lambda,\gamma)}\left(\frac{y}{1+x}\right), \quad k = 0, 1, \ldots, n. \tag{49}$$

The set $\left\{ {}_{21}E_{n,k}^{(\lambda,\gamma)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{21}(x,y) = (1-x)^{\lambda}(1+x)^{\lambda+\gamma}y^{\gamma}(1+x+y)^{-(\lambda+\gamma)}$$

over the domain $\Omega_{21} = \{(x,y) : -1 < x < 1, 0 < y < \infty\}$ for $\lambda > 2N+1$, $\gamma > -1$. Indeed,

$$\int_{-1}^{1}\int_{0}^{\infty}(1-x)^{\lambda}(1+x)^{\lambda+\gamma}y^{\gamma}(1+x+y)^{-(\lambda+\gamma)}\,{}_{21}E_{n,k}^{(\lambda,\gamma)}(x,y)\,{}_{21}E_{r,s}^{(\lambda,\gamma)}(x,y)dydx$$
$$=\frac{2^{2k+\lambda+\gamma+2}k!\Gamma(n-k+\lambda+1)\Gamma(n+k+\gamma+2)\Gamma(\lambda-k)\Gamma(k+\gamma+1)\delta_{n,r}\delta_{k,s}}{(n-k)!(2n+\lambda+\gamma+2)(\lambda-2k-1)\Gamma(n+k+\lambda+\gamma+2)\Gamma(\lambda+\gamma-k)}$$

for $n,r=0,1,\ldots,N<\frac{\lambda-1}{2},\gamma>-1,N=max\{n,r\}$.

Theorem 69. *The polynomials given by (49) satisfy the relation (28) for $\widetilde{E}_{n,k}(x,y)={}_{21}E_{n,k}^{(\lambda,\gamma)}(x,y)$, with coefficients:*

$$A_{n,k}=2(n-k+1)(n+k+\lambda+\gamma+2)(2n+\lambda+\gamma+1),$$
$$B_{n,k}=(2n+\lambda+\gamma+1)_3,\quad C_{n,k}=(2n+\lambda+\gamma+2)\left(\lambda^2-(\gamma+2k+1)^2\right),$$
$$D_{n,k}=2(n-k+\lambda)(n+k+\gamma+1)(2n+\lambda+\gamma+3).$$

Theorem 70. *The polynomials ${}_{21}E_{n,k}^{(\lambda,\gamma)}(x,y)$ satisfy the partial differential equation:*

$$y(1+x+y)E_{yy}-((\lambda-2)y-(\gamma+1)(1+x))E_y+k(\lambda-k-1)E=0.$$

Theorem 71. *The polynomials ${}_{21}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)$ are generated by*

$$\sum_{n,k=0}^{\infty}{}_{21}E_{n+k,k}^{(\lambda,\gamma-2k)}(x,y)\frac{t^{n+k}}{k!}=\frac{2^{\lambda+\gamma+1}(1+\xi)^{\gamma+1}(1+\eta)^{-(\lambda+\gamma-1)}(1-t+R)^{-\lambda}}{R(1+t+R)^{\gamma+1}(1+2\xi-2\eta-\xi\eta)}$$

where $\xi=-\dfrac{t(1+x+y)(1+\eta)^3}{1+\xi}$, $\eta=-\dfrac{ty(1+\eta)^3}{1+\xi}$, *and* $R=\sqrt{1-2xt+t^2}$.

Lemma 18. *If we substitute $x\to\frac{2x}{\lambda}-1$ and $y\to\frac{2y}{\lambda^2}$ and take the limit as $\lambda\to\infty$ in Definition (49), we obtain*

$$\lim_{\lambda\to\infty}\left[\left(\frac{\lambda}{2}\right)^k{}_{21}E_{n,k}^{(\lambda,\gamma)}\left(\frac{2x}{\lambda}-1,\frac{2y}{\lambda^2}\right)\right]=\lim_{\lambda\to\infty}\left[P_{n-k}^{(\lambda,\gamma+2k+1)}\left(\frac{2x}{\lambda}-1\right)x^kM_k^{(\lambda,\gamma)}\left(\frac{y}{\lambda x}\right)\right]$$
$$=(-1)^nk!L_{n-k}^{(\gamma+2k+1)}(x)x^kL_k^{(\gamma)}\left(\frac{y}{x}\right)=(-1)^nk!R_{n,k}^{(\gamma,\gamma)}(x,y),$$

where $R_{n,k}^{(\gamma,\gamma)}(x,y)$ is defined in [26].

3.22. The Set of Polynomials ${}_{22}E_{n,k}^{(\lambda)}(x,y)$

Definition 22. *Let us define*

$$ {}_{22}E_{n,k}^{(\lambda)}(x,y)=N_{n-k}^{(\lambda-2k-1)}(x)x^kH_k\left(\frac{y}{x}\right),\qquad k=0,1,\ldots,n. \tag{50}$$

The set $\left\{{}_{22}E_{n,k}^{(\lambda)}(x,y)\right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{22}(x,y)=\frac{x^{-\lambda}}{e^{\frac{1}{x}+\left(\frac{y}{x}\right)^2}}$$

over the domain $\Omega_{22}=\{(x,y):0<x<\infty,-\infty<y<\infty\}$ for $\lambda>2N+2$. The orthogonality relation is given by

$$\int_0^\infty \int_{-\infty}^\infty \frac{x^{-\lambda}}{e^{\frac{x+y^2}{x^2}}} \, {}_{22}E_{n,k}^{(\lambda)}(x,y) \, {}_{22}E_{r,s}^{(\lambda)}(x,y) dy dx = \frac{(n-k)!k!\sqrt{\pi}\Gamma(\lambda-n-k-1)}{2^{-k}(\lambda-2n-2)} \delta_{n,r}\delta_{k,s},$$

for $n, r = 0, 1, \ldots, N < \dfrac{\lambda-2}{2}$, $N = \max\{n, r\}$.

Theorem 72. *The polynomials* ${}_{22}E_{n,k}^{(\lambda)}(x,y)$ *defined in* (50) *satisfy the recurrence relations:*

$${}_{22}E_{n,k}^{(\lambda)}(x,y) = 2y \, {}_{22}E_{n-1,k-1}^{(\lambda-2)}(x,y) - 2(k-1)x^2 \, {}_{22}E_{n-2,k-2}^{(\lambda-4)}(x,y), \qquad n \geq 2,$$

$$\begin{aligned} &{}_{22}E_{n,k}^{(\lambda)}(x,y) - ((\lambda-2k-3)x-1) \, {}_{22}E_{n-1,k}^{(\lambda-2)}(x,y) \\ &\qquad + (n-k-1)(\lambda-n-k-3)x^2 \, {}_{22}E_{n-2,k}^{(\lambda-4)}(x,y) = 0, \qquad n \geq 2, \quad 0 \leq k \leq n-2, \end{aligned}$$

the differential relations:

$$\frac{\partial^j}{\partial y^j} \, {}_{22}E_{n,k}^{(\lambda)}(x,y) = 2^j (k-j+1)_j \, {}_{22}E_{n-j,k-j}^{(\lambda-2j)}(x,y), \qquad 0 \leq j \leq k \leq n$$

as well as Relation (28) *for* $\widetilde{E}_{n,k}(x,y) = {}_{22}E_{n,k}^{(\lambda)}(x,y)$, *with coefficients explicitly given by* $A_{n,k} = (\lambda-n-k-2)(\lambda-2n-1)$, $B_{n,k} = (\lambda-2n-3)_3$, $C_{n,k} = (2n-\lambda+2)(\lambda-2k-1)$, *and* $D_{n,k} = (n-k)(\lambda-2n-3)$.

Theorem 73. *The polynomials* ${}_{22}E_{n,k}^{(\lambda)}(x,y)$ *defined in* (50) *satisfy the partial differential equations:*

$$\begin{aligned} &x^3 E_{xx} + 2x^2 y E_{xy} + xy^2 E_{yy} - x((\lambda-3)x-1)E_x \\ &\qquad - y((\lambda-3)x-1)E_y + (n(\lambda-n-2)x-k)E = 0, \end{aligned}$$

and

$$x^2 E_{yy} - 2yE_y + 2kE = 0.$$

Theorem 74. *For the polynomials* ${}_{22}E_{n+k,k}^{(\lambda+2k)}(x,y)$, *we have the generating function:*

$$\sum_{n,k=0}^{\infty} {}_{22}E_{n+k,k}^{(\lambda+2k)}(x,y) \frac{t^{n+k}}{n!k!} = \frac{(1+v)^{\lambda-1} \exp(2yt - t^2x^2)}{(1+2v)\exp(v/x)}$$

where $v = \frac{tx}{1+v}$, $v(0) = 0$.

3.23. The Set of Polynomials ${}_{23}E_{n,k}^{(\lambda)}(x,y)$

Definition 23. *Let us define*

$${}_{23}E_{n,k}^{(\lambda)}(x,y) = I_{n-k}^{(\lambda-k-1/2)}(x)\left(1+x^2\right)^{k/2} H_k\left(\frac{y}{\sqrt{1+x^2}}\right), \qquad k = 0, 1, \ldots, n. \tag{51}$$

The set $\left\{ {}_{23}E_{n,k}^{(\lambda)}(x,y) \right\}_{k,n=0}^{n,N}$ is orthogonal with respect to the weight function:

$$w_{23}(x,y) = \frac{e^{-y^2/\left(1+x^2\right)}}{(1+x^2)^{\lambda-1/2}}$$

over the domain $\Omega_{23} = \{(x,y) : -\infty < x, y < \infty\}$ for $\lambda > N + 3/2$. The orthogonality relation corresponding to these polynomials is:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+x^2\right)^{-(\lambda-1/2)}\exp\left(-\frac{y^2}{1+x^2}\right){}_{23}E_{n,k}^{(\lambda)}(x,y)\,{}_{23}E_{r,s}^{(\lambda)}(x,y)dydx$$
$$=\frac{2^{2\lambda-k-3}(n-k)!k!\sqrt{\pi}\Gamma^2(\lambda-k-1/2)}{(\lambda-n-3/2)\Gamma(2\lambda-n-k-2)}\delta_{n,r}\delta_{k,s},$$

for $n,r=0,1,\ldots,N<\lambda-\frac{3}{2}$, $N=\max\{n,r\}$.

Theorem 75. *The polynomials defined in* (51) *satisfy the recurrence relations:*

$${}_{23}E_{n,k}^{(\lambda)}(x,y)=2y\,{}_{23}E_{n-1,k-1}^{(\lambda-1)}(x,y)-2(k-1)\left(1+x^2\right){}_{23}E_{n-2,k-2}^{(\lambda-2)}(x,y),$$

$$4(n-k-1)(\lambda-k-5/2)_2\left(1+x^2\right){}_{23}E_{n-2,k}^{(\lambda-2)}(x,y)$$
$$-4(\lambda-k-3/2)(\lambda-k-2)x\,{}_{23}E_{n-1,k}^{(\lambda-1)}(x,y)=(n+k+3-2\lambda)\,{}_{23}E_{n,k}^{(\lambda)}(x,y),$$

for $n\geq 2$, $0\leq k\leq n-2$, *the differential properties:*

$$\frac{\partial^j}{\partial y^j}\,{}_{23}E_{n,k}^{(\lambda)}(x,y)=2^j(k-j+1)_j\,{}_{23}E_{n-j,k-j}^{(\lambda-j)}(x,y),\qquad 0\leq j\leq k\leq n$$

and the relation (28) *for* $\widetilde{E}_{n,k}(x,y)={}_{23}E_{n,k}^{(\lambda)}(x,y)$, $A_{n,k}=1$, $B_{n,k}=2\lambda-2n-3$, $C_{n,k}=0$, *and* $D_{n,k}=(n-k)(2\lambda-n-k-2)$.

Theorem 76. *The polynomials* ${}_{23}E_{n,k}^{(\lambda)}(x,y)$ *defined in* (51) *satisfy the partial differential equations:*

$$\left(1+x^2\right)^2E_{xx}+2xy\left(1+x^2\right)E_{xy}+x^2y^2E_{yy}+2(2-\lambda)x\left(1+x^2\right)E_x+y\left(1-2(\lambda-2)x^2\right)E_y$$
$$-\left(n+(n-k)(n+k+2-2\lambda)+n(n+3-2\lambda)x^2\right)E=0,$$

and

$$\left(1+x^2\right)E_{yy}-2yE_y+2kE=0.$$

Theorem 77. *For the polynomials* ${}_{23}E_{n+k,k}^{(\lambda+k)}(x,y)$, *we have the following generating function:*

$$\sum_{n,k=0}^{\infty}{}_{23}E_{n+k,k}^{(\lambda+k)}(x,y)\frac{t^{n+k}}{n!k!}=\left(1+2tx-t^2\right)^{\lambda-3/2}\exp\left(2yt-t^2\left(1+x^2\right)\right).$$

Lemma 19. *If we substitute* $x\to\frac{x}{\sqrt{\lambda}}$ *and take the limit as* $\lambda\to\infty$ *in Definition* (51), *we obtain*

$$\lim_{\lambda\to\infty}\left[\lambda^{-\frac{n-k}{2}}\,{}_{23}E_{n,k}^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}},y\right)\right]$$
$$=\lim_{\lambda\to\infty}\left[\lambda^{-\frac{n-k}{2}}I_{n-k}^{(\lambda-k-1/2)}\left(\frac{x}{\sqrt{\lambda}}\right)\left(1+\frac{x^2}{\lambda}\right)^{k/2}H_k\left(\frac{y}{\sqrt{1+\frac{x^2}{\lambda}}}\right)\right]$$
$$=H_{n-k}(x)H_k(y)=H_{n,k}(x,y)$$

where $H_{n,k}(x,y)$ *are the Hermite-Hermite polynomials defined in [23–25].*

4. Conclusions

Classical univariate orthogonal polynomials with respect to a positive weight function have been deeply analyzed since the works of Laplace in 1810. They include the Jacobi, Laguerre, and Hermite polynomials. As for the latter family, they were studied by

Chebyshev in 1959 and by Hermite in 1964. Hermite introduced in 1865 for the first time a sequence of multidimensional orthogonal polynomials. Later, a number of properties and characterizations have been considered, which enlarged the univariate families to the Bessel polynomials if we consider definite weights. Very recently, new families emerged under the name of exceptional families, which provide a vast extension of the first mentioned families in many areas of mathematics, in particular the "time-and-band limiting" commutative property found and exploited by D. Slepian, H. Landau, and H. Pollak at Bell Labs in the 1960s [27]. On the other hand, finite families of orthogonal polynomials in the univariate case have been considered since the works of Romanovski connected with the analysis of probability distribution functions in statistics [28].

As for the finite bivariate case, up to now, there were only 15 classes defined by Güldoğan et al., and these were obtained from the product of two finite univariate polynomials. In this paper, 23 finite bivariate orthogonal polynomials were obtained from the product of a finite and an infinite univariate orthogonal polynomials. Therefore, the study fills a gap in the literature, providing a way to generalize to other dimensions. Once we have these new finite families, Fourier transforms can be calculated or q-analogues can be studied, which shall be considered in future works.

Author Contributions: Both authors equally contributed to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The work of I.A. was partially supported by MCIN/AEI/10.13039/501100011033 Grant PID2020-113275GB-I00 and by the European Union. The work of E.G.L. was partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK) (Grant Number 2218-122C240).

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the three reviewers for helpful suggestions and comments, which improved a preliminary version of this work.

Conflicts of Interest: The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Bochner, S. Über Sturm–Liouvillesche Polynomsysteme. *Math. Z.* **1929**, *29*, 730–736. [CrossRef]
2. Koekoek, R.; Lesky, P.A.; Swarttouw, R.F. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*; Springer: Berlin/Heidelberg, Germany, 2010.
3. Nikiforov, A.F.; Suslov, S.K.; Uvarov, V.B. *Classical Orthogonal Polynomials of a Discrete Variable*; Springer: Berlin/Heidelberg, Germany, 1991.
4. Masjed-Jamei, M. Three finite classes of hypergeometric orthogonal polynomials and their application in functions approximation. *Integral Transform. Spec. Funct.* **2002**, *13*, 169–190. [CrossRef]
5. Atakishiyev, N.M.; Rahman, M.; Suslov, S.K. On classical orthogonal polynomials. *Constr. Approx.* **1995**, *11*, 181–223. [CrossRef]
6. Andrews, G.E.; Askey, R. Classical orthogonal polynomials. In *Polynômes Orthogonaux et Applications*; Lecture Notes in Mathematics; Brezinski, C., Ed.; Springer: Berlin/Heidelberg, Germany, 1985; Volume 1171.
7. Labelle, J.; d'Askey, T. *Polynômes Orthogonaux et Applications*; Lecture Notes in Mathematics; Brezinski, C., Draux, A., Magnus, A.P., Maroni, P., Ronveaux, A., Eds.; Springer: Berlin/Heidelberg, Germany, 1985; Volume 1171.
8. Güldoğan, E.; Aktaş, R.; Masjed-Jamei, M. On Finite Classes of Two-Variable Orthogonal Polynomials. *Bull. Iran. Math. Soc.* **2020**, *46*, 1163–1194. [CrossRef]
9. Koornwinder, T.H. Two-variable analogues of the classical orthogonal polynomials. In *Theory and Application of Special Functions*; Askey, R., Ed.; Academic Press: New York, NY, USA, 1975; pp. 435–495.
10. Güldoğan Lekesiz, E.; Aktaş, R.; Masjed-Jamei, M. Fourier Transforms of Some Finite Bivariate Orthogonal Polynomials. *Symmetry* **2021**, *13*, 452. [CrossRef]
11. Wolfram Research, Inc. *Mathematica*, version 13.3; Wolfram Research, Inc.: Champaign, IL, USA, 2023.
12. Rainville, E.D. *Special Functions*; The Macmillan Company: New York, NY, USA, 1960.
13. Srivastava, H.M.; Manocha, H.L. *A Treatise on Generating Functions*; Halsted Press (Ellis Horwood Limited, Chichester): New York, NY, USA, 1984.
14. Wilf, H.S. *Generatingfunctionology*; Academic Press, Inc.: Cambridge, MA, USA, 1994.
15. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Dover Publications Inc.: New York, NY, USA, 1965.

16. Szegö, G. *Orthogonal Polynomials, American Mathematical Society Colloquium Publications*, 4th ed.; American Mathematical Society: Rhode Island, NY, USA, 1975; Volume 23.
17. Mastroianni, G.; Milovanovic, G.V. *Interpolation Processes-Basic Theory and Applications*; Springer: Berlin/Heidelberg, Germany, 2008.
18. Froehlich, J. Parameter derivatives of the Jacobi polynomials and the gaussian hypergeometric function. *Integral Transform. Spec.* **1994**, *2*, 253–266. [CrossRef]
19. Ronveaux, A.; Zarzo, A.; Area, I.; Godoy, E. Classical orthogonal polynomials: Dependence of parameters. *J. Comput. Appl. Math.* **2000**, *121*, 95–112. [CrossRef]
20. Carlitz, L. Some generating functions for Laguerre polynomials. *Duke Math. J.* **1968**, *35*, 825–827. [CrossRef]
21. Koepf, W. Identities for families of orthogonal polynomials and special functions. *Integral Transform. Spec. Funct.* **1997**, *5*,69–102. [CrossRef]
22. Güldoğan Lekesiz, E.; Aktaş, R. Some limit relationships between some two-variable finite and infinite sequences of orthogonal polynomials. *J. Diff. Equations Appl.* **2021**, *27*, 1692–1722. [CrossRef]
23. Dunkl, C.F.; Xu, Y. Orthogonal Polynomials of Several Variables. In *Encyclopedia of Mathematics and Its Applications*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2014; Volume 155.
24. Krall, H.L.; Sheffer, I.M. Orthogonal polynomials in two variables. *Ann. Mat. Pura Appl.* **1967**, *76*, 325–376. [CrossRef]
25. Suetin, P.K. *Orthogonal Polynomials in Two Variables*, 1st ed.; Gordon and Breach Science Publishers: Moscow, Russia, 1988.
26. Fernández, L.; Pérez, T.E.; Piñar, M.A. On Koornwinder classical orthogonal polynomials in two variables. *J. Comput. Appl. Math.* **2012**, *236*, 3817–3826. [CrossRef]
27. Castro, M.M.; Grünbau, F.A. A new commutativity property of exceptional orthogonal polynomials. *arXiv* **2023**, arXiv:2210.13928.
28. Romanovski, V. Sur quelques classes nouvelles de polynomes orthogonaux. *C. R. Acad. Sci. Paris* **1929**, *188*, 1023–1025.