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# $\mathcal{F}$-Contractions Endowed with Mann's Iterative Scheme in Convex $\mathcal{G}_b$-Metric Spaces

**Amna Naz [1], Samina Batul [1], Dur-e-Shehwar Sagheer [1], Irshad Ayoob [2] and Nabil Mlaiki [2,*]**

[1] Department of Mathematics, Capital University of Science and Technology, Islamabad 44000, Pakistan; dmt211002@cust.pk (A.N.); samina.batul@cust.edu.pk (S.B.); d.e.shehwar@cust.edu.pk (D.-e.-S.S.)
[2] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; iayoub@psu.edu.sa
* Correspondence: nmlaiki2012@gmail.com or nmlaiki@psu.edu.sa

**Abstract:** Recently, Ji et al. established certain fixed-point results using Mann's iterative scheme tailored to $\mathcal{G}_b$-metric spaces. Stimulated by the notion of the $\mathcal{F}$-contraction introduced by Wardoski, the contraction condition of Ji et al. was generalized in this research. Several fixed-point results with Mann's iterative scheme endowed with $\mathcal{F}$-contractions in $\mathcal{G}_b$-metric spaces were proven. One non-trivial example was elaborated to support the main theorem. Moreover, for application purposes, the existence of the solution to an integral equation is provided by using the axioms of the proven result. The obtained results are generalizations of several existing results in the literature. Furthermore, the results of Ji. et al. are the special case of theorems provided in the present research.

**Keywords:** fixed point ($\mathfrak{fp}$); metric space ($\mathfrak{M}\mathcal{S}$); $b$-metric spaces ($b$-$\mathfrak{M}\mathcal{S}$); $\mathcal{G}$-metric space ($\mathcal{G}$-$\mathfrak{M}\mathcal{S}$); $\mathcal{G}_b$-metric space ($\mathcal{G}_b$-$\mathfrak{M}\mathcal{S}$); Cauchy sequence ($\mathfrak{cs}$); convex structure ($\mathcal{CST}$)

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## 1. Introduction

Fixed-point ($\mathfrak{fp}$) theory is a dominant branch of functional analysis, which has an extraordinary role in non-linear analysis. Certain non-linear equations, such as non-linear integral and differential equations, model several problems in science and engineering. Banach [1] introduced a famous theorem known as the Banach contraction principle, in 1922, which has many applications in the Mathematical and Physical sciences. Due to these applications, Banach's theorem became a triggering point for researchers. Banach established an iterative scheme to find the fixed point of a mapping, which inspired researchers to employ this contraction principle to establish the existence of solutions of differential equations, integral equations, and certain dynamic programming equations. The important task of analyzing the existence of solutions to these equations can be resolved by converting them into an equivalent $\mathfrak{fp}$ problem. An operator equation $\mathcal{G}\zeta = 0$ can be expressed in terms of the $\mathfrak{fp}$ equation $\mathcal{Q}\zeta = \zeta$ with self-mapping $\mathcal{Q}$ and a suitable domain. For example, to find the roots of $f(\zeta) = 3\zeta^3 - 4\zeta^2 + \zeta + 3 = 0$, one can reformulate the problem into the form $\mathcal{Q}\zeta = \zeta$, where $\mathcal{Q}\zeta = -3\zeta^3 + 4\zeta^2 - 3$, and find the fixed point of the mapping $\mathcal{Q}$. It is clear that, if the mappings $\mathcal{Q}$ have $\mathfrak{fp}$, then the solution of the corresponding equation exists. Due to this equivalence of the existence of a solution of a non-cooperative equilibrium and a couple fixed point, the existence problem of the non-cooperative equilibrium of two-person games was clarified by Dechboon et al. [2], applying some coupled fixed-point theorems in partial metric spaces. Younis et al. [3] established some novel results concerning graph contractions in a more-generalized setting and, to arouse more interest, provided an application for the existence of a solution to fourth-order two-point boundary value problems describing deformations of an elastic beam, the ascending motion of a rocket, and a class of integral equations. It is important to note that every Banach contraction is continuous. In 1981, Vul'pe [4] investigated the idea of $b$-metric spaces ($b$-$\mathfrak{MS}$) and their topological features. It would be a real benefit if this article is considered for further research. Czerwik [5] defined this concept formally in 1993 by introducing a condition that was weaker than the third property of $\mathfrak{MS}$. Czerwik and many other researchers generalized the Banach contraction principle using these spaces (see [6–9]). Mustafa and Sims [10] presented the idea of $\mathcal{G}$-$\mathfrak{MS}$ in 2006, which was further generalized by Aghajani et al. [11] by presenting the concept of $\mathcal{G}_b$-$\mathfrak{MS}$.

$\mathfrak{fp}$ iterative schemes present an attractive method to compute the $\mathfrak{fp}$s of any arbitrary non-linear algebraic function accurately and efficiently. In 1890, Picard introduced the simplest iterative scheme where $\sum\limits_{p=0}^{+\infty}\{\zeta_p\}$ is a Picard sequence with initial point $\zeta_0$. In 1953, Mann [12] presented the iterative scheme defined as

$$\zeta_{p+1} = (1-\theta_p)\zeta_p + \theta_p \mathcal{Q}\zeta_p,$$

where $\sum\limits_{p=0}^{+\infty}\{\theta_p\}$ denotes a sequence based on the real numbers in $[0,1]$. This iterative scheme turns into Picard's scheme by replacing $\theta_p = 1$. Iterative methods are often used to solve the different non-linear equations that can be converted into a fixed-point equation $\mathcal{Q}\zeta = \zeta$. Mann's iterative method has been proven to be a powerful method for solving non-linear operator equations involving non-expensive mapping, asymptotically non-expensive mapping, and other kinds of non-linear mappings (see [12,13]). Ishikawa [14] generalized the Mann iterative scheme in the following manners:

$$\begin{cases} \eta_p = (1-\phi_p)\zeta_p + \phi_p \mathcal{Q}\zeta_p, \\ \zeta_{p+1} = (1-\theta_p)\zeta_p + \theta_p \mathcal{Q}\eta_p, \end{cases}$$

where $\sum\limits_{p=0}^{+\infty}\{\theta_p\}$ and $\sum\limits_{p=0}^{+\infty}\{\phi_p\}$ denote sequences based on the real numbers in $[0,1]$. This Mann iterative scheme turns into Picard's scheme by replacing $\phi_p = 0$. Let $\mathcal{S}$ be a non-empty closed convex and bounded subset of a uniformly convex Banach space and $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$ be a non-expensive mapping.

$$\|\mathcal{Q}\zeta - \mathcal{Q}\Psi\| \leq \|\zeta - \Psi\| \quad \text{for each } \zeta, \Psi \in \mathcal{S}.$$

Then, $\zeta^* \in \mathcal{S}$ is an $\mathfrak{fp}$ of $\mathcal{Q}$ (see [15]). Unlike in the case of the Banach contraction mapping principle, trivial examples show that the sequence of successive approximations $\zeta_{p+1} = \mathcal{Q}\zeta_p$, $\zeta_0 \in \mathcal{S}$, $p \geq 0$, for a non-expensive map $\mathcal{Q}$ even with a unique $\mathfrak{fp}$ may fail to converge to $\mathfrak{fp}$. It is sufficient, for example, to take, for $\mathcal{Q}$, a rotation of the unit ball in the plane around the origin of the coordinates. Krasnoselski [16] showed that, in this example, one can obtain a convergent sequence of successive approximations if, instead of $\mathcal{Q}$, one takes the auxiliary non-expensive mapping $\frac{1}{2}(I+\mathcal{Q})$, where $I$ denotes the identity transformation of the plane, i.e., if the sequence of successive approximations is defined, for arbitrary $\zeta_0 \in \mathcal{S}$, by

$$\zeta_{p+1} = \frac{1}{2}(\zeta_p + \mathcal{Q}\zeta_p), \quad p \geq 0. \tag{1}$$

It is easy to see that the mapping $\mathcal{Q}$ and $\frac{1}{2}(I+\mathcal{Q})$ have the same set of $\mathfrak{fp}$s, so that the limit of the convergent sequence defined by (1) is necessarily an $\mathfrak{fp}$ of $\mathcal{Q}$. In 2017, Karakaya et al. [17] presented the idea of a three-step iterative scheme for the first time as follows:

Consider a mapping $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a convex and closed subset of a normed space $E$; the sequence $\sum_{p=0}^{+\infty}\{\zeta_p\} \subseteq \mathcal{S}$ is defined by:

$$\begin{cases} w_p = \mathcal{Q}\zeta_p, \\ v_p = (1-\theta_p)w_p + \theta_p \mathcal{Q} w_p, \\ \zeta_{p+1} = \mathcal{Q} v_p, \qquad \text{where } \zeta_0 \in \mathcal{S} \end{cases}$$

where $\sum_{p=0}^{+\infty}\{\theta_p\}$ is a sequence based on $[0,1] \in \mathbb{R}$.

Inspired by this reality, Sharma et al. [18] presented a new three-step iteration scheme with better characteristics. This scheme is defined as follows:

For each real number $m > 0, \zeta_0 \in E$, the sequence $\sum_{p=0}^{+\infty}\{\zeta_p\}$ in $E$ is defined by $\begin{cases} z_p = \frac{m\zeta_p + \mathcal{Q}\zeta_p}{m+1}, \\ \eta_p = \mathcal{Q} z_p, \\ \zeta_{p+1} = \mathcal{Q}\eta_p. \end{cases}$

The above scheme is based on the scheme suggested by Kanwar et al. [19] as follows:

$$\zeta_{p+1} = \frac{m\zeta_p + \mathcal{Q}\zeta_p}{m+1},$$

where $m$ is any real number greater than zero. Notice that it transforms into the Picard iteration when $m = 0$.

Some more literature on such iterative schemes can be seen in [20–27]. Ji et al. [28] presented the idea of a convex $\mathcal{G}_b$-$\mathfrak{MS}$ employing the convex structure introduced by Takahashi [29]. Then, they proved the existence and uniqueness theorem by generalizing the Mann algorithm to $\mathcal{G}_b$-$\mathfrak{MS}$.

Wardowski [30] presented the concept of the $\mathcal{F}$-contraction in 2012 and proved an $\mathfrak{fp}$ theorem using this new idea. Afterward, many generalizations have been made to produce interesting results using the $\mathcal{F}$-contraction. One of them was the generalization of the $\mathcal{F}$-contraction into the Hardy–Rogers-type $\mathcal{F}$-contraction presented by Cosentino et al. [31]. After that, Asif et al. [32] introduced the $\mathcal{F}$-Reich contraction by removing the third and fourth condition of the $\mathcal{F}$-contraction of Nadler's type, defined by Cosentino.

This manuscript is organized in the following manner. In Section 2, preliminaries and some basic definitions are given for the optimum understanding of the current article. Section 3 examines the existence and uniqueness of $\mathfrak{fp}$ theorems with the help of the $\mathcal{F}$-contraction. To stimulate more interest, one example is provided to support our result. Finally, the well-posedness of an $\mathfrak{fp}$ problem is proven. In Section 4, an application is provided that ensures the existence of a solution to an integral equation by using the axioms of the provided theorem. The last section is dedicated to the conclusions of the research.

## 2. Preliminaries

**Definition 1** ([5])**.** *Let $\mathcal{S} \neq \phi$ and $d : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ be a mapping, which fulfills the subsequent properties for every $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{S}$:*

*(1):* $d(\zeta_1, \zeta_2) = 0$ *if and only if* $\zeta_1 = \zeta_2$;
*(2):* $d(\zeta_1, \zeta_2) = d(\zeta_2, \zeta_1)$;
*(3):* $d(\zeta_1, \zeta_3) \leq \mathfrak{s}[d(\zeta_1, \zeta_2) + d(\zeta_2, \zeta_3)]$ *for every* $\mathfrak{s} \geq 1$.

*Then, for every $\mathfrak{s} \geq 1$, $d$ and $(\mathcal{S}, d)$ represent the b-metric and b-$\mathfrak{MS}$, respectively.*

**Definition 2** ([11])**.** *Let $\mathcal{S} \neq \phi$ and $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ be a mapping, which fulfills the subsequent properties for all $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{S}$:*

*(1):* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_3) = 0$ *if* $\zeta_1 = \zeta_2 = \zeta_3$;
*(2):* $\mathcal{G}(\zeta_1, \zeta_1, \zeta_2) > 0$ *for every* $\zeta_1, \zeta_2 \in \mathcal{S}$ *with* $\zeta_1 \neq \zeta_2$;
*(3):* $\mathcal{G}(\zeta_1, \zeta_1, \zeta_2) \leq \mathcal{G}(\zeta_1, \zeta_2, \zeta_3)$ *for every* $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{S}$ *with* $\zeta_2 \neq \zeta_3$;

*(4):* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_3) = \mathcal{G}(\zeta_1, \zeta_3, \zeta_2) = \mathcal{G}(\zeta_3, \zeta_1, \zeta_2) = \ldots$;
*(5):* *there exists a real number* $\mathfrak{s} \geq 1$ *such that* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_3) \leq \mathfrak{s}[\mathcal{G}(\zeta_1, \eta, \eta) + \mathcal{G}(\eta, \zeta_2, \zeta_3)]$ *for every* $\zeta_1, \zeta_2, \zeta_3, \eta \in \mathcal{S}$.

*Then,* $\mathcal{G}$ *and* $(\mathcal{S}, \mathcal{G})$ *are called the* $\mathcal{G}_b$*-metric and* $\mathcal{G}_b$*-*$\mathfrak{MS}$*, respectively.*

**Remark 1** ([11])**.** *It is notable that* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *and b-*$\mathfrak{MS}$ *are equivalent topologically. By utilizing this fact, we can carry many results of b-*$\mathfrak{MS}$ *into* $\mathcal{G}_b$*-*$\mathfrak{MS}$*.*

**Proposition 1** ([11])**.** *Consider a* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}, \mathcal{G})$*. Then, for every* $\zeta_1, \zeta_2, \zeta_3, \eta \in \mathcal{S}$*, we have:*

*(1):* *If* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_3) = 0$*, then* $\zeta_1 = \zeta_2 = \zeta_3$;
*(2):* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_3) \leq \mathfrak{s}(\mathcal{G}(\zeta_1, \zeta_1, \zeta_2) + \mathcal{G}(\zeta_1, \zeta_1, \zeta_3))$;
*(3):* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_2) \leq 2\mathfrak{s}\mathcal{G}(\zeta_2, \zeta_1, \zeta_1)$;
*(4):* $\mathcal{G}(\zeta_1, \zeta_2, \zeta_3) \leq \mathfrak{s}(\mathcal{G}(\zeta_1, \eta, \zeta_3) + \mathcal{G}(\eta, \zeta_2, \zeta_3))$.

**Definition 3** ([33])**.** *Consider a* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}, \mathcal{G})$*. We say that* $\{\zeta_p\} \subseteq \mathcal{S}$ *is a* $\mathcal{G}$*-Cauchy sequence (*$\mathfrak{cs}$*) if, for all* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that, for all* $\mathfrak{l}, \mathfrak{m}, \mathfrak{n} \geq N$*,* $\mathcal{G}(\zeta_{\mathfrak{l}}, \zeta_{\mathfrak{m}}, \zeta_{\mathfrak{n}}) < \epsilon$*.*

**Definition 4** ([11])**.** *Consider a* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}, \mathcal{G})$*. If there exists* $\zeta' \in \mathcal{S}$ *such that* $\lim\limits_{p,k \to +\infty} \mathcal{G}(\zeta_p, \zeta_k, \zeta') = 0$*, then a sequence* $\{\zeta_p\} \subseteq \mathcal{S}$ *is said to be convergent in* $\mathcal{S}$*.*

**Remark 2.** $(\mathcal{S}, \mathcal{G})$ *is called a complete* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *if every* $\mathfrak{cs}$ *in* $\mathcal{S}$ *is convergent.*

**Proposition 2** ([11])**.** *Consider a* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}, \mathcal{G})$*. Then, the following are equivalent:*

*(1)* *The sequence* $\{\zeta_p\}$ *is a* $\mathfrak{cs}$*.*
*(2)* *For all* $\epsilon > 0$*, there exists* $p_0 \in \mathbb{N}$ *such that* $\mathcal{G}(\zeta_p, \zeta_k, \zeta_k) < \epsilon$ *for any* $p, k \geq p_0$*.*

**Definition 5** ([11])**.** *A* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *is called symmetric if* $\mathcal{G}(\zeta_p, \zeta_k, \zeta_k) = \mathcal{G}(\zeta_k, \zeta_p, \zeta_p)$ *for every* $\zeta_p, \zeta_k \in \mathcal{S}$*.*

**Definition 6** ([10])**.** *Consider two* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}_1, \mathcal{G}_1)$ *and* $(\mathcal{S}_2, \mathcal{G}_2)$*. Then,* $f : (\mathcal{S}_1, \mathcal{G}_1) \to (\mathcal{S}_2, \mathcal{G}_2)$ *is* $\mathcal{G}$*-continuous at a point* $\zeta' \in \mathcal{S}$ *if, for every* $\zeta_1, \zeta_2 \in \mathcal{S}$ *and* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $\mathcal{G}_1(\zeta', \zeta_1, \zeta_2) < \delta$ *implies* $\mathcal{G}_2(f\zeta', f\zeta_1, f\zeta_2) < \epsilon$*.*

**Proposition 3** ([11])**.** *Consider two* $\mathcal{G}_b$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}_1, \mathcal{G}_1)$ *and* $(\mathcal{S}_2, \mathcal{G}_2)$*. Then,* $f : (\mathcal{S}_1, \mathcal{G}_1) \to (\mathcal{S}_2, \mathcal{G}_2)$ *is* $\mathcal{G}$*-continuous at a point* $\zeta' \in \mathcal{S}$ *if and only if* $f(\zeta_p)$ *is* $\mathcal{G}$*-convergent to* $f(\zeta')$ *whenever* $\{\zeta_p\}$ *is* $\mathcal{G}$*-convergent to* $\zeta'$*.*

The convex structure ($\mathcal{CST}$) in $\mathcal{G}$-$\mathfrak{MS}$ was presented by Norouzian et al. [34].

**Definition 7** ([34])**.** *Consider a* $\mathcal{G}$*-*$\mathfrak{MS}$ *defined as* $(\mathcal{S}, \mathcal{G})$*. A mapping* $v : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \times [0,1] \times [0,1] \times [0,1] \to \mathcal{S}$ *is a* $\mathcal{CST}$ *on* $\mathcal{S}$ *if, for each* $(\zeta_1, \zeta_2, \zeta_3; \mu_1, \mu_2, \mu_3) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S} \times [0,1] \times [0,1] \times [0,1]$ *with* $\mu_1 + \mu_2 + \mu_3 = 1$*, then* $v(\zeta_1, \zeta_2, \zeta_3; \mu_1, \mu_2, \mu_3) \in \mathcal{S}$*, where* $v(\zeta_1, \zeta_2, \zeta_3; \mu_1, \mu_2, \mu_3) = \mu_1\zeta_1 + \mu_2\zeta_2 + \mu_3\zeta_3$*. If* $v$ *is a* $\mathcal{CST}$ *on* $\mathcal{S}$*, then* $(\mathcal{S}, \mathcal{G}, v)$ *is a convex* $\mathcal{G}$*-*$\mathfrak{MS}$*.*

**Definition 8** ([28])**.** *Let* $(\mathcal{S}, \mathcal{G})$ *be a* $\mathcal{G}_b$*-* $\mathfrak{MS}$ *and a mapping* $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$*. We say that* $\{\zeta_p\}$ *is a Mann sequence if*

$$\zeta_{p+1} = v(\zeta_p, \mathcal{Q}\zeta_p; \theta_p), \; p \in \mathbb{N}_0,$$

*where* $\zeta_0 \in \mathcal{S}$ *and* $\theta_p \in [0,1]$*.*

However, iterative methods are important for finding the $\mathfrak{fp}$s of non-expansive mappings. In particular, the Mann iteration is one of the numerous methods of $\mathfrak{fp}$ to find the approximations of the $\mathfrak{fp}$ problems using iterative schemes. Mann's iterative scheme is defined as

$$\zeta_{p+1} = \theta_p \zeta_p + (1-\theta_p)\mathcal{Q}\zeta_p, \quad \theta_p \in [0,1].$$

**Definition 9** ([28]). *Let $(\mathcal{S},\mathcal{G})$ be a $\mathcal{G}_b$- $\mathfrak{MS}$ with constant $\mathfrak{s} \geq 1$ and $I = [0,1]$. A mapping $v : \mathcal{S} \times \mathcal{S} \times I \to \mathcal{S}$ is called a $\mathcal{CST}$ on $\mathcal{S}$ if, for all $\zeta_1, \zeta_2, \zeta_3, \eta \in \mathcal{S}$ and $\theta \in I$,*

$$\mathcal{G}(\eta,\xi,v(\zeta_1,\zeta_2;\theta)) \leq \theta\mathcal{G}(\eta,\xi,\zeta_1) + (1-\theta)\mathcal{G}(\eta,\xi,\zeta_2). \tag{2}$$

*$(\mathcal{S},\mathcal{G},v)$ is said to be a convex $\mathcal{G}_b$-$\mathfrak{MS}$.*

Next, we give some examples of a convex $\mathcal{G}_b$-$\mathfrak{MS}$.

**Example 1.** *Let $\mathcal{S} = \mathbb{R}^{\mathfrak{n}}$, and define a b-metric $d : \mathcal{S} \times \mathcal{S} \to [0,+\infty)$ for all $\zeta, \Psi \in \mathcal{S}$ by*

$$d(\zeta,\Psi) = \sum_{i=1}^{\mathfrak{n}} (\zeta_i - \Psi_i)^2,$$

*for each $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{\mathfrak{n}}) \in \mathcal{S}$ and $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_{\mathfrak{n}}) \in \mathcal{S}$ and define the mapping $v : \mathcal{S} \times \mathcal{S} \times [0,1] \to \mathcal{S}$ by*

$$v(\Psi,\zeta;\mu) = \frac{\Psi+\zeta}{2}.$$

*Then, $(\mathcal{S},d)$ is a convex b-$\mathfrak{MS}$ with $\mathfrak{s} = 2$. Define a $\mathcal{G}_b$-metric $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to [0,+\infty)$ by*

$$\mathcal{G}(\Psi,\zeta,\eta) = \max\{d(\Psi,\zeta), d(\Psi,\eta), d(\eta,\zeta)\} \quad \text{for all } \Psi,\zeta,\eta \in \mathcal{S}.$$

*For each $\zeta, \Psi, \alpha, \beta \in \mathcal{S}$, we have*

$$\begin{aligned}
\mathcal{G}(\zeta,\Psi,v(\alpha,\beta;\theta)) =& \max\{d(\zeta,\Psi), d(\zeta,v(\alpha,\beta;\theta)), d(\Psi,v(\alpha,\beta;\theta))\} \\
\leq& \max\{d(\zeta,\Psi), \theta d(\zeta,\alpha) + (1-\theta)d(\zeta,\beta), \theta d(\Psi,\alpha) + (1-\theta)d(\Psi,\beta)\} \\
\leq& \theta\max\{d(\zeta,\Psi), d(\zeta,\alpha), d(\Psi,\alpha)\} + (1-\theta)\max\{d(\zeta,\Psi), d(\zeta,\beta), d(\Psi,\beta)\} \\
=& \theta\mathcal{G}(\zeta,\Psi,\alpha) + (1-\theta)\mathcal{G}(\zeta,\Psi,\beta).
\end{aligned}$$

*Hence, $(\mathcal{G},\mathcal{S},v)$ is a convex $\mathcal{G}_b$-$\mathfrak{MS}$ with $\mathfrak{s} = 2^{p-1}$.*

**Example 2.** *Let $\mathcal{S} = \mathbb{R}$, and define a $\mathcal{G}_b$-metric $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to [0,+\infty)$ by*

$$\mathcal{G}(\zeta,\Psi,\eta) = \left[\frac{1}{3}(|\zeta-\Psi| + |\psi-\eta| + |\zeta-\eta|)\right]^2 \text{ for all } \zeta,\Psi,\eta \in \mathcal{S}$$

*and also the mapping $v : \mathcal{S} \times \mathcal{S} \times [0,1] \to \mathcal{S}$ by*

$$v(\zeta,\Psi;\theta) = \theta\zeta + (1-\theta)\Psi.$$

*For each $\zeta, \Psi, \alpha, \beta \in \mathcal{S}$, we have*

$$\begin{aligned}
\mathcal{G}(\zeta, \Psi, v(\alpha, \beta; \theta)) =& \frac{1}{9} \times \Big( |\zeta - \Psi| + |\psi - \theta\alpha - (1-\theta)\beta| + |\zeta - \theta\alpha - (1-\theta)\beta| \Big)^2 \\
\leq& \frac{1}{9} \times \Big[ \theta|\zeta - \Psi| + (1-\theta)|\zeta - \Psi| + \theta|\Psi - \alpha| + (1-\theta)|\Psi - \beta| + \theta|\zeta - \alpha| \\
&+ (1-\theta)|\zeta - \beta| \Big]^2 \\
=& \frac{1}{9} \times \Big[ \theta(|\zeta - \Psi| + |\Psi - \alpha| + |\zeta - \alpha|) + (1-\theta)(|\zeta - \Psi| + |\Psi - \beta| + |\zeta - \beta|) \Big]^2 \\
\leq& \frac{1}{9} \times \Big[ \theta^2(|\zeta - \Psi| + |\Psi - \alpha| + |\zeta - \alpha|)^2 + (1-\theta)^2(|\zeta - \Psi| + |\Psi - \beta| + |\zeta - \beta|)^2 \\
&+ 2\theta(1-\theta)(|\zeta - \Psi| + |\Psi - \alpha| + |\zeta - \alpha|)^2 \Big] \\
\leq& \frac{1}{9} \times \Big[ \theta(|\zeta - \Psi| + |\Psi - \alpha| + |\zeta - \alpha|)^2 + (1-\theta)(|\zeta - \Psi| + |\Psi - \beta| + |\zeta - \beta|)^2 \Big] \\
=& \theta\mathcal{G}(\zeta, \Psi, \alpha) + (1-\theta)\mathcal{G}(\zeta, \Psi, \beta).
\end{aligned}$$

*Hence, $(\mathcal{G}, \mathcal{S}, v)$ is a convex $\mathcal{G}_b$-$\mathfrak{MS}$ with $\mathfrak{s} = 2$.*

**Remark 3.** *A convex $\mathcal{G}_b$- $\mathfrak{MS}$ reduces a convex $\mathcal{G}$-$\mathfrak{MS}$ for $\mathfrak{s} = 1$.*

Wardowski [30] presented the idea of $\mathcal{F}$-contractions in 2012, which has a crucial role in the recent trend of research in the field of $\mathfrak{fp}$ theory.

**Definition 10** ([30])**.** *Consider a mapping $\mathcal{F} : (0, +\infty) \to \mathbb{R}$, which satisfies the subsequent conditions:*

*$(F_1)$: $\mathcal{F}$ is increasing strictly.*
*$(F_2)$: For every sequence $\{\alpha_p\}_{p \in \mathbb{N}}$ of positive numbers $\lim_{p \to +\infty} \alpha_p = 0$ iff $\lim_{p \to +\infty} \mathcal{F}(\alpha_p) = -\infty$.*
*$(F_3)$: There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$.*

**Definition 11** ([30])**.** *Consider an $\mathfrak{MS}$ defined as $(\mathcal{S}, d)$. A mapping $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$ is said to be an $\mathcal{F}$-contraction if there exists $\tau > 0$ such that $d(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_2) > 0$ implies*

$$\tau + \mathcal{F}(d(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_2)) \leq \mathcal{F}(d(\zeta_1, \zeta_2)) \quad \textit{for every } \zeta_1, \zeta_2 \in \mathcal{S}.$$

Popescu and Stan [35] proved fixed-point results by applying weaker symmetrical conditions on the self-map of a complete metric space, Wadowski's control function $\mathcal{F}$, and the contractions defined by Wardowski. Vujakovic et al. [36] proved Wardowski-type results within $\mathcal{G}$-$\mathfrak{MS}$ using only the condition $F_1$. Fabiano et al. [37] presented a beautiful survey on $\mathcal{F}$ mappings and suggested some improvements on the conditions of an $\mathcal{F}$ mapping involved in the contractive condition.
We now state a property [36,37] of the function $\mathcal{F}$, which is the consequence of the condition $F_1$. This paper is the third chapter of the book (see [38]):

- At each point $d \in (0, \infty)$, there exist its left and right limits $\lim_{\zeta \to d^-} \mathcal{F}(\zeta) = \mathcal{F}(d^-)$ and $\lim_{\zeta \to d^+} \mathcal{F}(\zeta) = \mathcal{F}(d^+)$. Moreover, for the function $\mathcal{F}$, one of the following two properties hold: $\mathcal{F}(0^+) = m \in \mathbb{R}$ or $\mathcal{F}(0^+) = -\infty$.

In 2021, Huang et al. [39] presented the concept of a convex $\mathcal{F}$-contraction in $b$-$\mathfrak{MS}$ to obtain $\mathfrak{fp}$ results in $b$-$\mathfrak{MS}$.

**Definition 12** ([39])**.** *Consider a self-mapping $\mathcal{Q}$ on $\mathcal{S}$ and a complete $b$-$\mathfrak{MS}$ defined as $(\mathcal{S}, d, \mathcal{F})$. We say that $\mathcal{Q}$ is a convex $\mathcal{F}$-contraction if there exists a function $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ such that $\mathcal{Q}$ satisfies $(F_1)$, $(F_2)$, $(F_3)$ and also the following:*

$(F_4^{\mu})$ *There exists* $\tau > 0$ *and* $\mu \in [0,1)$ *such that*

$$\tau + \mathcal{F}(d_p) \leq \mathcal{F}(\mu d_p + (1-\mu)d_{p-1}) \quad \text{for all } d_p > 0,\ p \in \mathbb{N}.$$

Throughout, the next discussion, the collection of functions that satisfy condition $F_1$ will be denoted by $\mathbb{F}$.

**Proposition 4** ([28])**.** *Consider a convex $\mathcal{G}_b$-$\mathfrak{MS}$ defined as $(\mathcal{S}, \mathcal{G}, v)$. Then, the $\mathcal{G}_b$-metric is $\mathcal{G}$-symmetric if $\theta \in (0,1)$, i.e. $\mathcal{G}(\zeta_1, \zeta_1, \zeta_2) = \mathcal{G}(\zeta_1, \zeta_2, \zeta_2)$.*

## 3. Main Results

Throughout this article, by convex $\mathcal{F}$-contraction, we mean a mapping that satisfies both $F_4^{\mu}$ and $F_1$. First, we generalize the results of Ji et al. [28] regarding the $\mathcal{F}$-contraction.

**Theorem 1.** *Let $(\mathcal{S}, \mathcal{G}, v)$ be a complete convex $\mathcal{G}_b$- $\mathfrak{MS}$ with constant $\mathfrak{s} \geq 1$ and $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$ be a convex $\mathcal{F}$-contraction. Furthermore, assume that the sequence $\{\zeta_p\}$ is generated by the Mann iterative scheme and $\mathfrak{s}_0 \in \mathcal{S}$. If the sequence $\{\theta_p\} \in (0,1)$ converges to $\theta$, then $\mathcal{Q}$ has a unique $\mathfrak{fp}$ $\zeta^* \in \mathcal{S}$. Moreover, $\mathcal{Q}$ is $\mathcal{G}$-continuous at $\zeta^*$.*

**Proof.** For every $p \in \mathbb{N}_0$, we obtain

$$\begin{aligned} \mathcal{G}(\zeta_p, \zeta_p, \zeta_{p+1}) =& \mathcal{G}(\zeta_p, \zeta_p, v(\zeta_p, \mathcal{Q}\zeta_p; \theta_p)) \\ \leq& (1-\theta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p). \end{aligned} \tag{3}$$

From the condition $F_4^{\mu}$, we have

$$\begin{aligned} \mathcal{F}(\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)) \leq& \tau + \mathcal{F}(\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)) \\ \leq& \mathcal{F}\Big(\mu\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + (1-\mu)\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})\Big). \end{aligned}$$

Then, using $F_1$, we have

$$\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) \leq \mu\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + (1-\mu)\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}),$$

then,

$$0 < \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) < \mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) \tag{4}$$

for each $p \in \mathbb{N}$. Next, we show that

$$\tau + \mathcal{F}(d_p) = \tau + \mathcal{F}(\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)) \leq \mathcal{F}(\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})) = \mathcal{F}(d_{p-1}) \quad \text{for all } p \in \mathbb{N}. \tag{5}$$

Indeed, if (5) is not true, then

$$\tau + \mathcal{F}(\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)) > \mathcal{F}(\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})) \quad \text{for all } p \in \mathbb{N}.$$

Thus, it establishes that

$$\begin{aligned} \mathcal{F}(\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})) <& \tau + \mathcal{F}(\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)) \quad \text{for all } p \in \mathbb{N} \\ \leq& \mathcal{F}(\mu\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + (1-\mu)\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})). \end{aligned}$$

Using condition $F_1$, we obtain

$$\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) < \mu\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + (1-\mu)\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1});$$

that is,

$$\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) < \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p), \tag{6}$$

$d_{p-1} < d_p$, which is a contradiction to (4). Hence, (5) holds.

$$\mathcal{F}(d_p) < \mathcal{F}(d_{p-1}) - \tau \quad \text{for all } p \in \mathbb{N}. \tag{7}$$

Since $\mathcal{F}$ is strictly increasing, then $d_p < d_{p-1}$. Thus, we conclude that the sequence $\{d_p\}$ is strictly decreasing, so there exists $\lim\limits_{p\to+\infty} d_p = d$. Suppose that $d > 0$. Since $\mathcal{F}$ is an increasing mapping, there exists $\lim\limits_{\zeta\to d^+} \mathcal{F}(\zeta) = \mathcal{F}(d^+)$, so taking the limit as $p \to +\infty$ in Inequality (7), we obtain

$$\tau + \mathcal{F}(d^+) \leq \mathcal{F}(d^+),$$

a contradiction. Therefore, $\lim\limits_{p\to+\infty} d_p = 0$,

$$\lim_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) = 0. \tag{8}$$

By using Equation (3), we have

$$\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p+1}) \leq (1-\theta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) \leq \eta\, \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p),$$

where $\eta < 1$. Therefore, $\lim\limits_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta_p, \zeta_{p+1}) = 0$. Thus, for each $p, q \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p+q}) =& \mathcal{G}(\zeta_p, \zeta_{p+q}, \zeta_{p+q}) \\
\leq& \mathfrak{s}\mathcal{G}(\zeta_p, \zeta_{p+1}, \zeta_{p+1}) + \mathfrak{s}\mathcal{G}(\zeta_{p+1}, \zeta_{p+q}, \zeta_{p+q}) \\
\leq& \mathfrak{s}\mathcal{G}(\zeta_p, \zeta_{p+1}, \zeta_{p+1}) + \mathfrak{s}^2\mathcal{G}(\zeta_{p+1}, \zeta_{p+2}, \zeta_{p+2}) + \mathfrak{s}^2\mathcal{G}(\zeta_{p+2}, \zeta_{p+q}, \zeta_{p+q}) \\
\leq& \mathfrak{s}\mathcal{G}(\zeta_p, \zeta_{p+1}, \zeta_{p+1}) + \mathfrak{s}^2\mathcal{G}(\zeta_{p+1}, \zeta_{p+2}, \zeta_{p+2}) + \cdots + \mathfrak{s}^q\mathcal{G}(\zeta_{p+q-1}, \zeta_{p+q}, \zeta_{p+q}) \\
\leq& \mathfrak{s}\eta\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + \mathfrak{s}^2\eta\mathcal{G}(\zeta_{p+1}, \zeta_{p+1}, \mathcal{Q}\zeta_{p+1}) + \cdots + \mathfrak{s}^q\eta\mathcal{G}(\zeta_{p+q-1}, \zeta_{p+q-1}, \mathcal{Q}\zeta_{p+q-1}) \\
\leq& (\eta\mathfrak{s} + \eta\mathfrak{s}^2 + \eta\mathfrak{s}^3 + \cdots)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) \\
\leq& \eta\mathfrak{s}(1 + \mathfrak{s} + \mathfrak{s}^2 + \cdots)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) \\
<& \frac{s}{1-\mathfrak{s}}\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p).
\end{aligned}$$

implying that $\lim\limits_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta_p, \zeta_{p+q}) = 0$, which reveals that $\{\zeta_p\}$ is a $\mathfrak{cs}$ in $\mathcal{S}$. Since $(\mathcal{S}, \mathcal{G}, v)$ is a complete convex $\mathcal{G}_b$-$\mathfrak{M}\mathcal{S}$, there exists $\zeta' \in \mathcal{S}$ such that

$$\lim_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta_p, \zeta') = 0. \tag{9}$$

Notice that

$$\begin{aligned}
\mathcal{G}(\zeta', \mathcal{Q}\zeta', \mathcal{Q}\zeta') \leq& \mathfrak{s}\Big(\mathcal{G}(\zeta', \zeta_p, \zeta_p) + \mathcal{G}(\zeta_p, \mathcal{Q}\zeta', \mathcal{Q}\zeta')\Big) \\
\leq& \mathfrak{s}\mathcal{G}(\zeta', \zeta_p, \zeta_p) + \mathfrak{s}^2\Big(\mathcal{G}(\zeta_p, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) + \mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\zeta', \mathcal{Q}\zeta')\Big) \\
\leq& \mathfrak{s}\mathcal{G}(\zeta', \zeta_p, \zeta_p) + \mathfrak{s}^2\mathcal{G}(\zeta_p, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) + \mathfrak{s}^3\Big(\mathcal{G}(\mathcal{Q}\zeta_p, \zeta', \zeta') + \mathcal{G}(\zeta', \mathcal{Q}\zeta', \mathcal{Q}\zeta')\Big) \\
\implies (1-\mathfrak{s}^3)\mathcal{G}(\zeta', \mathcal{Q}\zeta', \mathcal{Q}\zeta') \leq& \mathfrak{s}\mathcal{G}(\zeta', \zeta_p, \zeta_p) + \mathfrak{s}^2\mathcal{G}(\zeta_p, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) + \mathfrak{s}^3\mathcal{G}(\mathcal{Q}\zeta_p, \zeta', \zeta') \\
\leq& \mathfrak{s}\mathcal{G}(\zeta_p, \zeta_p, \zeta') + \mathfrak{s}^2\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + \mathfrak{s}^4\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) + \mathfrak{s}^4\mathcal{G}(\zeta_p, \zeta_p, \zeta').
\end{aligned}$$

Letting $\lim_{p\to+\infty}$ in the above inequality and by using (8) and (9), we deduce $\lim_{p\to+\infty}\mathcal{G}(\zeta',\mathcal{Q}\zeta',\mathcal{Q}\zeta')=0$, implying that $\mathcal{Q}\zeta'=\zeta'$. Thus, $\mathfrak{fp}$ of $\mathcal{Q}$ is $\zeta'$.
Assume that $\zeta',\eta'\in\mathcal{S}$ are two different $\mathfrak{fp}$s of $\mathcal{Q}$. Then,

$$\begin{aligned}0<\mathcal{G}(\zeta',\zeta',\eta')=&\mathcal{G}(\mathcal{Q}\zeta',\mathcal{Q}\eta',\mathcal{Q}\eta')\\ \leq&\mathfrak{s}\mathcal{G}(\mathcal{Q}\zeta',\zeta',\zeta')+\mathfrak{s}\mathcal{G}(\zeta',\mathcal{Q}\eta',\mathcal{Q}\eta')\\ =&\mathfrak{s}\mathcal{G}(\zeta',\zeta',\eta'),\end{aligned}$$

which is impossible. Therefore, $\mathcal{G}(\zeta',\zeta',\eta')$=0. To observe that $\mathcal{Q}$ is $\mathcal{G}$-continuous at an $\mathfrak{fp}$ $\zeta'$, consider a sequence $\{\eta_p\}$ such that $\lim_{p\to+\infty}\eta_p=\zeta'$. Then,

$$\mathcal{G}(\zeta',\mathcal{Q}\eta_p,\mathcal{Q}\eta_p)=\mathcal{G}(\mathcal{Q}\zeta',\mathcal{Q}\eta_p,\mathcal{Q}\eta_p)\leq\mathfrak{s}\mathcal{G}(\mathcal{Q}\zeta',\zeta',\zeta')+\mathfrak{s}\mathcal{G}(\zeta',\mathcal{Q}\eta_p,\mathcal{Q}\eta_p).$$

Taking limit as $p\to+\infty$, we have $\lim_{p\to+\infty}\mathcal{G}(\zeta',\mathcal{Q}\eta_p,\mathcal{Q}\eta_p)=0$, which implies that $\lim_{p\to+\infty}\mathcal{Q}\eta_p=\zeta'=\mathcal{Q}\zeta'$. By combining this with Proposition 4, it is derived that $\mathcal{Q}$ is $\mathcal{G}$-continuous at $\zeta'$. □

**Theorem 2.** *Let $(\mathcal{S},\mathcal{G},v)$ be a complete convex $\mathcal{G}_b$-$\mathfrak{MS}$ with $\mathfrak{s}\geq 1$. Let $\mathcal{Q}:\mathcal{S}\to\mathcal{S}$ be a mapping such that, for each $\zeta_1,\zeta_2,\zeta_3\in\mathcal{S}$ and $\mathcal{F}\in\mathbb{F}$.*

$$\begin{aligned}\tau+\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1,\mathcal{Q}\zeta_2,\mathcal{Q}\zeta_3))\leq&\mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta_1,\zeta_1,\zeta_2)\mathcal{G}(\zeta_2,\zeta_2,\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_2)}+\mu_2\frac{\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_2)\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_2)}\\ &+\mu_3\frac{\mathcal{G}(\zeta_2,\zeta_2,\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\zeta_2)}{\mathcal{M}(\zeta_2,\zeta_3)}+\mu_4\frac{\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_2)}{\mathcal{M}(\zeta_2,\zeta_3)}\\ &+\mu_5\frac{\mathcal{G}(\zeta_1,\zeta_1,\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_3)}+\mu_6\frac{\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_3)}\Big),\end{aligned}\tag{10}$$

*where*

$$\begin{aligned}\mathcal{M}(\zeta_1,\zeta_2)=&\max\{\zeta,\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_1),\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_2)\},\\ \mathcal{M}(\zeta_1,\zeta_3)=&\max\{\zeta,\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_1),\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_3)\},\\ \mathcal{M}(\zeta_2,\zeta_3)=&\max\{\zeta,\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_2),\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_3)\}\end{aligned}$$

*and $\mu_1+\mu_3+\mu_5\leq\frac{1}{5\mathfrak{s}^2}$ and $\mu_2+\mu_4+\mu_6\leq\frac{1}{5\mathfrak{s}^2}$. Assume that the sequence $\{\zeta_p\}$ is generated by the Mann iteration and $\mathfrak{s}_0\in\mathcal{S}$. If $\{\theta_p\}\in[0,\frac{1}{2\mathfrak{s}^2}]$, then an $\mathfrak{fp}$ of $\mathcal{Q}$ exists, that is $\mathcal{F}(\mathcal{Q})\neq\phi$.*

**Proof.** For any $p\in\mathbb{N}_0$, we have

$$\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p+1})=\mathcal{G}(\zeta_p,\zeta_p,v(\zeta_p,\mathcal{Q}\zeta_p;\theta_p))\leq(1-\theta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p).\tag{11}$$

If $\zeta_p=\zeta_{p+1}$, then

$$\mathcal{G}(\zeta_p,\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)=\mathcal{G}(\zeta_{p+1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)\leq\theta_p\mathcal{G}(\zeta_p,\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p),$$

which implies that $\zeta_p=\mathcal{Q}\zeta_p$ and $\zeta_p$ is an $\mathfrak{fp}$ of $\mathcal{Q}$. Therefore, assume that $\zeta_p\neq\zeta_{p+1}$ and $\zeta_p\neq\mathcal{Q}\zeta_p$. In view of Definition 9 and Proposition 4, it follows that

$$\begin{aligned}\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)=&\mathcal{G}(\zeta_p,\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)\\ \leq&\mathfrak{s}[\mathcal{G}(\zeta_p,\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)]\\ \leq&\mathfrak{s}[\theta_{p-1}\mathcal{G}(\zeta_{p-1},\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)].\end{aligned}\tag{12}$$

Using symmetry, we have the following six possible cases for $\{\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)\}$:

- Case 1: For any $p \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p)) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_2 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + \mu_3 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_4 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + \mu_5 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} + \mu_6 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)}\Big) \\
\Longrightarrow\ &\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p)) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_2 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + \mu_3 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_4 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + \mu_5 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} + \mu_6 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)}\Big) - \tau \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_2 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + \mu_3 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_4 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + \mu_5 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} + \mu_6 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)}\Big) \\
&\leq \mathcal{F}\Big((\mu_1 + \mu_3) \frac{(1-\theta_{p-1})^2 \mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})^2}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&\quad + (\mu_2 + \mu_4) \frac{\theta_{p-1}\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_6 \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\Big) \\
&\leq \mathcal{F}\Big([(1-\theta_{p-1})^2(\mu_1 + \mu_3) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2 + \mu_4)]\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) \\
&\quad + [\theta_{p-1}\mathfrak{s}(\mu_2 + \mu_4) + \mu_6]\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\Big).
\end{aligned}$$

Using $\mathcal{F}_1$, we obtain

$$\begin{aligned}
\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) \leq &[(1-\theta_{p-1})^2(\mu_1 + \mu_3) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2 + \mu_4)]\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) \\
&+ [\theta_{p-1}\mathfrak{s}(\mu_2 + \mu_4) + \mu_6]\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p).
\end{aligned}$$

- Case 2: For any $p \in \mathbb{N}_0$, we have

$$
\begin{aligned}
&\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) \\
&= \tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p)) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_{p-1})} + \mu_2 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_{p-1})} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} + \mu_4 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_6 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)}\Big) \\
\Longrightarrow\ &\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_{p-1})} + \mu_2 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_{p-1})} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} + \mu_4 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_6 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)}\Big) - \tau \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_{p-1})} + \mu_2 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_{p-1})} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} + \mu_4 \frac{\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p, \zeta_p)} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} + \mu_6 \frac{\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1}, \zeta_p)}\Big) \\
&\leq \mathcal{F}\Big((\mu_1 + \mu_5) \frac{(1-\theta_{p-1})^2 \mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})^2}{\mathcal{M}(\zeta_{p-1}, \zeta_p)} \\
&+ (\mu_2 + \mu_6) \frac{\theta_{p-1}\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_p)\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_p, \zeta_{p-1})} + \mu_4 \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathcal{F}\Big([(1-\theta_{p-1})^2(\mu_1 + \mu_5) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2 + \mu_4)]\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) \\
&+ [\theta_{p-1}\mathfrak{s}(\mu_2 + \mu_6) + \mu_4]\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p)\Big).
\end{aligned}
$$

Applying $\mathcal{F}_1$, we obtain

$$
\begin{aligned}
\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) \leq &[(1-\theta_{p-1})^2(\mu_1 + \mu_5) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2 + \mu_4)]\mathcal{G}(\zeta_{p-1}, \zeta_{p-1}, \mathcal{Q}\zeta_{p-1}) \\
&+ [\theta_{p-1}\mathfrak{s}(\mu_2 + \mu_6) + \mu_4]\mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p).
\end{aligned}
$$

- Case 3: For any $p \in \mathbb{N}_0$, we have

$$
\begin{aligned}
&\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p)) \\
&= \tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_{p-1})) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)} + \mu_2 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_4 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_6 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}\Big) \\
\Longrightarrow\ &\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p)) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)} + \mu_2 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_4 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_6 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}\Big) - \tau \\
&\leq \mathcal{F}\Big([(1-\theta_{p-1})^2(\mu_3+\mu_5) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_4+\mu_6)]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}) \\
&+ [\theta_{p-1}\mathfrak{s}(\mu_4+\mu_6)+\mu_2]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\Big).
\end{aligned}
$$

Applying $\mathcal{F}_1$, we obtain

$$
\begin{aligned}
\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) \leq& [(1-\theta_{p-1})^2(\mu_3+\mu_5) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_4+\mu_6)]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}) \\
&+ [\theta_{p-1}\mathfrak{s}(\mu_4+\mu_6)+\mu_2]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p).
\end{aligned}
$$

Since

$$\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) = \mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_{p-1}),$$

proceeding in the same way, we obtain the following.

- Case 4: For any $p \in \mathbb{N}_0$, we have

$$
\begin{aligned}
&\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_{p-1})) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_2 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_p,\zeta_{p-1})} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_4 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})} + \mu_6 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}\Big) \\
\Longrightarrow\ &\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\zeta_{p-1}, \mathcal{Q}\zeta_{p-1})) \\
&\leq \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_2 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_p,\zeta_{p-1})} \\
&+ \mu_3 \frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} + \mu_4 \frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})} \\
&+ \mu_5 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})} + \mu_6 \frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}\Big) - \tau \\
&\leq \mathcal{F}\Big([(1-\theta_{p-1})^2(\mu_1+\mu_3) + \theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2+\mu_4)+\mu_6]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}) \\
&+ [\theta_{p-1}\mathfrak{s}(\mu_4+\mu_6)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\Big).
\end{aligned}
$$

Using $\mathcal{F}_1$, we obtain

$$\begin{aligned}\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p) \leq&[(1-\theta_{p-1})^2(\mu_1+\mu_3)+\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2+\mu_4)+\mu_6]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\&+[\theta_{p-1}\mathfrak{s}(\mu_4+\mu_6)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p).\end{aligned}$$

- Case 5: For any $p\in\mathbb{N}_0$, we have

$$\begin{aligned}&\tau+\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_{p-1}))\\&\leq\mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}+\mu_2\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}\\&+\mu_3\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}+\mu_4\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}\\&+\mu_5\frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}+\mu_6\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}\Big)\\&\Longrightarrow\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_{p-1}))\\&\leq\mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}+\mu_2\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}\\&+\mu_3\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}+\mu_4\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}\\&+\mu_5\frac{\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}+\mu_6\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_{p-1})}\Big)-\tau\end{aligned}$$

$$\begin{aligned}\leq&\mathcal{F}\Big([(1-\theta_{p-1})^2(\mu_1+\mu_5)+\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2+\mu_6)+\mu_4]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\&+[\theta_{p-1}\mathfrak{s}(\mu_2+\mu_6)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\Big).\end{aligned}$$

Using $\mathcal{F}_1$, we obtain

$$\begin{aligned}\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p) \leq&[(1-\theta_{p-1})^2(\mu_1+\mu_5)+\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_2+\mu_6)+\mu_4]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\&+[\theta_{p-1}\mathfrak{s}(\mu_2+\mu_6)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p).\end{aligned}$$

- Case 6: For any $p\in\mathbb{N}_0$, we have

$$\begin{aligned}&\tau+\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_{p-1}))\\&\leq\mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}+\mu_2\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}\\&+\mu_3\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}+\mu_4\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}\\&+\mu_5\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}+\mu_6\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}\Big)\\&\Longrightarrow\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_{p-1}))\\&\leq\mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}+\mu_2\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_{p-1})}\\&+\mu_3\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}+\mu_4\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}\\&+\mu_5\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}+\mu_6\frac{\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_{p-1})}{\mathcal{M}(\zeta_{p-1},\zeta_p)}\Big)-\tau\\&\leq\mathcal{F}\Big([(1-\theta_{p-1})^2(\mu_3+\mu_5)+\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_4+\mu_6)+\mu_2]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\&+[\theta_{p-1}\mathfrak{s}(\mu_4+\mu_6)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\Big).\end{aligned}$$

Using $\mathcal{F}_1$, we obtain

$$\begin{aligned}\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)\leq&[(1-\theta_{p-1})^2(\mu_3+\mu_5)+\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}(\mu_4+\mu_6)+\mu_2]\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\&+[\theta_{p-1}\mathfrak{s}(\mu_4+\mu_6)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p).\end{aligned}$$

Combining all the above cases, we obtain

$$\begin{aligned}\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)\leq&\frac{1}{6}\Big\{[4(1-\theta_{p-1})^2(\mu_1+\mu_3+\mu_5)+(4\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}+1)(\mu_2+\mu_4+\mu_6)]\\&\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+(4\theta_{p-1}\mathfrak{s}+1)(\mu_2+\mu_4+\mu_6)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\Big\}.\end{aligned}\tag{13}$$

Using (12) and (13), we obtain

$$\begin{aligned}&\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\\&\leq\mathfrak{s}[\theta_{p-1}\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\frac{1}{6}\Big\{[4(1-\theta_{p-1})^2(\mu_1+\mu_3+\mu_5)+\\&(4\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}+1)(\mu_2+\mu_4+\mu_6)]\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\&+(4\theta_{p-1}\mathfrak{s}+1)(\mu_2+\mu_4+\mu_6)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\Big\}\\&\leq\mathfrak{s}[\theta_{p-1}+\frac{4(1-\theta_{p-1})^2(\mu_1+\mu_3+\mu_5)+(4\theta_{p-1}(1-\theta_{p-1})\mathfrak{s}+1)(\mu_2+\mu_4+\mu_6)}{6}]\\&\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\frac{(4\theta_{p-1}\mathfrak{s}+1)(\mu_2+\mu_4+\mu_6)}{6}\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\\&\leq\mathfrak{s}[\frac{1}{2\mathfrak{s}^2}+\frac{4(1-\frac{1}{2\mathfrak{s}^2})^2(\frac{1}{5\mathfrak{s}^2})+(4\frac{1}{2\mathfrak{s}^2}(1-\frac{1}{2\mathfrak{s}^2})\mathfrak{s}+1)(\frac{1}{5\mathfrak{s}^2})}{6}]\\&\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\frac{(4\frac{1}{2\mathfrak{s}^2}\mathfrak{s}+1)(\frac{1}{5\mathfrak{s}^2})}{6}\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\\&\leq\frac{1}{\mathfrak{s}}[\frac{1}{2}+\frac{4(\frac{1}{5})+(\frac{2}{\mathfrak{s}}+1)(\frac{1}{5})}{6}]\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\frac{(\frac{2}{\mathfrak{s}}+1)(\frac{1}{5})}{6}\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\\&=\frac{1}{\mathfrak{s}}(\frac{11}{15})\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})+\frac{1}{10}\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p),\end{aligned}$$

implying

$$\begin{aligned}(1-\frac{1}{10})\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\leq&\frac{1}{\mathfrak{s}}(\frac{11}{15})\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\\Longrightarrow\ \mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\leq&\frac{1}{\mathfrak{s}}(\frac{11\times10}{15\times9})\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\=&\frac{1}{\mathfrak{s}}(\frac{110}{135})\times\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\end{aligned}$$

Let $\eta=\frac{1}{\mathfrak{s}}\times\frac{110}{135}$. Then, $\eta<\frac{1}{\mathfrak{s}}$. This implies that

$$\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\leq\eta\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}).\tag{14}$$

Moreover, Equation (11) implies

$$\begin{aligned}\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p+1})\leq&(1-\theta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\\\leq&(1-\theta_p)\eta\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1})\\\leq&\eta\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}).\end{aligned}\tag{15}$$

Therefore, by using (10), (12) and (14), we have

$$\begin{aligned}\mathcal{F}(\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)) \leq& \mathcal{F}(\eta\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}))-\tau \\ \leq& \mathcal{F}(\mathcal{G}(\zeta_{p-1},\zeta_{p-1},\mathcal{Q}\zeta_{p-1}))-\tau.\end{aligned}$$

$$\mathcal{F}(d_p) < \mathcal{F}(d_{p-1})-\tau \quad \text{for all } p\in\mathbb{N}. \tag{16}$$

Since $\mathcal{F}$ is strictly increasing, then $d_p < d_{p-1}$. Thus, we conclude that the sequence $\{d_p\}$ is strictly decreasing, so there exists $\lim\limits_{p\to+\infty} d_p = d$. Suppose that $d > 0$. Since $\mathcal{F}$ is an increasing mapping, there exists $\lim\limits_{\zeta\to d^+} \mathcal{F}(\zeta) = \mathcal{F}(d^+)$, so taking the limit as $p\to+\infty$ in Inequality (16), we obtain

$$\tau+\mathcal{F}(d^+) \leq \mathcal{F}(d^+),$$

a contradiction. Therefore, $\lim\limits_{p\to+\infty} d_p = 0$,

$$\lim_{p\to+\infty} \mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p) = 0. \tag{17}$$

Therefore, by using Equations (15) and (17), we deduce

$$\lim_{p\to+\infty} \mathcal{G}(\zeta_p,\zeta_p,\zeta_{p+1}) = 0.$$

Thus, for each $p,q\in\mathbb{N}$,

$$\begin{aligned}\mathcal{G}(\zeta_p,\zeta_p,\zeta_{p+q}) =& \mathcal{G}(\zeta_p,\zeta_{p+q},\zeta_{p+q}) \\ \leq& \mathfrak{s}\mathcal{G}(\zeta_p,\zeta_{p+1},\zeta_{p+1})+\mathfrak{s}\mathcal{G}(\zeta_{p+1},\zeta_{p+q},\zeta_{p+q}) \\ \leq& \mathfrak{s}\mathcal{G}(\zeta_p,\zeta_{p+1},\zeta_{p+1})+\mathfrak{s}^2\mathcal{G}(\zeta_{p+1},\zeta_{p+2},\zeta_{p+2})+\mathfrak{s}^2\mathcal{G}(\zeta_{p+2},\zeta_{p+q},\zeta_{p+q}) \\ \leq& \mathfrak{s}\mathcal{G}(\zeta_p,\zeta_{p+1},\zeta_{p+1})+\mathfrak{s}^2\mathcal{G}(\zeta_{p+1},\zeta_{p+2},\zeta_{p+2}))+\ldots.+\mathfrak{s}^q\mathcal{G}(\zeta_{p+q-1},\zeta_{p+q},\zeta_{p+q}) \\ \leq& \mathfrak{s}\eta^p\mathcal{G}(\zeta_0,\zeta_0,\mathcal{Q}\zeta_0)+\mathfrak{s}^2\eta^{p+1}\mathcal{G}(\zeta_0,\zeta_0,\mathcal{Q}\zeta_0) \\ &+\ldots..+\mathfrak{s}^p\eta^{p+q-1}\mathcal{G}(\zeta_0,\zeta_0,\mathcal{Q}\zeta_0) \\ \leq& \eta^p(\mathfrak{s}+\mathfrak{s}^2\eta+\mathfrak{s}^3\eta^2+\ldots\ldots.)\mathcal{G}(\zeta_0,\zeta_0,\mathcal{Q}\zeta_0) \\ \leq& \frac{1}{1-\mathfrak{s}\eta}\mathfrak{s}\eta^p\mathcal{G}(\zeta_0,\zeta_0,\mathcal{Q}\zeta_0).\end{aligned}$$

Letting $p\to+\infty$, we obtain that

$$\lim_{p\to+\infty} \mathcal{G}(\zeta_p,\zeta_p,\zeta_{p+q}) \leq \lim_{p\to+\infty} \frac{1}{1-\mathfrak{s}\eta}\mathfrak{s}\eta^p\mathcal{G}(\zeta_0,\zeta_0,\mathcal{Q}\zeta_0),$$

implying that $\lim\limits_{p\to+\infty} \mathcal{G}(\zeta_p,\zeta_p,\zeta_{p+q}) = 0$, which reveals that $\{\zeta_p\}$ is a $\mathfrak{cs}$ in $\mathcal{S}$. Since $(\mathcal{S},\mathcal{G},v)$ is a complete convex $\mathcal{G}_b$-$\mathfrak{MS}$, there exists $\zeta'\in\mathcal{S}$ such that $\lim\limits_{p\to+\infty} \mathcal{G}(\zeta_p,\zeta_p,\zeta') = 0$. Notice that

$$
\begin{aligned}
\tau+\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta',\zeta',\zeta')) \leq& \tau+\mathcal{F}\Big(\mathfrak{s}[\mathcal{G}(\mathcal{Q}\zeta',\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)+\mathfrak{s}\mathcal{G}(\mathcal{Q}\zeta_p,\zeta_p,\zeta_p)+\mathfrak{s}\mathcal{G}(\zeta_p,\zeta',\zeta')]\Big)\\
\leq& \mathcal{F}\Big(\mathfrak{s}[(\mu_1+\mu_5)\frac{\mathcal{G}(\zeta',\zeta',\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta')}{\mathcal{M}(\zeta',\zeta_p)}\\
&+(\mu_2+\mu_6)\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta_p)}\\
&+\mu_4\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)}]\\
&+\mathfrak{s}^2\mathcal{G}(\mathcal{Q}\zeta_p,\zeta_p,\zeta_p)+\mathfrak{s}^2\mathcal{G}(\zeta_p,\zeta',\zeta')\Big)\\
\implies \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta',\zeta',\zeta')) \leq& \mathcal{F}\Big(\mathfrak{s}[(\mu_1+\mu_5)\frac{\mathcal{G}(\zeta',\zeta',\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta')}{\mathcal{M}(\zeta',\zeta_p)}\\
&+(\mu_2+\mu_6)\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta_p)}\\
&+\mu_4\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)}]\\
&+\mathfrak{s}^2\mathcal{G}(\mathcal{Q}\zeta_p,\zeta_p,\zeta_p)+\mathfrak{s}^2\mathcal{G}(\zeta_p,\zeta',\zeta')\Big)-\tau\\
\leq& \mathcal{F}\Big(\mathfrak{s}[(\mu_1+\mu_5)\frac{\mathcal{G}(\zeta',\zeta',\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta')}{\mathcal{M}(\zeta',\zeta_p)}\\
&+(\mu_2+\mu_6)\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta_p)}\\
&+\mu_4\frac{\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)}{\mathcal{M}(\zeta_p,\zeta_p)}]\\
&+\mathfrak{s}^2\mathcal{G}(\mathcal{Q}\zeta_p,\zeta_p,\zeta_p)+\mathfrak{s}^2\mathcal{G}(\zeta_p,\zeta',\zeta')\Big)\\
\leq& \mathcal{F}\Big(\mathfrak{s}[(\mu_1+\mu_5)\mathcal{G}(\zeta',\zeta',\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta')\\
&+(\mu_2+\mu_6)\zeta[\mathcal{G}(\zeta',\zeta_p,\zeta_p)+\mathcal{G}(\zeta_p,\mathcal{Q}\zeta_p,\mathcal{Q}\zeta_p)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta')\\
&+\mu_4\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)]+\mathfrak{s}^2\mathcal{G}(\mathcal{Q}\zeta_p,\zeta_p,\zeta_p)+\mathfrak{s}^2\mathcal{G}(\zeta_p,\zeta',\zeta')\Big).
\end{aligned}
$$

Using $F_1$, we can write

$$
\begin{aligned}
\mathcal{G}(\mathcal{Q}\zeta',\zeta',\zeta') \leq& \mathfrak{s}[(\mu_1+\mu_5)\mathcal{G}(\zeta',\zeta',\zeta_p)\mathcal{G}(\zeta_p,\zeta_p,\zeta')\\
&+(\mu_2+\mu_6)\mathfrak{s}[\mathcal{G}(\zeta',\zeta_p,\zeta_p)+\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)]\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta')\\
&+\mu_4\mathcal{G}(\zeta_p,\zeta_p,\mathcal{Q}\zeta_p)]+\mathfrak{s}^2\mathcal{G}(\mathcal{Q}\zeta_p,\zeta_p,\zeta_p)+\mathfrak{s}^2\mathcal{G}(\zeta_p,\zeta_p,\zeta').
\end{aligned}
$$

Letting $p\to+\infty$, we obtain that $\lim\limits_{p\to+\infty}\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')=0$, which implies that $\zeta'=\mathcal{Q}\zeta'$. Thus, the $\mathfrak{fp}$ of $\mathcal{Q}$ is $\zeta'$. $\square$

**Remark 4.** *The next example shows that Theorem 2 does not ensure the uniqueness of the* $\mathfrak{fp}$.

**Example 3.** *Assume that* $\mathcal{S}=\{1,2,3\}$ *and* $\mathcal{G}:\mathcal{S}\times\mathcal{S}\times\mathcal{S}\to[0,+\infty)$ *is a mapping for each* $\zeta_1,\zeta_2,\zeta_3\in\mathcal{S}$ *such that* $\mathcal{G}(\zeta_1,\zeta_2,\zeta_3)=\mathcal{G}(\zeta_2,\zeta_1,\zeta_3)=\mathcal{G}(\zeta_3,\zeta_2,\zeta_1)=\ldots.$ *and* $\mathcal{G}(1,1,1)=\mathcal{G}(2,2,2)=\mathcal{G}(3,3,3)=0$, $\mathcal{G}(1,1,2)=\mathcal{G}(2,2,1)=3$, $\mathcal{G}(1,1,3)=\mathcal{G}(3,3,1)=4$, $\mathcal{G}(2,2,3)=\mathcal{G}(3,3,2)=5$, $\mathcal{G}(1,2,3)=6$. *Then,* $(\mathcal{S},\mathcal{G})$ *is a complete* $\mathcal{G}_b$-$\mathfrak{MS}$ *with* $\mathfrak{s}=1$. *Define a mapping* $\mathcal{Q}$ *such that* $\mathcal{Q}\zeta=\zeta$ *for any* $\zeta\in\mathcal{S}$. *For any* $\zeta_1,\zeta_2,\zeta_3\in\mathcal{S}$, *we obtain*

$$
\begin{aligned}
\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3)) \leq & \mathcal{F}\Bigg( \mu_1 \frac{\mathcal{G}(\zeta_1,\zeta_1,\zeta_2)\mathcal{G}(\zeta_2,\zeta_2,\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_2)} + \mu_2 \frac{\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_2)\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_2)} \\
& + \mu_3 \frac{\mathcal{G}(\zeta_2,\zeta_2,\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\zeta_2)}{\mathcal{M}(\zeta_2,\zeta_3)} + \mu_4 \frac{\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_2)}{\mathcal{M}(\zeta_2,\zeta_3)} \\
& + \mu_5 \frac{\mathcal{G}(\zeta_1,\zeta_1,\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_3)} + \mu_6 \frac{\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_3)} \Bigg) \\
\implies \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3)) \leq & \mathcal{F}\Bigg( \mu_1 \frac{\mathcal{G}(\zeta_1,\zeta_1,\zeta_2)\mathcal{G}(\zeta_2,\zeta_2,\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_2)} + \mu_2 \frac{\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_2)\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_2)} \\
& + \mu_3 \frac{\mathcal{G}(\zeta_2,\zeta_2,\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\zeta_2)}{\mathcal{M}(\zeta_2,\zeta_3)} + \mu_4 \frac{\mathcal{G}(\zeta_2,\zeta_2,\mathcal{Q}\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_2)}{\mathcal{M}(\zeta_2,\zeta_3)} \\
& + \mu_5 \frac{\mathcal{G}(\zeta_1,\zeta_1,\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_3)} + \mu_6 \frac{\mathcal{G}(\zeta_1,\zeta_1,\mathcal{Q}\zeta_3)\mathcal{G}(\zeta_3,\zeta_3,\mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1,\zeta_3)} \Bigg) - \tau \\
\leq & \mathcal{F}\Big( (\mu_1+\mu_2)\mathcal{G}(\zeta_1,\zeta_1,\zeta_2) + (\mu_3+\mu_4)\mathcal{G}(\zeta_2,\zeta_2,\zeta_3) \\
& + (\mu_5+\mu_6)\mathcal{G}(\zeta_1,\zeta_1,\zeta_3) \Big).
\end{aligned}
$$

*Using $F_1$, we obtain*

$$
\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3) \leq (\mu_1+\mu_2)(\mathcal{G}(\zeta_1,\zeta_1,\zeta_2))^2 + (\mu_3+\mu_4)(\mathcal{G}(\zeta_2,\zeta_2,\zeta_3))^2 + (\mu_5+\mu_6)(\mathcal{G}(\zeta_1,\zeta_1,\zeta_3))^2.
$$

*Similarly,*

$$
\mathcal{G}(\mathcal{Q}\zeta_3, \mathcal{Q}\zeta_1, \mathcal{Q}\zeta_2) \leq (\mu_3+\mu_4)(\mathcal{G}(\zeta_1,\zeta_1,\zeta_2))^2 + (\mu_5+\mu_6)(\mathcal{G}(\zeta_2,\zeta_2,\zeta_3))^2 + (\mu_1+\mu_2)(\mathcal{G}(\zeta_1,\zeta_1,\zeta_3))^2,
$$

$$
\mathcal{G}(\mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3, \mathcal{Q}\zeta_1) \leq (\mu_5+\mu_6)(\mathcal{G}(\zeta_1,\zeta_1,\zeta_2))^2 + (\mu_1+\mu_2)(\mathcal{G}(\zeta_2,\zeta_2,\zeta_3))^2 + (\mu_3+\mu_4)(\mathcal{G}(\zeta_1,\zeta_1,\zeta_3))^2.
$$

*Combining the above equations,*

$$
\begin{aligned}
3\mathcal{G}(\mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3, \mathcal{Q}\zeta_1) \leq & \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_1,\zeta_1,\zeta_2))^2 + \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_2,\zeta_2,\zeta_3))^2 + \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_1,\zeta_1,\zeta_3))^2 \\
\implies \mathcal{G}(\mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3, \mathcal{Q}\zeta_1) \leq & \frac{1}{3}\Big\{ \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_1,\zeta_1,\zeta_2))^2 + \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_2,\zeta_2,\zeta_3))^2 + \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_1,\zeta_1,\zeta_3))^2 \Big\}.
\end{aligned}
$$

*Applying $\mathcal{F}_1$, we can write*

$$
\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_2, \mathcal{Q}\zeta_3, \mathcal{Q}\zeta_1) \leq \mathcal{F}\Big( \frac{1}{3}\Big\{ \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_1,\zeta_1,\zeta_2))^2 + \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_2,\zeta_2,\zeta_3))^2 + \sum_{i=1}^{6} \mu_i (\mathcal{G}(\zeta_1,\zeta_1,\zeta_3))^2 \Big\} \Big). \tag{18}
$$

*Consider Equation (18):*

- *Case 1: If* $\zeta_1 = 1, \zeta_2 = 2$ *and* $\zeta_3 = 3$*, then*

$$\begin{aligned}\mathcal{F}(\mathcal{G}(\mathcal{Q}2,\mathcal{Q}3,\mathcal{Q}1)) \leq & \mathcal{F}\Big(\frac{1}{3}\Big\{\sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,2))^2 + \sum_{i=1}^{6}\mu_i(\mathcal{G}(2,2,3))^2 \\ & + \sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,3))^2\Big\}\Big) \\ = & \mathcal{F}\Big(\frac{1}{3}\sum_{i=1}^{6}\mu_i(9+25+16)\Big) \\ = & \mathcal{F}(11.1111) \\ \Longrightarrow \mathcal{F}(\mathcal{G}(\mathcal{Q}2,\mathcal{Q}3,\mathcal{Q}1)) \leq & \mathcal{F}(11.1111).\end{aligned}$$

*Then,*

$$\frac{1}{10} + \mathcal{F}(6) \leq \mathcal{F}(11.11) \implies \frac{1}{10} + \ln(6) \leq \ln(11.11) \implies 1.8918 \leq 2.4079.$$

- *Case 2: If* $\zeta_1 = \zeta_2 = 1$ *and* $\zeta_3 = 2$*, then*

$$\begin{aligned}\mathcal{F}(\mathcal{G}(\mathcal{Q}1,\mathcal{Q}1,\mathcal{Q}2)) \leq & \mathcal{F}\Big(\frac{1}{3}\Big\{\sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,1))^2 + \sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,2))^2 \\ & + \sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,2))^2\Big\}\Big) \\ = & \mathcal{F}\Big(\sum_{i=1}^{6}\mu_i(9+9)\Big) \\ = & \mathcal{F}(4) \\ \Longrightarrow \mathcal{F}(\mathcal{G}(\mathcal{Q}1,\mathcal{Q}1,\mathcal{Q}2)) \leq & \mathcal{F}(4).\end{aligned}$$

*Then*

$$\frac{1}{10} + \mathcal{F}(\mathcal{G}(\mathcal{Q}1,\mathcal{Q}1,\mathcal{Q}2)) \leq \mathcal{F}(4) \implies \frac{1}{10} + \ln(3) \leq \ln(4) \implies 1.1986 \leq 1.3863.$$

- *Case 3: If* $\zeta_1 = \zeta_2 = 1 and \zeta_3 = 2$*, then*

$$\begin{aligned}\mathcal{F}(\mathcal{G}(\mathcal{Q}1,\mathcal{Q}1,\mathcal{Q}3)) \leq & \mathcal{F}\Big(\frac{1}{3}\Big\{\sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,1))^2 + \sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,3))^2 \\ & + \sum_{i=1}^{6}\mu_i(\mathcal{G}(1,1,3))^2\Big\}\Big) \\ = & \mathcal{F}\Big(\frac{1}{3}\sum_{i=1}^{6}\mu_i(16+16)\Big) \\ = & \mathcal{F}(\frac{64}{9}) \\ \Longrightarrow \mathcal{F}(\mathcal{G}(\mathcal{Q}1,\mathcal{Q}1,\mathcal{Q}3)) \leq & \mathcal{F}(7.1111).\end{aligned}$$

*Then*

$$\frac{1}{10} + \mathcal{F}(\mathcal{G}(\mathcal{Q}1,\mathcal{Q}1,\mathcal{Q}3)) \leq \mathcal{F}(7.1111) \implies \frac{1}{10} + \ln(4) \leq \ln(7.1111) \implies 1.4862 \leq 1.9617.$$

- *Case 4: If $\zeta_1 = \zeta_2 = 2 and \zeta_3 = 3$, then*

$$\begin{aligned}
\mathcal{F}(\mathcal{G}(\mathcal{Q}2, \mathcal{Q}2, \mathcal{Q}3)) \leq & \mathcal{F}\Big(\frac{1}{3}\Big\{\sum_{i=1}^{6}\mu_i(\mathcal{G}(2,2,2))^2 + \sum_{i=1}^{6}\mu_i(\mathcal{G}(2,2,3))^2 \\
& + \sum_{i=1}^{6}\mu_i(\mathcal{G}(2,2,3))^2\Big\}\Big) \\
= & \mathcal{F}\Big(\sum_{i=1}^{6}\mu_i(25+25)\Big) \\
= & \mathcal{F}(\frac{100}{9}) \\
\implies \mathcal{F}(\mathcal{G}(\mathcal{Q}1, \mathcal{Q}1, \mathcal{Q}3)) \leq & \mathcal{F}(11.1111).
\end{aligned}$$

*Then*

$$\frac{1}{10} + \mathcal{F}(\mathcal{G}(1,1,3)) \leq \mathcal{F}(11.1111) \implies \frac{1}{10} + \ln(4) \leq \ln(11.1111) \implies 1.7094 \leq 2.4079.$$

*Therefore, the contraction condition is satisfied for every $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{S}$. Hence, Theorem 1 is satisfied and $F(\mathcal{Q}) = \{1,2,3\}$.*

**Definition 13.** *Consider a $\mathcal{G}_b$-$\mathfrak{MS}$ defined as $(\mathcal{S}, \mathcal{G})$ and a self mapping $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$. The $\mathcal{FP}$ problem for $\mathcal{Q}$ is called well-posed if:*

*(1): $\zeta' \in \mathcal{S}$ is a unique $\mathfrak{fp}$ of $\mathcal{Q}$.*

*(2): For any sequence $\{\zeta_p\} \in \mathcal{S}$, if $\lim_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta_p, \mathcal{Q}\zeta_p) = 0$, then $\lim_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta_p, \zeta') = 0$ or if $\lim_{p\to+\infty} \mathcal{G}(\zeta_p, \mathcal{Q}\zeta_p, \mathcal{Q}\zeta_p) = 0$, then $\lim_{p\to+\infty} \mathcal{G}(\zeta_p, \zeta', \zeta') = 0$.*

**Theorem 3.** *Assume that all assumptions of Theorem (2) hold. If*

$$\sum_{i=1}^{6}\mu_i \leq \max\{\mu_1 + \mu_2, \mu_3 + \mu_4, \mu_5 + \mu_6\}, \tag{19}$$

*then $\mathcal{Q}$ has a well-posed $\mathfrak{fp}$ problem.*

**Proof.** By Theorem 2, there exists an $\mathfrak{fp}$ of $\mathcal{Q}$, say $\zeta' \in \mathcal{S}$. To prove the uniqueness, we proceed by a contradiction. Assume that $\Psi'$ is also an $\mathfrak{fp}$ of $\mathcal{Q}$. By using the hypothesis, let $\sum_{i=1}^{6}\mu_i \leq \mu_1 + \mu_2$, which is possible only if $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$. Then, by using the contraction condition

$$\begin{aligned}
\tau + \mathcal{F}(\mathcal{G}(\zeta', \zeta', \Psi')) \leq & \mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta',\zeta',\zeta')\mathcal{G}(\zeta',\zeta',\zeta')}{\mathcal{M}(\zeta',\zeta')} + \mu_2\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta')}\Big) \\
\implies \mathcal{F}(\mathcal{G}(\zeta', \zeta', \Psi')) \leq & \mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta',\zeta',\zeta')\mathcal{G}(\zeta',\zeta',\zeta')}{\mathcal{M}(\zeta',\zeta')} + \mu_2\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta')}\Big) - \tau \\
\leq & \mathcal{F}\Big(\mu_1\frac{\mathcal{G}(\zeta',\zeta',\zeta')\mathcal{G}(\zeta',\zeta',\zeta')}{\mathcal{M}(\zeta',\zeta')} + \mu_2\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta')}\Big).
\end{aligned}$$

Using $F_1$, we have

$$\begin{aligned}
\mathcal{G}(\zeta', \zeta', \Psi') \leq & \mu_1\frac{\mathcal{G}(\zeta',\zeta',\zeta')\mathcal{G}(\zeta',\zeta',\zeta')}{\mathcal{M}(\zeta',\zeta')} + \mu_2\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta')} \\
= & \mu_2\frac{\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')\mathcal{G}(\zeta',\zeta',\mathcal{Q}\zeta')}{\mathcal{M}(\zeta',\zeta')} = 0,
\end{aligned}$$

that is $\mathcal{G}(\zeta', \zeta', \Psi') = 0$, which is a contradiction. The remaining cases are very easy to verify, as they also produce a contradiction. This implies that $\zeta'$ is a unique $\mathfrak{fp}$. Consider a sequence $\{\Psi_p\} \in \mathcal{S}$ such that $\lim_{p\to+\infty} \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p) = 0$. Further, we consider the subsequent cases:

**Case 1** : If $\sum_{i=1}^{6} \mu_i \leq \mu_1 + \mu_2$, then $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$, then by using (19),

$$
\begin{aligned}
\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) =& \tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \mathcal{Q}\Psi_p, \mathcal{Q}\zeta') \\
\leq& \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_2 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) \\
\implies \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) \leq& \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_2 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) - \tau \\
\leq& \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_2 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) \\
=& \mathcal{F}\Big(\mu_2 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big).
\end{aligned}
$$

Applying $F_1$, we have

$$
\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') \leq \mu_2 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}.
$$

Letting $p \to +\infty$, we obtain $\lim_{p\to+\infty} \mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') = 0$.

**Case 2**: If $\sum_{i=1}^{6} \mu_i \leq \mu_3 + \mu_4$, then $\mu_1 = \mu_2 = \mu_5 = \mu_6 = 0$,

$$
\begin{aligned}
\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) =& \tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \mathcal{Q}\zeta', \mathcal{Q}\Psi_p)) \\
\leq& \mathcal{F}\Big(\mu_3 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_4 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) \\
\implies \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) \leq& \mathcal{F}\Big(\mu_3 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_4 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) - \tau \\
\leq& \mathcal{F}\Big(\mu_3 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_4 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) \\
=& \mathcal{F}(\mu_4 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)).
\end{aligned}
$$

Applying $F_1$, we have $\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') \leq \mu_4 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)$.
If $p \to +\infty$, we obtain $\lim_{p\to+\infty} \mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') = 0$.

**Case 3**: If $\sum_{i=1}^{6} \mu_i \leq \mu_5 + \mu_6$, then $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$,

$$
\begin{aligned}
\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) =& \tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta', \mathcal{Q}\Psi_p, \mathcal{Q}\Psi_p)) \\
\leq& \mathcal{F}\Big(\mu_5 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_6 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) \\
\implies \mathcal{F}(\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) \leq& \mathcal{F}\Big(\mu_5 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_6 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) - \tau \\
\leq& \mathcal{F}\Big(\mu_5 \frac{\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)} + \mu_6 \frac{\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)}{\mathcal{M}(\Psi_p, \Psi_p)}\Big) \\
=& \mathcal{F}(\mu_6 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)).
\end{aligned}
$$

Using $F_1$, we have $\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') \leq \mu_6 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)$.
By letting $p \to +\infty$, we obtain $\lim_{p\to+\infty} \mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') = 0$.
Combining all the above cases, we obtain

$$\mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta') \leq \frac{1}{3}\Big(\mu_2 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p) + \mu_4 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p) + \mu_6 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p)\Big).$$

Then,

$$\begin{aligned}
\mathcal{G}(\Psi_p, \zeta', \zeta') \leq & \mathfrak{s}(\mathcal{G}(\Psi_p, \mathcal{Q}\Psi_p, \mathcal{Q}\Psi_p) + \mathcal{G}(\mathcal{Q}\Psi_p, \zeta', \zeta')) \\
& \leq \mathfrak{s}\Big(\mathcal{G}(\Psi_p, \mathcal{Q}\Psi_p, \mathcal{Q}\Psi_p) + \frac{1}{3}(\mu_2 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p) + \mu_4 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p) \\
& + \mu_6 \mathcal{G}(\Psi_p, \Psi_p, \mathcal{Q}\Psi_p))\Big) \\
& \leq \mathfrak{s}(1 + \frac{1}{3}(\mu_2 + \mu_4 + \mu_6)) \mathcal{G}(\Psi_p, \mathcal{Q}\Psi_p, \mathcal{Q}\Psi_p).
\end{aligned}$$

Taking the limit as $p \to +\infty$, we obtain that $\lim_{p\to+\infty} \mathcal{G}(\Psi_p, \Psi_p, \zeta') = 0$. □

According to Jeong and Rhoades [40], a map $\mathcal{Q}$ has the $P$-property if it fulfills

$$\mathcal{F}(\mathcal{Q}) = \mathcal{F}(\mathcal{Q}^p) \qquad \text{for all } p \in \mathbb{N}_0.$$

It is important to note that if $\zeta'$ is an $\mathfrak{fp}$ of $\mathcal{Q}$, then it is an $\mathfrak{fp}$ of $\mathcal{Q}^p$ also, but its converse does not hold. This point is named the periodic point. Rahimi, H. et al. [41] proved some periodic point theorems for the $T$-contraction of two maps on cone metric spaces.

**Theorem 4.** *Consider a $\mathcal{G}_b$-$\mathfrak{MS}$ defined as $(\mathcal{S}, \mathcal{G})$ with coefficient $\mathfrak{s} \geq 1$ and a mapping $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$ with $\mathcal{F}(\mathcal{Q}) \neq 0$ satisfying*

$$\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1, \mathcal{Q}^2\zeta_1)) \leq \mathcal{F}(\eta \mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}\zeta_1)), \tag{20}$$

*for any $\zeta_1 \in \mathcal{S}$, $\eta \in [0, 1)$. Then, $\mathcal{Q}$ has the $P$ property.*

**Proof.** Let $p > 1$ and $\zeta_3 = \mathcal{Q}^p\zeta_3$ for every $p > 1$. We have

$$\begin{aligned}
\tau + \mathcal{F}(\mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3)) =& \mathcal{F}(\mathcal{G}(\mathcal{Q}\mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}^2\mathcal{Q}^{p-1}\zeta_3)) \\
\leq & \mathcal{F}(\eta \mathcal{G}(\mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-1}\zeta_3)) \\
\implies \mathcal{F}(\mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3)) \leq & \mathcal{F}(\eta \mathcal{G}(\mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-1}\zeta_3)) - \tau \\
< & \mathcal{F}(\eta \mathcal{G}(\mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}^{p-1}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-1}\zeta_3)) \\
= & \mathcal{F}(\eta \mathcal{G}(\mathcal{Q}\mathcal{Q}^{p-2}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-2}\zeta_3, \mathcal{Q}^2\mathcal{Q}^{p-2}\zeta_3)) \\
\leq & \mathcal{F}(\eta^2(\mathcal{G}(\mathcal{Q}^{p-2}\zeta_3, \mathcal{Q}^{p-2}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-2}\zeta_3))) - \tau \\
< & \mathcal{F}(\eta^2(\mathcal{G}(\mathcal{Q}^{p-2}\zeta_3, \mathcal{Q}^{p-2}\zeta_3, \mathcal{Q}\mathcal{Q}^{p-2}\zeta_3))) \\
& . \\
& . \\
\leq & \mathcal{F}(\eta^p \mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3)) - \tau \\
< & \mathcal{F}(\eta^p \mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3)).
\end{aligned}$$

Using $\mathcal{F}_1$, we can write $\mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3) \leq \eta^p \mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3)$. By taking the limit as $p \to +\infty$, we obtain $\lim_{p\to+\infty} \mathcal{G}(\zeta_3, \zeta_3, \mathcal{Q}\zeta_3) = 0$, which implies that $\zeta_3 = \mathcal{Q}\zeta_3$. □

**Theorem 5.** *Assume that all assumptions of Theorem 2 are fulfilled, then $\mathcal{Q}$ has the $p$ property.*

**Proof.** For each $\zeta_1 \in \mathcal{S}$, we obtain

$$\begin{aligned}\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1, \mathcal{Q}^2\zeta_1)) =& \tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1, \mathcal{Q}\mathcal{Q}\zeta_1)) \\ \leq& \mathcal{F}\Big(\mu_1 \frac{\mathcal{G}(\zeta_1, \zeta_1, \zeta_1)\mathcal{G}(\zeta_1, \zeta_1, \zeta_1)}{\mathcal{M}(\zeta_1, \zeta_1)} + \mu_2 \frac{\mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}\zeta_1)\mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}\zeta_1)}{\mathcal{M}(\zeta_1, \zeta_1)} \\ &+ (\mu_3 + \mu_5) \frac{\mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}\zeta_1)\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1, \zeta_1)}{\mathcal{M}(\mathcal{Q}\zeta_1, \zeta_1)} \\ &+ (\mu_4 + \mu_6) \frac{\mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}^2\zeta_1)\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1)}{\mathcal{M}(\mathcal{Q}\zeta_1, \zeta_1)}\Big) \\ \leq& \mathcal{F}((\mu_2 + \mu_3 + \mu_5)\mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}\zeta_1)).\end{aligned}$$

Note that $\mu_2 + \mu_3 + \mu_5 < 1$. Let $\mu_2 + \mu_3 + \mu_5 = \eta$, then

$$\tau + \mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_1, \mathcal{Q}\zeta_1, \mathcal{Q}^2\zeta_1)) \leq \mathcal{F}(\eta\mathcal{G}(\zeta_1, \zeta_1, \mathcal{Q}\zeta_1)).$$

This is the same as (20). By Theorem 4, $\mathcal{Q}$ has the $P$ property. □

Due to the many applications of integral equations in many real-life problems, the solution of integral equations and their existence have become important topics for researchers. A huge literature is present on the existence of the solution to such integral equations using the fixed-point technique. Gnanaprakasam et al. [42] applied their results to prove the existence of the solution to the integral equation by incorporating the F-Khan contraction. Similarly, Panda et al. [43] presented fixed-point results and their application to find the solution of the Volterra integral equations to verify their results on the platform of dislocated extended $b$-metric spaces. Recently, many works can be seen in the perspective of applying the fixed-point results to ensure the existence of the solution to certain integral equations. In 2022, Gupta et al. [44] applied their results to find the solution of the Fredholm integral equation in the framework of $\mathcal{G}_b$-$\mathfrak{MS}$, and in 2023, Joseph et al. [45] observed the solution of an integral equation using the fixed point technique in the $\mathcal{G}$-metric space.

## 4. Application

To ensure the existence of a solution to the subsequent integral equation, we apply Theorem 1.

$$\zeta_p(\mathfrak{u}) = f(\mathfrak{u}) + \gamma \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{u}, \Psi)\mathfrak{K}_1(\Psi, \zeta_p(\Psi))d\Psi \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{u}, \Psi)\mathfrak{K}_2(\Psi, \zeta_p(\Psi))d\Psi \quad \text{for all } p \in \mathbb{N} \tag{21}$$

for any $\mathfrak{u} \in [\mathfrak{l}_1, \mathfrak{l}_2]$, where $f : [\mathfrak{l}_1, \mathfrak{l}_2] \to \mathbb{R}$, $w : [\mathfrak{l}_1, \mathfrak{l}_2] \times [\mathfrak{l}_1, \mathfrak{l}_2] \to \mathbb{R}$, and $\mathfrak{K}_1, \mathfrak{K}_2 : [\mathfrak{l}_1, \mathfrak{l}_2] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $\mathcal{S} = C([\mathfrak{l}_1, \mathfrak{l}_2], \mathbb{R})$ represent the space of continuous functions on $[\mathfrak{l}_1, \mathfrak{l}_2]$. Define

$$\mathcal{G}(\zeta_p, \beta_p, \eta_p) = \Big(\sup_{\mathfrak{u} \in [\mathfrak{l}_1, \mathfrak{l}_2]} |\zeta_p(\mathfrak{u}) - \beta_p(\mathfrak{u})| + \sup_{\mathfrak{u} \in [\mathfrak{l}_1, \mathfrak{l}_2]} |\beta_p(\mathfrak{u}) - \eta_p(\mathfrak{u})| + \sup_{\mathfrak{u} \in [\mathfrak{l}_1, \mathfrak{l}_2]} |\zeta_p(\mathfrak{u}) - \eta_p(\mathfrak{u})|\Big)^2, \quad \text{for all } p \in \mathbb{N}$$

while the function $v : \mathcal{S} \times \mathcal{S} \times (0, 1) \to \mathcal{S}$ is presented as $v(\zeta_p, \beta_p; \theta) = \theta\zeta_p + (1 - \theta)\beta_p$. Then, $(\mathcal{S}, \mathcal{G}, v)$ represents a complete convex $\mathcal{G}_b$-$\mathfrak{MS}$ with $\mathfrak{s} = 2$. Consider a mapping $\mathcal{Q} : \mathcal{S} \to \mathcal{S}$ by

$$\mathcal{Q}\zeta_p(\mathfrak{u}) = f(\mathfrak{u}) + \gamma \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{u}, \Psi)\mathfrak{K}_1(\Psi, \zeta_p(\Psi))d\Psi \int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{u}, \Psi)\mathfrak{K}_2(\Psi, \zeta_p(\Psi))d\Psi. \tag{22}$$

It is obvious that $\mathcal{Q}$ is well-defined. To obtain the solution for (21), it is equivalent to finding an $\mathfrak{fp}$ of $\mathcal{Q}$. Next, we state the subsequent theorem.

**Theorem 6.** *Suppose that the subsequent conditions are fulfilled:*

*(1):* $\gamma \leq 1$;

*(2):* $\int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{u},\Psi)d(\Psi) \leq 1$;

*(3):* $|\mathfrak{K}_i(\Psi,\zeta_p(\Psi)) - \mathfrak{K}_i(\Psi,\beta_p(\Psi))| \leq \frac{\sqrt{5}}{5}|\mathcal{Q}\zeta_{p-1}(\Psi) - \mathcal{Q}\beta_{p-1}(\Psi)|$, $i = 1,2$, $p \in \mathbb{N}$, *and*

$$\int_{\mathfrak{l}_1}^{\mathfrak{l}_2} w(\mathfrak{u},\Psi)|\mathfrak{K}_1(\Psi,\beta_p(\Psi)) + \mathfrak{K}_2(\Psi,\zeta_p(\Psi))|d\Psi \leq 1.$$

*Then, the unique solution of Equation (21) exists.*

**Proof.** Clearly , any $\mathfrak{fp}$ of (22) is a solution of (21). Using Conditions (1)–(3), we have

$$\begin{aligned}
&\mathcal{G}(\mathcal{Q}\zeta_p,\mathcal{Q}\beta_p,\mathcal{Q}\beta_p)\\
&=\Big(2\sup_{u\in[\mathfrak{l}_1,\mathfrak{l}_2]}|\mathcal{Q}\zeta_p(\mathfrak{u})-\mathcal{Q}\beta_p(\mathfrak{u})|\Big)^2\\
&=\gamma^2\Bigg(2\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Bigg|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_1(\Psi,\zeta_p(\Psi))d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_2(\Psi,\zeta_p(\Psi))d\Psi\\
&-\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_1(\Psi,\beta_p(\Psi))d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_2(\Psi,\beta_p(\Psi))d\Psi\Bigg|\Bigg)^2
\end{aligned}$$

$$\begin{aligned}
&\leq\gamma^2\Bigg(2\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Bigg|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\Big|\mathfrak{K}_1(\Psi,\zeta_p(\Psi))d\Psi-\mathfrak{K}_1(\Psi,\beta_p(\Psi))\Big|d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_2(\Psi,\zeta_p(\Psi))d\Psi\\
&+\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_1(\Psi,\beta_p(\Psi))d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\Big|\mathfrak{K}_2(\Psi,\zeta_p(\Psi))d\Psi-\mathfrak{K}_2(\Psi,\beta_p(\Psi))\Big|d\Psi\Bigg|\Bigg)^2\\
&\leq4\gamma^2\Bigg(\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\sup_{\Psi\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Big|\mathfrak{K}_1(\Psi,\zeta_p(\Psi))d\Psi-\mathfrak{K}_1(\Psi,\beta_p(\Psi))\Big|\Bigg|\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)d\Psi\\
&\quad\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_2(\Psi,\zeta_p(\Psi))d\Psi\Bigg|+\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\sup_{\Psi\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Big|\mathfrak{K}_2(\Psi,\zeta_p(\Psi))-\mathfrak{K}_2(\Psi,\beta_p(\Psi))\Big|\\
&\quad\Bigg|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_1(\Psi,\beta_p(\Psi)d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)d\Psi\Bigg|\Bigg)^2\\
&\leq4\gamma^2\Bigg(\frac{\sqrt{5}}{5}\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}|\mathcal{Q}\zeta_{p-1}-\mathcal{Q}\beta_{p-1}|\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Bigg|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_2(\Psi,\zeta_p(\Psi))d\Psi\\
&+\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\mathfrak{K}_1(\Psi,\beta_p(\Psi))d\Psi\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)d\Psi\Bigg|\Bigg)^2\\
&\leq\frac{4}{5}\gamma^2\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Big(\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)d\Psi\Big)^2\Bigg(\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}|\mathcal{Q}\zeta_{p-1}-\mathcal{Q}\beta_{p-1}|\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Bigg|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\Big|\mathfrak{K}_2(\Psi,\zeta_p(\Psi))d\Psi\\
&+\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)\Big|\mathfrak{K}_1(\Psi,\zeta_p(\Psi))d\Psi\Bigg|\Bigg)^2\\
&\leq\frac{4}{5}\gamma^2\Bigg(2\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}|\mathcal{Q}\zeta_{p-1}-\mathcal{Q}\beta_{p-1}|\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}\Bigg|\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}w(\mathfrak{u},\Psi)|\mathfrak{K}_1(\Psi,\beta_p(\Psi))+\mathfrak{K}_2(\Psi,\zeta_p(\Psi))|d\Psi\Bigg|\Bigg)^2\\
&\leq\frac{1}{5}\Big(2\sup_{\mathfrak{u}\in[\mathfrak{l}_1,\mathfrak{l}_2]}|\mathcal{Q}\zeta_{p-1}-\mathcal{Q}\beta_{p-1}|\Big)^2\\
&=\frac{1}{5}\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\beta_{p-1},\mathcal{Q}\beta_{p-1}).
\end{aligned}$$

Hence

$$\tau+\mathcal{F}(\mathcal{G}(\mathcal{Q}\zeta_p,\mathcal{Q}\beta_p,\mathcal{Q}\beta_p))\leq\mathcal{F}(\frac{1}{5}\mathcal{G}(\mathcal{Q}\zeta_{p-1},\mathcal{Q}\beta_{p-1},\mathcal{Q}\beta_{p-1})),$$

where $F(t) = \ln t$ and $\tau \in \left(0, \ln\left(\frac{1}{5}\frac{\mathcal{G}(\mathcal{Q}\zeta_{p-1}, \mathcal{Q}\beta_{p-1}, \mathcal{Q}\beta_{p-1})}{\mathcal{G}(\mathcal{Q}\zeta_p, \mathcal{Q}\beta_p, \mathcal{Q}\beta_p)}\right)\right)$.

All conditions of Theorem 1 with $\beta_p = \zeta_p$ are satisfied, which enables us to know that a fixed point for $\mathcal{Q}$ exists. Thus, the solution of the integral equation exists. Hence, we obtain that Equation (21) gives a unique solution, where the sequence satisfies the convex condition with $\theta_p \in (0, \frac{1}{4})$. □

## 5. Conclusions

In 2023, Ji et al. [28] presented an article on fixed-point results using Mann's iterative scheme tailored to $\mathcal{G}_b$-metric spaces. Wardowski introduced the idea of the $\mathcal{F}$-contraction using an increasing function as a control function. Incorporating both concepts, in this manuscript, the existence and uniqueness of the fixed points were presented with Mann's iteration scheme in convex $\mathcal{G}_b$-metric spaces using the $\mathcal{F}$-contraction. This task was achieved by further weakening the conditions of the $\mathcal{F}$ mappings presented by Wardowski. An example was provided to support our results. Eventually, an application was given for the validity of our results. The obtained results are generalizations of several existing results in the literature. Furthermore, the results of Ji. et al. are the special case of these theorems.

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