axioms

MDPI

Article

Generalized Refinements of Reversed AM-GM Operator Inequalities for Positive Linear Maps

Yonghui Ren

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China; yonghuiren1992@163.com

Abstract: We shall present some more generalized and further refinements of reversed AM-GM operator inequalities for positive linear maps due to Xue's and Ali's publications.

Keywords: AM-GM; operator inequalities; positive linear maps

MSC: 47A64; 47A30; 47B02

1. Introduction

Let m, M be scalars and I be the identity operator. $B(\mathcal{H})$ denote all bounded linear operators acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. In addition, $A \geq 0$ means the operator A is positive. We say $A \geq B$ if $A - B \geq 0$. A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is called positive (strictly positive) if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$.

If $A, B \in B(\mathcal{H})$ are two positive operators, then the operator weighted arithmetic mean and geometric mean are defined as

$$A\nabla_v B = (1-v)A + vB \quad \text{and} \quad A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}$$

for $v \in [0,1]$, respectively. Denoted $A\nabla_v B$ by $A\nabla B$ and $A\sharp_v B$ by $A\sharp B$ when $v = \frac{1}{2}$ for brevity. The Kantorovich constant and Specht's ratio are defined by $K(h,2) = \frac{(h+1)^2}{4h}$ and $S(h) = \frac{h^{\frac{1}{h-1}}}{e\ln h^{\frac{1}{h-1}}}$ when $h > 0$. If there is no special explanations, we always default to $a, b > 0$ and $v \in [0,1]$ in this paper.

It is well known that the famous Young's inequality reads

$$a^{1-v}b^v \leq (1-v)a + vb. \tag{1}$$

Furuichi [1] improved (1) with Specht's ratio

$$S\left(\left(\frac{b}{a}\right)^r\right)a^{1-v}b^v \leq (1-v)a + vb, \tag{2}$$

where $r = \min\{v, 1-v\}$. Zuo et al. [2] further improved (2) as

$$S\left(\left(\frac{b}{a}\right)^r\right)a^{1-v}b^v \leq K^r\left(\frac{b}{a}\right)a^{1-v}b^v \leq (1-v)a + vb. \tag{3}$$

Sababheh and Moslehian [3] gave a refinement of (3) in the following form

$$K\left(\sqrt[2^N]{h}, 2\right)^{\beta_N(v)}a^{1-v}b^v + S_N(v;b,a) \leq (1-v)a + vb, \tag{4}$$

check for updates

Citation: Ren, Y. Generalized Refinements of Reversed AM-GM Operator Inequalities for Positive Linear Maps. *Axioms* **2023**, *12*, 977. https://doi.org/10.3390/axioms12100977

Academic Editor: Christophe Chesneau

Received: 9 September 2023
Revised: 5 October 2023
Accepted: 9 October 2023
Published: 17 October 2023

where $N \in \mathbb{N}^+$, $h = \frac{a}{b}$, $\beta_N(v) = \min\{\alpha_N(v), 1 - \alpha_N(v)\}$ with $\alpha_N(v) = 1 + [2^N v] - 2^N v$ and

$$S_N(v; b, a) = \sum_{j=1}^{N} s_j(v) \left(\sqrt[2^j]{a^{2^{j-1}-k_j(v)} b^{k_j(v)}} - \sqrt[2^j]{b^{k_j(v)+1} a^{2^{j-1}-k_j(v)-1}} \right)^2 \tag{5}$$

for $j = 1, 2, \cdots, N$, $k_j(v) = [2^{j-1}v]$, $r_j(v) = [2^j v]$ and $s_j(v) = (-1)^{r_j(v)} 2^{j-1} v + (-1)^{r_j(v)+1} [\frac{r_j(v)+1}{2}]$.

Taking $v = \frac{1}{2}$ in (1), we can get the following AM-GM operator inequality for any two positive operators A and B,

$$A \sharp B \leq A \nabla B. \tag{6}$$

Lin [4] gave a reverse of inequality (6) involving unital positive linear maps Φ :

$$\Phi(A \nabla B) \leq K(h, 2) \Phi(A \sharp B) \tag{7}$$

for $0 < mI \leq A, B \leq MI$ and $h = \frac{M}{m}$. In generally, for any two positive operators A, B and $P > 1$,

$$A \geq B \not\Rightarrow A^P \geq B^P. \tag{8}$$

For example, putting $P = 2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. However, Lin [4] showed that the inequality (7) can be squared under the same conditions as in it,

$$\Phi^2(A \nabla B) \leq K^2(h, 2) \Phi^2(A \sharp B) \tag{9}$$

and

$$\Phi^2(A \nabla B) \leq K^2(h, 2) \big(\Phi(A) \sharp \Phi(B)\big)^2. \tag{10}$$

Moreover, the author [4] found that Specht's ratio and Kantorovich constant have the following relations

$$S(h) \leq K(h, 2) \leq S^2(h) \tag{11}$$

for $h > 1$. Also, Lin [4] conjectured $K(h)$ can be replaced by $S(h)$ in (9) and (10). Xue [5] solved Lin's conjecture under some conditions: $0 < mI \leq A, B \leq MI$, $\sqrt{\frac{M}{m}} \leq 2.314$ and $h = \frac{M}{m}$, she got

$$\Phi^2(A \nabla B) \leq K(h, 2) \Phi^2(A \sharp B) \tag{12}$$

and

$$\Phi^2(A \nabla B) \leq K(h, 2) (\Phi(A) \sharp \Phi(B))^2. \tag{13}$$

Recently, Ali et al. [6] gave some refinements of inequalities (12) and (13) as follows:

Theorem 1. *Let $0 < m \leq M$, $\sqrt{\frac{M}{m}} \leq 2.314$. For every positive unital linear map Φ,*

(1) if $0 < mI \leq A, B \leq \frac{M+m}{2} I$, then

$$\Phi^2\left(A \nabla B + \frac{M+m}{2} m \left(\frac{A^{-1} + B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^2 \Phi^2(A \sharp_v B);$$

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}m\left(\frac{A^{-1}+B^{-1}}{2}-(A^{-1}\sharp B^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2(\Phi(A)\natural_v\Phi(B))^2;$$

(2) *if* $0<\frac{M+m}{2}I\leq A,B\leq MI$, *then*

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}M\left(\frac{A^{-1}+B^{-1}}{2}-(A^{-1}\sharp B^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2\Phi^2(A\natural_v B);$$

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}M\left(\frac{A^{-1}+B^{-1}}{2}-(A^{-1}\sharp B^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2(\Phi(A)\natural_v\Phi(B))^2;$$

(3) *if* $0<mI\leq A<\frac{M+m}{2}I\leq B\leq MI$, *then*

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}\left(\frac{mA^{-1}+MB^{-1}}{2}-(mA^{-1}\sharp MB^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2\Phi^2(A\natural_v B);$$

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}\left(\frac{mA^{-1}+MB^{-1}}{2}-(mA^{-1}\sharp MB^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2(\Phi(A)\natural_v\Phi(B))^2;$$

(4) *if* $0<mI\leq B\leq\frac{M+m}{2}I\leq A\leq MI$, *then*

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}\left(\frac{MA^{-1}+mB^{-1}}{2}-(MA^{-1}\sharp mB^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2\Phi^2(A\natural_v B);$$

$$\Phi^2\left(A\nabla B+\frac{M+m}{2}\left(\frac{MA^{-1}+mB^{-1}}{2}-(MA^{-1}\sharp mB^{-1})\right)\right)\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^2(\Phi(A)\natural_v\Phi(B))^2;$$

For more information about operator inequalities involving positive linear maps, we refer the readers to [7–11] and references therein.

In this paper, we shall give some more generalized and further refinements of reversed AM-GM operator inequalities for positive linear maps due to Ali's results.

2. Main Results

We give some lemmas to prove our main results.

Lemma 1. *Let* $0<mI\leq A\leq m'I<M'I\leq B\leq MI$, $h=\frac{M}{m}$, $h'=\frac{M'}{m'}$. *Then we have*

$$K(\sqrt[2^N]{h'},2)^{\beta_N(v)}A\natural_v B+\sum_{j=1}^{N}s_j(v)\left(A\natural_{\alpha_j(v)}B+A\natural_{\alpha_j(v)+2^{1-j}}B-2A\natural_{\alpha_j(v)+2^{-j}}B\right)\leq A\nabla_v B, \tag{14}$$

where $\alpha_j(v)=\frac{k_j(v)}{2^{j-1}}$, $k_j(v)=[2^{j-1}v]$ *and* $\beta_N(v)=\min\{1+[2^Nv]-2^Nv,2^Nv-[2^Nv]\}$, $r_j(v)=[2^jv]$ *and* $s_j(v)=(-1)^{r_j(v)}2^{j-1}v+(-1)^{r_j(v)+1}\left[\frac{r_j(v)+1}{2}\right]$.

Proof. Putting $a=1$ in (4), we obtain

$$K\left(\sqrt[2^N]{\frac{1}{b}},2\right)^{\beta_N(v)}b^v+\sum_{j=1}^{N}s_j(v)\left(b^{\alpha_j(v)}+b^{\alpha_j(v)+2^{1-j}}-2b^{\alpha_j(v)+2^{-j}}\right)\leq(1-v)+vb.$$

Taking $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, and then $Sp(X) \subseteq [h', h] \subseteq (1, +\infty)$. By standard functional calculus and the Kantorovich constant $K(\frac{1}{t}, 2)$ is a decreasing function on $\frac{1}{t} \in (0,1)$, we get

$$K\left(\sqrt[2^N]{\frac{1}{h'}}, 2\right)^{\beta_N(v)} X^v + \sum_{j=1}^{N} s_j(v)\left(X^{\alpha_j(v)} + X^{\alpha_j(v)+2^{1-j}} - 2X^{\alpha_j(v)+2^{-j}}\right) \leq (1-v) + vX. \quad (15)$$

Multiplying $A^{\frac{1}{2}}$ on both sides of inequality (15), with the fact $K(h,2) = K(\frac{1}{h}, 2)$, we can get (14) directly. □

Lemma 2. *Under the same conditions as in Lemma 1, we have*

$$A\nabla_v B + MmK\left(\sqrt[2^N]{h'}, 2\right)^{\beta_N(v)}(A\sharp_v B)^{-1} + MmS_N(v; B^{-1}, A^{-1}) \leq M + m, \quad (16)$$

where $S_N(v; B^{-1}, A^{-1}) = \sum_{j=1}^{N} s_j(v)\left(A^{-1}\sharp_{\alpha_j(v)}B^{-1} + A^{-1}\sharp_{\alpha_j(v)+2^{1-j}}B^{-1} - 2A^{-1}\sharp_{\alpha_j(v)+2^{-j}}B^{-1}\right)$.

Proof. If $0 < mI \leq A, B \leq MI$, then

$$(M-A)(A-m)A^{-1} \geq 0 \quad \text{and} \quad (M-B)(B-m)B^{-1} \geq 0,$$

that is

$$A + MmA^{-1} \leq M + m \quad \text{and} \quad B + MmB^{-1} \leq M + m. \quad (17)$$

So we have

$$A\nabla_v B + Mm(A^{-1}\nabla_v B^{-1}) \leq M + m. \quad (18)$$

Thus, we obtain

$$\begin{aligned}
&A\nabla_v B + MmS_N(v; B^{-1}, A^{-1}) + MmK\left(\sqrt[2^N]{h'}, 2\right)^{\beta_N(v)}(A\sharp_v B)^{-1} \\
&= A\nabla_v B + Mm\left(S_N(v; B^{-1}, A^{-1}) + K\left(\sqrt[2^N]{h'}, 2\right)^{\beta_N(v)}(A^{-1}\sharp_v B^{-1})\right) \\
&\leq A\nabla_v B + Mm\left(A^{-1}\nabla_v B^{-1}\right) \quad \text{(by (14))} \\
&\leq M + m. \quad \text{(by (18))}
\end{aligned}$$

□

Lemma 3 ([12])**.** *Let $A, B \geq 0$. Then the following norm inequality holds*

$$\|AB\| \leq \frac{1}{4}\|A + B\|^2. \quad (19)$$

Lemma 4 ([13])**.** *Let Φ be a unital positive linear map and $A > 0$. Then*

$$\Phi(A)^{-1} \leq \Phi(A^{-1}). \quad (20)$$

Lemma 5 ([14])**.** *(i) If $0 \leq P \leq 1$ and $A \geq B \geq 0$, then*

$$A^P \geq B^P. \quad (21)$$

(ii) Let Φ be a unital positive linear map and $A, B > 0$. For $v \in [0,1]$, we have

$$\Phi(A\sharp_v B) \leq \Phi(A)\sharp_v \Phi(B). \quad (22)$$

Lemma 6 ([15])**.** *Let $A, B \geq 0$. Then for $1 \leq P < +\infty$,*

$$\|A^P + B^P\| \leq \|(A+B)^P\|. \tag{23}$$

Theorem 2. *Let $0 < mI \leq MI$, $\sqrt{\frac{M}{m}} \leq 2.314$ and $S_N(v; B^{-1}, A^{-1})$ defined as in Lemma 2. Then for every positive unital linear map Φ and $P \in [0,2]$,*

(1) if $0 < mI \leq A \leq m_1' I < M_1' I \leq B \leq \frac{M+m}{2} I$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} m\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq \frac{K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_1'}, 2\big)} \Phi^P(A\sharp_v B); \tag{24}$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} m\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq \frac{K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_1'}, 2\big)} \big(\Phi(A)\sharp_v \Phi(B)\big)^P; \tag{25}$$

where $h = \frac{M}{m}$, $h_1' = \frac{M_1'}{m_1'}$ and $v \in [0,1]$.

(2) if $0 < \frac{M+m}{2} I \leq A \leq m_2' I < M_2' I \leq B \leq MI$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} M\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq \frac{K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_2'}, 2\big)} \Phi^P(A\sharp_v B); \tag{26}$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} M\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq \frac{K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_2'}, 2\big)} \big(\Phi(A)\sharp_v \Phi(B)\big)^P; \tag{27}$$

where $h = \frac{M}{m}$, $h_2' = \frac{M_2'}{m_2'}$ and $v \in [0,1]$.

(3) if $0 < mI \leq A \leq m_3' I < \frac{M+m}{2} I \leq B \leq MI$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} \big(S_N(v; MB^{-1}, mA^{-1})\big)\right) \leq \frac{\left(\frac{M}{m}\right)^{\frac{P}{2}(1-2v)} K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_3'}, 2\big)} \Phi^P(A\sharp_v B); \tag{28}$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} \big(S_N(v; MB^{-1}, mA^{-1})\big)\right) \leq \frac{\left(\frac{M}{m}\right)^{\frac{P}{2}(1-2v)} K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_3'}, 2\big)} \big(\Phi(A)\sharp_v \Phi(B)\big)^P; \tag{29}$$

where $h = \frac{M}{m}$, $h_3' = \frac{M+m}{2m_3'}$ and $v \in [0, \frac{1}{2}]$.

(4) if $0 < mI \leq B \leq \frac{M+m}{2} I < M_4' I \leq A \leq MI$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} \big(S_N(v; mB^{-1}, MA^{-1})\big)\right) \leq \frac{\left(\frac{M}{m}\right)^{\frac{P}{2}(2v-1)} K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_4'}, 2\big)} \Phi^P(A\sharp_v B); \tag{30}$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2} \big(S_N(v; mB^{-1}, MA^{-1})\big)\right) \leq \frac{\left(\frac{M}{m}\right)^{\frac{P}{2}(2v-1)} K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_4'}, 2\big)} \big(\Phi(A)\sharp_v \Phi(B)\big)^P; \tag{31}$$

where $h = \frac{M}{m}$, $h_4' = \frac{2M_4'}{M+m}$ and $v \in [\frac{1}{2}, 1]$.

Proof. If $0 < mI \leq A \leq m_1' I < M_1' I \leq B \leq \frac{M+m}{2} I$, we obtain

$$\begin{aligned}
&\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) \times \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} \Phi^{-1}(A\sharp_v B)\right|\right| \\
&\leq \frac{1}{4}\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) + \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} \Phi^{-1}(A\sharp_v B)\right|\right|^2 \\
&\leq \frac{1}{4}\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) + \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} \Phi\left((A\sharp_v B)^{-1}\right)\right|\right|^2 \\
&= \frac{1}{4}\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} (A\sharp_v B)^{-1} + \frac{M+m}{2} mS_N(v; B^{-1}, A^{-1})\right)\right|\right|^2 \\
&\leq \frac{1}{4}(\frac{M+m}{2} + m)^2,
\end{aligned} \tag{32}$$

where the first inequality is by (19), the second one is by (20), and the last inequality comes from (16). That is

$$\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right)\Phi^{-1}(A\sharp_v B)\right|\right| \leq \frac{(\frac{M+m}{2} + m)^2}{4\frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)}}.$$

Since $1 \leq \sqrt{\frac{M}{m}} \leq 2.314$, it follows that $\left(\sqrt{\frac{M}{m}} - 1\right)^2 \left[(\sqrt{\frac{M}{m}})^3 - \frac{2M}{m} + \sqrt{\frac{M}{m}} - 4\right] \leq 0$, which is equivalent to

$$\frac{\left(\frac{M+m}{2} + m\right)^2}{4\frac{M+m}{2} m} \leq \frac{M+m}{2\sqrt{Mm}}. \tag{33}$$

So we have

$$\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right)\Phi^{-1}(A\sharp_v B)\right|\right| \leq \frac{M+m}{2\sqrt{Mm}K(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)}}.$$

That is

$$\Phi^2\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) \leq \left(\frac{K(h, 2)}{K(\sqrt[2^N]{h_1'}, 2)^{2\beta_N(v)}}\right)\Phi^2(A\sharp_v B). \tag{34}$$

In addition, we can get

$$\begin{aligned}
&\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) \times \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} \left(\Phi(A)\sharp_v \Phi(B)\right)^{-1}\right|\right| \\
&\leq \frac{1}{4}\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) + \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} \left(\Phi(A)\sharp_v \Phi(B)\right)^{-1}\right|\right|^2 \\
&\leq \frac{1}{4}\left|\left|\Phi\left(A\nabla_v B + \frac{M+m}{2} m\left(S_N(v; B^{-1}, A^{-1})\right)\right) + \frac{M+m}{2} mK(\sqrt[2^N]{h_1'}, 2)^{\beta_N(v)} \Phi^{-1}(A\sharp_v B)\right|\right|^2 \\
&\leq \frac{1}{4}(\frac{M+m}{2} + m)^2,
\end{aligned}$$

where the first inequality is by (19), the second is by (22), and the third is by (32). That is

$$\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}m\left(S_N(v;B^{-1},A^{-1})\right)\right)\left(\Phi(A)\sharp_v\Phi(B)\right)^{-1}\right\|$$
$$\le\frac{\left(\frac{M+m}{2}+m\right)^2}{4\frac{M+m}{2}mK(\sqrt[2^N]{h_1'},2)^{\beta_N(v)}}\le\frac{K^{\frac{1}{2}}(h,2)}{K(\sqrt[2^N]{h_1'},2)^{\beta_N(v)}}.$$

So we have

$$\Phi^2\left(A\nabla_v B+\frac{M+m}{2}m\left(S_N(v;B^{-1},A^{-1})\right)\right)\le\left(\frac{K(h,2)}{K(\sqrt[2^N]{h_1'},2)^{2\beta_N(v)}}\right)\left(\Phi(A)\sharp_v\Phi(B)\right)^2. \tag{35}$$

We can get (24) and (25) by (34) and (35) with Lemma 5 (i), respectively.

Since $\frac{\left(\frac{M+m}{2}+M\right)^2}{4\frac{M+m}{2}M}\le\frac{\left(\frac{M+m}{2}+m\right)^2}{4\frac{M+m}{2}m}$, by 2nd case $0<\frac{M+m}{2}I\le A\le m_2'I<M_2'I\le B\le MI$, we can similarly obtain the inequalities (26) and (27) by (16), (17), (19) and (20). So we omit the details.

If $0<mI\le A\le m_3'I<\frac{M+m}{2}I\le B\le MI$ and $v\in[0,\frac{1}{2}]$, then we have

$$A+\frac{M+m}{2}mA^{-1}\le\frac{M+m}{2}+m\quad\text{and}\quad B+\frac{M+m}{2}MB^{-1}\le\frac{M+m}{2}+M.$$

So

$$A\nabla_v B+\frac{M+m}{2}\left((mA^{-1})\nabla_v(MB^{-1})\right)\le(v+\frac{1}{2})M+(\frac{3}{2}-v)m\le M+m. \tag{36}$$

Compute

$$\begin{aligned}&\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}\left(S_N(v;MB^{-1},mA^{-1})\right)\right)\times\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\Phi^{-1}(A\sharp_v B)\right\|\\
&\le\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}\left(S_N(v;MB^{-1},mA^{-1})\right)\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\Phi^{-1}(A\sharp_v B)\right\|^2\\
&\le\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}\left(S_N(v;MB^{-1},mA^{-1})\right)\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\Phi\left((A\sharp_v B)^{-1}\right)\right\|^2\\
&=\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}\left(S_N(v;MB^{-1},mA^{-1})+K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}\left((mA^{-1})\sharp_v(MB^{-1})\right)\right)\right)\right\|^2\\
&\le\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}\left((mA^{-1})\nabla_v(MB^{-1})\right)\right)\right\|^2\\
&\le\frac{(M+m)^2}{4}.\end{aligned} \tag{37}$$

where the first inequality is by (19), the second is by (20), the third is by (14), and the last one is by (36). That is

$$\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}\left(S_N(v;MB^{-1},mA^{-1})\right)\right)\Phi^{-1}(A\sharp_v B)\right\|$$
$$\le\frac{(M+m)^2}{4\frac{M+m}{2}m^{1-v}M^vK(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}}=\frac{\left(\frac{M}{m}\right)^{\frac{1}{2}-v}K^{\frac{1}{2}}(h,2)}{K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}}.$$

That is

$$\Phi^2\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\leq\frac{(\frac{M}{m})^{1-2v}K(h,2)}{K(\sqrt[2^N]{h_3'},2)^{2\beta_N(v)}}\Phi^2(A\sharp_v B).\tag{38}$$

Moreover,

$$\begin{aligned}&\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\times\frac{M+m}{2}m^{1-v}M^vK(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}(\Phi(A)\sharp_v\Phi(B))^{-1}\right\|\\&\leq\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}m^{1-v}M^vK(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}(\Phi(A)\sharp_v\Phi(B))^{-1}\right\|^2\\&\leq\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}m^{1-v}M^vK(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}\Phi^{-1}(A\sharp_v B)\right\|^2\\&\leq\frac{(M+m)^2}{4},\end{aligned}$$

where the first inequality is by (19), the second is by (22), and the third is by (37). That is

$$\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)(\Phi(A)\sharp_v\Phi(B))^{-1}\right\|\leq\frac{(\frac{M}{m})^{\frac{1}{2}-v}K^{\frac{1}{2}}(h,2)}{K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}},$$

so we have

$$\Phi^2\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\leq\frac{(\frac{M}{m})^{1-2v}K(h,2)}{K(\sqrt[2^N]{h_3'},2)^{2\beta_N(v)}}(\Phi(A)\sharp_v\Phi(B))^2.\tag{39}$$

We can get (28) and (29) by (38) and (39) with Lemma 5 (i), respectively.

We can similarly obtain the inequalities (30) and (31) under the conditions $0<mI\leq B\leq\frac{M+m}{2}I<M_3'I\leq A\leq MI$ and $v\in[\frac{1}{2},1]$. So we omit the details.

Here we complete the proof. □

Remark 1. *Putting $v=\frac{1}{2}$, $N=1$ and $P=2$ in Theorem 2, we can get Theorem 1.*

Next, we present the generalizations of Theorem 2 for $P\geq 2$.

Theorem 3. *Let $0<mI\leq MI$, $\sqrt{\frac{M}{m}}\leq 2.314$ and $S_N(v;B^{-1},A^{-1})$ defined as in Lemma 2. Then for every positive unital linear map Φ and $P\geq 2$,*

(i) if $0<mI\leq A\leq m_1'I<M_1'I\leq B\leq\frac{M+m}{2}I$, then

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)\leq\frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}(\sqrt[2^N]{h_1'},2)}\Phi^P(A\sharp_v B);\tag{40}$$

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)\leq\frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}(\sqrt[2^N]{h_1'},2)}(\Phi(A)\sharp_v\Phi(B))^P;\tag{41}$$

where $h=\frac{M}{m}$, $h_1'=\frac{M_1'}{m_1'}$ and $v\in[0,1]$.

(ii) if $0<\frac{M+m}{2}I\leq A\leq m_2'I<M_2'I\leq B\leq MI$, then

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}M\big(S_N(v;B^{-1},A^{-1})\big)\right)\leq\frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_2'},2\big)}\Phi^P(A\sharp_v B);\tag{42}$$

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}M\big(S_N(v;B^{-1},A^{-1})\big)\right)\leq\frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_2'},2\big)}\big(\Phi(A)\sharp_v\Phi(B)\big)^P;\tag{43}$$

where $h=\frac{M}{m}$, $h_2'=\frac{M_2'}{m_2'}$ *and* $v\in[0,1]$.

(iii) *if* $0<mI\leq A\leq m_3'I<\frac{M+m}{2}I\leq B\leq MI$, *then*

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}\big(S_N(v;MB^{-1},mA^{-1})\big)\right)\leq\frac{4^{P-2}\big(\frac{M}{m}\big)^{\frac{P}{2}(1-2v)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_3'},2\big)}\Phi^P(A\sharp_v B);\tag{44}$$

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}\big(S_N(v;MB^{-1},mA^{-1})\big)\right)\leq\frac{4^{P-2}\big(\frac{M}{m}\big)^{\frac{P}{2}(1-2v)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_3'},2\big)}\big(\Phi(A)\sharp_v\Phi(B)\big)^P;\tag{45}$$

where $h=\frac{M}{m}$, $h_3'=\frac{M+m}{2m_3'}$ *and* $v\in[0,\frac{1}{2}]$.

(iv) *if* $0<mI\leq B\leq\frac{M+m}{2}I<M_4'I\leq A\leq MI$, *then*

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}\big(S_N(v;mB^{-1},MA^{-1})\big)\right)\leq\frac{4^{P-2}\big(\frac{M}{m}\big)^{\frac{P}{2}(2v-1)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_4'},2\big)}\Phi^P(A\sharp_v B);\tag{46}$$

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}\big(S_N(v;mB^{-1},MA^{-1})\big)\right)\leq\frac{4^{P-2}\big(\frac{M}{m}\big)^{\frac{P}{2}(2v-1)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\big(\sqrt[2^N]{h_4'},2\big)}\big(\Phi(A)\sharp_v\Phi(B)\big)^P;\tag{47}$$

where $h=\frac{M}{m}$, $h_4'=\frac{2M_4'}{M+m}$ *and* $v\in[\frac{1}{2},1]$.

Proof. The proof of the line (ii) and (iv) are similar to the one presented in (i) and (iii), respectively, thus we omit them. Under the conditions i) $0<mI\leq A\leq m_1'I<M_1'I\leq B\leq\frac{M+m}{2}I$, we have

$$\begin{aligned}
&\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}m\big(S_N(v;B^{-1},A^{-1})\big)\right)\times\big(\frac{M+m}{2}\big)^{\frac{P}{2}}m^{\frac{P}{2}}K^{\frac{P}{2}\beta_N(v)}\big(\sqrt[2^N]{h_1'},2\big)\Phi^{-\frac{P}{2}}(A\sharp_v B)\right\|\\
&\leq\frac{1}{4}\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}m\big(S_N(v;B^{-1},A^{-1})\big)\right)+\big(\frac{M+m}{2}\big)^{\frac{P}{2}}m^{\frac{P}{2}}K^{\frac{P}{2}\beta_N(v)}\big(\sqrt[2^N]{h_1'},2\big)\Phi^{-\frac{P}{2}}(A\sharp_v B)\right\|^2\\
&\leq\frac{1}{4}\left\|\left(\Phi\left(A\nabla_v B+\frac{M+m}{2}m\big(S_N(v;B^{-1},A^{-1})\big)\right)+\frac{M+m}{2}mK\big(\sqrt[2^N]{h_1'},2\big)^{\beta_N(v)}\Phi^{-1}(A\sharp_v B)\right)^{\frac{P}{2}}\right\|^2\\
&=\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}m\big(S_N(v;B^{-1},A^{-1})\big)\right)+\frac{M+m}{2}mK\big(\sqrt[2^N]{h_1'},2\big)^{\beta_N(v)}\Phi^{-1}(A\sharp_v B)\right\|^P\\
&\leq\frac{1}{4}\big(\frac{M+m}{2}+m\big)^P.
\end{aligned}$$

where the first inequality is by (19), the second is by (23) and the third is by (32). That is

$$\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)\Phi^{-\frac{P}{2}}(A\sharp_v B)\right\|$$
$$\leq \frac{(\frac{M+m}{2}+m)^P}{4(\frac{M+m}{2})^{\frac{P}{2}}m^{\frac{P}{2}}K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_1'},2)}\leq \frac{4^{\frac{P}{2}-1}(\frac{M+m}{2\sqrt{Mm}})^{\frac{P}{2}}}{K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_1'},2)}=\frac{4^{\frac{P}{2}-1}K^{\frac{P}{4}}(h,2)}{K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_1'},2)},$$

where the second inequality is by (33). So we have

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)\leq \frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}(\sqrt[2^N]{h_1'},2)}\Phi^P(A\sharp_v B).$$

In addition, we can get

$$\begin{aligned}
&\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)\times(\frac{M+m}{2})^{\frac{P}{2}}m^{\frac{P}{2}}K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_1'},2)(\Phi(A)\sharp_v\Phi(B))^{-\frac{P}{2}}\right\|\\
&\leq\frac{1}{4}\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)+(\frac{M+m}{2})^{\frac{P}{2}}m^{\frac{P}{2}}K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_1'},2)(\Phi(A)\sharp_v\Phi(B))^{-\frac{P}{2}}\right\|^2\\
&\leq\frac{1}{4}\left\|\left(\Phi(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)+\frac{M+m}{2}mK(\sqrt[2^N]{h_1'},2)^{\beta_N(v)}((\Phi(A)\sharp_v\Phi(B))^{-1})^{\frac{P}{2}}\right\|^2\\
&=\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)+\frac{M+m}{2}mK(\sqrt[2^N]{h_1'},2)^{\beta_N(v)}(\Phi(A)\sharp_v\Phi(B))^{-1}\right\|^P\\
&\leq\frac{1}{4}\left\|\Phi(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)+\frac{M+m}{2}mK(\sqrt[2^N]{h_1'},2)^{\beta_N(v)}\Phi^{-1}(A\sharp_v B)\right\|^P\\
&\leq\frac{1}{4}(\frac{M+m}{2}+m)^P.
\end{aligned}$$

where the first inequality is by (19), the second is by (23), the third is by (22) and the last is by (32). That is

$$\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)(\Phi(A)\sharp_v\Phi(B))^{-\frac{P}{2}}\right\|\leq\frac{4^{\frac{P}{2}-1}K^{\frac{P}{4}}(h,2)}{K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_1'},2)},$$

so we have

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}m(S_N(v;B^{-1},A^{-1}))\right)\leq\frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}(\sqrt[2^N]{h_1'},2)}(\Phi(A)\sharp_v\Phi(B))^P,$$

as desired.

If $0<mI\leq A\leq m_3'I<\frac{M+m}{2}I\leq B\leq MI$ and $v\in[0,\frac{1}{2}]$, then

$$\begin{aligned}
&\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\times\left[\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\right]^{\frac{P}{2}}\Phi^{-\frac{P}{2}}(A\sharp_v B)\right\|\\
&\le\frac{1}{4}\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\left[\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\right]^{\frac{P}{2}}\Phi^{-\frac{P}{2}}(A\sharp_v B)\right\|^2\\
&\le\frac{1}{4}\left\|\left(\Phi(A\nabla_v B+\frac{M+m}{2}\left(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\Phi^{-1}(A\sharp_v B)\right)^{\frac{P}{2}}\right\|^2\\
&=\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\Phi^{-1}(A\sharp_v B)\right\|^P\\
&\le\frac{1}{4}(M+m)^P.
\end{aligned}$$

where the first inequality is by (19), the second is by (23) and the third is by (37). That is

$$\begin{aligned}
&\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\Phi^{-\frac{P}{2}}(A\sharp_v B)\right\|\\
&\le\frac{(M+m)^P}{4\left[\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\right]^{\frac{P}{2}}}=\frac{2^{P-2}\left(\frac{M}{m}\right)^{\frac{P}{4}(1-2v)}K^{\frac{P}{4}}(h,2)}{K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_3'},2)}.
\end{aligned}$$

So we have

$$\Phi^P\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\le\frac{4^{P-2}\left(\frac{M}{m}\right)^{\frac{P}{2}(1-2v)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}(\sqrt[2^N]{h_3'},2)}\Phi^P(A\sharp_v B).$$

At the same time, we can get

$$\begin{aligned}
&\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)\times\left[\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\right]^{\frac{P}{2}}(\Phi(A)\sharp_v\Phi(B))^{-\frac{P}{2}}\right\|\\
&\le\frac{1}{4}\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\left[\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\right]^{\frac{P}{2}}(\Phi(A)\sharp_v\Phi(B))^{-\frac{P}{2}}\right\|^2\\
&\le\frac{1}{4}\left\|\left(\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v(\Phi(A)\sharp_v\Phi(B))^{-1}\right)^{\frac{P}{2}}\right\|^2\\
&=\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v(\Phi(A)\sharp_v\Phi(B))^{-1}\right\|^P\\
&\le\frac{1}{4}\left\|\Phi\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)+\frac{M+m}{2}K(\sqrt[2^N]{h_3'},2)^{\beta_N(v)}m^{1-v}M^v\Phi^{-1}(A\sharp_v B)\right\|^P\\
&\le\frac{1}{4}(M+m)^P.
\end{aligned}$$

where the first inequality is by (19), the second is by (23), the third is by (22) and the last is by (37). That is

$$\begin{aligned}
&\left\|\Phi^{\frac{P}{2}}\left(A\nabla_v B+\frac{M+m}{2}(S_N(v;MB^{-1},mA^{-1}))\right)(\Phi(A)\sharp_v\Phi(B))^{-\frac{P}{2}}\right\|\\
&\le\frac{(M+m)^P}{4\left(\frac{M+m}{2}m^{1-v}M^v\right)^{\frac{P}{2}}K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_3'},2)}=\frac{2^{P-2}\left(\frac{M}{m}\right)^{\frac{P}{4}(1-2v)}K^{\frac{P}{4}}(h,2)}{K^{\frac{P}{2}\beta_N(v)}(\sqrt[2^N]{h_3'},2)}.
\end{aligned}$$

So we have

$$\Phi^P(A\nabla_v B + \frac{M+m}{2}(S_N(v; MB^{-1}, mA^{-1}))) \leq \frac{4^{P-2}(\frac{M}{m})^{\frac{P}{2}(1-2v)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}(\sqrt[2^N]{h'_3},2)}(\Phi(A)\sharp_v\Phi(B))^P.$$

Here we complete the proof. $\square$

Theorems 2 and 3 implies the following results.

Corollary 1. *Let $0 < m \leq M$, $\sqrt{\frac{M}{m}} \leq 2.314$ and $S_N(v; B^{-1}, A^{-1})$ defined as in Lemma 2. Then for every positive unital linear map Φ and $P \geq 0$,*

(i) if $0 < mI \leq A \leq m'_1 I < M'_1 I \leq B \leq \frac{M+m}{2}I$, $v \in [0,1]$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}m\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq W_1\Phi^P(A\sharp_v B);$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}m\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq W_1\big(\Phi(A)\sharp_v\Phi(B)\big)^P;$$

where $h = \frac{M}{m}$, $h'_1 = \frac{M'_1}{m'_1}$, $W_1 = \max\left\{\frac{K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(\sqrt[2^N]{h'_1},2\right)}, \frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(\sqrt[2^N]{h'_1},2\right)}\right\}$.

(ii) if $0 < \frac{M+m}{2}I \leq A \leq m'_2 I < M'_2 I \leq B \leq MI$, $v \in [0,1]$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}M\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq W_2\Phi^P(A\sharp_v B);$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}M\big(S_N(v; B^{-1}, A^{-1})\big)\right) \leq W_2\big(\Phi(A)\sharp_v\Phi(B)\big)^P;$$

where $h = \frac{M}{m}$, $h'_2 = \frac{M'_2}{m'_2}$, $W_2 = \max\left\{\frac{K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(\sqrt[2^N]{h'_2},2\right)}, \frac{4^{P-2}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(\sqrt[2^N]{h'_1},2\right)}\right\}$.

(iii) if $0 < mI \leq A \leq m'_3 I < \frac{M+m}{2}I \leq B \leq MI$, $v \in [0, \frac{1}{2}]$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}\big(S_N(v; MB^{-1}, mA^{-1})\big)\right) \leq W_3\Phi^P(A\sharp_v B);$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}\big(S_N(v; MB^{-1}, mA^{-1})\big)\right) \leq W_3\big(\Phi(A)\sharp_v\Phi(B)\big)^P;$$

where $h = \frac{M}{m}$, $h'_3 = \frac{M+m}{2m'_3}$, $W_3 = \max\left\{\frac{\left(\frac{M}{m}\right)^{\frac{P}{2}(1-2v)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(\sqrt[2^N]{h'_3},2\right)}, \frac{4^{P-2}\left(\frac{M}{m}\right)^{\frac{P}{2}(1-2v)}K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(\sqrt[2^N]{h'_3},2\right)}\right\}$.

(iv) if $0 < mI \leq B \leq \frac{M+m}{2}I < M'_4 I \leq A \leq MI$, $v \in [\frac{1}{2}, 1]$, then

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}\big(S_N(v; mB^{-1}, MA^{-1})\big)\right) \leq W_4\Phi^P(A\sharp_v B);$$

$$\Phi^P\left(A\nabla_v B + \frac{M+m}{2}\big(S_N(v; mB^{-1}, MA^{-1})\big)\right) \leq W_4\big(\Phi(A)\sharp_v\Phi(B)\big)^P;$$

where $h = \frac{M}{m}$, $h'_4 = \frac{2M'_4}{M+m}$, $W_4 = \max\left\{ \frac{\left(\frac{M}{m}\right)^{\frac{P}{2}(2v-1)} K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(2^N\sqrt{h'_4},2\right)}, \frac{4^{P-2}\left(\frac{M}{m}\right)^{\frac{P}{2}(2v-1)} K^{\frac{P}{2}}(h,2)}{K^{P\beta_N(v)}\left(2^N\sqrt{h'_4},2\right)} \right\}$.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Furuichi, S. Refined Young inequalities with Specht's ratio. *J. Egypt. Math. Soc.* **2012**, *20*, 46–49. [CrossRef]
2. Zuo, H.L.; Shi, G.H.; Fujii, M. Refined Young inequality with Kantorovich constant. *J. Math. Inequal.* **2011**, *5*, 551–556. [CrossRef]
3. Sababheh, M.; Moslehian, M.S. Advanced refinements of Young and Heinz inequalities. *J. Number Theory* **2017**, *172*, 178–199. [CrossRef]
4. Lin, M.H. Squaring a reverse AM-GM inequality. *Studia Math.* **2013**, *215*, 187–194. [CrossRef]
5. Xue, J.M. Some refinements of operator reverse AM-GM inequality. *J. Inequal. Appl.* **2017**, *2017*, 283. [CrossRef] [PubMed]
6. Ali, I.; Shakoor, A.; Rehman, A. More refinements of the operator reverse AM-GM inequality for positive linear maps. *J. Math. Inequal.* **2019**, *13*, 287–300. [CrossRef]
7. Ighachane, M.A.; Akkouchi, M. Further refinements of Young's type inequality for positive linear maps. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **2021**, *115*, 19. [CrossRef]
8. Karim, S.; Ali, I.; Mushtaq, M. Improvements of operator reverse AM-GM inequality involving positive linear maps. *J. Inequal. Appl.* **2020**, *73*, 12. [CrossRef]
9. Karim, S.; Ali, I.; Mushtaq, M. New developments on operator reverse AM-GM inequality for positive linear maps. *Acta Math. Sin.* **2022**, *38*, 419–430. [CrossRef]
10. Srivastava, H.M.; Chandel, R.C.S.; Kumar, H. Some general Hurwitz-Lerch type Zeta functions associated with the Srivastava-Daoust multiple hypergeometric functions. *J. Nonlinear Var. Anal.* **2022**, *6*, 299–315.
11. Yang, C.S.; Li, Y. On the further refinement of some operator inequalities for positive linear map. *J. Math. Inequal.* **2021**, *15*, 781–797. [CrossRef]
12. Bhatia, R.; Kittaneh, F. Notes on matrix arithmetic-geometric mean inequalities. *Linear Algebra Appl.* **2000**, *308*, 203–211. [CrossRef]
13. Bhatia, R. *Positive Definite Matrices*; Princeton University Press: Princeton, NJ, USA, 2007.
14. Pečarić, J.; Furuta, T.; Hot, J.; Seo, Y. *Mond-Pečarić Method in Operator Inequalities*; Element: Zagreb, Croatia, 2005.
15. Ando, T.; Zhan, X. Norm inequalities related to operator monotone functions. *Math. Ann.* **1999**, *315*, 771–780. [CrossRef]