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Ramsey Chains in Linear Forests

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Abstract: Every red–blue coloring of the edges of a graph G results in a sequence $G_1, G_2, \ldots, G_\ell$ of pairwise edge-disjoint monochromatic subgraphs G_i $(1 \le i \le \ell)$ of size i, such that G_i is isomorphic to a subgraph of G_{i+1} for $1 \le i \le \ell - 1$. Such a sequence is called a Ramsey chain in G, and $AR_c(G)$ is the maximum length of a Ramsey chain in G, with respect to a red–blue coloring c. The Ramsey index $AR(G)$ of G is the minimum value of $AR_c(G)$ among all the red–blue colorings c of G. If G has size m, then $\binom{k+1}{2} \le m < \binom{k+2}{2}$ for some positive integer k. It has been shown that there are infinite classes S of graphs, such that for every graph G of size m in S, $AR(G) = k$ if and only if $\binom{k+1}{2} \le m < \binom{k+2}{2}$. Two of these classes are the matchings mK_2 and paths P_{m+1} of size m. These are both subclasses of linear forests (a forest of which each of the components is a path). It is shown that if F is any linear forest of size m with $\binom{k+1}{2} < m < \binom{k+2}{2}$, then $AR(F) = k$. Furthermore, if F is a linear forest of size $\binom{k+1}{2}$, where $k \ge 4$, that has at most $\binom{k-1}{2}$ components, then $AR(F) = k$, while for each integer t with $\binom{k-1}{2} < t < \binom{k+1}{2}$ there is a linear forest F of size $\binom{k+1}{2}$ with t components, such that $AR(F) = k - 1$.

Keywords: red–blue edge coloring; Ramsey chain; Ramsey index; linear forest

MSC: 05C05; 05C15; 05C35; 05C55; 05C70

1. Introduction

For every graph G of size m, there is a unique positive integer k, such that $\binom{k+1}{2} \le m < \binom{k+2}{2}$. The graph G is said to have an *ascending subgraph decomposition* $\{G_1, G_2, \ldots, G_k\}$ into k (pairwise edge-disjoint) subgraphs of G if G_i is isomorphic to a proper subgraph of G_{i+1} for $i = 1, 2, \ldots, k - 1$. The following conjecture was stated in [1].

The Ascending Subgraph Decomposition Conjecture *Every graph has an ascending subgraph decomposition.*

This conjecture has drawn the attention of many researchers but has never been proved or disproved. There are many papers dealing with this conjecture (see [2–9], for example). Among the several classes of graphs for which the conjecture has been verified are regular graphs (see [3,10]). Two classes of graphs for which the conjecture can easily be verified are matchings mK_2 (consisting of m components K_2) and stars $K_{1,m}$: that is, for every positive integer m, where $\binom{k+1}{2} \le m < \binom{k+2}{2}$, there is an ascending subgraph decomposition $\{G_1, G_2, \ldots, G_k\}$ of the graph G if either $G = mK_2$ or $G = K_{1,m}$, such that G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le k - 1$. If $G = mK_2$, the subgraphs G_i are matchings and if $G = K_{1,m}$, each subgraph G_i is a star.

By a *red–blue edge coloring* (or simply a *red–blue coloring*) of a graph G, every edge of G is colored red or blue. Such an edge coloring is also referred to as a 2-*edge coloring*. In [11], the concept of ascending subgraph decomposition was extended to graphs possessing a red–blue coloring. Suppose that a red–blue coloring of a graph $G = mK_2$ or $G = K_{1,m}$ is given, where $\binom{k+1}{2} \le m < \binom{k+2}{2}$. It was shown in [12] that there is not only an ascending subgraph decomposition $\{G_1, G_2, \ldots, G_k\}$ of G but one in which each subgraph

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is monochromatic as well. This (perhaps unexpected) observation led to another concept, which is related to some topics in Ramsey Theory, named for the British mathematician Frank Ramsey [13]. Ramsey theory is one of the most studied areas in combinatorics and graph theory, with many highly nontrivial and beautiful results (see [14–22], for example). We refer to the books [11,23] for basic definitions and notation in graph theory that are not defined here.

Let G be a graph (without isolated vertices) of size m with a red–blue coloring c. A subgraph H of G is *monochromatic* if all edges of H are colored the same. A *Ramsey chain* of G *with respect to* c is a sequence $G_1, G_2, \ldots, G_\ell$ of pairwise edge-disjoint subgraphs of G, such that each subgraph G_i $(1 \le i \le \ell)$ is monochromatic of size i and G_i is isomorphic to a subgraph of G_{i+1} for $1 \le i \le \ell - 1$. Each subgraph G_i $(1 \le i \le \ell)$ in a Ramsey chain is called a *link* of the chain. The maximum length of a Ramsey chain of G with respect to c is the (ascending) *Ramsey index* $AR_c(G)$ of G. The *Ramsey index* $AR(G)$ of G is defined by $AR(G) = \min\{AR_c(G) : c \text{ is a red–blue edge coloring of } G\}$. Consequently, if $AR(G) = k$ for some graph G, then for every red–blue coloring of G, there is a Ramsey chain of length k in G, while there exists at least one red–blue coloring for which there is no Ramsey chain of length greater than k. These concepts were introduced and studied in [11,12], using somewhat different technology, and they were studied further in [24,25]. An immediate observation on Ramsey indexes of graphs was presented in [12].

Observation 1 ([12])**.** *If G is a graph of size m where $2 \le m < \binom{k+2}{2}$ for a positive integer k, then $AR(G) \le k$.*

The result obtained on matchings and stars can therefore be stated as follows:

Theorem 1 ([12])**.** *If $G \in \{mK_2, K_{1,m}\}$, where $m \ge 3$, then*

$$AR(G) = k \text{ if and only if } \binom{k+1}{2} \le m < \binom{k+2}{2}.$$

In [24], the question was posed as to whether there are other infinite classes S of graphs, such that for every sufficiently large integer m and each graph G of size m in S, it follows that $AR(G) = k$ if and only if $\binom{k+1}{2} \le m < \binom{k+2}{2}$. In [24,25], this concept was studied for cycles and paths. As the emphasis here is on the size of a graph, let C_m denote a cycle of size m and Q_m a path of size m: that is, Q_m is a path of order $m + 1$.

Theorem 2 ([12,25])**.** *If $G \in \{C_m, Q_m\}$, where $m \ge 3$, then*

$$AR(G) = k \text{ if and only if } \binom{k+1}{2} \le m < \binom{k+2}{2}.$$

A *linear forest* is a forest of which every component is a path. Here, we are only concerned with linear forests without isolated vertices. Paths and matchings are linear forests, namely, the linear forests with the minimum and maximum number of components. The goal here is to determine whether Theorems 1 and 2 can be extended to include linear forests distinct from paths and matchings.

2. Ramsey Indexes of Linear Forests

We saw in Theorems 1 and 2 that if $G = mK_2$ or $G = Q_m$ for a positive integer m, then $AR(G) = k$ if and only if $\binom{k+1}{2} \le m < \binom{k+2}{2}$. As mK_2 and Q_m are both linear forests, this raises the question whether the same result holds for all linear forests of size m. This question can be answered immediately. The linear forest $F = Q_2 + (m-2)K_2$ consisting of $m - 1$ components where $m = \binom{k+1}{2}$ with $k \ge 3$ has Ramsey index $k - 1$, not k. The red–blue coloring of F that assigns red to both edges of Q_2 and blue to all other edges does not result in a Ramsey chain of length k, since a Ramsey chain $F_1, F_2, \ldots, F_k$ of length k would require that $F_2 = Q_2$ and $F_3 = 3K_2$; however, $F_2 \not\subseteq F_3$. On the other hand, $\binom{k}{2}K_2 \subseteq F$ and $AR(\binom{k}{2}K_2) = k - 1$ by Theorem 1; therefore, $AR(F) = k - 1$ by Observation 2.

Each of the linear forests $Q_2 + 8K_2$, $Q_3 + 7K_2$, $Q_4 + 6K_2$, $2Q_3 + 4K_2$, $Q_4 + Q_3 + 3K_2$, $Q_7 + 3K_2$ has size $10 = \binom{k+1}{2}$, where $k = 4$. In each of these linear forests, a red–blue coloring is given in Figure 1 that shows that its Ramsey index is $3 = k - 1$. In Figure 1, a bold edge indicates a red edge and a thin edge indicates a blue edge.

Each of the six linear forests in Figure 1 has t components for $t = 4, 5, \ldots, 9$. The examples in Figure 1 suggest that determining $AR(F)$ for a linear forest F may depend not only on its size but the number of components of F as well. First, we present a result that gives the Ramsey index of linear forests of size m, where $m \neq \binom{k+1}{2}$ for any positive integer k. Prior to doing this, we state some useful information from three results presented in [24,25].

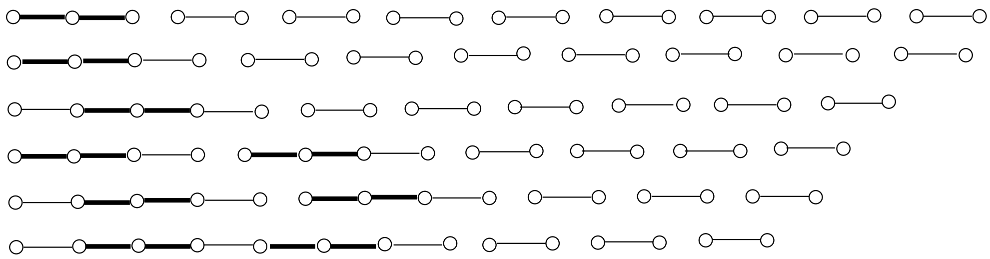

Figure 1. Linear forests of size 10 with Ramsey index 3.

Observation 2 ([24]). *If H and G are graphs, such that $H \subseteq G$, then $AR(H) \leq AR(G)$. Consequently, if $AR(H) \geq k$ for each graph H of size m, then $AR(G) \geq k$ for every graph G of size $m + 1$.*

Theorem 3 ([25]). *Let $n \geq 5$ be an integer. For every set $\{n_1, n_2, \ldots, n_t\}$ of t integers, such that $1 \leq n_1 < n_2 < \cdots < n_t \leq \lceil n/2 \rceil$ and $\sum_{i=1}^{t} n_i = n$, every linear forest of size n can be decomposed into the matchings n_1K_2, n_2K_2, …, n_tK_2.*

Proposition 1 ([25]). *Let $m = \binom{k+1}{2}$ for some integer $k \geq 5$. For every two positive integers m_1 and m_2 with $m = m_1 + m_2$ and $m_1, m_2 \notin \{2, 4\}$, there exists a partition of $[k] = \{1, 2, \ldots, k\}$ into two subsets $A = \{a_1, a_2, \ldots, a_{k_1}\}$ and $B = \{b_1, b_2, \ldots, b_{k_2}\}$, where $k_1 + k_2 = k$, $a_1 < a_2 < \cdots < a_{k_1} \leq \lceil \frac{m_1}{2} \rceil$, and $b_1 < b_2 < \cdots < b_{k_2} \leq \lceil \frac{m_2}{2} \rceil$, such that $\sum_{i=1}^{k_1} a_i = m_1$ and $\sum_{i=1}^{k_2} b_i = m_2$.*

We now apply Observations 1 and 2, Theorem 3, and Proposition 1, to establish the following result.

Theorem 4. *If H is a linear forest of size m, where $\binom{k+1}{2} < m < \binom{k+2}{2}$ for some integer $k \geq 3$, then $AR(H) = k$.*

Proof. As $AR(H) \leq k$ by Observation 1, it remains only to verify that $AR(H) \geq k$: that is, to verify that H has a Ramsey chain of length k for every 2-edge coloring of H. It suffices to assume that $m = \binom{k+1}{2} + 1$ by Observation 2. Let c be a 2-edge coloring of H using the colors 1 and 2. We show that there is a Ramsey chain of length k in H. For $i = 1, 2$, let H_i be the linear forest in H induced by the edges of H colored i. Let H_i have size m_i where $1 \leq m_1 \leq m_2$. Therefore, $m_1 + m_2 = m = \binom{k+1}{2} + 1$. We now consider five cases, according to whether $m_1 \in \{1, 2, 3, 4\}$ or $m_1 \geq 5$.

Case 1: $m_1 = 1$; thus, $m_2 = \binom{k+1}{2}$. As $k \geq 3$, it follows that $k \leq \left\lceil \frac{\binom{k+1}{2}}{2} \right\rceil$. Because $1 + 2 + \cdots + k = \binom{k+1}{2}$, it follows by Theorem 3 that H_2 can be decomposed into $K_2, 2K_2, 3K_2, \cdots, kK_2$, which is a Ramsey chain of length k in H.

Case 2: $m_1 = 2$; thus, $m_2 = \binom{k+1}{2} - 1$. As $k \geq 3$, it follows that $k \leq \left\lceil \frac{\binom{k+1}{2} - 1}{2} \right\rceil$. Because $2 + 3 + \cdots + k = \binom{k+1}{2} - 1$, it follows by Theorem 3 that H_2 can be decomposed into $2K_2, 3K_2, \cdots, kK_2$. As $K_2 \subseteq H_1$, there is a Ramsey chain of length k in H.

Case 3: $m_1 = 3$; thus, $m_2 = \binom{k+1}{2} - 2$. Suppose first that $k = 3$; then, $m_2 = 4$. The linear forest H_1 can be decomposed into K_2 and $2K_2$, while the linear forest H_2 contains either $3K_2$ or $Q_2 + K_2$. In either case, H contains a Ramsey chain of length $k = 3$. Hence, we may assume that $k \geq 4$. Let F be a linear forest of size $m' = \binom{k+1}{2} - 3$ in H_2. As $k \geq 4$, it follows that $k \leq \left\lceil \frac{\binom{k+1}{2}-3}{2} \right\rceil$. Because $3 + 4 + \cdots + k = \binom{k+1}{2} - 3$, it follows by Theorem 3 that F can be decomposed into $3K_2, 4K_4, \ldots, kK_2$. As H_1 can be decomposed into K_2 and $2K_2$, there is a Ramsey chain of length k in H.

Case 4: $m_1 = 4$; thus, $m_2 = \binom{k+1}{2} - 3 \geq 7$, and so, $k \geq 4$. Therefore, $k \leq \left\lceil \frac{\binom{k+1}{2}-3}{2} \right\rceil$. As $3 + 4 + \cdots + k = \binom{k+1}{2} - 3$, it follows by Theorem 3 that H_2 can be decomposed into $3K_2, 4K_4, \ldots, kK_2$. Let F be a linear forest of size 3 in H_1. As F can be decomposed into K_2 and $2K_2$, there is a Ramsey chain of length k in H.

Case 5: $m_1 \geq 5$; thus, $m_2 = \binom{k+1}{2} + 1 - m_1$ and $k \geq 5$. Let F be a linear forest of size $m_2' = m_2 - 1 \geq 5$ in H_2. Consequently, $m_1 \notin \{2, 4\}$, $m_2' \notin \{2, 4\}$, and $m_1 + m_2' = \binom{k+1}{2}$. By Proposition 1, there exists a partition of $[k] = \{1, 2, \ldots, k\}$ into two subsets $A = \{a_1, a_2, \ldots, a_{k_1}\}$ and $B = \{b_1, b_2, \ldots, b_{k_2}\}$, where $k_1 + k_2 = k$, $a_1 < a_2 < \cdots < a_{k_1} \leq \left\lceil \frac{m_1}{2} \right\rceil$ and $b_1 < b_2 < \cdots < b_{k_2} \leq \left\lceil \frac{m_2'}{2} \right\rceil$, such that $\sum_{i=1}^{k_1} a_i = m_1$ and $\sum_{i=1}^{k_2} b_i = m_2'$. Hence, H_1 can be decomposed into $a_1K_2, a_2K_2, \ldots a_{k_1}K_2$ and F can be decomposed into $b_1K_2, b_2K_2, \ldots b_{k_2}K_2$, resulting in a Ramsey chain $K_2, 2K_2, \ldots, kK_2$ of length k in H. □

We illustrate the proof of Theorem 4 for the linear forest $Q_5 + Q_6$ of size $m = 11$ consisting of paths of sizes 5 and 6. Then, $k = 4$ and $\binom{4+1}{2} < 11 < 15 = \binom{4+2}{2}$. Five red–blue colorings of $Q_5 + Q_6$ are given in Figure 2, to reflect the five cases for $m_1 = 1, 2, 3, 4, 5$ in the proof, where a bold edge indicates a red edge and a thin edge indicates a blue edge. For $i = 1, 2, 3, 4$, an edge labeled i belongs to the link G_i in a Ramsey chain $R : G_1, G_2, G_3, G_4$ of length 4 in $Q_5 + Q_6$, and the unlabeled edge is not used in R.

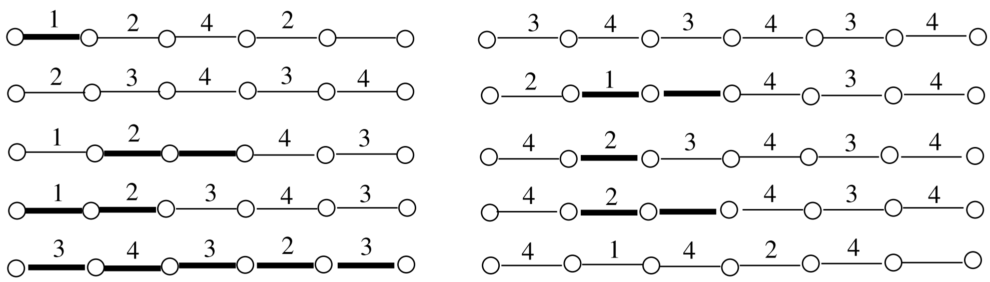

Figure 2. Red–blue colorings of $Q_5 + Q_6$ of size 11.

3. Binomial Linear Forests

From the results obtained in Section 2, it follows that investigating $AR(F)$ for a linear forest F of size m, we need only be concerned when $m = \binom{k+1}{2}$ for some integer $k \geq 3$. Thus, it is convenient to introduce terminology for this class of linear forests. A linear forest is *k-binomial* (or simply *binomial*) if its size is $\binom{k+1}{2}$ for some positive integer k. For example, a 3-binomial linear forest has size 6, a 4-binomial linear forest has size 10, and a 5-binomial linear forest has size 15. We begin with an observation that is a consequence of Theorem 4 and Observation 2.

Corollary 1. *If H is a k-binomial linear forest where $k \geq 3$, then $AR(H) \in \{k-1, k\}$. Furthermore, both values $k-1$ and k are attainable.*

Proof. First, $AR(H) \leq k$ by Observation 1. By Theorem 4, every linear forest of size $\binom{k+1}{2} - 1$ has Ramsey index $k-1$. Hence, $AR(H) \geq k-1$ by Observation 2. Therefore, $AR(H) \in \{k-1, k\}$. As $AR(Q_2 + (\binom{k+1}{2} - 2)K_2) = k-1$ and $AR(Q_{\binom{k+1}{2}}) = k$, both values $k-1$ and k are attainable. □

By Theorem 4 and Corollary 1, if H is a 3-binomial linear forest, then $AR(H) \in \{2,3\}$. We now determine the exact value of the Ramsey index of every 3-binomial linear forest. All 3-binomial linear forests are listed below:

$$Q_6,\, Q_5+K_2,\, Q_4+Q_2,\, Q_4+2K_2,\, 2Q_3,\, Q_3+Q_2+K_2,$$
$$Q_3+3K_2,\, 3Q_2,\, 2Q_2+2K_2,\, Q_2+4K_2,\, 6K_2.$$

Proposition 2. *Let H be a 3-binomial linear forest. Then, $AR(H)=2$ if and only if*

$$H \in \{Q_4+2K_2, Q_3+3K_2, Q_2+4K_2\}.$$

Proof. Let $X=\{Q_4+2K_2, Q_3+3K_2, Q_2+4K_2\}$. First, observe that the linear forests H in X are the only 3-binomial linear forests containing a subgraph $F=Q_2$, such that $H-E(F)=4K_2$ and, consequently, contains no copies of Q_2. Let c be a 2-edge coloring of H that assigns the color 1 to the two edges of F and the color 2 to all other edges of H. We claim that there is no Ramsey chain of length 3 in H. Assume, to the contrary, that there is a Ramsey chain G_1, G_2, G_3 of length 3 in H. As the size of H is 6, it follows that $\{G_1, G_2, G_3\}$ is a decomposition of H. Necessarily, $G_2=F=Q_2$, and so, $Q_2 \subseteq G_3$. As $H-E(F)=4K_2$, this is impossible. Therefore, $AR(H)=2$ by Corollary 1.

For the converse, suppose that $H \notin X$. We show that $AR(H)=3$. As $AR(H) \le 3$ by Observation 1, it suffices to show that $AR(H) \ge 3$. Let c be a 2-edge coloring of H using the colors 1 and 2, where m_i is the number of the edges in the subgraph H_i of H colored i for $i=1,2$ and $m_1 \le m_2$. Thus, $1 \le m_1 \le 3$ and $m_1+m_2=6$. We consider all possible pairs (m_1, m_2):

- ⋆ If $(m_1,m_2)=(1,5)$, then $G_1=H_1=K_2$ and H_2 is a linear forest of size 5. As $2+3=5$, it follows by Theorem 3 that H_2 can be decomposed into $2K_2, 3K_2$. Thus, $K_2, 2K_2, 3K_2$ is a Ramsey chain of length 3 in H.
- ⋆ If $(m_1,m_2)=(2,4)$, then $G_2=H_1 \in \{Q_2, 2K_2\}$. First, suppose that $H_1=2K_2$. Then, H_2 is a linear forest of size 4, and so, H_2 can be decomposed into K_2 and a subgraph G_3 of size 3. As every graph of size 3 contains $2K_2$, it follows that $K_2, H_1=2K_2, G_3$ is a Ramsey chain of length 3 in H. Next, suppose that $H_1=Q_2$. As $H \notin X$, it follows that $H_2 \ne 4K_2$. Thus, $1 \le k(H_2) \le 3$. If $k(H_2)=1$, then $H_2=Q_4$ can be decomposed into K_2 and Q_3. Thus, K_2, Q_2, Q_3 is a Ramsey chain of length 3 in H. If $k(H_2)=2$, then $H_2=2Q_2$ or $H_2=Q_3+K_2$. As H_2 can be decomposed into K_2 and Q_2+K_2, it follows that K_2, Q_2, Q_2+K_2 is a Ramsey chain of length 3 in H. If $k(H_2)=3$, then $H_2=Q_2+2K_2$. As H_2 can be decomposed into K_2 and Q_2+K_2, it follows that K_2, Q_2, Q_2+K_2 is a Ramsey chain of length 3 in H.
- ⋆ If $(m_1,m_2)=(3,3)$, then H_1 can be decomposed into K_2 and $2K_2$ and H_2 contains $2K_2$. Thus, $K_2, 2K_2, H_2$ is a Ramsey chain of length 3 in H.

□

We now consider k-binomial linear forests for integers $k \ge 4$. The following two results provide useful information on 2-edge colorings of k-binomial linear forests for $k \ge 4$.

Proposition 3. *For a k-binomial linear forest H where $k \ge 4$, let c be a 2-edge coloring of H using the colors 1 and 2. For $i=1,2$, let H_i be the subgraph of H induced by the edges of H colored i, where H_i has size m_i and $1 \le m_1 \le m_2$. If $H_1 \notin \{Q_2, 2Q_2, Q_4\}$, then $K_2, 2K_2, \ldots kK_2$ is a Ramsey chain of length k in H and $AR_c(H)=k$.*

Proof. We consider five cases, depending on whether $m_1 \in \{1,2,3,4\}$ or $m_1 \ge 5$.

Case 1: $m_1=1$. Let $H_1=K_2$. As $k \le \left\lfloor \frac{\binom{k+1}{2}-1}{2} \right\rfloor$ for $k \ge 4$ and $2+3+\cdots+k=\binom{k+1}{2}-1$, it follows by Theorem 3 that H_2 can be decomposed into $2K_2, 3K_2, \ldots, kK_2$. Thus, $K_2, 2K_2, \ldots kK_2$ is a Ramsey chain of length k in H and $AR_c(H)=k$.

Case 2: $m_1=2$. Thus, $H_1 \in \{2K_2, Q_2\}$ and $m_2=\binom{k+1}{2}-2$. As $H_1 \ne Q_2$, it follows that $H_1=2K_2$. Because H_2 is a linear forest of size $\binom{k+1}{2}-2$ and $1+3+4\cdots+k=\binom{k+1}{2}-2$

and $k \le \left\lfloor \frac{\binom{k+1}{2}-2}{2} \right\rfloor$ when $k \ge 4$, it follows by Theorem 3 that H_2 can be decomposed into $K_2, 3K_2, 4K_2, \cdots, kK_2$; thus, $K_2, 2K_2, 3K_2, \ldots, kK_2$ is a Ramsey chain of length k in H and $AR_c(H) = k$.

Case 3: $m_1 = 3$. The subgraph H_1 can be decomposed into K_2 and $2K_2$. As $k \le \left\lfloor \frac{\binom{k+1}{2}-3}{2} \right\rfloor$ for $k \ge 4$ and $3 + \cdots + k = \binom{k+1}{2} - 3$, it follows by Theorem 3 that H_2 can be decomposed into $3K_2, 4K_2, \ldots, kK_2$. Thus, $K_2, 2K_2, \ldots kK_2$ is a Ramsey chain of length k in H and $AR_c(H) = k$.

Case 4: $m_1 = 4$. Then, $H_1 \in \{Q_4, Q_3 + K_2, 2Q_2, Q_2 + 2K_2, 4K_2\}$ and $m_2 = \binom{k+1}{2} - 4$. As $H_1 \notin \{2Q_2, Q_4\}$, it follows that $H_1 \in \{Q_2 + 2K_2, Q_3 + K_2, 4K_2\}$:

- ⋆ Let $k = 4$. Then, H_2 is a linear forest of size 6. As $1 + 2 + 3 = 6$, it follows by Theorem 3 that H_2 can be decomposed into $K_2, 2K_2, 3K_2$. Thus, $K_2, 2K_2, 3K_2, H_1$ is a Ramsey chain of length 4 in H.
- ⋆ Let $k \ge 5$. Then, H_1 can be decomposed into K_2 and $3K_2$. As $2 + 4 + 5 + \cdots + k = \binom{k+1}{2} - 4$ and $k \le \left\lfloor \frac{\binom{k+1}{2}-4}{2} \right\rfloor$ when $k \ge 5$, it follows by Theorem 3 that H_2 can be decomposed into $2K_2, 4K_2, 5K_2, \cdots, kK_2$. Thus, $K_2, 2K_2, \ldots, kK_2$ is a Ramsey chain of length k in H and $AR_c(H) = k$.

Case 5: $m_1 \ge 5$. By Proposition 1, there exists a partition of $[k] = \{1, 2, \ldots, k\}$ into two sets $A = \{a_1, a_2, \ldots, a_{k_1}\}$ and $B = \{b_1, b_2, \ldots, b_{k_2}\}$, where $k_1 + k_2 = k$, $a_1 < a_2 < \cdots < a_{k_1} \le \lceil \frac{m_1}{2} \rceil$ and $b_1 < b_2 < \cdots < b_{k_2} \le \lceil \frac{m_2}{2} \rceil$, such that $\sum_{i=1}^{k_1} a_i = m_1$ and $\sum_{i=1}^{k_2} b_i = m_2$. As $m_2 \ge m_1 \ge 5$, it follows by Theorem 3 that H_1 can be decomposed into the matchings a_1K_2, a_2K_2, …, $a_{k_1}K_2$ and H_2 can be decomposed into the matchings b_1K_2, b_2K_2, …, $b_{k_2}K_2$. Consequently, $K_2, 2K_2, \ldots kK_2$ is a Ramsey chain of length k in H and $AR_c(H) = k$. □

Proposition 4. *For a k-binomial linear forest H where $k \ge 4$, let c be a 2-edge coloring of H using the colors 1 and 2. For $i = 1, 2$, let H_i be the subgraph of H induced by the edges of H colored i, where H_i has size m_i and $1 \le m_1 \le m_2$. If*

(a) $H_1 = Q_2$ and H_2 has at least $k - 2$ pairwise edge-disjoint copies of Q_2 or
(b) $H_1 \in \{Q_4, 2Q_2\}$ and H_2 has at least $k - 3$ pairwise edge-disjoint copies of Q_2,

then Q_1, $G_2 \in \{Q_2, 2Q_1\}$, $Q_2 + Q_1$, $Q_2 + 2Q_1$, …, $Q_2 + (k-2)Q_1$ is a Ramsey chain of length k in H.

Proof. First, suppose that $H_1 = Q_2$ and H_2 has at least $k - 2$ pairwise edge-disjoint copies of Q_2, which are denoted by A_3, A_4, …, A_k. Then, the number of edges of H_2 not belonging to any A_i ($3 \le i \le k$) is

$$\binom{k+1}{2} - 2 - 2(k-2) = \binom{k+1}{2} - 2k + 2 = \binom{k-1}{2} + 1,$$

and so, these $\binom{k-1}{2} + 1$ edges of H_2 can be decomposed into

$$B_1 = Q_1, B_3 = Q_1, B_4 = 2Q_1, B_5 = 3Q_1, \ldots, B_k = (k-2)Q_1,$$

in such a way that $A_i + B_i = Q_2 + (i-2)Q_1$ for $3 \le i \le k$. Consequently, Q_1, Q_2, $Q_2 + Q_1$, $Q_2 + 2Q_1$, …, $Q_2 + (k-2)Q_1$ is a Ramsey chain of length k in H.

Next, suppose that $H_1 \in \{Q_4, 2Q_2\}$. Then, H_1 can be decomposed into $G_1 = Q_1$ and $G_3 = Q_2 + Q_1$. The subgraph H_2 has at least $k - 3$ pairwise edge-disjoint copies of Q_2, which are denoted by A_4, A_5, …, A_k. Then, the number of edges of H_2 not belonging to any A_i ($4 \le i \le k$) is

$$\binom{k+1}{2} - 4 - 2(k-3) = \binom{k+1}{2} - 2k + 2 = \binom{k-1}{2} + 1,$$

and so, these $\binom{k-1}{2} + 1$ edges of H_2 can be decomposed into

$$B_2 = 2Q_1, B_4 = 2Q_1, B_5 = 3Q_1, \ldots, B_k = (k-2)Q_1,$$

in such a way that $A_i + B_i = Q_2 + (i-2)Q_1$ for $4 \le i \le k$. Consequently, $Q_1, 2Q_1, Q_2 + Q_1, Q_2 + 2Q_1, \ldots, Q_2 + (k-2)Q_1$ is a Ramsey chain of length k in H. $\square$

With the aid of Observation 1 and Propositions 3 and 4, we are able to establish the following result.

Theorem 5. *If H is a k-binomial linear forest, where $k \ge 4$ with at most $\binom{k-1}{2}$ components, then $AR(H) = k$.*

Proof. As $AR(H) \le k$ by Observation 1, it remains to show that $AR(H) \ge k$. Thus, we show that every 2-edge coloring of H produces a Ramsey chain of length k in H. Let c be a 2-edge coloring of H using the colors 1 and 2. For $i = 1, 2$, let H_i be the linear forest in H induced by the edges of H colored i, where H_i has size m_i with $1 \le m_1 \le m_2$ and $k(H_i)$ components. By Proposition 3, if $H_1 \notin \{Q_2, 2Q_2, Q_4\}$, then there is a Ramsey chain of length k in H. Thus, we may assume that $H_1 \in \{Q_2, 2Q_2, Q_4\}$. We consider these two cases, depending on whether $H_1 = Q_2$ or $H_1 \in \{Q_4, 2Q_2\}$.

Case 1: $H_1 = Q_2$. Then, H_2 is a linear forest of size $\binom{k+1}{2} - 2$. Let $k(H_2) = \ell$. Then, $\ell \le k(H) + 1 \le \binom{k-1}{2} + 1$. Let $J_1, J_2, \ldots, J_\ell$ be the components of H_2, where J_i has size q_i for $1 \le i \le \ell$. Thus, $\sum_{i=1}^{\ell} q_i = m_2 = \binom{k+1}{2} - 2$. Let p be the maximum number of pairwise edge-disjoint copies of Q_2 in H_2. In each component J_i ($1 \le i \le \ell$), the maximum number of pairwise edge-disjoint copies of Q_2 is $\lfloor q_i/2 \rfloor$, and so, at most one edge of J_i does not belong to these $\lfloor q_i/2 \rfloor$ pairwise edge-disjoint copies of Q_2 in J_i. Hence, at most one edge in each J_i ($1 \le i \le \ell$) does not belong to any of these p pairwise edge-disjoint copies of Q_2 in H_2. As $\ell \le \binom{k-1}{2} + 1$, it follows that

$$p \ge \frac{1}{2}\left[\binom{k+1}{2} - 2 - \ell\right] \ge \frac{1}{2}\left[\binom{k+1}{2} - 2 - \binom{k-1}{2} - 1\right] = k - 2.$$

Therefore, there are at least $k-2$ pairwise edge-disjoint copies of Q_2 in H_2. By Proposition 4, there is a Ramsey chain of length k in H.

Case 2: $H_1 \in \{Q_4, 2Q_2\}$. Then, H_1 can be decomposed into $G_1 = Q_1$ and $G_3 = Q_2 + Q_1$. Here, H_2 is a linear forest of size $\binom{k+1}{2} - 4$. Let $k(H_2) = \ell$. Then, $\ell \le k(H) + 2 \le \binom{k-1}{2} + 2$. Let $J_1, J_2, \ldots, J_\ell$ be the components of H_2, where J_i has size q_i for $1 \le i \le \ell$. Thus, $\sum_{i=1}^{\ell} q_i = m_2 = \binom{k+1}{2} - 4$. Let ℓ' be the number of these components having odd size. If J_i has even size, then let $J_i' = J_i$. If J_i has odd size, then let J_i' be the subgraph of J_i obtained by removing a pendant edge from J_i, where J_i' is empty if $q_i = 1$. Hence, every subgraph J_i' has even size q_i' for $1 \le i \le \ell$ and every nonempty linear forest J_i' can be decomposed into $\frac{q_i'}{2}$ copies of Q_2. The size of the linear forest H_2' consisting of $J_1', J_2', \cdots, J_\ell'$ is, therefore, $\sum_{i=1}^{\ell} q_i' = \binom{k+1}{2} - 4 - \ell'$, which is an even number. As $\ell \le k(H) + 2 \le \binom{k-1}{2} + 2$, it follows that

$$\binom{k+1}{2} - 4 - \ell' \ge \binom{k+1}{2} - 4 - \ell \ge \binom{k+1}{2} - 4 - \binom{k-1}{2} - 2 = 2k - 7.$$

As $\binom{k+1}{2} - 4 - \ell'$ is even, it follows that $\binom{k+1}{2} - 4 - \ell' \ge 2k - 6$. Therefore, the number of pairwise edge-disjoint copies of Q_2 in H_2' (and in H_2 as well) is at least $\frac{1}{2}(2k-6) = k-3$. By Proposition 4, there is a Ramsey chain of length k in H. $\square$

Next, we illustrate Theorem 5 for some 4-binomial forests of size 10, for which the number of components is 1, 2, or $3 = \binom{4-1}{2}$. Figure 3 shows red–blue colorings of the linear forests Q_{10}, $Q_3 + Q_7$, and $Q_2 + Q_3 + Q_5$ of size 10, where four bold edges are red edges and six thin edges are blue edges. For $i = 1, 2, 3, 4$, an edge labeled i belongs to the link G_i in a Ramsey chain G_1, G_2, G_3, G_4 of length 4 in the linear forest.

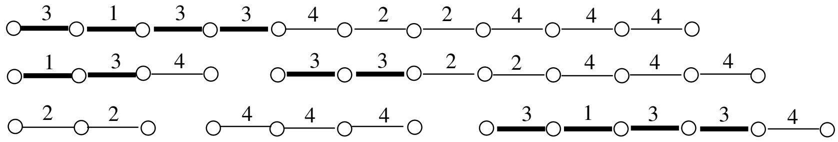

Figure 3. Red–blue colorings of three 4-binomial linear forests of size 10.

4. Binomial Linear Forests with an Intermediate Number of Components

By Theorems 1 and 5, every k-binomial linear forest where $k \geq 4$ with t components, such that either $t = \binom{k+1}{2}$ or $1 \leq t \leq \binom{k-1}{2}$, has Ramsey index k. A natural question concerns whether these bounds on t can be improved. As we will see, no such improvement is possible. First, we provide a necessary and sufficient condition for a k-binomial linear forest to have Ramsey index $k-1$.

Theorem 6. *A k-binomial linear forest H, where $k \geq 4$ has Ramsey index $k-1$ if and only if*

(*a*) *H contains a subgraph $F = Q_2$, such that $H - E(F)$ has at most $k-3$ pairwise edge-disjoint copies of Q_2 or*

(*b*) *H contains a subgraph $F \in \{Q_4, 2Q_2\}$, such that $H - E(F)$ has at most $k-4$ pairwise edge-disjoint copies of Q_2.*

Proof. First, we show that if H is a k-binomial linear forest where $k \geq 4$, such that neither (a) nor (b) holds, then $AR(H) = k$. As $AR(H) \leq k$ by Observation 1, it remains to show that $AR(H) \geq k$. Let c be a 2-edge coloring of H using the colors 1 and 2. For $i = 1, 2$, let H_i be the linear forest in H induced by the edges of H colored i. Let H_i have size m_i, where $1 \leq m_1 \leq m_2$. By Proposition 3, there is a Ramsey chain of length k in H if $H_1 \notin \{Q_2, 2Q_2, Q_4\}$. Thus, we may assume that $H_1 \in \{Q_2, 2Q_2, Q_4\}$:

- ⋆ If $H_1 = Q_2$, let $F = H_1 = Q_2$. As (a) does not occur, it follows that $H - E(F)$ has at least $k-2$ pairwise edge-disjoint copies of Q_2. Hence, H has a Ramsey chain of length k by Proposition 4.
- ⋆ If $H_1 \in \{2Q_2, Q_4\}$, let $F = H_1$. As (b) does not occur, it follows that $H - E(F)$ has at least $k-3$ pairwise edge-disjoint copies of Q_2. Hence, H has a Ramsey chain of length k by Proposition 4.

For the converse, suppose that H is a k-binomial linear forest where $k \geq 4$, such that either (a) or (b) occurs. We show that $AR(H) = k-1$. As $AR(H) \geq k-1$ by Corollary 4, it remains to show that $AR(H) \leq k-1$. We consider two cases, according to whether (a) or (b) occurs.

Case 1: (*a*) *occurs.* Let $F = Q_2$ be a subgraph of H, such that $H - E(F)$ has at most $k-3$ pairwise edge-disjoint copies of Q_2. Let c be a 2-edge coloring of H that assigns the color 1 to the two edges of F and the color 2 to all other edges of H. Then $H_1 = F = Q_2$ and $H_2 = H - E(F)$. We claim that there is no Ramsey chain of length k with respect to c. Assume, to the contrary, that there is a Ramsey chain $G_1, G_2, \ldots, G_k$ of length k in H. As the size of H is $\binom{k+1}{2}$, it follows that $\{G_1, G_2, \ldots, G_k\}$ is a decomposition of H. Necessarily, $G_2 = H_1 = F = Q_2$, and so, $Q_2 \subseteq G_i$ for $3 \leq i \leq k$, which implies that $H - E(F)$ contains at least $k-2$ pairwise edge-disjoint copies of Q_2, which contradicts the fact that $H - E(F)$ has at most $k-3$ pairwise edge-disjoint copies of Q_2. Thus, $AR(H) \leq k-1$, and so, $AR(H) = k-1$.

Case 2: (*b*) *occurs.* Let $F \in \{Q_4, 2Q_2\}$ be a subgraph of H, such that $H - E(F)$ has at most $k-4$ pairwise edge-disjoint copies of Q_2. Let c be a 2-edge coloring of H that assigns the color 1 to the four edges of F and the color 2 to all other edges of H. Then, $H_1 = F \in \{Q_4, 2Q_2\}$ and $H_2 = H - E(F)$. We claim that there is no Ramsey chain of length k, with respect to c. Assume, to the contrary, that there is a Ramsey chain $G_1, G_2, \ldots, G_k$ of length k in H. As $\{G_1, G_2, \ldots, G_k\}$ is a decomposition of H, it follows that either (i) H_1 is decomposed into $G_1 = Q_1$ and $G_3 \in \{Q_3, Q_2 + Q_1\}$ or (ii) $G_4 = H_2 = F$. If (i) occurs, then $Q_2 \subseteq G_i$ for $4 \leq i \leq k$, which implies that $H - E(F)$ contains at least

$k-3$ pairwise edge-disjoint copies of Q_2, which contradicts the fact that $H-E(F)$ has at most $k-4$ pairwise edge-disjoint copies of Q_2. If (ii) occurs, then G_3 contains a copy of Q_2 and G_i contains two edge-disjoint copies of Q_2 for $5 \le i \le k$, which implies that $H-E(F)$ contains at least $1+2(k-4) = 2k-7 > k-4$ pairwise edge-disjoint copies of Q_2, which contradicts the fact that $H-E(F)$ has at most $k-4$ pairwise edge-disjoint copies of Q_2. Thus, $AR(H) \le k-1$, and so, $AR(H) = k-1$. $\square$

We have now seen for a linear forest H of size m where $\binom{k+1}{2} \le m < \binom{k+2}{2}$ and $k \ge 4$ that by Theorems 1, 4, and 5, if (i) $\binom{k+1}{2} < m < \binom{k+2}{2}$ or (ii) $m = \binom{k+1}{2}$ and H has t components, where $t = \binom{k+1}{2}$ or $1 \le t \le \binom{k-1}{2}$, then $AR(H) = k$. Consequently, it remains only to consider those k-binomial linear forests with an intermediate number t of components, namely, $\binom{k-1}{2} < t < \binom{k+1}{2}$. Let c be any 2-edge coloring of H using the colors 1 and 2, resulting in the monochromatic subgraphs H_1 and H_2 of sizes m_1 and m_2, respectively, where $m_1 \le m_2$. By Proposition 3, if $H_1 \notin \{Q_2, 2Q_2, Q_4\}$, then $AR_c(H) = k$. By Proposition 4 and Theorem 6, if $H_1 \in \{Q_2, 2Q_2, Q_4\}$, then $AR_c(H) \in \{k, k-1\}$. Furthermore, by Theorem 6, if $H_1 = Q_2$, then $AR_c(H) = k$ only when H_2 contains at least $k-2$ pairwise edge-disjoint copies of Q_2; otherwise, $AR_c(H) = k-1$. Moreover, if $H_1 \in \{2Q_2, Q_4\}$, then $AR_c(H) = k$ only when H_2 has at least $k-3$ pairwise edge-disjoint copies of Q_2; otherwise, $AR_c(H) = k-1$. All of these suggest the need for only considering those 2-edge colorings of H for which $H_1 \in \{Q_2, 2Q_2, Q_4\}$ and determining whether there is (a) any 2-edge coloring of H where $H_1 = Q_2$, such that H_2 has fewer than $k-2$ pairwise edge-disjoint copies of Q_2 or (b) any 2-edge coloring of H where $H_1 \in \{2Q_2, Q_4\}$, such that H_2 has fewer than $k-3$ pairwise edge-disjoint copies of Q_2. If there is a 2-edge coloring of H resulting in (a) or (b), then $AR(H) = k-1$; otherwise, $AR(H) = k$. Therefore, to determine the Ramsey index of a k-binomial linear forest H where $k \ge 4$ with an intermediate number t of components with $\binom{k-1}{2} < t < \binom{k+1}{2}$, it suffices to study the 2-edge colorings of H, such that $H_1 \in \{Q_2, 2Q_2, Q_4\}$, such that $H_2 = H - E(H_1)$ possesses the minimum number of pairwise edge-disjoint copies of Q_2.

Each k-binomial linear forest H with t of components can be expressed as $Q_{q_1} + Q_{q_2} + \cdots + Q_{q_t}$, where $q_1 \ge q_2 \ge \cdots \ge q_t \ge 1$ and $\sum_{i=1}^{t} q_i = \binom{k+1}{2}$. The maximum number of pairwise edge-disjoint copies of Q_2 in H is $s = \sum_{i=1}^{t} \lfloor \frac{q_i}{2} \rfloor$. Now, let c be any 2-edge coloring of H using the colors 1 and 2 resulting in the monochromatic subgraphs and H_2 of sizes m_1 and m_2, respectively, where $m_1 \le m_2$ and $H_1 \in \{Q_2, 2Q_2, Q_4\}$. First, we make some observations. If $H_1 = Q_2$, then the maximum number of pairwise edge-disjoint copies of Q_2 in H_2 is either $s-1$ or $s-2$; while, if $H_1 \in \{Q_4, 2Q_2\}$, then the maximum number of pairwise edge-disjoint copies of Q_2 in H_2 is $s-2$, $s-3$, or $s-4$. We are now prepared to present the following result.

Theorem 7. *Let $H = Q_{q_1} + Q_{q_2} + \cdots + Q_{q_t}$ be a k-binomial linear forest of size $\sum_{i=1}^{t} q_i = \binom{k+1}{2}$ for some integer $k \ge 4$ with t components, where $\binom{k-1}{2} < t < \binom{k+1}{2}$, where $s = \sum_{i=1}^{t} \lfloor \frac{q_i}{2} \rfloor$ is the maximum number of pairwise edge-disjoint copies of Q_2 in H:*

(1) *If $s > k$, then $AR(H) = k$.*
(2) *If $s = k$ and H contains two components of even size 4 or more, then $AR(H) = k-1$; otherwise, $AR(H) = k$.*
(3) *If $s = k-1$ and H contains at least one component of even size 4 or more, then $AR(H) = k-1$; otherwise, $AR(H) = k$.*
(4) *If $s \le k-2$, then $AR(H) = k-1$.*

Proof. We first verify (1). Suppose that $s > k$. By Observation 1, it suffices to show that $AR_c(H) = k$ for every 2-edge coloring c of H. Let c be a 2-edge coloring of H using the colors 1 and 2. For $i = 1, 2$, let H_i be the linear forest in H induced by the edges of H colored i where H_i has size m_i and $1 \le m_1 \le m_2$. By Proposition 3, we may assume that $H_1 \in \{Q_2, 2Q_2, Q_4\}$. If $H_1 = Q_2$, then the number of pairwise edge-disjoint copies of Q_2 in H_2 is at least $s-2$. As $s-2 > k-3$, it follows that $AR_c(H) = k$ by Theorem 6. If

$H_1 \in \{Q_4, 2Q_2\}$, then the number of pairwise edge-disjoint copies of Q_2 in H_2 is at least $s-4$. As $s-4 > k-4$, it follows that $AR_c(H) = k$ by Theorem 6. Therefore, $AR(H) = k$.

Next, we verify (2). Suppose that $s = k$:

- ⋆ First, assume that H contains two components, Q_x and Q_y, where $x, y \geq 4$ are both even. Define a 2-edge coloring c of H using the colors 1 and 2, such that $H_1 = 2Q_2$ and H_1 is placed in H in such a way that each of Q_x and Q_y contains a copy of Q_2 of H_1 and the number of pairwise edge-disjoint copies of Q_2 in H_2 is $s-4 = k-4$. Thus, $AR_c(H) = k-1$ by Theorem 6, and so, $AR(H) = k-1$.
- ⋆ Next, assume that H contains at most one component of even order 4 or more. Let c be any 2-edge coloring of H using the colors 1 and 2. For $i = 1, 2$, let H_i be the linear forest in H induced by the edges of H colored i where H_i has size m_i and $1 \leq m_1 \leq m_2$. By Proposition 3, we may assume that $H_1 \in \{Q_2, 2Q_2, Q_4\}$. If $H_1 = Q_2$, then the number of pairwise edge-disjoint copies of Q_2 in H_2 is at least $s-2 = k-2$, and so, $AR_c(H) = k$ by Theorem 6. If $H_1 \in \{Q_4, 2Q_2\}$, then, since H contains at most one component of even order 4 or more, any placement of H_1 in H produces at least $s-3 = k-3$ pairwise edge-disjoint copies of Q_2 in H_2. It follows by Theorem 6 that $AR_c(H) = k$. Therefore, $AR(H) = k$.

We now verify (3). Suppose that $s = k-1$:

- ⋆ First, assume that H contains at least one component Q_x where $x \geq 4$ is even. Define a 2-edge coloring c of H using the colors 1 and 2, such that $H_1 = Q_2$ and H_1 is placed in Q_x, in such a way that the number of pairwise edge-disjoint copies of Q_2 in H_2 is $s-2 = k-3$. It follows by Theorem 6 that $AR_c(H) = k-1$, and so, $AR(H) = k-1$.
- ⋆ Next, assume that H contains no component of even order 4 or more. Let c be any 2-edge coloring of H using the colors 1 and 2, where H_i is the linear forest of size m_i in H induced by the edges of H colored i and $1 \leq m_1 \leq m_2$. By Proposition 3, we may assume that $H_1 \in \{Q_2, 2Q_2, Q_4\}$. If $H_1 = Q_2$, then the number of pairwise edge-disjoint copies of Q_2 in H_2 is at least $s-1 = k-2$, and so, $AR_c(H) = k$ by Theorem 6. If $H_1 \in \{Q_4, 2Q_2\}$, then, as H contains no component of even order 4 or more, any placement of H_1 in H produces at least $s-2 = k-3$ pairwise edge-disjoint copies of Q_2 in H_2. It follows by Theorem 6 that $AR_c(H) = k$. Therefore, $AR(H) = k$.

Finally, we verify (4). Suppose that $s \leq k-2$. Define a 2-edge coloring c of H using the colors 1 and 2, such that $H_1 \in \{Q_2, Q_4, 2Q_2\}$. Then, the number of pairwise edge-disjoint copies of Q_2 in H_2 is at least $s-1$. As $s-1 \leq k-3$, it follows by Theorem 6 that $AR_c(H) = k-1$, and so, $AR(H) = k-1$. □

We have seen (by Theorems 1 and 5) that if H is a k-binomial linear forest where $k \geq 4$ with t components, whether $t = \binom{k+1}{2}$ or $1 \leq t \leq \binom{k-1}{2}$, then $AR(H) = k$. With the aid of Theorem 7, we are now able to show that these bounds on t cannot be improved.

Theorem 8. *For every two integers t and k where $\binom{k-1}{2} < t < \binom{k+1}{2}$ and $k \geq 4$, there is a k-binomial linear forest H with t components, such that $AR(H) = k-1$.*

Proof. For each integer i with $1 \leq i \leq 2k-2$ where $k \geq 4$, we construct a k-binomial linear forest F_i with $t = \binom{k-1}{2} + i$ components, such that $AR(F_i) = k-1$. In particular, F_1 has $t = \binom{k-1}{2} + 1$ components and F_{2k-2} has $t = \binom{k-1}{2} + (2k-2) = \binom{k+1}{2} - 1$ components.

For an integer $k \geq 4$, let $F_1 = Q_x + Q_y + \left[\binom{k-1}{2} - 1\right] Q_1$, where $x = y = k$ if k is even and $x = k+1$ and $y = k-1$ if $k \geq 5$ is odd. Thus, $x, y \geq 4$ are both even. As the size of F_1 is $\binom{k+1}{2}$, it follows that F_1 is a k-binomial linear forest with $t = \binom{k-1}{2} + 1$ components. The maximum number of pairwise edge-disjoint copies of Q_2 in F_1 is $s = \lfloor \frac{x}{2} \rfloor + \lfloor \frac{y}{2} \rfloor = k$. As $s = k$ and F_1 has two components of even size 4 or more, it follows by Theorem 7 that $AR(F_1) = k-1$.

Next, let $F_2 = Q_{x-1} + Q_y + \binom{k-1}{2}Q_1$. Then, F_2 is a k-binomial linear forest with $t = \binom{k-1}{2} + 2$ components. The maximum number of pairwise edge-disjoint copies of Q_2 in F_2 is $s = \left\lfloor \frac{x-1}{2} \right\rfloor + \left\lfloor \frac{y}{2} \right\rfloor = k-1$ and F_2 has the component Q_y of even size 4 or more. By Theorem 7, $AR(F_2) = k-1$. Next, let $F_3 = Q_{x-2} + Q_y + \left[\binom{k-1}{2} + 1\right]Q_1$. Then, F_3 is a k-binomial linear forest with $t = \binom{k-1}{2} + 3$ components. The maximum number of pairwise edge-disjoint copies of Q_2 in F_3 is $s = \left\lfloor \frac{x-2}{2} \right\rfloor + \left\lfloor \frac{y}{2} \right\rfloor = k-1$ and F_3 has the component Q_y of even size 4 or more. By Theorem 7, $AR(F_3) = k-1$.

Next, let $F_4 = Q_{x-2} + Q_{y-1} + \left[\binom{k-1}{2} + 2\right]Q_1$. Then, F_4 is a k-binomial linear forest with $t = \binom{k-1}{2} + 4$ components. The maximum number of pairwise edge-disjoint copies of Q_2 in F_4 is $s = \left\lfloor \frac{x-2}{2} \right\rfloor + \left\lfloor \frac{y-1}{2} \right\rfloor = k-2$. We continue this procedure, reducing the size of components greater than 1 until we arrive at

$$F_{2k-2} = Q_2 + Q_1 + \left[\binom{k-1}{2} + 2k - 4\right]Q_1 = Q_2 + \left[\binom{k-1}{2} + 2k - 3\right]Q_1 = Q_2 + \left[\binom{k+1}{2} - 2\right]Q_1.$$

Here, F_{2k-2} is a k-binomial linear forest with $t = \binom{k+1}{2} - 1$ components. For each integer i with $4 \le i \le 2k-2$, the maximum number of pairwise edge-disjoint copies of Q_2 in F_i is $s \le k-2$, and so, $AR(F_i) = k-1$ by Theorem 7. $\square$

To illustrate Theorem 7, we consider $k = 5$ and $\binom{4}{2} < 7 \le t \le 14 < \binom{6}{2}$. For $i = 1, 2, \ldots, 8$, we construct a 5-binomial linear forest F_i with $t = \binom{5-1}{2} + i = 6 + i$ components, such that $AR(F_i) = 4$:

- ⋆ Let $F_1 = Q_6 + Q_4 + 5Q_1$, where $t = 7$ and $s = 5$. As F_1 has 2 components Q_6 and Q_4 of even size 4 or more, $AR(F_1) = 4$ by Theorem 7.
- ⋆ Let $F_2 = Q_5 + Q_4 + 6Q_1$, where $t = 8$ and $s = 4$. As F_2 has the component Q_4 of size 4, it follows by Theorem 7 that $AR(F_2) = 4$.
- ⋆ Let $F_3 = Q_4 + Q_4 + 7Q_1$, where $t = 9$ and $s = 4$. As F_3 has the component Q_4 of size 4, it follows by Theorem 7 that $AR(F_3) = 4$.
- ⋆ For $4 \le i \le 8$, let
  $F_4 = Q_4 + Q_3 + 8Q_1$, where $t = 10$ and $s = 3$,
  $F_5 = Q_3 + Q_3 + 9Q_1$, where $t = 11$ and $s = 2$,
  $F_6 = Q_3 + Q_2 + 10Q_1$, where $t = 12$ and $s = 2$,
  $F_7 = Q_2 + Q_2 + 11Q_1$, where $t = 13$ and $s = 2$, and
  $F_8 = Q_2 + Q_1 + 12Q_1$, where $t = 14$ and $s = 1$.
  As $s \le 5 - 2 = 3$, it follows by Theorem 7 that $AR(F_i) = 4$ for $4 \le i \le 8$.

We saw that the 5-binomial linear forest $F_1 = Q_6 + Q_4 + 5Q_1$ has 7 components and $AR(F_1) = 4$. This does not imply that every 5-binomial linear forest with 7 components has Ramsey index 4. For example, $F = 4Q_3 + 3Q_1$ is also a 5-binomial linear forest with 7 components and the maximum number of pairwise edge-disjoint copies of Q_2 in F is $s = 4$. As F has no component of even size 4 or more, it follows by Theorem 7 that $AR(F) = 5$.

5. Closing Comments

From the information obtained on Ramsey chains of linear forest, a question remains, namely that of determining information on Ramsey chains of other familiar classes of graphs. For every graph G of size m that has been investigated where $\binom{k+1}{2} \le m < \binom{k+2}{2}$, it has been shown that either $AR(G) = k$ or $AR(G) = k-1$. This leads to the following problem:

Problem 1. *Let G be a graph of size m with $\binom{k+1}{2} \le m < \binom{k+2}{2}$ for some positive integer k. Is it true that either $AR(G) = k$ or $AR(G) = k-1$?*

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