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Multiplication Operators on Weighted Zygmund Spaces of the First Cartan Domain

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Abstract: Inspired by some recent studies of the multiplication operators on holomorphic function spaces of the classical domains such as the open unit disk, the unit ball and the unit polydisk, the purpose of the present paper is to study just the operators that are defined on weighted Zygmund spaces of the first Cartan domain. We obtain some necessary conditions and sufficient conditions for the operators to be bounded and compact.

Keywords: multiplication operator; first Cartan domain; weighted Zygmund space; boundedness; compactness

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1. Introduction

In 1921, Bergman introduced the concept of the Bergman kernel function when he studied the orthogonal expansion on the domain of the complex plane. It is well known that there exists a unique Bergman kernel function for any bounded domain in $\mathbb{C}^n$. But for which domains can the Bergman kernel function be calculated explicitly? This is a natural question. The variety of domains for which an explicit expression for the Bergman kernel function can be calculated is not large. However, Bergman kernel functions can be explicitly calculated for some special domains. For example, Loo-keng Hua obtained Bergman kernel functions with explicit formulas for four types of irreducible symmetric classical domains in [1]. In this paper, we will use the first irreducible symmetric classical domain usually called the first Cartan domain. This domain is defined by

$$\Re_I(m,n) = \left\{ Z = (z_{ij})_{m\times n} \in \mathbb{C}^{m\times n} : I - Z\bar{Z}^T > 0 \right\},$$

where $\bar{Z}$ is the conjugate of the matrix Z, Z^T is the transpose of Z, and m, n are positive integers.

Let $\mathbb{B}^N = \{z \in \mathbb{C}^N : |z| < 1\}$ be the open unit ball of $\mathbb{C}^N$. When $N = 1$, $\mathbb{B}^N$ is the open unit disk denoted by $\mathbb{D}$. Since $\Re_I(1,N) = \mathbb{B}^N$, $\Re_I(m,n)$ can be regarded as a generalization of $\mathbb{B}^N$. For the sake of convenience, $\Re_I(m,n)$ is written by $\Re_I$.

Let $H(\Re_I)$ be the set of all holomorphic functions on $\Re_I$. For $\alpha \geq 0$, the weighted-type space $H^\infty_\alpha(\Re_I)$ on $\Re_I$ consists of all $f \in H(\Re_I)$ such that

$$\|f\|_{H^\infty_\alpha(\Re_I)} = \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha |f(Z)| < \infty.$$

The little weighted-type space $H^\infty_{\alpha,0}(\Re_I)$ on $\Re_I$ consists of all $f \in H(\Re_I)$ such that

$$\lim_{Z\to\partial\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha |f(Z)| = 0.$$

If $\alpha = 0$, then $H^\infty_\alpha(\Re_I)$ and $H^\infty_{\alpha,0}(\Re_I)$ are denoted by $H^\infty(\Re_I)$ and $H^\infty_0(\Re_I)$, respectively. The weighted-type spaces on the unit disk and the unit ball are frequently discussed in the literature, see [2–5].

Let $H(\mathbb{B}^N)$ be the space of all holomorphic functions on $\mathbb{B}^N$. The weighted Zygmund space on $\mathbb{B}^N$ denoted by $\mathcal{Z}^\alpha(\mathbb{B}^N)$ consists of all $f \in H(\mathbb{B}^N)$ such that

$$s(f) = \sup_{z\in\mathbb{B}^N} (1-|z|^2)^\alpha \left|\Re^2 f(z)\right| < \infty,$$

where $\Re f$ is the radial derivative

$$\Re f(z) = \sum_{j=1}^{N} z_j \frac{\partial f}{\partial z_j}(z),$$

and $\Re^2 f(z) = \Re(\Re f(z))$. It is well known that $s(f)$ is a seminorm of $\mathcal{Z}^\alpha(\mathbb{B}^N)$. For each $f \in \mathcal{Z}^\alpha(\mathbb{B}^N)$, we define $\|f\|_{\mathcal{Z}^\alpha(\mathbb{B}^N)} = |f(0)| + s(f)$. Then, $\|\cdot\|_{\mathcal{Z}^\alpha(\mathbb{B}^N)}$ is a norm on $\mathcal{Z}^\alpha(\mathbb{B}^N)$, and $\mathcal{Z}^\alpha(\mathbb{B}^N)$ is a Banach space with this norm. We also usually use this space defined on the unit disk (see [6]). For composition and product-type operators on or between the weighted Zygmund spaces, see, for example, refs. [7–9] and the references therein.

For $f \in H(\Re_I)$, we define

$$\Re f(Z) = \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}\frac{\partial f(Z)}{\partial z_{ij}},$$

$$\nabla f(Z) = \left(\frac{\partial f(Z)}{\partial z_{11}}, \frac{\partial f(Z)}{\partial z_{12}}, \cdots, \frac{\partial f(Z)}{\partial z_{mn}}\right),$$

and

$$|\nabla f(Z)|^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{\partial f(Z)}{\partial z_{ij}}\right|^2.$$

We say that $f \in H(\Re_I)$ is in the weighted Zygmund space $\mathcal{Z}^\alpha(\Re_I)$, if

$$s_1(f) = \sup_{Z\in\Re_I}\left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|\Re^2 f(Z)\right| < \infty.$$

If $\alpha = 1$, $\mathcal{Z}^\alpha(\Re_I)$ is called the Zygmund space denoted by $\mathcal{Z}(\Re_I)$. $\mathcal{Z}^\alpha(\Re_I)$ is a Banach space with the norm

$$\|f\|_{1,\mathcal{Z}^\alpha(\Re_I)} = \left|f(0)\right| + s_1(f).$$

On $\mathcal{Z}^\alpha(\Re_I)$ we also can define the following quantity:

$$s_2(f) = \sup_{Z\in\Re_I}\left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|\nabla\Re f(Z)\right|.$$

The quantity $s_2(f)$ is a seminorm on $\mathcal{Z}^\alpha(\Re_I)$, and

$$\|f\|_{2,\mathcal{Z}^\alpha(\Re_I)} = \left|f(0)\right| + s_2(f)$$

is a norm of $\mathcal{Z}^\alpha(\Re_I)$. From the proof of Theorem 3.1 in [10], we see that these two norms are equivalent. Therefore, we no longer need to distinguish them, uniformly denoted by $\|f\|_{\mathcal{Z}^\alpha(\Re_I)}$. The little weighted Zygmund space on $\Re_I$ denoted by $\mathcal{Z}^\alpha_0(\Re_I)$ consists of all $f \in H(\Re_I)$ such that

$$\lim_{Z\to\partial\Re_I}\left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|\Re^2 f(Z)\right| = 0.$$

It is not difficult to see that $\mathcal{Z}^\alpha_0(\Re_I)$ is a closed subspace of $\mathcal{Z}^\alpha(\Re_I)$.

Let X be a function space on $\Re_I$ and ψ a function defined on $\Re_I$. The function ψ is called a multiplier on X, if $\psi \cdot f \in X$ for all $f \in X$. The operator

$$M_\psi : f \mapsto \psi \cdot f$$

is usually called a multiplication operator on X. Generally speaking, there may exist some function $f \in X$ such that $\psi \cdot f$ does not belong to X. Now, we will explain this phenomenon. To this end, we consider the Bloch space $\mathcal{B}(\mathbb{D}^2)$, which consists of all $f \in H(\mathbb{D}^2)$ such that

$$\sup_{z\in\mathbb{D}^2} \Big[(1-|z_1|^2)\Big|\frac{\partial f}{\partial z_1}(z)\Big| + (1-|z_2|^2)\Big|\frac{\partial f}{\partial z_2}(z)\Big|\Big] < \infty,$$

where $\mathbb{D}^2 = \{z = (z_1, z_2) : z_1 \in \mathbb{D}, z_2 \in \mathbb{D}\}$. On $\mathbb{D}^2$ define the function $\psi(z) = z_1$. If we choose the function

$$f(z_1, z_2) = \ln \frac{1}{1-z_1} + \ln \frac{1}{1-z_2},$$

then f belongs to $\mathcal{B}(\mathbb{D}^2)$. But, it follows from a direct calculation that $\psi \cdot f$ does not belong to $\mathcal{B}(\mathbb{D}^2)$. This shows that $\psi(z) = z_1$ is not a multiplier on $\mathcal{B}(\mathbb{D}^2)$.

Multipliers and multiplication operators on function spaces have been studied for a long time. For example, Taylor started the study of the multipliers in [11] in 1966. Stegenga studied the multipliers of the Dirichlet space in [12] in 1980. Now, multipliers and multiplication operators on holomorphic function spaces of the unit disk $\mathbb{D}$ and the unit ball $\mathbb{B}^N$ have been studied (see, [13–16]). In addition, there is a great interest in some related operators for multiplication operators such as weighted composition operators, see, [17–20]. Recently, Su et al. in [21] obtained the necessary condition and sufficient condition for the boundedness and compactness of the composition operators from u-Bloch space to v-Bloch space on the first Hua domain. Su et al. in [22] gave the necessary condition and sufficient condition for the boundedness and compactness of the composition operators from p-Bloch space to q-Bloch space on the first Cartan-Hartogs domain. The author characterized the bounded and compact weighted composition operators on the weighted Bers-type spaces of the Hua domains in [23]. It must be mentioned that these domains are defined by the first Cartan domain. On the other hand, we do not find any result about the multiplication operators that are defined on weighted Zygmund spaces of the first Cartan domain. Therefore, motivated by the above-mentioned studies and facts, the natural tendency is to extend the related studies to the first Cartan domain. For this purpose, we study just multiplication operators that are defined on weighted Zygmund spaces of the first Cartan domain in this paper. We obtain some necessary conditions and sufficient conditions for the boundedness and compactness of the multiplication operators.

We write $|Z|^2 = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} |z_{ij}|^2$ for $Z = (z_{ij})_{m\times n} \in \mathbb{C}^{m\times n}$. Throughout the paper, real positive constants are denoted by C, and they may vary from place to place.

2. Some Elementary Lemmas

First, we obtain the following result from a direct calculation.

Lemma 1. *Let $\psi \in H(\Re_I)$. Then for each $f \in H(\Re_I)$ and $Z \in \Re_I$, the following statement holds.*

$$\Re^2(M_\psi f)(Z) = f(Z)\Re^2\psi(Z) + 2\Re f(Z)\Re\psi(Z) + \Re^2 f(Z)\psi(Z).$$

To arrive at the point evaluation estimate for the functions in $\mathcal{Z}^\alpha(\Re_I)$, we need the following result (see [22]).

Lemma 2. *Let $Z \in \Re_I$. Then there exist two unitary matrices U and V such that*

$$Z = U\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix}V,$$

where $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ *and* $\lambda_1^2, \ldots, \lambda_m^2$ *are eigenvalues of* $Z\bar{Z}^T$.

From the calculations, we obtain the following result.

Lemma 3. *(a)* *If* $0 \leq m\alpha < 1$, *then*

$$\int_0^1 \frac{\lambda_1 dt}{(1-\lambda_1 t)^{m\alpha}} \leq \frac{1}{1-m\alpha}.$$

(b) *If* $m\alpha = 1$, *then*

$$\int_0^1 \frac{\lambda_1 dt}{(1-\lambda_1 t)^{m\alpha}} = \ln\frac{1}{1-\lambda_1}.$$

(c) *If* $m\alpha > 1$, *then*

$$\int_0^1 \frac{\lambda_1 dt}{(1-\lambda_1 t)^{m\alpha}} \leq \frac{1}{m\alpha-1}\frac{1}{(1-\lambda_1)^{m\alpha-1}}.$$

Lemma 4. *(a)* *If* $0 \leq m\alpha < 1$, *then there exists a positive constant C independent of* $f \in \mathcal{Z}^\alpha(\Re_I)$ *and* $Z \in \Re_I$ *such that*

$$\left|\Re f(Z)\right| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.$$

(b) *If* $m\alpha = 1$, *then there exists a positive constant C independent of* $f \in \mathcal{Z}^\alpha(\Re_I)$ *and* $Z \in \Re_I$ *such that*

$$\left|\Re f(Z)\right| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \ln\frac{2e}{det(I-Z\bar{Z}^T)}.$$

(c) *If* $m\alpha > 1$, *then there exists a positive constant C independent of* $f \in \mathcal{Z}^\alpha(\Re_I)$ *and* $Z \in \Re_I$ *such that*

$$\left|\Re f(Z)\right| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \frac{1}{[det(I-Z\bar{Z}^T)]^{m\alpha-1}}.$$

Proof. We prove all three statements simultaneously. If $Z = 0$, then the lemma obviously holds. Now, assume that $Z = (z_{ij})_{m\times n} \neq 0$. It follows from Lemma 2 that there exist two unitary matrices U and V such that

$$Z = U\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix}V, \tag{1}$$

where $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and $\lambda_1^2, ..., \lambda_m^2$ are eigenvalues of $Z\bar{Z}^T$. From (1), we have

$$I - t^2 Z\bar{Z}^T = U\begin{pmatrix} 1-t^2\lambda_1^2 & 0 & \cdots & 0 \\ 0 & 1-t^2\lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-t^2\lambda_m^2 \end{pmatrix}\bar{U}^T.$$

Let $t \in [0,1]$. Since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$, for each $i \in \{1, 2, ..., m\}$ we have

$$1 - t^2\lambda_i^2 = (1 - t\lambda_i)(1 + t\lambda_i) \geq 1 - t\lambda_i \geq 1 - t\lambda_1.$$

From this, we have

$$\left[\det(I - t^2 Z\bar{Z}^T)\right]^\alpha = \prod_{i=1}^m (1 - t^2\lambda_i^2)^\alpha \geq (1 - t\lambda_1)^{m\alpha}. \tag{2}$$

In particular, from (2) we have

$$1 - \lambda_1 \geq \frac{1}{2}(1 - \lambda_1^2) \geq \frac{1}{2}\det(I - Z\bar{Z}^T). \tag{3}$$

From the facts

$$Z\bar{Z}^T = U \begin{pmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^2 \end{pmatrix} \bar{U}^T$$

and $|Z|^2 = \operatorname{tr}(Z\bar{Z}^T)$, we obtain

$$|Z|^2 = \sum_{i=1}^m \lambda_i^2 \leq m\lambda_1^2,$$

which shows

$$|Z| \leq \sqrt{m}\lambda_1. \tag{4}$$

Then, from Lemma 3 and (2)–(4), it follows that

$$\begin{aligned}
\left|\frac{\partial f(Z)}{\partial z_{ij}}\right| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial f}{\partial z_{ij}}(tZ)\Big)\mathrm{d}t + \frac{\partial f}{\partial z_{ij}}(0)\right| \\
&= \left|\int_0^1 \Big[\sum_{u=1}^m \sum_{v=1}^n z_{uv}\frac{\partial^2 f}{\partial z_{ij}\partial z_{uv}}(tZ)\Big]\mathrm{d}t + \frac{\partial f}{\partial z_{ij}}(0)\right| \\
&\leq \int_0^1 \Big[\sum_{u=1}^m \sum_{v=1}^n |z_{uv}|\Big|\frac{\partial^2 f}{\partial z_{ij}\partial z_{uv}}(tZ)\Big|\Big]\mathrm{d}t + \left|\frac{\partial f}{\partial z_{ij}}(0)\right| \\
&\leq \int_0^1 \frac{|Z|\mathrm{d}t}{\left[\det(I - t^2 Z\bar{Z}^T)\right]^\alpha}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} + \left|\frac{\partial f}{\partial z_{ij}}(0)\right| \\
&= \int_0^1 \frac{|Z|\mathrm{d}t}{\prod_{j=1}^m (1 - t^2\lambda_j^2)^\alpha}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} + \left|\frac{\partial f}{\partial z_{ij}}(0)\right| \\
&\leq \Big[\int_0^1 \frac{\sqrt{m}\lambda_1}{(1 - \lambda_1 t)^{m\alpha}}\mathrm{d}t + 1\Big]\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \\
&\leq \begin{cases} \big(\frac{\sqrt{m}}{1 - m\alpha} + 1\big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & 0 \leq m\alpha < 1 \\ \sqrt{m}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \ln\frac{e}{1 - \lambda_1}, & m\alpha = 1 \\ \big(\frac{\sqrt{m}}{m\alpha - 1} + 1\big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\frac{1}{(1 - \lambda_1)^{m\alpha - 1}}, & m\alpha > 1 \end{cases} \\
&\leq \begin{cases} \big(\frac{\sqrt{m}}{1 - m\alpha} + 1\big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & 0 \leq m\alpha < 1 \\ \sqrt{m}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \ln\frac{2e}{\det(I - Z\bar{Z}^T)}, & m\alpha = 1 \\ \big(\frac{\sqrt{m}}{m\alpha - 1} + 1\big)2^{m\alpha - 1}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\frac{1}{[\det(I - Z\bar{Z}^T)]^{m\alpha - 1}}, & m\alpha > 1. \end{cases}
\end{aligned} \tag{5}$$

Then, from (5) and the fact

$$\left|\Re f(Z)\right| \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{\partial f(Z)}{\partial z_{ij}}\right|,$$

it follows that

$$\left|\Re f(Z)\right| \leq \begin{cases} \left(\frac{\sqrt{m}}{1-m\alpha}+1\right)mn\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & 0 \leq m\alpha < 1 \\ \sqrt{m}mn\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\ln\frac{2e}{\det(I-Z\bar{Z}^T)}, & m\alpha = 1 \\ \left(\frac{\sqrt{m}}{m\alpha-1}+1\right)2^{m\alpha-1}mn\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\frac{1}{[\det(I-Z\bar{Z}^T)]^{m\alpha-1}}, & m\alpha > 1. \end{cases} \tag{6}$$

From (6), the desired result follows. The proof is complete. □

In order to prove Lemma 6, we need the following result.

Lemma 5. *(a)* *If $0 \leq m\alpha < 2$, then*

$$\int_0^1\int_0^1 \frac{\lambda_1 dsdt}{(1-\lambda_1 ts)^{m\alpha}} \leq C,$$

where

$$C = \max\left\{\frac{1}{1-m\alpha}\int_0^1 \frac{1}{t}\left[1-(1-t)^{1-m\alpha}\right]dt, \int_0^1 \frac{1}{t}\ln\frac{1}{1-t}dt, \frac{1}{m\alpha-1}\int_0^1\frac{1}{t}\left[\frac{1}{(1-t)^{m\alpha-1}}-1\right]dt\right\}.$$

(b) *If $m\alpha = 2$, then*

$$\int_0^1\int_0^1 \frac{\lambda_1 dsdt}{(1-\lambda_1 ts)^{m\alpha}} = \ln\frac{1}{1-\lambda_1}.$$

(c) *If $m\alpha > 2$, then*

$$\int_0^1\int_0^1 \frac{\lambda_1 dsdt}{(1-\lambda_1 ts)^{m\alpha}} \leq \frac{1}{(1-\lambda_1)^{m\alpha-2}}.$$

Proof. (a). We divide into three cases to prove the statement (a).

Case 1. *Assume that $0 \leq m\alpha < 1$. Since the limit*

$$\lim_{t\to 0^+}\frac{1-(1-t)^{1-m\alpha}}{t}$$

exists, we see that

$$\int_0^1 \frac{1}{t}[1-(1-t)^{1-m\alpha}]dt$$

is a definite integral. From this, it follows that $C_1 = \frac{1}{1-m\alpha}\int_0^1 \frac{1}{t}[1-(1-t)^{1-m\alpha}]dt$ is a positive constant. Then, we have

$$\begin{aligned}\int_0^1\int_0^1 \frac{\lambda_1 dsdt}{(1-\lambda_1 ts)^{m\alpha}} &= \int_0^1\frac{1}{t}\int_0^1 \frac{\lambda_1 tdsdt}{(1-\lambda_1 ts)^{m\alpha}} = \frac{1}{1-m\alpha}\int_0^1\frac{1}{t}\left[1-(1-\lambda_1 t)^{1-m\alpha}\right]dt \\ &\leq \frac{1}{1-m\alpha}\int_0^1\frac{1}{t}\left[1-(1-t)^{1-m\alpha}\right]dt = C_1.\end{aligned}$$

Case 2. *Assume that* $m\alpha = 1$. *Let* $x = \frac{1}{1-t}$. *We have*

$$\int_0^1 \frac{1}{t} \ln \frac{1}{1-t} dt = \int_1^{+\infty} \frac{\ln x}{x(x-1)} dx = \int_1^2 \frac{\ln x}{x(x-1)} dx + \int_2^{+\infty} \frac{\ln x}{x(x-1)} dx = I_1 + I_2.$$

Since the limit

$$\lim_{x \to 1^+} \frac{\ln x}{x(x-1)}$$

exists, I_1 *is a definite integral. On the other hand, since*

$$\lim_{x \to +\infty} \frac{\ln x}{x^{\frac{1}{2}}} = 0$$

implies that $\ln x \leq x^{\frac{1}{2}}$ *for sufficiently large* x,

$$\frac{\ln x}{x(x-1)} \leq \frac{x^{\frac{1}{2}}}{x(x-1)} \leq \frac{1}{(x-1)^{\frac{3}{2}}} \tag{7}$$

for sufficiently large x. *From (7), we obtain that* I_2 *is convergent. So,* $C_2 := \int_0^1 \frac{1}{t} \ln \frac{1}{1-t} dt$ *is finite. Then, we have*

$$\int_0^1 \int_0^1 \frac{\lambda_1 dsdt}{1 - st\lambda_1} = \int_0^1 \frac{1}{t} \int_0^1 \frac{t\lambda_1 dsdt}{1 - st\lambda_1} = \int_0^1 \frac{1}{t} \ln \frac{1}{1-\lambda_1 t} dt \leq \int_0^1 \frac{1}{t} \ln \frac{1}{1-t} dt = C_2.$$

Case 3. *Assume that* $1 < m\alpha < 2$. *Let* $x = \frac{1}{1-t}$. *Then, we see that*

$$\int_0^1 \frac{1}{t} [\frac{1}{(1-t)^{m\alpha-1}} - 1] dt = \int_1^{+\infty} \frac{x^{m\alpha-1} - 1}{x(x-1)} dx$$

is convergent. Write $C_3 = \frac{1}{m\alpha-1} \int_0^1 \frac{1}{t} [\frac{1}{(1-t)^{m\alpha-1}} - 1]$. *Therefore, we have*

$$\begin{aligned}\int_0^1 \int_0^1 \frac{\lambda_1 dsdt}{(1 - st\lambda_1)^{m\alpha}} &= \int_0^1 \frac{1}{t} \int_0^1 \frac{t\lambda_1 dsdt}{(1 - st\lambda_1)^{m\alpha}} = \frac{1}{m\alpha - 1} \int_0^1 \frac{1}{t} \Big[\frac{1}{(1-\lambda_1 t)^{m\alpha-1}} - 1\Big] dt \\ &\leq \frac{1}{m\alpha - 1} \int_0^1 \frac{1}{t} \Big[\frac{1}{(1-t)^{m\alpha-1}} - 1\Big] dt = C_3.\end{aligned}$$

Combining the above three cases, we complete the proof of (a).

(b). From the calculations, it follows that

$$\int_0^1 \int_0^1 \frac{\lambda_1 \mathrm{d}s\mathrm{d}t}{(1-\lambda_1 ts)^2} = \int_0^1 \frac{1}{t} \int_0^1 \frac{\lambda_1 t \mathrm{d}s\mathrm{d}t}{(1-\lambda_1 ts)^2} = \ln \frac{1}{1-\lambda_1}.$$

(c). Since

$$\frac{\lambda_1}{(1-\lambda_1 ts)^{m\alpha}} \leq \frac{1}{(1-\lambda_1)^{m\alpha-2}(1-t)^2}$$

for all $s, t \in [0,1]$, we have

$$\int_0^1 \int_0^1 \frac{\lambda_1 \mathrm{d}s\mathrm{d}t}{(1-\lambda_1 ts)^{m\alpha}} \leq \frac{1}{(1-\lambda_1)^{m\alpha-2}} \int_0^1 \int_0^1 \frac{\mathrm{d}s\mathrm{d}t}{(1-t)^2} = \frac{1}{(1-\lambda_1)^{m\alpha-2}}.$$

$\square$

Lemma 6. *(a)* *If $0 \le m\alpha < 2$, then there exists a positive constant C independent of $f \in \mathcal{Z}^\alpha(\Re_I)$ and $Z \in \Re_I$ such that*

$$\big|f(Z)\big| \le C\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.$$

(b) *If $m\alpha = 2$, then there exists a positive constant C independent of $f \in \mathcal{Z}^\alpha(\Re_I)$ and $Z \in \Re_I$ such that*

$$\big|f(Z)\big| \le C\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \ln \frac{2e}{det(I - Z\bar{Z}^T)}.$$

(c) *If $m\alpha > 2$, then there exists a positive constant C independent of $f \in \mathcal{Z}^\alpha(\Re_I)$ and $Z \in \Re_I$ such that*

$$\big|f(Z)\big| \le C\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \frac{1}{[det(I - Z\bar{Z}^T)]^{m\alpha-2}}.$$

Proof. We prove all three statements simultaneously. Similar to the proof of Lemma 4, for $s, t \in [0,1]$ we have

$$\big[\det(I - s^2t^2 Z\bar{Z}^T)\big]^\alpha = \prod_{i=1}^m (1 - s^2t^2\lambda_i^2)^\alpha \ge (1 - st\lambda_1)^{m\alpha}. \tag{8}$$

From (3), (4), (8) and Lemma 5, for each $f \in \mathcal{Z}^\alpha(\Re_I)$ and $Z \in \Re_I$, we have

$$\begin{aligned}
\big|f(Z)\big| &= \Big|f(0) + \int_0^1 \frac{1}{t}\Re f(tZ)\mathrm{d}t\Big| \le \big|f(0)\big| + \Big|\int_0^1 \frac{1}{t}\int_0^1 \langle tZ, \nabla\Re f(stZ)\rangle \mathrm{d}s\mathrm{d}t\Big| \\
&\le \big|f(0)\big| + \int_0^1 \frac{1}{t}\int_0^1 |tZ||\nabla\Re f(stZ)|\mathrm{d}s\mathrm{d}t \\
&\le \Big(1 + \int_0^1\int_0^1 \frac{|Z|\mathrm{d}s\mathrm{d}t}{\big[\det(I - s^2t^2 Z\bar{Z}^T)\big]^\alpha}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \\
&\le \Big(1 + \sqrt{m}\int_0^1\int_0^1 \frac{\lambda_1 \mathrm{d}s\mathrm{d}t}{(1 - st\lambda_1)^{m\alpha}}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \\
&\le \begin{cases} (1+\sqrt{m}C)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & 0 \le m\alpha < 2 \\ \sqrt{m}\ln\frac{e}{1-\lambda_1}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & m\alpha = 2 \\ (1+\sqrt{m})\frac{1}{(1-\lambda_1)^{m\alpha-2}}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & m\alpha > 2 \end{cases} \\
&\le \begin{cases} (1+\sqrt{m}C)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}, & 0 \le m\alpha < 2 \\ \sqrt{m}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\ln\frac{2e}{\det(I - Z\bar{Z}^T)}, & m\alpha = 2 \\ (1+\sqrt{m})2^{m\alpha-2}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\frac{1}{[\det(I - Z\bar{Z}^T)]^{m\alpha-2}}, & m\alpha > 2, \end{cases}
\end{aligned}$$

where C is the constant in (a) of Lemma 5. The proof is complete. □

Remark 1. *In Lemmas 4 and 6, we note the presence of the parameter m which is necessary and cannot be avoided. This maybe is the biggest difference from the corresponding results on $\mathcal{Z}^\alpha(\mathbb{B}^N)$ ([24]). Unfortunately, we do not find an effective method to avoid it. However, it is shown that Lemmas 4 and 6 can be regarded as the generalizations of the corresponding results on $\mathcal{Z}^\alpha(\mathbb{B}^N)$.*

Replacing $\Re^2 f$ by $\Re f$ in the definitions of the spaces $\mathcal{Z}^\alpha(\Re_I)$ and $\mathcal{Z}_0^\alpha(\Re_I)$, respectively, we obtain the weighted Bloch space and the little weighted Bloch space on $\Re_I$, denoted by $\mathcal{B}^\alpha(\Re_I)$ and $\mathcal{B}_0^\alpha(\Re_I)$, respectively. $\mathcal{B}^1(\Re_I)$ is usually called the Bloch space, denoted by $\mathcal{B}(\Re_I)$.

Let $Z \in \Re_I$, $0 \le \lambda_m \le \lambda_{m-1} \le \cdots \le \lambda_1 < 1$ and $\lambda_1^2, ..., \lambda_m^2$ be the eigenvalues of $Z\bar{Z}^T$. Then from the proof of Lemma 4, we see that

$$det(I - Z\bar{Z}^T) = \prod_{j=1}^{m}(1 - \lambda_j^2),$$

which shows that

$$det(I - Z\bar{Z}^T) \leq 1. \tag{9}$$

Proposition 1. *If* $0 \leq m\alpha < 1$, *then* $f \in \mathcal{Z}^\alpha(\Re_I)$ *implies* $f \in H_\alpha^\infty(\Re_I) \cap \mathcal{B}^\alpha(\Re_I)$.

Proof. Let $f \in \mathcal{Z}^\alpha(\Re_I)$. From (a) in Lemma 4, there exists a positive constant C such that

$$|\Re f(Z)| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \tag{10}$$

for all $Z \in \Re_I$. By (9) and (10),

$$[\det(I - Z\bar{Z}^T)]^\alpha |\Re f(Z)| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)}$$

for all $Z \in \Re_I$, which shows that $f \in \mathcal{B}^\alpha(\Re_I)$. On other hand, from (a) in Lemma 6, there exists a positive constant C such that

$$|f(Z)| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \tag{11}$$

for all $Z \in \Re_I$. By (9) and (11),

$$[\det(I - Z\bar{Z}^T)]^\alpha |f(Z)| \leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)}$$

for all $Z \in \Re_I$, which shows that $f \in H_\alpha^\infty(\Re_I)$. Therefore, we prove that $f \in H_\alpha^\infty(\Re_I) \cap \mathcal{B}^\alpha(\Re_I)$. The proof is complete. □

In order to characterize the compactness, we need the following result. Since the proof is similar to that of Proposition 3.11 in [25], we do not provide proof anymore.

Lemma 7. *Let* $\psi \in H(\Re_I)$. *Then the bounded operator* M_ψ *on* $\mathcal{Z}^\alpha(\Re_I)$ *is compact if and only if for every bounded sequence* $\{f_k\}$ *in* $\mathcal{Z}^\alpha(\Re_I)$ *such that* $f_k \to 0$ *uniformly on every compact subset of* $\Re_I$ *as* $k \to \infty$, *it follows that*

$$\lim_{k\to\infty} \|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0.$$

In the case of several complex variables, Loo-keng Hua found an inequality (usually called the Hua's inequality) in 1955. In [22], the authors obtained a generalization of the Hua's inequality on the first Cartan-Hartogs domain:

$$Y_I(N, m, n; K) = \{W \in \mathbb{C}^N, Z \in \Re_I(m, n) : |W|^{2K} < \det(I - Z\bar{Z}^T)\}.$$

Setting $W_1 = 0$ and $W_2 = 0$ in Theorem 1 in [22], we obtain the following inequality.

Lemma 8. *If* $A, B \in \Re_I$, *then*

$$det(I - A\bar{A}^T)det(I - B\bar{B}^T) \leq |det(I - A\bar{B}^T)|^2.$$

Let $S \in \Re_I$ and

$$\bar{S}^T = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1m} \\ s_{21} & s_{22} & \cdots & s_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ s_{n1} & s_{n2} & \cdots & s_{nm} \end{pmatrix}.$$

On $\Re_I$ we define the function

$$A_S(Z) = \det(I - Z\bar{S}^T).$$

If we write

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix} \in \Re_I,$$

then

$$A_S(Z) = \begin{vmatrix} 1 - \sum\limits_{k=1}^{n} s_{k1} z_{1k} & -\sum\limits_{k=1}^{n} s_{k2} z_{1k} & \cdots & -\sum\limits_{k=1}^{n} s_{km} z_{1k} \\ -\sum\limits_{k=1}^{n} s_{k1} z_{2k} & 1 - \sum\limits_{k=1}^{n} s_{k2} z_{2k} & \cdots & -\sum\limits_{k=1}^{n} s_{km} z_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum\limits_{k=1}^{n} s_{k1} z_{mk} & -\sum\limits_{k=1}^{n} s_{k2} z_{mk} & \cdots & 1 - \sum\limits_{k=1}^{n} s_{km} z_{mk} \end{vmatrix}.$$

For the sake of convenience, write

$$A_{S,ij}(Z) = \frac{\partial A_S(Z)}{\partial z_{ij}}$$

and

$$A_{S,ij,pq}(Z) = \frac{\partial^2 A_S(Z)}{\partial z_{ij} \partial z_{pq}}.$$

From the derivation rule of determinant functions, we obtain the following result.

Lemma 9. *For each $Z \in \Re_I$, we have*

$$A_{S,ij}(Z) = \begin{vmatrix} 1 - \sum\limits_{k=1}^{n} s_{k1} z_{1k} & -\sum\limits_{k=1}^{n} s_{k2} z_{1k} & \cdots & -\sum\limits_{k=1}^{n} s_{kj} z_{1k} & \cdots & -\sum\limits_{k=1}^{n} s_{km} z_{1k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -s_{j1} & -s_{j2} & \cdots & -s_{jj} & \cdots & -s_{jm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\sum\limits_{k=1}^{n} s_{k1} z_{mk} & -\sum\limits_{k=1}^{n} s_{k2} z_{mk} & \cdots & -\sum\limits_{k=1}^{n} s_{kj} z_{mk} & \cdots & 1 - \sum\limits_{k=1}^{n} s_{km} z_{mk} \end{vmatrix}$$

and

$$A_{S,ij,pq}(Z) = \begin{vmatrix} 1 - \sum\limits_{k=1}^{n} s_{k1} z_{1k} & -\sum\limits_{k=1}^{n} s_{k2} z_{1k} & \cdots & -\sum\limits_{k=1}^{n} s_{kj} z_{1k} & \cdots & -\sum\limits_{k=1}^{n} s_{km} z_{1k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -s_{q1} & -s_{q2} & \cdots & -s_{qq} & \cdots & -s_{qm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -s_{j1} & -s_{j2} & \cdots & -s_{jj} & \cdots & -s_{jm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\sum\limits_{k=1}^{n} s_{k1} z_{mk} & -\sum\limits_{k=1}^{n} s_{k2} z_{mk} & \cdots & -\sum\limits_{k=1}^{n} s_{kj} z_{mk} & \cdots & 1 - \sum\limits_{k=1}^{n} s_{km} z_{mk} \end{vmatrix}.$$

Next, the following result holds.

Lemma 10. *There exists a positive constant C independent of $Z \in \Re_I$ such that*

$$\sum_{i,p=1}^{m} \sum_{j,q=1}^{n} \left| A_{S,ij,pq}(Z) \right| \le C.$$

Proof. Since $Z, S \in \Re_I$, we have $I - Z\bar{Z}^T > 0$ and $I - S\bar{S}^T > 0$. Then, for each $1 \le i \le m$, it follows that

$$1 - \sum_{j=1}^{n} |z_{ij}|^2 > 0, \;\; 1 - \sum_{j=1}^{n} |s_{ij}|^2 > 0.$$

So, we have $|z_{ij}| < 1$ and $|s_{ij}| < 1$ for $1 \le i \le m$ and $1 \le j \le n$. Let $a_{11}, a_{12}, \ldots, a_{mn}$ denote the elements of the determinant in $A_{S,ij,pq}(Z)$. From Lemma 9, we see that $|a_{ij}| \le n+1$ for $1 \le i \le m$ and $1 \le j \le n$. With this and the definition of determinant, we obtain

$$\begin{aligned} \left|A_{S,ij,pq}(Z)\right| &= \Big| \sum_{j_1 j_2 \cdots j_m} (-1)^{\tau(j_1 j_2 \cdots j_m)} a_{1j_1} a_{2j_2} \cdots a_{mj_m} \Big| \\ &\le \sum_{j_1 j_2 \cdots j_m} |a_{1j_1}||a_{2j_2}| \cdots |a_{mj_m}| \\ &\le n!(n+1)^{m^2}, \end{aligned} \tag{12}$$

where $\tau(j_1 j_2 \cdots j_m)$ denotes the inverse ordinal of the arrangement $j_1 j_2 \cdots j_n$. Let $C = n!(n+1)^{m^2} m^2 n^2$. Then, from (12) the desired result follows. The proof is complete. $\square$

We can similarly prove the next two results. Therefore, the proofs are omitted.

Lemma 11. *There exists a positive constant C independent of $Z \in \Re_I$ such that*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left|A_{S,ij}(Z)\right| \le C.$$

Lemma 12. *There exists a positive constant C independent of $Z \in \Re_I$ such that*

$$\left|det(I - Z\bar{S}^T)\right| \le C.$$

Let S be a fixed matrix in $\Re_I$. If $\alpha = 1$, we define

$$f_S(Z) = \det(I - S\bar{S}^T) \ln \frac{2e}{\det(I - Z\bar{S}^T)}, \;\; Z \in \Re_I,$$

and if $\alpha \ne 1$, we define

$$g_S(Z) = \frac{\left[\det(I - S\bar{S}^T)\right]^{\alpha}}{\left[\det(I - Z\bar{S}^T)\right]^{2\alpha - 2}}, \;\; Z \in \Re_I.$$

Next, we prove that $f_S \in \mathcal{Z}(\Re_I)$ and $g_S \in \mathcal{Z}^{\alpha}(\Re_I)$.

Lemma 13. *(a) The function f_S belongs to $\mathcal{Z}(\Re_I)$. Moreover, there exists a positive constant C such that*

$$\sup_{S \in \Re_I} \|f_S\|_{\mathcal{Z}(\Re_I)} \le C. \tag{13}$$

(b) The function g_S belongs to $\mathcal{Z}^{\alpha}(\Re_I)$. Moreover, there exists a positive constant C such that

$$\sup_{S \in \Re_I} \|g_S\|_{\mathcal{Z}^{\alpha}(\Re_I)} \le C. \tag{14}$$

Proof. (a). From a direct calculation, we have

$$\frac{\partial f_S}{\partial z_{ij}}(Z) = -\frac{\det(I - S\bar{S}^T)}{\det(I - Z\bar{S}^T)} A_{S,ij}(Z). \tag{15}$$

From (15), we obtain

$$\Re f_S(Z) = -\frac{\det(I - S\bar{S}^T)}{\det(I - Z\bar{S}^T)} \sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} A_{S,ij}(Z). \tag{16}$$

From (16), it is easy to see that

$$\begin{aligned} \frac{\partial \Re f_S(Z)}{\partial z_{kl}} &= \frac{\det(I - S\bar{S}^T)}{[\det(I - Z\bar{S}^T)]^2} A_{S,kl}(Z) \sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} A_{S,ij}(Z) \\ &\quad - \frac{\det(I - S\bar{S}^T)}{\det(I - Z\bar{S}^T)} \Big[\sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} A_{S,ij,kl}(Z) + A_{S,kl}(Z) \Big]. \end{aligned} \tag{17}$$

Then, from (17) and $|z_{ij}| < 1$ for each i and j, we have

$$\begin{aligned} \big|\nabla \Re f_S(Z)\big| &\le \frac{\det(I - S\bar{S}^T)}{[\det(I - Z\bar{S}^T)]^2} \sum_{k=1}^{m} \sum_{l=1}^{n} |A_{S,kl}(Z)| \sum_{i=1}^{m} \sum_{j=1}^{n} \big|A_{S,ij}(Z)\big| \\ &\quad + \frac{\det(I - S\bar{S}^T)}{\det(I - Z\bar{S}^T)} \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} \Big[\big|A_{S,ij,kl}(Z)\big| + \big|A_{S,kl}(Z)\big| \Big]. \end{aligned} \tag{18}$$

From (18) and Lemmas 10 and 11, we have

$$\det(I - Z\bar{Z}^T)\big|\nabla \Re f_S(Z)\big| \le C. \tag{19}$$

It is easy to see that $|f_S(0)| = 1$. From this and (19), it follows that $f_S \in \mathcal{Z}(\Re_I)$ and (13) holds.

The statement (b) and (14) can be similarly proven, and the details are omitted. The proof is complete. □

Remark 2. *Since $\det(I - S\bar{S}^T)$ converges to zero as $S \to \partial\Re_I$, we see that $\{f_S\}$ and $\{g_S\}$ uniformly converge to zero on any compact subset of $\Re_I$ as $S \to \partial\Re_I$.*

3. Boundedness and Compactness of M_ψ on $\mathcal{Z}^\alpha(\Re_I)$

We begin to study the boundedness and compactness of the multiplication operators on $\mathcal{Z}^\alpha(\Re_I)$. We have the following result about the boundedness.

Theorem 1. *Let $\alpha \ge 0$ and $\psi \in H(\Re_I)$. Then the following statements hold.*

(a) For $0 \le m\alpha < 1$, if $\psi \in \mathcal{Z}^\alpha(\Re_I)$, then the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

(b) For $m\alpha = 1$, if $\psi \in \mathcal{Z}^\alpha(\Re_I)$ and

$$M_1 := \sup_{Z \in \Re_I} \big[\det(I - Z\bar{Z}^T)\big]^\alpha |\Re\psi(Z)| \ln \frac{2e}{\det(I - Z\bar{Z}^T)} < \infty,$$

then the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

(c) For $1 < m\alpha < 2$, if $\psi \in \mathcal{Z}^\alpha(\Re_I)$ and

$$M_2 := \sup_{Z \in \Re_I} \frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha - \alpha - 1}} < \infty,$$

then the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

(d) For $m\alpha = 2$, if $\psi \in H^\infty(\Re_I)$,

$$M_3 := \sup_{Z \in \Re_I} \frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{1-\alpha}} < \infty,$$

and

$$M_4 := \sup_{Z \in \Re_I} \left[det(I - Z\bar{Z}^T)\right]^{\alpha} |\Re^2 \psi(Z)| \ln \frac{2e}{det(I - Z\bar{Z}^T)} < \infty,$$

then the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

(e) *For $m\alpha > 2$, if $\psi \in H^\infty(\Re_I)$,*

$$M_5 := \sup_{Z \in \Re_I} \frac{|\Re^2 \psi(Z)|}{[det(I - Z\bar{Z}^T)]^{m\alpha - \alpha - 2}} < \infty,$$

and

$$M_6 := \sup_{Z \in \Re_I} \frac{|\Re \psi(Z)|}{[det(I - Z\bar{Z}^T)]^{m\alpha - \alpha - 1}} < \infty,$$

then the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

Proof. We prove the statement (a). For $0 \leq m\alpha < 1$ and $\psi \in \mathcal{Z}^\alpha(\Re_I)$, it follows from Proposition 1 that $\psi \in H^\infty_\alpha(\Re_I) \cap \mathcal{B}^\alpha(\Re_I)$. Then, for $f \in \mathcal{Z}^\alpha(\Re_I)$, it follows from (a) in Lemmas 4 and 6 that

$$\begin{aligned}
s_1(M_\psi f) &= \sup_{Z \in \Re_I} \left[\det(I - Z\bar{Z}^T)\right]^{\alpha} |\Re^2 (M_\psi f)(Z)| \\
&= \sup_{Z \in \Re_I} \left[\det(I - Z\bar{Z}^T)\right]^{\alpha} |f(Z)\Re^2\psi(Z) + 2\Re f(Z)\Re\psi(Z) + \Re^2 f(Z)\psi(Z)| \\
&\leq \sup_{Z \in \Re_I} \left[\det(I - Z\bar{Z}^T)\right]^{\alpha} \Big(|f(Z)\Re^2\psi(Z)| + 2|\Re f(Z)\Re\psi(Z)| + |\Re^2 f(Z)\psi(Z)|\Big) \qquad (20) \\
&\leq C\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} + C\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} + \|f\|_{\mathcal{Z}^\alpha(\Re_I)}\|\psi\|_{H^\infty_\alpha(\Re_I)} \\
&= \Big(C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} + C\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} + \|\psi\|_{H^\infty_\alpha(\Re_I)}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.
\end{aligned}$$

From (20) and the basic fact $|f(0)| \leq \|f\|_{\mathcal{Z}^\alpha(\Re_I)}$, we have

$$\|M_\psi f\|_{\mathcal{Z}^\alpha(\Re_I)} \leq \Big(|\psi(0)| + C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} + C\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} + \|\psi\|_{H^\infty_\alpha(\Re_I)}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}. \qquad (21)$$

It follows from (21) that the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

Now, we prove the statement (b). From (b) in Lemma 4 and (a) in Lemma 6, we have

$$\begin{aligned}
s_1(M_\psi f) &= \sup_{Z \in \Re_I} \left[\det(I - Z\bar{Z}^T)\right]^{\alpha} |\Re^2 (M_\psi f)(Z)| \\
&\leq \sup_{Z \in \Re_I} \left[\det(I - Z\bar{Z}^T)\right]^{\alpha} \Big(|f(Z)\Re^2\psi(Z)| + 2|\Re f(Z)\Re\psi(Z)| + |\Re^2 f(Z)\psi(Z)|\Big) \\
&\leq C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} + C\sup_{Z \in \Re_I} |\Re\psi(Z)| \left[\det(I - Z\bar{Z}^T)\right]^{\alpha} \ln\frac{2}{\det(I - Z\bar{Z}^T)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \qquad (22) \\
&\quad + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)} \\
&= \Big(C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} + CM_1 + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.
\end{aligned}$$

From (22) and the assumption, it follows that the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$.

Next, we prove the statement (c). From (c) in Lemma 4 and (a) in Lemma 6, we have

$$
\begin{aligned}
s_1(M_\psi f) &\le \sup_{Z\in\Re_I}\left[\det(I-Z\bar{Z}^T)\right]^\alpha\Big(\left|f(Z)\Re^2\psi(Z)\right|+2\left|\Re f(Z)\Re\psi(Z)\right|+\left|\Re^2 f(Z)\psi(Z)\right|\Big)\\
&\le C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}+C\sup_{Z\in\Re_I}\frac{|\Re\psi(Z)|}{[\det(I-Z\bar{Z}^T)]^{m\alpha-\alpha-1}}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\\
&\quad+C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\\
&=\Big(C\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}+CM_2+\|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.
\end{aligned}
\tag{23}
$$

From (23) and the assumption, it follows that the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$. We prove the statement (d). From (c) in Lemma 4 and (b) in Lemma 6, it follows that

$$
\begin{aligned}
s_1(M_\psi f) &\le \sup_{Z\in\Re_I}\left[\det(I-Z\bar{Z}^T)\right]^\alpha\Big(\left|f(Z)\Re^2\psi(Z)\right|+2\left|\Re f(Z)\Re\psi(Z)\right|+\left|\Re^2 f(Z)\psi(Z)\right|\Big)\\
&\le C\sup_{Z\in\Re_I}\left[\det(I-Z\bar{Z}^T)\right]^\alpha|\Re^2\psi(Z)|\ln\frac{2}{\det(I-Z\bar{Z}^T)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\\
&\quad+C\sup_{Z\in\Re_I}\frac{|\Re\psi(Z)|}{[\det(I-Z\bar{Z}^T)]^{1-\alpha}}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}+\|\psi\|_{H^\infty(\Re_I)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\\
&=\Big(CM_4+CM_3+\|\psi\|_{H^\infty(\Re_I)}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.
\end{aligned}
\tag{24}
$$

From (24) and the assumption, it follows that the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$. Finally, we prove the statement (e). From (c) in Lemmas 4 and 6, we have

$$
\begin{aligned}
s_1(M_\psi f) &\le \sup_{Z\in\Re_I}\left[\det(I-Z\bar{Z}^T)\right]^\alpha\Big(\left|f(Z)\Re^2\psi(Z)\right|+2\left|\Re f(Z)\Re\psi(Z)\right|+\left|\Re^2 f(Z)\psi(Z)\right|\Big)\\
&\le C\sup_{Z\in\Re_I}\frac{|\Re^2\psi(Z)|}{[\det(I-Z\bar{Z}^T)]^{m\alpha-\alpha-2}}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\\
&\quad+C\sup_{Z\in\Re_I}\frac{|\Re\psi(Z)|}{[\det(I-Z\bar{Z}^T)]^{m\alpha-\alpha-1}}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}+\|\psi\|_{H^\infty(\Re_I)}\|f\|_{\mathcal{Z}^\alpha(\Re_I)}\\
&=\Big(CM_5+CM_6+\|\psi\|_{H^\infty(\Re_I)}\Big)\|f\|_{\mathcal{Z}^\alpha(\Re_I)}.
\end{aligned}
\tag{25}
$$

From (25) and the assumption, it follows that the operator M_ψ is bounded on $\mathcal{Z}^\alpha(\Re_I)$. The proof is complete. □

Next, we consider the compactness of the operator M_ψ on $\mathcal{Z}^\alpha(\Re_I)$.

Theorem 2. *Let $\alpha \ge 0$, $\psi \in H(\Re_I)$ and the operator M_ψ be bounded on $\mathcal{Z}^\alpha(\Re_I)$. Then the following statements hold.*

(a) For $0 \le m\alpha < 1$, if $\psi \in H_0^\infty(\Re_I)\cap\mathcal{B}_0^\alpha(\Re_I)\cap\mathcal{Z}_0^\alpha(\Re_I)$, then the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

(b) For $m\alpha = 1$, if $\psi \in H_0^\infty(\Re_I)\cap\mathcal{Z}_0^\alpha(\Re_I)\cap\mathcal{B}^\alpha(\Re_I)$ and

$$
\lim_{Z\to\partial\Re_I}\left[det(I-Z\bar{Z}^T)\right]^\alpha|\Re\psi(Z)|\ln\frac{2e}{det(I-Z\bar{Z}^T)}=0,
\tag{26}
$$

then the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

(c) For $1 < m\alpha < 2$, if $\psi \in H_0^\infty(\Re_I)\cap\mathcal{Z}_0^\alpha(\Re_I)\cap\mathcal{B}^\alpha(\Re_I)$ and

$$
\lim_{Z\to\partial\Re_I}\frac{|\Re\psi(Z)|}{[det(I-Z\bar{Z}^T)]^{m\alpha-\alpha-1}}=0,
\tag{27}
$$

then the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

(d) *For $m\alpha = 2$, if $\psi \in H_0^\infty(\Re_I) \cap \mathcal{B}^\alpha(\Re_I)$,*

$$\lim_{Z\to\partial\Re_I} \frac{|\Re\psi(Z)|}{[det(I - Z\bar{Z}^T)]^{1-\alpha}} = 0, \tag{28}$$

and

$$\lim_{Z\to\partial\Re_I} \left[det(I - Z\bar{Z}^T)\right]^\alpha |\Re^2\psi(Z)| \ln \frac{2e}{det(I - Z\bar{Z}^T)} = 0, \tag{29}$$

then the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

(e) *For $m\alpha > 2$, if $\psi \in H_0^\infty(\Re_I) \cap \mathcal{B}^\alpha(\Re_I) \cap \mathcal{Z}^\alpha(\Re_I)$,*

$$\lim_{Z\to\partial\Re_I} \frac{|\Re^2\psi(Z)|}{[det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-1}} = 0, \tag{30}$$

and

$$\lim_{Z\to\partial\Re_I} \frac{|\Re\psi(Z)|}{[det(I - Z\bar{Z}^T)]^{m\alpha-1}} = 0, \tag{31}$$

then the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

Proof. We first prove the statement (a). Let $\{f_k\}$ be a bounded sequence in $\mathcal{Z}^\alpha(\Re_I)$ and $f_k \to 0$ uniformly on any compact subset of $\Re_I$ as $k \to \infty$. To prove that the bounded operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$, by Lemma 7 we only need to prove that

$$\lim_{k\to\infty} \|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0.$$

Since $\{f_k\}$ is bounded, we assume that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Z}^\alpha(\Re_I)} \le M$, where M is a positive number. From (a) in Lemmas 4 and 6, it follows that

$$\sup_{Z\in\Re_I} |f_k(Z)| \le C_1 M \text{ and } \sup_{Z\in\Re_I} |\Re f_k(Z)| \le C_2 M \tag{32}$$

for all $k \in \mathbb{N}$. Since $\psi \in H_0^\infty(\Re_I) \cap \mathcal{B}_0^\alpha(\Re_I) \cap \mathcal{Z}_0^\alpha(\Re_I)$, for arbitrary $\varepsilon > 0$ there exists an $\sigma > 0$ such that on $K = \{Z \in \Re_I : \text{dist}(Z, \partial\Re_I) < \sigma\}$ it follows that

$$|\psi(z)| < \varepsilon \quad \text{and} \quad [\det(I - Z\bar{Z}^T)]^\alpha \left|\Re^j\psi(Z)\right| < \varepsilon \tag{33}$$

for $j = 1, 2$. Then, from (32) and (33), it follows that

$$\begin{aligned}
s_1(M_\psi f_k) &= \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|\Re^2(M_\psi f_k)(Z)\right| \\
&= \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\le \Big(\sup_{Z\in K} + \sup_{Z\in\Re_I\setminus K}\Big) \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\le C_1 M \sup_{Z\in K} \left[\det(I - Z\bar{Z}^T)\right]^\alpha |\Re^2\psi(Z)| + 2C_2 M \sup_{Z\in K} \left[\det(I - Z\bar{Z}^T)\right]^\alpha |\Re\psi(Z)| \\
&\quad + M \sup_{Z\in K} |\psi(Z)| + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re^2 f_k(Z)\right| \\
&\le (C_1 M + 2C_2 M + M)\varepsilon + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re^2 f_k(Z)\right|.
\end{aligned} \tag{34}$$

It is obvious to see that $\{f_k\}$ converges to zero uniformly on any compact subset of $\Re_I$ as $k \to \infty$ implies that $\{|\Re f_k|\}$ and $\{|\Re^2 f_k|\}$ also perform the same convergence as $i \to \infty$. Since $\Re_I \setminus K$ is a compact subset of $\Re_I$ and the obvious fact $|\psi(0) f_k(0)| \to 0$ as $k \to \infty$, it follows from (34) that

$$\lim_{k\to\infty} \|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0,$$

which shows that the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

Now, we prove the statement (b). Assume that $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha(\Re_I)$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha(\Re_I)} \leq M$ and $f_i \to 0$ uniformly on any compact subset of $\Re_I$ as $i \to \infty$. Then by Lemma 7 we only need to prove that

$$\lim_{i\to\infty} \|M_\psi f_i\|_{\mathcal{Z}^\alpha(\Re_I)} = 0.$$

First, since the sequence $\{f_i\}$ is bounded, by (a) in Lemma 6 there exists a positive constant $\widehat{C}$ such that $\sup_{Z\in\Re_I} |f_i(Z)| \leq \widehat{C}$ for all $i \in \mathbb{N}$. From the conditions, we see that for arbitrary $\varepsilon > 0$ there exists an $\sigma > 0$ such that on $K = \{Z \in \Re_I : \mathrm{dist}(Z, \partial\Re_I) < \sigma\}$ it follows that

$$\big|\psi(Z)\big| < \varepsilon, \tag{35}$$

$$[\det(I - Z\bar{Z}^T)]^\alpha |\Re^2 \psi(Z)| < \varepsilon, \tag{36}$$

and

$$\big[\det(I - Z\bar{Z}^T)\big]^\alpha |\Re\psi(Z)| \ln \frac{2e}{\det(I - Z\bar{Z}^T)} < \varepsilon. \tag{37}$$

For above ε and σ, by using (35)–(37), (b) in Lemma 4 and (a) in Lemma 6, we have

$$\begin{aligned}
s_1(M_\psi f_i) &= \sup_{Z\in\Re_I} \big[\det(I - Z\bar{Z}^T)\big]^\alpha \big|\Re^2(M_\psi f_i)(Z)\big| \\
&= \sup_{Z\in\Re_I} \big[\det(I - Z\bar{Z}^T)\big]^\alpha \big|f_i(Z)\Re^2\psi(Z) + 2\Re f_i(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_i(Z)\big| \\
&\leq \big(\sup_{Z\in K} + \sup_{Z\in\Re_I\setminus K}\big) \big[\det(I - Z\bar{Z}^T)\big]^\alpha \big|f_i(Z)\Re^2\psi(Z) + 2\Re f_i(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_i(Z)\big| \\
&\leq \widehat{C} \sup_{Z\in K} \big[\det(I - Z\bar{Z}^T)\big]^\alpha |\Re^2\psi(Z)| + CM \sup_{Z\in K} \big[\det(I - Z\bar{Z}^T)\big]^\alpha |\Re\psi(Z)| \ln \frac{2e}{\det(I - Z\bar{Z}^T)} \\
&\quad + M \sup_{Z\in K} |\psi(Z)| + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_i(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} \big|\Re f_i(Z)\big| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K} \big|\Re^2 f_i(Z)\big| \\
&\leq (\widehat{C} + CM + M)\varepsilon + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_i(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} \big|\Re f_i(Z)\big| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K} \big|\Re^2 f_i(Z)\big|.
\end{aligned} \tag{38}$$

It is obvious to see that $\{f_i\}$ converges to zero uniformly on any compact subset of $\Re_I$ as $i \to \infty$ implies that $\{|\Re f_i|\}$ and $\{|\Re^2 f_i|\}$ also perform the same convergence as $i \to \infty$. From (38), the compactness of $\Re_I \setminus K$, and the obvious fact $|\psi(0) f_i(0)| \to 0$ as $i \to \infty$, it follows that

$$\lim_{i\to\infty} \|M_\psi f_i\|_{\mathcal{Z}^\alpha(\Re_I)} = 0,$$

which shows that the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

Next, we prove the statement (c). Assume that $\{f_k\}$ is a sequence in $\mathcal{Z}^\alpha(\Re_I)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Z}^\alpha(\Re_I)} \le M$ and $f_k \to 0$ uniformly on any compact subset of $\Re_I$ as $k\to\infty$. Then by Lemma 7 we only need to prove that

$$\lim_{i\to\infty}\|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0.$$

Since $\{f_k\}$ is bounded in $\mathcal{Z}^\alpha(\Re_I)$, by (a) in Lemma 6 there exists a positive constant C such that $\sup_{Z\in\Re_I}|f_k(Z)| \le C$ for all $k\in\mathbb{N}$. From (c) in Lemma 4, it follows that there exists a positive constant $\widehat{C}$ such that

$$\left|\Re f_k(Z)\right| \le \frac{\widehat{C}}{[\det(I - Z\bar{Z}^T)]^{m\alpha-1}}.$$

Since $\psi \in H_0^\infty(\Re_I)\cap\mathcal{Z}_0^\alpha(\Re_I)$ and the assumption (27) holds, for arbitrary $\varepsilon>0$ there exists an $\sigma>0$ such that on $K=\{Z\in\Re_I : \mathrm{dist}(Z,\partial\Re_I)<\sigma\}$ it follows that

$$\left|\psi(Z)\right| < \varepsilon, \quad \left[\det(I - Z\bar{Z}^T)\right]^\alpha\left|\Re^2\psi(Z)\right| < \varepsilon, \tag{39}$$

and

$$\frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-1}} < \varepsilon. \tag{40}$$

For above ε and η, by using (39), (40), (c) in Lemma 4 and (a) in Lemma 6, we have

$$\begin{aligned}
s_1(M_\psi f_k) &= \sup_{Z\in\Re_I}\left[\det(I - Z\bar{Z}^T)\right]^\alpha\left|\Re^2(M_\psi f_k)(Z)\right| \\
&= \sup_{Z\in\Re_I}\left[\det(I - Z\bar{Z}^T)\right]^\alpha\left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\le \Big(\sup_{Z\in K} + \sup_{Z\in\Re_I\setminus K}\Big)\left[\det(I - Z\bar{Z}^T)\right]^\alpha\left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\le C\sup_{Z\in K}\left[\det(I - Z\bar{Z}^T)\right]^\alpha|\Re^2\psi(Z)| + \widehat{C}\sup_{Z\in K}\frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-1}} \\
&\quad + M\sup_{Z\in K}|\psi(Z)| + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\sup_{Z\in\Re_I\setminus K}|f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)}\sup_{Z\in\Re_I\setminus K}\left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)}\sup_{Z\in\Re_I\setminus K}\left|\Re^2 f_k(Z)\right| \\
&\le (C + \widehat{C} + M)\varepsilon + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)}\sup_{Z\in\Re_I\setminus K}|f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)}\sup_{Z\in\Re_I\setminus K}\left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)}\sup_{Z\in\Re_I\setminus K}\left|\Re^2 f_k(Z)\right|.
\end{aligned} \tag{41}$$

Since $\{f_k\}$ converges to zero uniformly on any compact subset of $\Re_I$ as $k\to\infty$ implies that $\{|\Re f_k|\}$ and $\{|\Re^2 f_k|\}$ also perform the same convergence as $k\to\infty$. From (41), the compactness of $\Re_I\setminus K$, and the obvious fact $|\psi(0)f_k(0)|\to 0$ as $k\to\infty$, it follows that

$$\lim_{k\to\infty}\|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0,$$

which shows that the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

We prove the statement (d). Assume that $\{f_k\}$ is a sequence in $\mathcal{Z}^\alpha(\Re_I)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Z}^\alpha(\Re_I)} \le M$ and $f_k\to 0$ uniformly on any compact subset of $\Re_I$ as $k\to\infty$. Then by Lemma 7 we only need to prove that

$$\lim_{i\to\infty}\|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0.$$

Since $\{f_k\}$ is bounded in $\mathcal{Z}^\alpha(\Re_I)$, by (b) in Lemma 6 there exists a positive constant C such that

$$|f_k(Z)| \leq C \ln \frac{2e}{\det(I - Z\bar{Z}^T)} \tag{42}$$

for all $Z \in \Re_I$ and $k \in \mathbb{N}$. From (c) in Lemma 4, it follows that there exists a positive constant $\widehat{C}$ such that

$$\left|\Re f_k(Z)\right| \leq \frac{\widehat{C}}{\det(I - Z\bar{Z}^T)} \tag{43}$$

for all $Z \in \Re_I$ and $k \in \mathbb{N}$. Since $\psi \in H_0^\infty(\Re_I)$ and the assumptions (28) and (29) hold, for arbitrary $\varepsilon > 0$ there exists an $\sigma > 0$ such that on $K = \{Z \in \Re_I : \mathrm{dist}(Z, \partial\Re_I) < \sigma\}$ it follows that

$$\left|\psi(Z)\right| < \varepsilon, \quad \frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{1-\alpha}} < \varepsilon, \tag{44}$$

and

$$[\det(I - Z\bar{Z}^T)]^\alpha |\Re^2\psi(Z)| \ln \frac{2e}{\det(I - Z\bar{Z}^T)} < \varepsilon. \tag{45}$$

For above ε and η, by using (42)-(45), (c) in Lemma 4 and (b) in Lemma 6, we have

$$\begin{aligned}
s_1(M_\psi f_k) &= \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|\Re^2(M_\psi f_k)(Z)\right| \\
&= \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\leq \Big(\sup_{Z\in K} + \sup_{Z\in\Re_I\setminus K}\Big)\left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\leq CM \sup_{Z\in K}[\det(I - Z\bar{Z}^T)]^\alpha |\Re^2\psi(Z)| \ln \frac{2e}{\det(I - Z\bar{Z}^T)} + 2\widehat{C}M \sup_{Z\in K} \frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{1-\alpha}} \\
&\quad + M \sup_{Z\in K} |\psi(Z)| + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re^2 f_k(Z)\right| \\
&\leq (C + 2\widehat{C}M + M)\varepsilon + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K} \left|\Re^2 f_k(Z)\right|.
\end{aligned} \tag{46}$$

Since $\{f_k\}$ converges to zero uniformly on any compact subset of $\Re_I$ as $k \to \infty$ implies that $\{|\Re f_k|\}$ and $\{|\Re^2 f_k|\}$ also perform the same convergence as $k \to \infty$. From (46), the compactness of $\Re_I \setminus K$, and the obvious fact $|\psi(0)f_k(0)| \to 0$ as $k \to \infty$, it follows that

$$\lim_{i\to\infty} \|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0,$$

which shows that the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$.

Finally, we prove the statement (e). Assume that $\{f_k\}$ is a sequence in $\mathcal{Z}^\alpha(\Re_I)$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Z}^\alpha(\Re_I)} \leq M$ and $f_k \to 0$ uniformly on any compact subset of $\Re_I$ as $k \to \infty$. Then by Lemma 7 we only need to prove that

$$\lim_{i\to\infty} \|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0.$$

Since $\{f_k\}$ is bounded in $\mathcal{Z}^\alpha(\Re_I)$, by (c) in Lemma 6 there exists a positive constant C such that

$$|f_k(Z)| \le \frac{CM}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-2}} \tag{47}$$

for all $Z \in \Re_I$ and $k \in \mathbb{N}$. From (c) in Lemma 4, it follows that there exists a positive constant $\widehat{C}$ such that

$$\left|\Re f_k(Z)\right| \le \frac{\widehat{C}M}{[\det(I - Z\bar{Z}^T)]^{m\alpha-1}} \tag{48}$$

for all $Z \in \Re_I$ and $k \in \mathbb{N}$. Since $\psi \in H_0^\infty(\Re_I)$ and the assumptions (30) and (31) hold, for arbitrary $\varepsilon > 0$ there exists an $\sigma > 0$ such that on $K = \{Z \in \Re_I : \mathrm{dist}(Z, \partial\Re_I) < \sigma\}$ it follows that

$$\left|\psi(Z)\right| < \varepsilon, \quad \frac{|\Re^2\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-2}} < \varepsilon, \tag{49}$$

and

$$\frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-1}} < \varepsilon. \tag{50}$$

For above ε and η, by using (47)-(50), (c) in Lemmas 4 and 6, we have

$$\begin{aligned}
s_1(M_\psi f_k) &= \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|\Re^2(M_\psi f_k)(Z)\right| \\
&= \sup_{Z\in\Re_I} \left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\le \Big(\sup_{Z\in K} + \sup_{Z\in\Re_I\setminus K}\Big)\left[\det(I - Z\bar{Z}^T)\right]^\alpha \left|f_k(Z)\Re^2\psi(Z) + 2\Re f_k(Z)\Re\psi(Z) + \psi(Z)\Re^2 f_k(Z)\right| \\
&\le CM\sup_{Z\in K} \frac{|\Re^2\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-2}} + 2\widehat{C}M\sup_{Z\in K}\frac{|\Re\psi(Z)|}{[\det(I - Z\bar{Z}^T)]^{m\alpha-\alpha-1}} \\
&\quad + M\sup_{Z\in K}|\psi(Z)| + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K}\left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K}\left|\Re^2 f_k(Z)\right| \\
&\le (C + 2\widehat{C}M + M)\varepsilon + \|\psi\|_{\mathcal{Z}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K} |f_k(Z)| + 2\|\psi\|_{\mathcal{B}^\alpha(\Re_I)} \sup_{Z\in\Re_I\setminus K}\left|\Re f_k(Z)\right| \\
&\quad + \|\psi\|_{H^\infty(\Re_I)} \sup_{Z\in\Re_I\setminus K}\left|\Re^2 f_k(Z)\right|.
\end{aligned} \tag{51}$$

Since $\{f_k\}$ converges to zero uniformly on any compact subset of $\Re_I$ as $k \to \infty$ implies that $\{|\Re f_k|\}$ and $\{|\Re^2 f_k|\}$ also perform the same convergence as $k \to \infty$. From (51), the compactness of $\Re_I \setminus K$, and the obvious fact $|\psi(0)f_k(0)| \to 0$ as $k \to \infty$, it follows that

$$\lim_{i\to\infty} \|M_\psi f_k\|_{\mathcal{Z}^\alpha(\Re_I)} = 0,$$

which shows that the operator M_ψ is compact on $\mathcal{Z}^\alpha(\Re_I)$. The proof is complete. □

We have the following necessary conditions for the compactness of M_ψ on $\mathcal{Z}^\alpha(\Re_I)$, which can be easily obtained by using the functions f_S and g_S.

Theorem 3. *Let $\alpha \ge 0$ and $\psi \in H(\Re_I)$. Then the following statements hold.*

(a) *For $\alpha = 1$, if the operator M_ψ is compact on $\mathcal{Z}(\Re_I)$, then*

$$\lim_{S\to\partial\Re_I} det(I - S\bar{S}^T)\left|\Re^2 M_\psi f_S(S)\right| = 0.$$

(b) *For $\alpha \neq 1$, if the operator M_ψ is compact on $\mathcal{Z}(\Re_I)$, then*

$$\lim_{S\to\partial\Re_I} [det(I - S\bar{S}^T)]^\alpha\left|\Re^2 M_\psi g_S(S)\right| = 0.$$

4. Conclusions

In this paper, the author obtains some sufficient conditions and necessary conditions of the boundedness and compactness for the multiplication operators on weighted Zygmund spaces of the first Cartan domain. The author still has not obtained the necessary and sufficient conditions for bounded and compact multiplication operators in this space. In addition to multiplication operators, one can study other operators in this space, such as composition operators and weighted composition operators.

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