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Coefficient Bounds for a Family of *s*-Fold Symmetric Bi-Univalent Functions

Isra Al-shbeil [1,*], Nazar Khan [2], Fairouz Tchier [3], Qin Xin [4], Sarfraz Nawaz Malik [5] and Shahid Khan [2]

1 Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
2 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22500, Pakistan
3 Mathematics Department, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia
4 Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands, Denmark
5 Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan
* Correspondence: i.shbeil@ju.edu.jo

Abstract: We present a new family of *s*-fold symmetrical bi-univalent functions in the open unit disc in this work. We provide estimates for the first two Taylor–Maclaurin series coefficients for these functions. Furthermore, we define the Salagean differential operator and discuss various applications of our main findings using it. A few new and well-known corollaries are studied in order to show the connection between recent and earlier work.

Keywords: analytic functions; univalent functions; bi-univalent functions; *m*-fold symmetric functions; coefficient estimates

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1. Introduction

Let Ω represents the collection of all analytical functions $\digamma$ with the series representation

$$\digamma(\tau) = \tau + \sum_{n=2}^{\infty} \alpha_n \tau^n, \tag{1}$$

which are analytic in $E = \{\tau : |\tau| < 1\}$. The class $\mathcal{S} \subset \Omega$ we mean the set of all univalent functions. The inverse function ($\digamma^{-1}$) of a univalent function $\digamma \in \mathcal{S}$ be presented as:

$$\tau = \digamma^{-1}(\digamma(\tau)), \quad \tau \in E$$

and

$$\eta = \digamma(\digamma^{-1}(\eta)),\ r_0(\digamma) \geq \frac{1}{4},\ |\eta| < r_0(\digamma),$$

where

$$g_1(\eta) = \digamma^{-1}(\eta) = \eta - \alpha_2\eta^2 + (2\alpha_2^2 - \alpha_3)\eta^3 - (5\alpha_2^3 - 5\alpha_2\alpha_3 + \alpha_4)\eta^4 + \ldots . \tag{2}$$

We speak $\digamma$ a bi-univalent function in E if each of the two functions $\digamma$ and $\digamma^{-1}$ are univalent in E. The class of all bi-univalent functions is symbolized by Σ. Here are very few examples of bi-univalent functions that are drawn from Srivastava et al. [1] first research.

$$h_1(\tau) = \frac{\tau}{1-\tau},\ h_2(\tau) = -\log(1-\tau),\ h_3(\tau) = \frac{1}{2}\log\left(\frac{1+\tau}{1-\tau}\right),\ \tau \in E.$$

The famous Koebe function

$$k(\tau) = \frac{\tau}{(1-\tau)^2}, \ \tau \in E,$$

is not in the class Σ.

The question of establishing bounds on the coefficients has always been a significant one in geometric function theory. The size of their coefficients can influence a number of features of analytical functions, including univalency, rate of growth, and distortion. Several researchers used various methods to address the aforementioned issues. Just like for univalent functions, determining coefficient estimates for bi-univalent functions has drawn a lot of interest recently. Lewin [2], while studying a subclass of Σ, shown that the bound on the second coefficient of the functions falling under the class Σ is less than one half (i.e., $|\alpha_2| < 1.5$). Lewin's result was improved by Brannan and Clunie [3], where they shown that $|\alpha_2| \leq \sqrt{2}$ and then by Netanyahu [4] to $|\alpha_2| \leq \frac{4}{3}$ which is an even greater improvement. In 1984, Tan [5] derived certain initial coefficient estimates for the class Σ, while Brannan and Taha [6] addressed several intriguing characteristics of subfamilies of Σ. In general, over the past few years, mathematicians have been interested in discovering the initial coefficient bounds for functions belonging to Σ. Remarkably, in [1] as well as the references referred therein such as [7–12]. reinvigorate the study of coefficient problems pertaining to bi-univalent functions. Many researchers find bounds for $|\alpha_2|$ and $|\alpha_3|$, that is for the first and second coefficient in Taylor (Maclatuin) series for diverse subclasses of bi-univalent functions, see [1] including [13–17].

However, such general coefficient estimation concerns were taken into account by numerous writers in some particular bi-univalent functions subclasses by using the expansions of the Lucas, Chebyshev, Legendrae, Horadam, Fibonacci, and Gegenbauer polynomials expansions (see, for example, [18–23]). For specific special subfamilies of Σ connected to any of the preceding polynomials, impressive results on initial coefficient estimates were developed in [24–27]. However for each of the listed below coefficients, $|\alpha_n|$, ($n \in \mathbb{N} \backslash \{1,2,3\}$, $\mathbb{N} := 1,2,3,\ldots$), the coefficient problem is remains a question.

A domain D is referred to as s-fold symmetric if a rotation of the domain D about the origin with an angle of $\frac{2\pi}{s}$ maps D on itself. An analytical function $\digamma$ in a domain E is said to be s-fold symmetric if

$$\digamma\left(e^{\frac{2\pi i}{s}}\tau\right) = e^{\frac{2\pi i}{s}}\digamma(\tau).$$

where s is a positive integer. For every $\digamma \in \mathcal{S}$, the function

$$h(\tau) = \sqrt[s]{\digamma(\tau^s)}, \tag{3}$$

is univalent and maps a region with s-fold symmetry onto the unit disc. We use $\mathcal{S}_s$ to represent the family of s-fold symmetrical univalent functions in E.

Obviously

$$\mathcal{S}_1 = \mathcal{S}.$$

A series expansion for $\digamma \in \mathcal{S}_s$ is provided by

$$\digamma(\tau) = \tau + \sum_{k=1}^{\infty} \alpha_{sk+1}\tau^{sk+1}. \tag{4}$$

In 2014, Srivastava et al. [28], investigated the natural extension of $\mathcal{S}_s$ and presented Σ_s a set of symmetric s-fold bi-univalent functions. The following is the series expansion for $g(\eta) = \digamma^{-1}(\eta)$

$$\begin{aligned} g(\eta) &= F^{-1}(\eta) = \eta - \alpha_{s+1}\eta^{s+1} + \left\{(s+1)\alpha_{s+1}^2 - \alpha_{2s+1}\right\}\eta^{2s+1} \\ &\quad -\left\{\frac{1}{2}(s+1)(3s+2)\alpha_{s+1}^3 - (3s+2)\alpha_{s+1}\alpha_{2s+1} + \alpha_{3s+1}\right\}\eta^{3s+1}. \end{aligned} \tag{5}$$

For $s = 1$, the Equations (2) and (5) become equal. Some functions in the family Σ_s are given as follows:

$$\begin{aligned} \zeta_1(\tau) &= \left(\frac{\tau^s}{1-\tau^s}\right)^s, \ \zeta_2(\tau) = [\log(1-\tau^s)]^{\frac{-1}{s}}, \\ \zeta_3(\tau) &= \log\sqrt{\frac{1+\tau^s}{1-\tau^s}}, \ \ \tau \in E \end{aligned}$$

and their corresponding inverse functions are

$$\begin{aligned} \zeta_1^{-1}(\tau) &= \left(\frac{\eta^s}{1+\eta^s}\right)^{\frac{1}{s}}, \zeta_2^{-1}(\tau) = \left(\frac{e^{2\eta^s}-1}{e^{2\eta^s}+1}\right)^{\frac{1}{s}}, \\ \zeta_3^{-3}(\tau) &= \left(\frac{e^{\eta^s}-1}{e^{\eta^s}}\right)^{\frac{1}{s}}. \end{aligned}$$

The study of the family Σ_s has recently gained relevance due to the research [29,30] which become base for a significant number of articles on subclasses of Σ_s. In a brand new subclass of Σ_s Srivastava et al. [31] investigated initial coefficient estimations of the Taylor–Maclaurin series expansion. Moreover Sakar and Tasar [32] presented subclasses of Σ_s and developed initial coefficient bounds for the functions included in these families (see also [33–35]). The following articles [36–42]. revealed intriguing findings about the initial coefficient estimations for specific subfamilies of Σ_s.

In this part of the article, we presented a few novel subclasses of s–fold symmetric bi-univalent functions and derived initial coefficient bounds $|\alpha_{s+1}|$ and $|\alpha_{2s+1}|$. We have taken motivation from the earlier works of Ma and Minda [40] and Tang et al. [41].

We will assume the following values during this whole paper

$$\tau,\ \eta \in E,\ F^{-1} = g,\ s \in \mathbb{N},\ 0 < \varkappa \le 1,\ 0 \le \beta < 1,\ 0 \le \gamma \le 1,\ 0 \le \xi < 1 \text{ and } 0 \le \varrho < 1.$$

1.1. The Class $\mathcal{S}_{\Sigma_s}(\varkappa, \gamma, \varrho, \xi)$

Definition 1. *A function $F \in \Sigma_s$, is seen as being a member of the class $\mathcal{S}_{\Sigma_s}(\varkappa, \gamma, \varrho, \xi)$ if the criteria listed below are satisfied:*

$$\left|\arg\left(\frac{1-\gamma}{1-\varrho}\left(\frac{\tau F'(\tau)}{F(\tau)} - \varrho\right) + \frac{\gamma}{1-\xi}\left(1 - \xi + \frac{\tau F''(\tau)}{F'(\tau)}\right)\right)\right| < \frac{\varkappa\pi}{2} \tag{6}$$

and

$$\left|\arg\left(\frac{1-\gamma}{1-\varrho}\left(\frac{\eta g'(\eta)}{g(\eta)} - \varrho\right) + \frac{\gamma}{1-\xi}\left(1 - \xi + \frac{\eta g''(\eta)}{g'(\eta)}\right)\right)\right| < \frac{\varkappa\pi}{2}. \tag{7}$$

Remark 1. *By taking the different values of the parameter γ, ϱ, ξ, s we can obtain some known subclasses of analytic bi-univalent functions.*

(1): $\mathcal{S}_{\Sigma_1}(\varkappa, \gamma, 0, 0) = \mathcal{S}_{\Sigma}(\varkappa, \gamma)$ introduced by Ali et al. [43].
(2): $\mathcal{S}_{\Sigma_s}(\varkappa, 0, 0, 0) = \mathcal{S}_{\Sigma_s}(\varkappa)$ introduced by Altinkaya and Yalcinn [44].
(3): $\mathcal{S}_{\Sigma_1}(\varkappa, 0, 0, 0) = \mathcal{S}_{\Sigma}(\varkappa)$ introduced by Brannan and Taha [24].

1.2. The Class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi)$

Definition 2. *A function $F \in \Sigma_s$, is seen as being a member of the class $S_{\Sigma_s}(\beta,\gamma,\varrho,\xi)$ if the criteria listed below are satisfied:*

$$Re\left(\frac{1-\gamma}{1-\varrho}\left(\frac{\tau F'(\tau)}{F(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{\tau F''(\tau)}{F'(\tau)}\right)\right)>\beta \tag{8}$$

and

$$Re\left(\frac{1-\gamma}{1-\varrho}\left(\frac{\eta g'(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{\eta g''(\eta)}{g'(\eta)}\right)\right)>\beta. \tag{9}$$

Remark 2. *By taking different values of the parameter γ, ϱ, ξ, s, we can obtain some known subclasses of analytic bi-univalent functions.*

(1): $\mathcal{S}_{\Sigma_1}(\beta,\gamma,0,0)=\mathcal{S}_{\Sigma}(\beta,\gamma)$ introduced by Ali et al. [43].
(2): $\mathcal{S}_{\Sigma_s}(\beta,0,0,0)=\mathcal{S}_{\Sigma_s}(\beta)$ introduced by Altinkaya and Yalcinn [44].
(3): $\mathcal{S}_{\Sigma_1}(\beta,0,0,0)=\mathcal{S}_{\Sigma}(\beta)$ introduced by Brannan and Taha [24].
Given below are a few preliminary findings that will be used to produce the main findings.

Lemma 1 **([45]).** *Let $p \in P$, where P is the Caratheodary class of analytic functions p in E satisfying*

$$Re(p(\tau))>0$$

and

$$p(\tau)=1+c_1\tau+c_2\tau^2+\ldots,$$

then

$$|c_n|\le 2,\ \ n\in\mathbb{N}.$$

In Section 2, for functions from the classes $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi)$ and $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi)$, we establish constraints on the first two coefficients in the Taylor–Maclaurin expansion. We also highlight noteworthy cases of our key findings and discuss significant linkages to earlier findings. We investigate the Salagean differential operator in Section 3 and provide two new classes of s–fold symmetric bi-univalent functions. For functions belonging to the classes $S_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi,p)$ and $S_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi,p)$, we find bounds on $|\alpha_{s+1}|$ and $|\alpha_{2s+1}|$ in the Taylor–Maclaurin expansion.

2. Main Results

Theorem 1. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}|\le\frac{2\varkappa(1-\varrho)(1-\xi)}{\sqrt{s^2T_1(s,\varrho,\xi,\gamma)[2\varkappa(1-\varrho)(1-\xi)-(\varkappa-1)T_1(s,\varrho,\xi,\gamma)]}}$$

and

$$|\alpha_{2s+1}|\le\frac{\varkappa(1-\varrho)(1-\xi)}{s\{(1-\gamma)(1-\xi)+\gamma(1-\varrho)(1+2s)\}}+\frac{2\varkappa^2(1-\varrho)^2(1-\xi)^2(1+s)}{s^2\{T_1(s,\varrho,\xi,\gamma)\}^2},$$

where

$$T_1(s,\varrho,\xi,\gamma)=\{(1-\gamma)(1-\xi)+\gamma(1+s)(1-\varrho)\}. \tag{10}$$

Proof. If we assume that $F\in\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi)$, then

$$\frac{1-\gamma}{1-\varrho}\left(\frac{\tau F'(\tau)}{F(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{\tau F''(\tau)}{F'(\tau)}\right)=[p(\tau)]^{\varkappa} \tag{11}$$

moreover , we have for its inverse map $g = F^{-1}$

$$\frac{1-\gamma}{1-\varrho}\left(\frac{\eta g'(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{\eta g''(\eta)}{g'(\eta)}\right)=[q(\eta)]^{\varkappa}, \tag{12}$$

where p and q are expressed in the following series:

$$p(\tau) = 1 + p_s\tau^s + p_{2s}\tau^{2s} + \ldots \tag{13}$$

and

$$q(\eta) = 1 + q_s\eta^s + q_{2s}\eta^{2s} + \ldots. \tag{14}$$

So, by analysing the coefficients in (11) and (12) we acquire

$$s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)}{1-\xi}\right)\alpha_{s+1} = \varkappa p_s, \tag{15}$$

$$\begin{pmatrix} 2s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \\ -s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)^2}{1-\xi}\right)\alpha_{s+1}^2 \end{pmatrix} = \varkappa p_{2s} + \frac{\varkappa(\varkappa-1)}{2}p_s^2, \tag{16}$$

$$-s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)}{1-\xi}\right)\alpha_{s+1} = \varkappa q_s, \tag{17}$$

$$\begin{pmatrix} s\left(\frac{(1-\gamma)(1+2s)}{1-\varrho}+\frac{\gamma(1+s)(1+3s)}{1-\xi}\right)\alpha_{s+1}^2 \\ -2s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \end{pmatrix} = \varkappa q_{2s} + \frac{\varkappa(\varkappa-1)}{2}q_s^2. \tag{18}$$

From (15) and (17) we obtain

$$p_s = -q_s, \tag{19}$$

and

$$2s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)}{1-\xi}\right)^2\alpha_{s+1}^2 = \varkappa^2(p_s^2+q_s^2). \tag{20}$$

Furthermore, form (16), (18), and (20) we have

$$\begin{aligned} &\left(\frac{2s^2(1-\gamma)}{1-\varrho}+\frac{2\gamma s^2(1+s)}{1-\xi}\right)\alpha_{s+1}^2 \\ = \;& \varkappa(p_{2s}+q_{2s})+\frac{\varkappa(\varkappa-1)}{2}(p_s^2+q_s^2) \\ = \;& \left\{\varkappa(p_{2s}+q_{2s})+\frac{s^2(\varkappa-1)}{\varkappa}\left\{\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)}{1-\xi}\right\}^2\alpha_{s+1}^2\right\}. \end{aligned}$$

Therefore we have

$$\alpha_{s+1}^2 = \frac{\varkappa^2(1-\varrho)^2(1-\xi)^2(p_{2s}+q_{2s})}{s^2T_1(s,\varrho,\xi,\gamma)[2\varkappa(1-\varrho)(1-\xi)-(\varkappa-1)T_1(s,\varrho,\xi,\gamma)]}, \tag{21}$$

where

$$T_1(s,\varrho,\xi,\gamma) = \{(1-\gamma)(1-\xi)+\gamma(1+s)(1-\varrho)\}. \tag{22}$$

Lemma 1 in conjuction with Equation (21) produces

$$|\alpha_{s+1}| \le \frac{2\varkappa(1-\varrho)(1-\xi)}{\sqrt{s^2T_1(s,\varrho,\xi,\gamma)[2\varkappa(1-\varrho)(1-\xi)-(\varkappa-1)T_1(s,\varrho,\xi,\gamma)]}}.$$

Next, by subtracting (18) from (16), we can determine the bound on $|\alpha_{2s+1}|$, that is

$$\begin{bmatrix} 4s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \\ \\ -2s(1+s)\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{s+1}^2 \end{bmatrix} = \varkappa(p_{2s}-q_{2s})+\frac{\varkappa(\varkappa-1)}{2}(p_s^2-q_s^2). \tag{23}$$

After that, taking into account (19) and (20), and using the Lemma 1, on (23) for p_{2s}, q_{2s}, p_s and q_s we have

$$|\alpha_{2s+1}| \le \frac{\varkappa(1-\varrho)(1-\xi)}{s\{(1-\gamma)(1-\xi)+\gamma(1-\varrho)(1+2s)\}}+\frac{2\varkappa^2(1-\varrho)^2(1-\xi)^2(1+s)}{s^2\{T_1(s,\varrho,\xi,\gamma)\}^2}.$$

As a result, the Theorem 1 proof is achieved. □

For $s=1$, in Theorem 1, the additional corollary for the new class $\mathcal{S}_\Sigma(\varkappa,\gamma,\varrho,\xi)$ is as follows:

Corollary 1. *If F belongs to the class $\mathcal{S}_\Sigma(\varkappa,\gamma,\varrho,\xi)$ and has the series representation described in (4), then*

$$|\alpha_2| \le \frac{2\varkappa(1-\varrho)(1-\xi)}{\sqrt{s^2T_2(\varrho,\xi,\gamma)[2\varkappa(1-\varrho)(1-\xi)-(\varkappa-1)T_2(\varrho,\xi,\gamma)]}}$$

and

$$|\alpha_3| \le \frac{\varkappa(1-\varrho)(1-\xi)}{\{(1-\gamma)(1-\xi)+3\gamma(1-\varrho)\}}+\frac{4\varkappa^2(1-\varrho)^2(1-\xi)^2}{\{T_2(\varrho,\xi,\gamma)\}^2},$$

where

$$T_2(\varrho,\xi,\gamma)=\{(1-\gamma)(1-\xi)+2\gamma(1-\varrho)\}.$$

For $\varrho=0$, in Theorem 1, we have the following new corollary for the new class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\xi)$.

Corollary 2. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\xi)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \le \frac{2\varkappa(1-\xi)}{\sqrt{s^2T_3(s,\xi,\gamma)[2\varkappa(1-\xi)-(\varkappa-1)T_3(s,\xi,\gamma)]}}$$

and

$$|\alpha_{2s+1}| \le \frac{\varkappa(1-\xi)}{s\{(1-\gamma)(1-\xi)+\gamma(1+2s)\}}+\frac{2\varkappa^2(1-\xi)^2(1+s)}{s^2\{T_3(s,\xi,\gamma)\}^2},$$

where

$$T_3(s,\xi,\gamma)=\{(1-\gamma)(1-\xi)+\gamma(1+s)\}.$$

For $\xi=0$, in Theorem 1, we acquire at the new class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho)$, and a corollary listed below.

Corollary 3. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \le \frac{2\varkappa(1-\varrho)}{\sqrt{s^2T_4(s,\varrho,\gamma)[2\varkappa(1-\varrho)-(\varkappa-1)T_4(s,\varrho,\gamma)]}}$$

and

$$|\alpha_{2s+1}| \le \frac{\varkappa(1-\varrho)(1-\xi)}{s\{(1-\gamma)+\gamma(1-\varrho)(1+2s)\}}+\frac{2\varkappa^2(1-\varrho)^2(1-\xi)^2(1+s)}{s^2\{T_4(s,\varrho,\gamma)\}^2},$$

where

$$T_4(s,\varrho,\gamma) = \{(1-\gamma)+\gamma(1+s)(1-\varrho)\}.$$

The following new corollary for the new class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma)$ exists for $\varrho = 0$, and $\xi = 0$, in Theorem 1.

Corollary 4. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \frac{2\varkappa}{\sqrt{2s^2\varkappa\{(1-\gamma)+\gamma(1+s)\} - s^2(\varkappa-1)\{(1-\gamma)+\gamma(1+s)\}^2}},$$

and

$$|\alpha_{2s+1}| \leq \frac{\varkappa}{s\{(1-\gamma)+\gamma(1+2s)\}} + \frac{2\varkappa^2(1-\varrho)^2(1-\xi)^2(1+s)}{s^2\{(1-\gamma)+\gamma(1+s)\}^2}.$$

For $\varrho = 0$, $s = 1$ and $\xi = 0$, in Theorem 1, then we have the result as demonstrated by Ali et al. in [43].

Corollary 5 ([43])**.** *If F belongs to the class $\mathcal{S}_{\Sigma}(\varkappa,\gamma)$ and has the series representation described in (4), then*

$$|\alpha_2| \leq \frac{2\varkappa}{\sqrt{(1+\gamma)[2\varkappa-(\varkappa-1)(1+\gamma)]}}$$

and

$$|\alpha_3| \leq \frac{\varkappa}{\{1+2\gamma\}} + \frac{4\varkappa^2(1-\varrho)^2(1-\xi)^2}{\{1+\gamma\}^2}.$$

For $\gamma = 0$, $\varrho = 0$, and $\xi = 0$, in Theorem 1, then we have the result as demonstrated by Altinkaya and Yalcinn in [44].

Corollary 6 ([44])**.** *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\varkappa)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \frac{2\varkappa}{s\sqrt{\varkappa+1)}}$$

and

$$|\alpha_{2s+1}| \leq \frac{\varkappa}{s} + \frac{2\varkappa^2(1+s)}{s^2}.$$

For $\gamma = 0$, $s = 1$, $\varrho = 0$, and $\xi = 0$, in Theorem 1, then we have the result as demonstrated by Murugusundaramoorthy in [46].

Corollary 7 ([46])**.** *If F belongs to the class $\mathcal{S}_{\Sigma}(\varkappa)$ and has the series representation described in (4), then*

$$|\alpha_2| \leq \frac{2\varkappa}{\sqrt{\varkappa+1}}$$

and

$$|\alpha_3| \leq 4\varkappa^2 + \varkappa.$$

Theorem 2. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)(1-\xi)(1-\varrho)}{s^2 T_1(s,\varrho,\xi,\gamma)}},$$

and

$$|\alpha_{2s+1}| \leq \frac{(1-\xi)(1-\varrho)(1-\beta)}{s[(1-\xi)(1-\gamma)+\gamma(1-\varrho)(1+2s)]} + \frac{2(1+s)(1-\beta)^2(1-\xi)^2(1-\varrho)^2}{s^2[T_1(s,\varrho,\xi,\gamma)]^2},$$

where $T_1(s,\varrho,\xi,\gamma)$ *is given by (10).*

Proof. Let $F \in \mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi)$, then

$$\frac{1-\gamma}{1-\varrho}\left(\frac{\tau F'(\tau)}{F(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{\tau F''(\tau)}{F'(\tau)}\right)=\beta+(1-\beta)p(\tau) \tag{24}$$

moreover , we have for its inverse map $g = F^{-1}$

$$\frac{1-\gamma}{1-\varrho}\left(\frac{\eta g'(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{\eta g''(\eta)}{g'(\eta)}\right)=\beta+(1-\beta)q(\eta), \tag{25}$$

where the expressions for p and q are given in (13) and (14). The coefficients are now equalised in (24) and (25), we arrive at

$$s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)}{1-\xi}\right)\alpha_{s+1}=(1-\beta)p_s, \tag{26}$$

$$\begin{pmatrix} 2s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \\ -s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)^2}{1-\xi}\right)\alpha_{s+1}^2 \end{pmatrix}=(1-\beta)p_{2s}, \tag{27}$$

$$-s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+s)}{1-\xi}\right)\alpha_{s+1}=(1-\beta)q_s, \tag{28}$$

$$\begin{pmatrix} s\left(\frac{(1-\gamma)(1+2s)}{1-\varrho}+\frac{\gamma(1+s)(1+3s)}{1-\xi}\right)\alpha_{s+1}^2 \\ -2s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \end{pmatrix}=(1-\beta)q_{2s}. \tag{29}$$

From (26) and (28) we obtain

$$p_s=-q_s, \tag{30}$$

Adding (27) and (29), we have

$$\left\{\frac{2s^2(1-\gamma)}{1-\varrho}+\frac{2\gamma s^2(1+s)}{1-\xi}\right\}\alpha_{s+1}^2=(1-\beta)(p_{2s}+q_{2s}), \tag{31}$$

therefore we have

$$\alpha_{s+1}^2=\frac{(1-\beta)(1-\xi)(1-\varrho)(p_{2s}+q_{2s})}{2s^2[(1-\gamma)(1-\xi)+\gamma(1+s)(1-\varrho)]}. \tag{32}$$

Equation (32) in conjunction with Lemma 1 yields

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)(1-\xi)(1-\varrho)}{s^2[(1-\gamma)(1-\xi)+\gamma(1+s)(1-\varrho)]}}.$$

Next, by subtracting (29) from (27), we can determine the bound on $|\alpha_{2s+1}|$, that is

$$\begin{bmatrix} 4s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \\ -2s(1+s)\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{s+1}^2 \end{bmatrix}=(1-\beta)(p_{2s}-q_{2s}),$$

$$\begin{aligned} & 4s\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{2s+1} \\ = & (1-\beta)(p_{2s}-q_{2s})+2s(1+s)\left(\frac{1-\gamma}{1-\varrho}+\frac{\gamma(1+2s)}{1-\xi}\right)\alpha_{s+1}^2, \end{aligned} \tag{33}$$

After that, taking into account (30) and (31), and using the Lemma 1, on (23) for p_{2s}, q_{2s}, p_s and q_s, we arrive at

$$|\alpha_{2s+1}| \leq \frac{(1-\xi)(1-\varrho)(1-\beta)}{s[(1-\xi)(1-\gamma)+\gamma(1-\varrho)(1+2s)]}+\frac{2(1+s)(1-\beta)^2(1-\xi)^2(1-\varrho)^2}{s^2[T_1(s,\varrho,\xi,\gamma)]^2},$$

As a result, the Theorem 2 proof is achieved. □

For $s=1$, in Theorem 2, the subsequent new corollary of new class $\mathcal{S}_\Sigma(\beta,\gamma,\varrho,\xi)$ is produced.

Corollary 8. *If F belongs to the class $\mathcal{S}_\Sigma(\beta,\gamma,\varrho,\xi)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)(1-\xi)(1-\varrho)}{(1-\xi)(1-\gamma)+2\gamma(1-\varrho)}},$$

and

$$|\alpha_{2s+1}| \leq \frac{(1-\xi)(1-\varrho)(1-\beta)}{[(1-\xi)(1-\gamma)+3\gamma(1-\varrho)]}+\frac{4(1-\beta)^2(1-\xi)^2(1-\varrho)^2}{[(1-\xi)(1-\gamma)+2\gamma(1-\varrho)]^2}.$$

For $\varrho=0$, in Theorem 2, the subsequent new corollary for a class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi)$ is produced.

Corollary 9. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\xi)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)(1-\xi)}{s^2[(1-\gamma)(1-\xi)+\gamma(1+s)]}},$$

and

$$|\alpha_{2s+1}| \leq \frac{(1-\xi)(1-\beta)}{s[(1-\xi)(1-\gamma)+\gamma(1+2s)]}+\frac{2(1+s)(1-\beta)^2(1-\xi)^2}{s^2[(1-\xi)(1-\gamma)+\gamma(1+s)]^2}.$$

For $\xi=0$, in Theorem 2, the subsequent new corollary for a class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho)$ is produced.

Corollary 10. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)(1-\varrho)}{s^2[(1-\gamma)+\gamma(1+s)(1-\varrho)]}},$$

and

$$|\alpha_{2s+1}| \leq \frac{(1-\varrho)(1-\beta)}{s[(1-\gamma)+\gamma(1-\varrho)(1+2s)]}+\frac{2(1+s)(1-\beta)^2(1-\varrho)^2}{s^2[(1-\gamma)+\gamma(1-\varrho)(1+s)]^2}.$$

For $\varrho=0$, and $\xi=0$, in Theorem 2, the subsequent new corollary for a class $\mathcal{S}_{\Sigma_s}(\beta,\gamma)$ is produced.

Corollary 11. *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\beta,\gamma)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)}{s^2[(1-\gamma)+\gamma(1+s)]}},$$

and

$$|\alpha_{2s+1}| \leq \frac{(1-\beta)}{s[(1-\gamma)+\gamma(1+2s)]} + \frac{2(1+s)(1-\beta)^2}{s^2[(1-\gamma)+\gamma(1+s)]^2}.$$

For $\varrho = 0$, $s = 1$ and $\xi = 0$, in Theorem 2, the subsequent known result is achieved that was proved by Ali et al. in [43].

Corollary 12 ([43]). *If F belongs to the class $\mathcal{S}_\Sigma(\beta,\gamma)$ and has the series representation described in (4), then*

$$|\alpha_2| \leq \sqrt{\frac{2(1-\beta)}{(1+\gamma}},$$

and

$$|\alpha_3| \leq \frac{(1-\beta)}{(1+2\gamma)} + \frac{4(1-\beta)^2}{(1+\gamma)^2}.$$

For $\varrho = 0$, $\xi = 0$, and $\gamma = 0$, in Theorem 2, then we have following known result proved by Altinkaya and Yalcinn [44].

Corollary 13 ([44]). *If F belongs to the class $\mathcal{S}_{\Sigma_s}(\beta)$ and has the series representation described in (4), then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)}{s}},$$

and

$$|\alpha_{2s+1}| \leq \frac{(1-\beta)}{s} + \frac{2(1+s)(1-\beta)^2}{s^2}.$$

For $\varrho = 0$, $\xi = 0, s = 1$ and $\gamma = 0$, in Theorem 2, then the subsequent known result is achieved that was proved by Murugusundaramoorthy in [46].

Corollary 14 ([46]). *If F belongs to the class $\mathcal{S}_\Sigma(\beta)$ and has the series representation described in (4), then*

$$|\alpha_2| \leq \sqrt{2(1-\beta)},$$

and

$$|\alpha_3| \leq 4(1-\beta)^2 + (1-\beta).$$

3. Applications of Salagean Differential Operator

In 1983, Salagean [47] defined the differential operator know as Salagean differential operator for analytic functions. By extending this idea, we define the Salagean differential operator for symmetric functions and discuss some of its applications for our main results.

Definition 3. *For $p \in \mathbb{N}$, the Salagean differential operator for $F \in \Sigma_s$ given in (4) is defined by*

$$\begin{aligned} S^0F(\tau) &= F(\tau),\ S^1F(\tau) = \tau F'(\tau), \dots, \\ S^pF(\tau) &= \tau\left(S^{p-1}F(\tau)\right)' = \left(\tau + \sum_{n=1}^{\infty}[sn+1]^p\alpha_{sn+1}\tau^{sn+1}\right), \\ &= \tau + \sum_{n=1}^{\infty}[sn+1]^p\alpha_{sn+1}\tau^{sn+1}. \end{aligned} \tag{34}$$

Remark 3. *For $s = 1$, we have the Salagean differential operator for analytic functions proved in [47].*

3.1. The Class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi,p)$

Definition 4. *A function $F \in \Sigma_s$ is referred to as belonging to class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi,p)$ if the following criteria are met:*

$$\left|\arg\left(\frac{1-\gamma}{1-\varrho}\left(\frac{S^pF(\tau)}{F(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}F(\tau)}{S^pF(\tau)}\right)\right)\right|<\frac{\varkappa\pi}{2}$$

and

$$Re\left(\frac{1-\gamma}{1-\varrho}\left(\frac{\tau S^pg(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}g(\eta)}{S^pg(\eta)}\right)\right)<\frac{\varkappa\pi}{2}.$$

3.2. The Class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi,p)$

Definition 5. *A function $F\in\Sigma_s$ is referred to as belonging to class $S_{\Sigma_s}(\beta,\gamma,\varrho,\xi,p)$ if the following criteria are met:*

$$Re\left(\frac{1-\gamma}{1-\varrho}\left(\frac{S^pF(\tau)}{F(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}F(\tau)}{S^pF(\tau)}\right)\right)>\beta$$

and

$$\left|\arg\left(\frac{1-\gamma}{1-\varrho}\left(\frac{S^pg(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}g(\eta)}{S^pg(\eta)}\right)\right)\right|>\beta.$$

Theorem 3. *Let F given by (4) be in the class $\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi,p)$, then*

$$|\alpha_{s+1}|\le\frac{2\varkappa}{\sqrt{[\varkappa(Q_1(\gamma,\varrho,s,p)+\varkappa Q_2(\gamma,\varrho,s,p))-(\varkappa-1)(Q_3(\gamma,\varrho,s,p))]}}$$

and

$$|\alpha_{2s+1}|\le\frac{4\varkappa}{Q_4(\gamma,\varrho,s,p)}+\frac{4\varkappa^2Q_5(\gamma,\varrho,s,p)}{Q_4(\gamma,\varrho,s,p)Q_3(\gamma,\varrho,s,p)},$$

where,

$$Q_1(\gamma,\varrho,s,p) = \frac{1-\gamma}{1-\varrho}\{(s+1)([1+2s]^p-1)-2([1+s]^p-1)\}, \tag{35}$$

$$Q_2(\gamma,\varrho,s,p) = \frac{\gamma}{1-\xi}\Big(2s(s+1)([1+2s]^p)-2s[1+s]^{2p}\Big), \tag{36}$$

$$Q_3(\gamma,\varrho,s,p) = \left(\left(\frac{1-\gamma}{1-\varrho}\right)([s+1]^p-1)+\frac{\gamma s([1+s]^p)}{1-\xi}\right)^2, \tag{37}$$

$$Q_4(\gamma,\varrho,s,p) = \left(2\left(\frac{1-\gamma}{1-\varrho}\right)([1+2s]^p-1)+\frac{4\gamma s([1+2s]^p}{1-\xi}\right), \tag{38}$$

$$Q_5(\gamma,\varrho,s,p) = \left(\frac{1-\gamma}{1-\varrho}(s+1)([1+2s]^p-1)-\frac{2\gamma s(s+1)(1+2s)^p}{1-\xi}\right). \tag{39}$$

Proof. Let $F\in\mathcal{S}_{\Sigma_s}(\varkappa,\gamma,\varrho,\xi,p)$, then

$$\frac{1-\gamma}{1-\varrho}\left(\frac{S^pF(\tau)}{F(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}F(\tau)}{S^pF(\tau)}\right)=[p(\tau)]^{\varkappa} \tag{40}$$

moreover , we have for its inverse map $g=F^{-1}$

$$\frac{1-\gamma}{1-\varrho}\left(\frac{S^pg(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}g(\eta)}{S^pg(\eta)}\right)=[q(\eta)]^{\varkappa}, \tag{41}$$

where the expressions for $p(\tau)$ and $q(\eta)$ are given in (13) and (14). The coefficients are now equalised in (40) and (41) we arrive at

$$\left(\left(\frac{1-\gamma}{1-\varrho}\right)\left([s+1]^p-1\right)+\frac{\gamma s([1+s]^p)}{1-\xi}\right)\alpha_{s+1}=\varkappa r_s, \tag{42}$$

$$\begin{bmatrix} \left(\left(\frac{1-\gamma}{1-\varrho}\right)\left([2s+1]^p-1\right)+\frac{2\gamma s[1+2s]^p}{1-\xi}\right)\alpha_{2s+1} \\ -\left(\frac{1-\gamma}{1-\varrho}\left([s+1]^p-1\right)+\frac{\gamma s([1+s]^{2p}}{1-\xi}\right)\alpha_{s+1}^2 \end{bmatrix}=\varkappa r_{2s}+\frac{\varkappa(\varkappa-1)}{2}r_s^2, \tag{43}$$

$$-\left(\left(\frac{1-\gamma}{1-\varrho}\right)\left([s+1]^p-1\right)+\frac{\gamma s([1+s]^p)}{1-\xi}\right)\alpha_{s+1}=\varkappa q_s, \tag{44}$$

$$\begin{bmatrix} \left[\frac{(1-\gamma)}{1-\varrho}\left[(s+1)\{(1+2s)^p-1\}-\{(1+s)^p-1\}\right]\right. \\ \left.+\frac{\gamma}{1-\xi}\left\{2s(1+s)(2s+1)^p-s(s+1)^{2p}\right\}\right]\alpha_{s+1}^2 \\ -\left\{\frac{1-\gamma}{1-\varrho}\left([2s+1]^p-1\right)+\frac{2\gamma s([1+2s]^p)}{1-\xi}\right\}\alpha_{2s+1} \end{bmatrix}=\varkappa q_{2s}+\frac{\varkappa(\varkappa-1)}{2}q_s^2. \tag{45}$$

From (42) and (44) we obtain

$$r_s=-q_s \tag{46}$$

and

$$2\left(\left(\frac{1-\gamma}{1-\varrho}\right)\left([s+1]^p-1\right)+\frac{\gamma s([1+s]^p)}{1-\xi}\right)^2\alpha_{s+1}^2=\varkappa^2(r_s^2+q_s^2). \tag{47}$$

Furthermore, form (43), (45), and (47) we have

$$\begin{aligned} &\frac{1-\gamma}{1-\varrho}\left\{(s+1)\left([1+2s]^p-1\right)-2\left([1+s]^p-1\right)\right\} \\ &+\frac{\gamma}{1-\xi}\left(2s(s+1)\left([1+2s]^p\right)-2s[1+s]^{2p}\right)\alpha_{s+1}^2 \\ =\;&\varkappa(r_{2s}+q_{2s})+\frac{\varkappa(\varkappa-1)}{2}(r_s^2+q_s^2) \\ =\;&\varkappa(r_{2s}+q_{2s})+\frac{(\varkappa-1)}{\varkappa}\begin{pmatrix}\left(\frac{1-\gamma}{1-\varrho}\right)\left([s+1]^p-1\right) \\ +\frac{\gamma s([1+s]^p)}{1-\xi}\end{pmatrix}^2\alpha_{s+1}^2. \end{aligned}$$

This can be written as

$$\begin{aligned} &(Q_1(\gamma,\varrho,s,p)+Q_2(\gamma,\varrho,s,p))\alpha_{s+1}^2 \\ =\;&\varkappa(r_{2s}+q_{2s})+\frac{\varkappa-1)}{\varkappa}(Q_3(\gamma,\varrho,s,p))\alpha_{s+1}^2, \\ &[\varkappa(Q_1(\gamma,\varrho,s,p)+\varkappa Q_2(\gamma,\varrho,s,p))-(\varkappa-1)(Q_3(\gamma,\varrho,s,p))] \\ =\;&\varkappa^2(r_{2s}+q_{2s}) \end{aligned}$$

Therefore we have

$$\alpha_{s+1}^2=\frac{\varkappa^2(r_{2s}+q_{2s})}{[\varkappa(Q_1(\gamma,\varrho,s,p)+\varkappa Q_2(\gamma,\varrho,s,p))-(\varkappa-1)(Q_3(\gamma,\varrho,s,p))]}, \tag{48}$$

where Q_1, Q_2 and Q_3 are given by (35)–(37).

Lemma 1 in conjuction with Equation (48) yields

$$|\alpha_{s+1}| \leq \frac{2\varkappa}{\sqrt{[\varkappa(Q_1(\gamma,\varrho,s,p)+\varkappa Q_2(\gamma,\varrho,s,p))-(\varkappa-1)(Q_3(\gamma,\varrho,s,p))]}}.$$

Next, by subtracting (45) from (44), we can determine the bound on $|\alpha_{2s+1}|$, that is

$$\begin{aligned} &\left[\begin{array}{c} 2\left\{\left(\frac{1-\gamma}{1-\varrho}\right)\left((1+2s)^p-1\right)+\frac{4\gamma s([1+2s]^p}{1-\xi}\right\}\alpha_{2s+1} \\ -\left\{\frac{1-\gamma}{1-\varrho}(s+1)\left((1+2s)^p-1\right)-\frac{2\gamma s(s+1)(1+2s)^p}{1-\xi}\right\}\alpha_{s+1}^2 \end{array}\right] \\ =&\ \varkappa(r_{2s}-q_{2s})+\frac{\varkappa(\varkappa-1)}{2}(r_s^2-q_s^2), \end{aligned}$$

or

$$Q_4(\gamma,\varrho,s,p)\alpha_{2s+1}+Q_5(\gamma,\varrho,s,p)\alpha_{s+1}^2=\varkappa(r_{2s}-q_{2s})+\frac{\varkappa(\varkappa-1)}{2}(r_s^2-q_s^2), \tag{49}$$

where Q_4 and Q_5 are given in (38) and (39).

After that, taking into account (46), (47) and using Lemma 1, on (49), we arrive at

$$|\alpha_{2s+1}| \leq \frac{4\varkappa}{Q_4(\gamma,\varrho,s,p)}+\frac{4\varkappa^2 Q_5(\gamma,\varrho,s,p)}{Q_4(\gamma,\varrho,s,p)Q_3(\gamma,\varrho,s,p)}.$$

This completes the proof. □

Theorem 4. *Let $\digamma$ given by (4) be in the class $\mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi,p)$, then*

$$|\alpha_{s+1}| \leq \sqrt{\frac{2(1-\beta)}{Q_3(\gamma,\varrho,s,p)}},$$

and

$$|\alpha_{2s+1}| \leq \frac{4(1-\beta)}{Q_4(\gamma,\varrho,s,p)}+\frac{4(1-\beta)Q_5(\gamma,\varrho,s,p)}{Q_3(\gamma,\varrho,s,p)},$$

where, $Q_1(\gamma,\varrho,s,p)$, $Q_2(\gamma,\varrho,s,p)$, $Q_3(\gamma,\varrho,s,p)$, $Q_4(\gamma,\varrho,s,p)$, and $Q_5(\gamma,\varrho,s,p)$ are given by (35)–(39).

Proof. Let $\digamma \in \mathcal{S}_{\Sigma_s}(\beta,\gamma,\varrho,\xi,p)$, then

$$\frac{1-\gamma}{1-\varrho}\left(\frac{S^p\digamma(\tau)}{\digamma(\tau)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}\digamma(\tau)}{S^p\digamma(\tau)}\right)=\beta+(1-\beta)-p(\tau), \tag{50}$$

moreover , we have for its inverse map $g=\digamma^{-1}$

$$\frac{1-\gamma}{1-\varrho}\left(\frac{S^pg(\eta)}{g(\eta)}-\varrho\right)+\frac{\gamma}{1-\xi}\left(1-\xi+\frac{S^{p+1}g(\eta)}{S^p(\eta)}\right)=\beta+(1-\beta)q(\eta), \tag{51}$$

where the expressions for $p(\tau)$ and $q(\eta)$ are given in (13) and (14). The coefficients are now equalised in (50) and (51), we obtain

$$\left(\left(\frac{1-\gamma}{1-\varrho}\right)\left((s+1)^p-1\right)+\frac{\gamma s((1+s)^p)}{1-\xi}\right)\alpha_{s+1}=(1-\beta)r_s, \tag{52}$$

$$\left[\begin{array}{c} \left(\frac{1-\gamma}{1-\varrho}\right)\left\{\left((2s+1)^p-1\right)+\frac{2\gamma s[1+2s]^p}{1-\xi}\right\}\alpha_{2s+1} \\ -\left(\frac{1-\gamma}{1-\varrho}\left((s+1)^p-1\right)+\frac{\gamma s([1+s]^{2p}}{1-\xi}\right)\alpha_{s+1}^2 \end{array}\right]=(1-\beta)r_{2s}, \tag{53}$$

$$-\left(\frac{1-\gamma}{1-\varrho}\right)\left\{\left((s+1)^p-1\right)+\frac{\gamma s((1+s)^p)}{1-\xi}\right\}\alpha_{s+1}=(1-\beta)q_s, \tag{54}$$

$$\begin{bmatrix} \left\{\frac{(1-\gamma)}{1-\varrho}\left((s+1)\left((1+2s)^p-1\right)-\left((1+s)^p-1\right)\right)\right. \\ \left.+\frac{\gamma}{1-\xi}\left(2s(1+s)(2s+1)^p-s(s+1)^{2p}\right)\right\}\alpha_{s+1}^2 \\ -\frac{1-\gamma}{1-\varrho}\left\{\left((2s+1)^p-1\right)+\frac{2\gamma s([1+2s]^p)}{1-\xi}\right\}\alpha_{2s+1} \end{bmatrix}=(1-\beta)q_{2s}. \tag{55}$$

From (52) and (54) we obtain

$$r_s=-q_s, \tag{56}$$

Adding (53) and (55), we have

$$2\left\{\left(\frac{1-\gamma}{1-\varrho}\right)\left((s+1)^p-1\right)+\frac{\gamma s([1+s]^p)}{1-\xi}\right\}^2\alpha_{s+1}^2=(1-\beta)(r_{2s}+q_{2s}), \tag{57}$$

therefore we have

$$\alpha_{s+1}^2=\frac{(1-\beta)(r_{2s}+q_{2s})}{2\left[\left(\frac{1-\gamma}{1-\varrho}\right)\left((s+1)^p-1\right)+\frac{\gamma s([1+s]^p)}{1-\xi}\right]^2}. \tag{58}$$

Lemma 1, in conjuction with Equation (58) produced

$$|\alpha_{s+1}|\leq\sqrt{\frac{2(1-\beta)}{Q_3(\gamma,\varrho,s,p)}}.$$

Next, by subtracting (55) from (53), we can determine the bound on $|\alpha_{2s+1}|$, that is

$$Q_4(\gamma,\varrho,s,p)\alpha_{2s+1}+Q_5(\gamma,\varrho,s,p)\alpha_{s+1}^2=(1-\beta)(r_{2s}-q_{2s}), \tag{59}$$

After that, taking into account (56) and (57), and using Lemma 1, on Equation (59), we have

$$|\alpha_{2s+1}|\leq\frac{4(1-\beta)}{Q_4(\gamma,\varrho,s,p)}+\frac{4(1-\beta)Q_5(\gamma,\varrho,s,p)}{Q_3(\gamma,\varrho,s,p)}.$$

As a result, the Theorem 4 proof is achieved. □

4. Conclusions

In the open unit disc E, we introduced a new family of bi-univalent functions that are s-fold symmetric, and we discovered the upper bounds $|\alpha_{s+1}|$ and $|\alpha_{2s+1}|$ for the functions falling within the newly defined classes. In Section 3, we also established the coefficient estimates $|\alpha_{s+1}|$ and $|\alpha_{2s+1}|$ for a novel family of symmetric bi-univalent functions that are connected with the Salagean differential operator. We examine a number of unique cases from this family, and our findings generalize those from [43,44,48–50].

This research glanced at a new family that may lead to further research into a variety of topics, including some unique families of bi-univalent functions using the Hohlov operator connected to the Legendre polynomial [51], the integro-differential operator [52], the q-derivative operator [53], the Fractional q-difference operator [20], the Faber polynomial [41], Modified sigmoid activated function and k-Fibonacci numbers [21], Horadam polynomials involving modified sigmoid [22], Pascal distribution series and Gegenbauer polynomials [23], Gegenbauer polynomial [18], Hankel and Symmetric Toeplitz Determinants for a New Subclass of q-Starlike Functions [54] and so on.

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