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Communication

Unification Theories: Rings, Boolean Algebras and Yang–Baxter Systems

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Abstract: This paper continues a series of papers on unification constructions. After a short discussion on the Euler's relation, we introduce a matrix version of the Euler's relation, $E^{I\,\pi} + U = O$. We refer to a related equation, the Yang–Baxter equation, and to Yang–Baxter systems. The most consistent part of the paper is on the unification of rings and Boolean algebras. These new structures are related to the Yang–Baxter equation and to Yang–Baxter systems.

Keywords: Euler's relation; Yang–Baxter equation; rings; Boolean algebras

MSC: 03G05; 06E05; 16B50; 16B70; 16T25; 32A05

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1. Introduction and Motivations

One of the most famous formulas in mathematics is the Euler's relation:

$$e^{i\,\pi} + 1 = 0\,. \tag{1}$$

It is related to an inequality, which has a geometrical interpretation (see the Figure 1):

$$|\,e^{i} - \pi\,| > e\,. \tag{2}$$

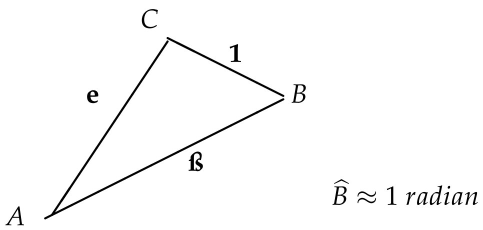

Figure 1. The inequality $|e^{i} - \pi| > e$, could also be read as $|e^{i} - \pi| \approx e$.

The Formulas (1) and (2) were unified in [1], and there exists a matrix version of Formula (1) in Remark 4:

$$E^{I\,\pi} + \mathbb{U} = \mathbb{O}\,, \tag{3}$$

where

$$E = \begin{pmatrix} e & i\sinh 1 \\ 0 & e^{-1} \end{pmatrix},\quad I = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix},\quad \mathbb{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

On the other hand, the Yang–Baxter equation [2] which appeared in theoretical physics and statistical mechanics (see [3–5]) has led to many applications in these fields and is also related to the Euler's formula. This equation can be understood as a unifying equation (see, for example, [6–8]).

Some results obtained under the guidance of Stefan Papadima were presented at *The 14th International Workshop on Differential Geometry and Its Applications, Ploiesti, 2019* (DGA 14). It was observed during that workshop that some results from [9] can be extended for the virtual braid groups from [10]. Moreover, the techniques of [11] can be considered in the framework of certain unification structures (from the article [6]). The Yang–Baxter equation plays an important role in knot theory. Dăscălescu and Nichita have shown in [12] how to associate a Yang–Baxter operator to an arbitrary associative algebra. Turaev has described in [13] a general scheme to derive an invariant of oriented links from a Yang–Baxter operator, provided this one can be "enhanced". The invariant which was obtained from these Yang–Baxter operators is the Alexander polynomial of knots. Thus, in a way, the Alexander polynomial is the knot invariant corresponding to the axioms of (unitary associative) algebras. (This result was highly appreciated by Stefan Papadima.) More recently, Cristina Anghel-Palmer has constructed a polynomial which unifies the Alexander and the Jones polynomials of knots (see [14,15]).

The main characters of our current paper are rings, Boolean algebras and Yang–Baxter systems. The rings and Boolean algebras are usually thought of as complementary structures because they arise from different theories, but sometimes it is useful to compare their axioms and properties. Our approach in this paper is somewhat enhanced from the usual point of view. Thus, we introduce structures which unify Boolean algebras and rings, called B–rings (*B* from Boolean algebras followed by *ring*).

After the definition of B–rings, we give examples, study the implications of this new approach and present some properties of these new structures.

2. Yang–Baxter Equations and Yang–Baxter Systems

Before presenting the unification of rings and Boolean algebras, we need to recall some facts about the Yang–Baxter equation and related structures. Moreover, a certain type of solution to the "colored" Yang–Baxter equation will lead us to a matrix version of the Euler's relation (3).

The following notations and terminology will be used in this section: For the moment, we will work over a field k, and the tensor products will be defined over this field. Let V be a vector space over k. Let $I_V : V \to V$ be the identity map of the space V. We denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$.

For $R : V \otimes V \to V \otimes V$ a k-linear map, let $R^{12} = R \otimes I_V$, $R^{23} = I_V \otimes R : V \otimes V \otimes V \to V \otimes V \otimes V$. In a similar manner, we denote by R^{13} a linear map acting on the first and third component of $V \otimes V \otimes V$.

Definition 1. *A* ***Yang–Baxter operator*** *is a k-linear map $R : V \otimes V \to V \otimes V$, which satisfies the braid condition (the Yang–Baxter equation):*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{5}$$

We also require that the map R is invertible.

Remark 1. *An important observation is that, if R satisfies (5), then both $R \circ \tau$ and $\tau \circ R$ satisfy the QYBE (the quantum Yang–Baxter equation):*

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{6}$$

So, the Equations (5) and (6) are equivalent.

Remark 2. *The following construction of Yang–Baxter operators is described in [12].*

If A is a k-algebra, then, for all non-zero $r, s \in k$, the linear map

$$R^A_{r,s} : A \otimes A \to A \otimes A, \quad a \otimes b \mapsto sab \otimes 1 + r1 \otimes ab - sa \otimes b \tag{7}$$

is a Yang–Baxter operator.

Remark 3. *For a certain matrix J (see [1]), the colored operator $R(x) = e^{x\,J} = \cos x + \sin x J$ satisfies the following version of the Yang–Baxter equation:*

$$R^{12}(x) \circ R^{23}(x+y) \circ R^{12}(y) \ = R^{23}(y) \circ R^{12}(x+y) \circ R^{23}(x)\,. \tag{8}$$

Remark 4. *With the formula $R(x) = e^{x\,J} = \cos x + \sin x J$ in mind, we will return to Formula (3). By A^B, we understand $e^{B\ \ln A}$, in which we use the Taylor expansion of the function $f(x) = \ln x$.*

Let us now recall that

$$E = \begin{pmatrix} e & i \sinh 1 \\ 0 & e^{-1} \end{pmatrix}, \quad I = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix}. \tag{9}$$

We observe that $E = e^{(-i)I}$ (this follows from the Taylor expansion of the function e^x). Now, $E^{\pi I} = [e^{(-i)I}]^{\pi\ I} = e^{\pi i \mathbb{U}} = (\cos \pi)\ \mathbb{U}\ +\ (i\mathbb{U}) \sin \pi = -\mathbb{U}$. So, $E^{\pi I} +\ \mathbb{U} =\ \mathbb{O}$.

Remark 5. *There is a similar terminology for the set-theoretical Yang–Baxter equation. In this case, V is replaced by a set X and the tensor product by the Cartesian product. For example, let $R : X \times X \to X \times X$, be defined by $R(p,\ q) = (p \vee q,\ p \wedge q)$ in a Boolean algebra. It follows that*

$$(R \times I_X) \circ (I_X \times R) \circ (R \times I_X) = (I_X \times R) \circ (R \times I_X) \circ (I_X \times R)\,. \tag{10}$$

Remark 6. Example of solution to the set-theoretical Yang–Baxter equation. *Another interesting solution to the set-theoretical Yang–Baxter equation is given by the Boolean map $R : X \times X \to X \times X,\ \ (p,\ q) \mapsto (p \to q,\ p)$.*

We now present the Yang–Baxter systems theory following the paper [16]. Yang–Baxter systems were introduced in [17] as a spectral-parameter-independent generalization of quantum Yang–Baxter. They are conveniently defined in terms of *Yang–Baxter commutators*. Consider three vector spaces V, V', V'' and three linear maps $R : V \otimes V' \to V \otimes V'$, $S : V \otimes V'' \to V \otimes V''$ and $T : V' \otimes V'' \to V' \otimes V''$.

Then a *Yang–Baxter commutator* is a map $[R,S,T] : V \otimes V' \otimes V'' \to V \otimes V' \otimes V''$, defined by

$$[R,S,T] = R_{12} \circ S_{13} \circ T_{23} - T_{23} \circ S_{13} \circ R_{12}\,. \tag{11}$$

In terms of a Yang–Baxter commutator, the quantum Yang–Baxter equation (6) is expressed simply as $[R,R,R] = 0$.

Definition 2. *Let V and V' be vector spaces. A system of linear maps*

$$W : V \otimes V \to V \otimes V, \quad Z : V' \otimes V' \to V' \otimes V', \quad X : V \otimes V' \to V \otimes V'$$

is called a WXZ-system *or a* Yang–Baxter system, *provided the following equations are satisfied:*

$$[W,W,W] \ = \ 0\,, \tag{12}$$

$$[Z,Z,Z] \ = \ 0\,, \tag{13}$$

$$[W,X,X] \ = \ 0\,, \tag{14}$$

$$[X,X,Z] \ = \ 0\,. \tag{15}$$

There are several algebraic origins and applications of WXZ-systems (see [18]).

3. Introduction to B–Ring Theory

The rings [19] and Boolean algebras [19] are usually thought of as complementary structures because they arise from different theories, but sometimes it is useful to compare their axioms and properties. Our approach in this section is different from that point of view. Thus, we introduce structures which unify Boolean algebras and rings, called B–rings (*B* from Boolean algebras followed by *ring*). After the definition of B–rings, we give examples and counterexamples. Then, we study the implications of this new approach and some properties of these new structures.

Definition 3. *A* ***B–ring*** *is a 6–tuple* $(X, \vee, 0, \cdot, 1, \hat{})$, *where* X *is a set,* $\vee$ *and* $\cdot$ *are binary operations on* X, $0, 1 \in X$ *and* $\hat{}$ *is a unary operation on* X ($\hat{x} \in X$).
These obey the following axioms.

- $(X, \vee, 0)$ *is a commutative monoid:*

(1) $(x \vee y) \vee z = x \vee (y \vee z) \quad \forall x, y, z \in X;$
(2) $x \vee y = y \vee x \quad \forall x, y \in X;$
(3) $x \vee 0 = x \quad \forall x \in X.$

- $(X, \cdot, 1)$ *is a monoid:*

(4) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall x, y, z \in X;$
(5) $x \cdot 1 = x = 1 \cdot x \quad \forall x \in X.$

- 0 *is an absorbing element:*

(6) $x \cdot 0 = 0 = 0 \cdot x \quad \forall x \in X.$

- *The second operation is distributive with regard to the first operation:*

(7) $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \quad \forall x, y, z \in X;$
(8) $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z) \quad \forall x, y, z \in X.$

- *The unary operation has some basic properties:*

(9) $\hat{\hat{x}} = x \quad \forall x \in X;$
(10) $x \vee \hat{x} = 1 \quad \forall x \in X;$
(11) $\hat{x} \vee x = 1 \quad \forall x \in X;$
(12) $x \cdot \hat{x} = \hat{x} \cdot x \quad \forall x \in X.$

If a B–ring satisfies the following De Morgan laws, we call it a **B–ring with DM***:*

(DM 1) $\hat{x} \vee \hat{y} = \widehat{(x \cdot y)} \vee (\hat{x} \cdot \hat{y}) \quad \forall x, y \in X;$

(DM 2) $\hat{x} \cdot \hat{y} = \widehat{\hat{x} \vee \widehat{x \cdot y}} \vee y \quad \forall x, y \in X.$

Remark 7. *Several properties in a B–ring follow right away:* $0 \vee x = x$ *(from the commutativity of* $\vee$*),* $\hat{0} = 1$ *and* $\hat{1} = 0$ *(since* $\hat{0} = \hat{0} \vee 0 = 1$*).*

Notice that we do not require $x \cdot \hat{x} = 0 \ \forall x \in X$*; also,* $x \vee \hat{x} = \hat{x} \vee x \ \forall x \in X$.

Definition 4. *We define new operations in a B–ring:* $x \nabla y = \widehat{\hat{x} \cdot \hat{y}}$ *and* $x \triangle y = \widehat{\hat{x} \vee \hat{y}}$*. In addition, let* $x \to y := \hat{x} \nabla y$.

At this moment it is convenient to present some properties of B–rings which resemble the properties of Boolean algebras. We have the following table of operations in a B–ring (see Figure 2):

x	y	$x \vee y$	$x \cdot y$	$x \nabla y$	$x \to y$
0	0	0	0	0	1
0	1	1	0	1	1
1	0	1	0	1	0
1	1	$1 \vee 1$	1	1	1

Figure 2. A table of operations.

Remark 8. Examples of B-rings. *Any Boolean algebra is a B–ring with DM. In this case the second operation is* $\wedge$*. The hardest part is to verify the De Morgan laws. In addition, in this case,* $x \nabla y = x \vee y$*.*

Remark 9. Examples of B-rings which are not Boolean algebras.
Any ring is a B–ring with DM. The first operation is just the addition (usually denoted by $+$), and the unary operation is $\hat{x} = 1 - x$. For example, in this case, $x \nabla y = x + y - xy$.

Remark 10. An example of a B-ring which is neither a Boolean algebra nor a ring.
One could introduce a B–ring structure on $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ which is neither a ring nor a Boolean algebra.

Thus, 0 is the neutral element for the commutative operation $\vee$:

$1 \vee 1 = 2, \ 1 \vee 2 = 3, \ 1 \vee 3 = 3, \ 2 \vee 2 = 0, \ 2 \vee 3 = 1, \ 3 \vee 3 = 1.$

The neutral element for the commutative operation $\cdot$ is 3 (0 is an absorbing element):
$1 \cdot 1 = 1, \ 1 \cdot 2 = 0, \ 1 \cdot 3 = 1, \ 2 \cdot 2 = 2, \ 2 \cdot 3 = 2, \ 3 \cdot 3 = 3.$

This structure is in fact a Cartesian product between $\mathbb{Z}_2$ viewed as a ring and $\mathbb{Z}_2$ viewed as a Boolean algebra.

Theorem 1. *(**Duality Theorem**) For a B–ring (with DM), $(X, \vee, 0, \cdot, 1, \hat{\ })$, one could associate another B–ring (with DM), namely $(X, \triangle, 1, \nabla, 0, \hat{\ })$.*

Moreover, this association is functorial, and applying the corresponding functor twice leads to the initial B–ring (see the Figure 3).

Proof. The proof is direct for B–rings, and it is left for the reader. However, the question if the dual of a B–ring with DM is a B–ring with DM is harder. We will now check some axioms of the new B–ring.

$(x \triangle y) \triangle z = \widehat{\hat{x} \vee \hat{y} \vee \hat{z}} = x \triangle (y \triangle z)$ (the axiom (1)).

$x \vee \hat{x} = 1$ imply $\widehat{x \vee \hat{x}} = \hat{1}$; so, $\hat{x} \triangle x = 0$.

From (DM 1), we have $\hat{x} \vee \hat{y} = \widehat{(x \cdot y)} \vee (\hat{x} \cdot \hat{y})$; so, $\widehat{\hat{x} \vee \hat{y}} = \widehat{\widehat{(x \cdot y)} \vee (\hat{x} \cdot \hat{y})}$; so,
$x \triangle y = (x \cdot y) \triangle (x \nabla y)$; so, $\hat{x} \triangle \hat{y} = \widehat{x \nabla y} \triangle (\hat{x} \nabla \hat{y})$. □

Remark 11. *For a better understanding of the implications of the above theorem, we consider the following example. We start with the usual ring ($\mathbb{Z}$, +, 0, $\times$, 1). In view of the above theorem, we obtain a new ring ($\mathbb{Z}$, $\oplus$, 1, $\otimes$, 0), where $n \oplus m = n + m - 1$ and $n \otimes m = n + m - n \times m$. It can be checked by hand that: $n \oplus 1 = n$, $n \otimes 0 = n$, $(n \oplus m) \otimes p = n \otimes p \oplus m \otimes p$, $n \oplus (2 - n) = 1$, etc.*

We now consider a question somewhat complementary to the main definition (our Definition 3).

Remark 12. *Our results so far could be summarized by the following diagram:*

$$\begin{tikzcd}
\mathbf{B-rings} \arrow[rr, shift left, "\mathrm{F}"] & \text{Theorem 1} & \mathbf{B-rings} \arrow[ll, shift left] & \mathrm{F(F(B))} \ = \ \mathrm{B} \\
\textbf{Boolean algebras} \arrow[u] & & \textbf{Rings} \arrow[u] &
\end{tikzcd}$$

Figure 3. B–rings are structures which generalize both rings and Boolean algebras. In other words, a ring (respectively, a Boolean algebra) could be understood as a B–ring.

The question is to find structures which are, to some extent, both rings and Boolean algebras. In other words, such a structure is a ring and a Boolean algebra at the same time.

Of course, a structure with one element $(\{*\}, \vee, *, \cdot, *, \hat{\ })$ could be thought of both as a ring and a Boolean algebra.

Another answer would refer to a 7–tuple $(X, \vee, +, 0, \cdot, 1, \hat{})$, *in which we consider a mixed structure: a Boolean algebra (* $(X, \vee, 0, \cdot, 1, \hat{})$*) and a ring (* $(X, +, 0, \cdot, 1)$*). For example, the subsets of a given set* X *(i.e.,* 2^X*), could be thought of both as a Boolean algebra (with the union and intersection) and as a ring (with the symmetric difference and intersection).*

For other applications of B-rings, we will use the concepts of the Yang–Baxter equation and Yang–Baxter system.

Theorem 2. *We have the following solution to the set-theoretical braid condition in a B–ring:*
$R : X \times X \to X \times X, \ (p, q) \mapsto (q, \beta p \vee \hat{\beta} q)$.

Proof. We compute the left-hand side of the braid condition, then the right-hand side, and then we use the axioms (7), (10) and (12).
$R^{12} \circ R^{23} \circ R^{12}(x, z, w) = R^{12} \circ R^{23}(z, \beta x \vee \hat{\beta} z, w) = R^{12}(z, w, \beta(\beta x \vee \hat{\beta} z) \vee \hat{\beta} w) = (w, \beta z \vee \hat{\beta} w, \beta(\beta x \vee \hat{\beta} z) \vee \hat{\beta} w)$;
$R^{23} \circ R^{12} \circ R^{23}(x, z, w) = R^{23} \circ R^{12}(x, w, \beta z \vee \hat{\beta} w) = R^{23}(w, \beta x \vee \hat{\beta} w, \beta z \vee \hat{\beta} w) = (w, \beta z \vee \hat{\beta} w, \beta(\beta x \vee \hat{\beta} w) \vee \hat{\beta}(\beta z \vee \hat{\beta} w))$. □

Theorem 3. *We have the following solution to the set-theoretical Yang–Baxter system in a B–ring B:*

$$W, \ X, \ Z : B \times B \to B \times B \,,$$
$$W(a, \ b) = (a \vee b, \ 0) \,,$$
$$X(a, \ b) = (a \cdot b, \ b) \,,$$
$$Z(a, \ b) = (b, \ a) \,.$$

Proof. The first two conditions from the definition of Yang–Baxter systems, $[W, W, W] = 0 = [Z, Z, Z]$, follow from classical theory (see the Remarks 1 and 2). A direct check could easily be performed also.

We now compute the left-hand side of the condition (11) from the definition of Yang–Baxter systems, then the right-hand side, and then we use the axiom (7).

$$W^{12} \circ X^{13} \circ X^{23}(a, b, c) = W^{12} \circ X^{13}(a, \ b \cdot c, \ c) = W^{12}(a \cdot c, \ b \cdot c, \ c) = ((a \cdot c) \vee \ (b \cdot c), \ 0, \ c);$$
$$X^{23} \circ X^{13} \circ W^{12}(a, b, c) = X^{23} \circ X^{13}(a \vee b, \ 0, \ c) = X^{23}((a \vee b) \cdot c, \ 0, \ c) = ((a \vee b) \cdot c, \ 0, \ c).$$

The last condition is left to the reader. □

4. Conclusions

Andre Weil explained that the unification of certain areas of modern mathematics was very important (see, for example, [20]). But the idea of unification has proved to be fruitful not only in other areas of science, but also in art and poetry (see, for example, [21]).

The Yang–Baxter equation unifies the algebra structures and the coalgebra structures, and it could be interpreted as a quadratic equation in the ring of matrices. Euler's formula is related to the Yang–Baxter equation, and it is natural to look for matrix versions of it.

Rings and Boolean algebras are unified in the category of B–rings in the current paper. Thus, the Cartesian (or even a semi-direct) product exists in the category of B–rings leading to future constructions. B–rings are related to the Yang–Baxter equation, and possibly to some kinds of Euler's formulas.

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