axioms

Article

Well-Posedness Scheme for Coupled Fixed-Point Problems Using Generalized Contractions

Dur-e-Shehwar Sagheer [1], Isma Urooj [1], Samina Batul [1], Ahmad Aloqaily [2,3] and Nabil Mlaiki [3,*]

[1] Department of Mathematics, Capital University of Science and Technology, Islamabad 44000, Pakistan; d.e.shehwar@cust.edu.pk (D.-e.-S.S.); mmt203011@cust.pk (I.U.)
[2] School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney 2150, Australia; maloqaily@psu.edu.sa
[3] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
* Correspondence: nmlaiki2012@gmail.com or nmlaiki@psu.edu.sa

Abstract: In this study, we present a more general class of rational-type contractions in the domain of Hilbert spaces, along with a novel coupled implicit relation. We develop several intriguing results and consequences for the existence of unique coupled fixed points. Further, we investigate a necessary condition that guarantees the well-posedness of a coupled fixed-point problem of self-mappings in Hilbert spaces. Some new observations proposed in this research broaden and extend previously published results in the literature.

Keywords: Hilbert spaces; asymptotically regular sequences; coupled implicit relations; coupled fixed points; common coupled fixed points; well-posedness

check for updates

Citation: Sagheer, D.-e.-S.; Urooj, I.; Batul, S.; Aloqaily, A.; Mlaiki, N. Well-Posedness Scheme for Coupled Fixed-Point Problems Using Generalized Contractions. *Axioms* **2023**, *12*, 523. https://doi.org/10.3390/axioms12060523

Academic Editor: Hsien-Chung Wu

Received: 7 April 2023
Revised: 24 May 2023
Accepted: 25 May 2023
Published: 26 May 2023

1. Introduction

Fixed-point theory is an established and classic mathematical subject with many applications. It is now an enormously developing and fascinating mathematical discipline with considerable implications in a variety of fields. Establishing a strong well-posedness strategy for the existence of solutions to complicated problems is one of the most powerful breakthroughs in this subject. This research study aims to offer an improved and effective approach for identifying such suitable scenarios to ensure the solution of coupled problems via fixed-point theory tools.

Banach [1] proposed the iconic Banach contraction principle (BCP) in 1922, later acknowledged as an efficient approach for obtaining unique fixed points. Researchers have been attempting to expand this idea by either embellishing the contraction condition or altering the properties of metric space in numerous contexts. Interested readers are encouraged to look at some recent extensions of BCP for finding fixed points and coupled fixed points in work by [2–14].

Inner product spaces are a subclass of normed spaces that are significantly older than ordinary normed spaces. Their theory is more comprehensive and retains many aspects of Euclidean space, including orthogonality as a core term. A remarkable work by D. Hilbert [15] on integral equations sparked the entire theory of these special spaces. Many people started working in this field using fixed-point theory techniques (one can see some useful results in [16–20]). Recently, S. Kim [21] presented some results in Hilbert spaces using coupled implicit relations, inspired by M Pitchaimani [22]. After giving some useful results, he constructed a scheme for the well-posedness of a coupled fixed-point problem. He was inspired by recent efforts by [23–27].

Apart from the preceding works, further research is needed in Hilbert spaces using rational contraction conditions via implicit relation. Following the previous findings, we investigate coupled fixed-point theorems for a class of self-mappings in Hilbert space, adopting asymptotically regular settings in sequences. We investigate feasible conditions for the existence of solutions to coupled fixed-point problems and designed a technique for guaranteeing the well-posedness of a coupled fixed-point problem.

In this document, $\mathcal{M}$ denotes a non-empty set, $\mathbb{N}$ represents the set of natural numbers, and $\mathbb{R}$ is the collection of real numbers. Let us look at some core ideas and preliminary facts which will set the stage for developing our main results.

2. Preliminaries

Definition 1. *Let $\mathcal{M}$ be a real linear space and define $\|.\| : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying,*

1. $\|s\| \geq 0$;
2. $\|s\| = 0$ *if and only if* $s = 0$;
3. $\|\alpha s\| = \mid \alpha \mid \|s\|$;
4. $\|s + t\| \leq \|s\| + \|t\|$;

for all $s, t \in \mathcal{M}$ and scalers α, then $\|.\|$ is called a norm and the pair $(\mathcal{M}, \|.\|)$ is called a linear normed space. An inner product on $\mathcal{M}$ defines a norm on it given by

$$\|s\| = \sqrt{\langle s, s \rangle}.$$

Definition 2. *Let $(\mathcal{M}, \|.\|)$ be a linear normed space. A sequence $\{s_\zeta\}$ is said to be convergent at $s \in \mathcal{M}$, if*

$$\lim_{\zeta \to \infty} \|s_\zeta - s\| = 0.$$

Definition 3. *In a linear normed space $(\mathcal{M}, \|.\|)$, $\{s_\zeta\}$ is called a Cauchy sequence if*

$$\lim_{\zeta, \eta \to \infty} \|s_\zeta - s_\eta\| = 0,$$

for all $\zeta, \eta > \zeta_0 \in \mathbb{N}$. If every Cauchy sequence is convergent in $\mathcal{M}$, then $(\mathcal{M}, \|.\|)$ is called a Banach space, and Hilbert spaces are Banach spaces.

Definition 4. *A mapping $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is said to be continuous at some $(s, t) \in \mathcal{M} \times \mathcal{M}$ if for any sequences $\{s_\zeta\}$ and $\{t_\zeta\}$, we have*

$$\mathcal{F}(s_\zeta, t_\zeta) \to \mathcal{F}(s, t) \;\; \textit{and} \;\; \mathcal{F}(t_\zeta, s_\zeta) \to \mathcal{F}(t, s) \;\; \textit{as} \;\; \zeta \to \infty,$$

where $s_\zeta \to s$ and $t_\zeta \to t$.

The notion of coupled fixed points for continuous and discontinuous nonlinear operators was given by D. Guo and V. Lakshmikantham [28] in 1987. Some basic ideas regarding coupled fixed points are recalled below:

Definition 5. *A point $s \in \mathcal{M}$ is a called common fixed point of $\mathcal{F}, \mathcal{G} : \mathcal{M} \to \mathcal{M}$ if $\mathcal{F}s = s = \mathcal{G}s$.*

Definition 6. *A point $(s, t) \in \mathcal{M} \times \mathcal{M}$ is called a coupled fixed point of $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ if*

$$\mathcal{F}(s, t) = s \;\; \textit{and} \;\; \mathcal{F}(t, s) = t.$$

The set of all coupled fixed points of $\mathcal{F}$ in $\mathcal{M} \times \mathcal{M}$ is denoted by $\mathcal{C}(\mathcal{F}, \mathcal{M} \times \mathcal{M})$.

Definition 7. *(see [23]) A point $(s, t) \in \mathcal{M} \times \mathcal{M}$ is called a common coupled fixed point of $\mathcal{F}, \mathcal{G} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ if,*

$$\mathcal{F}(s, t) = s = \mathcal{G}(s, t),$$

and

$$\mathcal{F}(t, s) = t = \mathcal{G}(t, s).$$

Definition 8. *In a Hilbert space,* $\{s_\zeta\}$ *is called an asymptotically* $\mathcal{F}$*-regular sequence if it fulfills the following condition,*

$$\lim_{\zeta\to\infty} \|s_\zeta - \mathcal{F}(s_\zeta)\| = 0.$$

Definition 9. *If* $\{s_\zeta\}$ *and* $\{t_\zeta\}$ *be two sequences in a Hilbert space, then the pair* $(\{s_\zeta\},\{t_\zeta\}) \in \mathcal{M}\times\mathcal{M}$ *is called coupled asymptotically* $\mathcal{F}$*-regular if*

$$\lim_{\zeta\to\infty} \|s_\zeta - \mathcal{F}(s_\zeta,t_\zeta)\| = 0 \ \text{ and } \ \lim_{\zeta\to\infty} \|t_\zeta - \mathcal{F}(t_\zeta,s_\zeta)\| = 0.$$

In 1999, V. Popa [29] incorporated the notion of implicit relation in the framework of fixed-point theory. He described several helpful characteristics of implicit relation and outlined some consequences. Later on, these characteristics were extended to various spaces. Encouraged by their work, we provide a new criterion for such relation as follows:

Definition 10. *Let* $\psi : \mathbb{R}^4 \to \mathbb{R}^+$ *be a continuous function and it is non-decreasing in the fourth argument; then, the following relation is called a coupled implicit relation for all* $s,t,\omega_1,\omega_2 > 0$ *if,*

1. $s \le \psi\left(\frac{\omega_1+\omega_2}{2}, \frac{s+\omega_1}{2}, \frac{t+\omega_2}{2}, t+\omega_2\right)$ *and* $t \le \psi\left(\frac{\omega_1+\omega_2}{2}, \frac{t+\omega_2}{2}, \frac{s+\omega_1}{2}, s+\omega_1\right)$,
   or
2. $s \le \psi\left(\frac{\omega_1+\omega_2}{2}, 0, 0, \omega_2\right)$ *and* $t \le \psi\left(\frac{\omega_1+\omega_2}{2}, 0, 0, \omega_1\right)$,

then there exists a real number $\lambda \in (0,1)$ *such that* $s+t \le \lambda(\omega_1+\omega_2)$.

From now on, any function satisfying this implicit relation will be a member of Ψ*-family.*

Definition 11. *In a Hilbert space the pair* $(\mathcal{F},\mathcal{G})$ *is said to satisfy a* ψ*-contraction if for all* $s,t,s^{'},t^{'} \in \mathcal{M}$*, we have*

$$\begin{aligned} \|\mathcal{F}(s,t)-\mathcal{G}(s^{'},t^{'})\|^2 \le \psi\Big(&\frac{\|s-s^{'}\|^2+\|t-t^{'}\|^2}{2}, \frac{\|s-\mathcal{F}(s,t)\|^2+\|s^{'}-\mathcal{G}(s^{'},t^{'})\|^2}{2}, \\ &\frac{\|t-\mathcal{F}(t,s)\|^2+\|t^{'}-\mathcal{G}(t^{'},s^{'})\|^2}{2}, \frac{\|t^{'}-\mathcal{G}(t,s)\|^2+\|t-\mathcal{F}(t^{'},s^{'})\|^2}{2}\Big), \end{aligned} \tag{1}$$

where $\psi \in \Psi$*-family.*

3. Main Results

In this section, we present some results for the existence and uniqueness of coupled fixed points of a self-mapping using rational type contractions endowed with implicit relation. Furthermore, we offer a fine condition for locating coupled fixed points for a sequence of self-mappings in Hilbert spaces.

Lemma 1. *(see [21]) Let* $\mathcal{M}$ *be a Hilbert space, then for any positive integer* c*, we have*

$$(s_1+s_2+\ldots+s_c)^2 \le c(s_1^2+s_2^2+\ldots+s_c^2),$$

for all $s_\zeta \in \mathcal{M}$ *where* $\zeta = 1,2,3....,c$.

Theorem 1. *Let* $\mathcal{K}$ *be a closed subset of a Hilbert space* $\mathcal{M}$ *and define* $\mathcal{F},\mathcal{G} : \mathcal{K}\times\mathcal{K} \to \mathcal{K}$ *such that,*

1. $\mathcal{F}$ *and* $\mathcal{G}$ *are continuous;*
2. *the pair* $(\mathcal{F},\mathcal{G})$ *satisfy* ψ*-contraction,*

Then, $\mathcal{F}$ *and* $\mathcal{G}$ *have a common coupled fixed point in* $\mathcal{K}\times\mathcal{K}$.

Proof. Let $s_0, t_0 \in \mathcal{K}$ such that for $\mathcal{F} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, we have

$$\mathcal{F}(s_0, t_0) = s_1 \text{ and } \mathcal{F}(t_0, s_0) = t_1,$$

and

$$\mathcal{G}(s_1, t_1) = s_2 \text{ and } \mathcal{G}(t_1, s_1) = t_2,$$

iteratively, we will obtain the following sequences in $\mathcal{K} \times \mathcal{K}$

$$\mathcal{F}(s_\zeta, t_\zeta) = s_{\zeta+1} \text{ and } \mathcal{F}(t_\zeta, s_\zeta) = t_{\zeta+1}, \text{ for all } \zeta \in \{0\} \cup \mathbb{N},$$

and

$$\mathcal{G}(s_{\zeta+1}, t_{\zeta+1}) = s_{\zeta+2} \text{ and } \mathcal{G}(t_{\zeta+1}, s_{\zeta+1}) = t_{\zeta+2}, \text{ for all } \zeta \in \{0\} \cup \mathbb{N},$$

Consider,

$$\begin{aligned}
\|s_{\zeta+1} - s_\zeta\|^2 &= \|\mathcal{F}(s_\zeta, t_\zeta) - \mathcal{G}(s_{\zeta-1}, t_{\zeta-1})\|^2 \\
&\leq \psi\bigg(\frac{\|s_\zeta - s_{\zeta-1}\|^2 + \|t_\zeta - t_{\zeta-1}\|^2}{2}, \frac{\|s_\zeta - \mathcal{F}(s_\zeta, t_\zeta)\|^2 + \|s_{\zeta-1} - \mathcal{G}(s_{\zeta-1}, t_{\zeta-1})\|^2}{2}, \\
&\quad \frac{\|t_\zeta - \mathcal{F}(t_\zeta, s_\zeta)\|^2 + \|t_{\zeta-1} - \mathcal{G}(t_{\zeta-1}, s_{\zeta-1})\|^2}{2}, \\
&\quad \frac{\|t_{\zeta-1} - \mathcal{G}(t_\zeta, s_\zeta)\|^2 + \|t_\zeta - \mathcal{F}(t_{\zeta-1}, s_{\zeta-1})\|^2}{2} \bigg) \\
&= \psi\bigg(\frac{\|s_\zeta - s_{\zeta-1}\|^2 + \|t_\zeta - t_{\zeta-1}\|^2}{2}, \frac{\|s_\zeta - s_{\zeta+1}\|^2 + \|s_{\zeta-1} - s_\zeta\|^2}{2}, \\
&\quad \frac{\|t_\zeta - t_{\zeta+1}\|^2 + \|t_{\zeta-1} - t_\zeta\|^2}{2}, \frac{\|t_{\zeta-1} - t_{\zeta+1}\|^2 + \|t_\zeta - t_\zeta\|^2}{2} \bigg) \\
&\leq \psi\bigg(\frac{\|s_\zeta - s_{\zeta-1}\|^2 + \|t_\zeta - t_{\zeta-1}\|^2}{2}, \frac{\|s_\zeta - s_{\zeta+1}\|^2 + \|s_{\zeta-1} - s_\zeta\|^2}{2}, \\
&\quad \frac{\|t_\zeta - t_{\zeta+1}\|^2 + \|t_{\zeta-1} - t_\zeta\|^2}{2}, \|t_\zeta - t_{\zeta+1}\|^2 + \|t_{\zeta-1} - t_\zeta\|^2 \bigg).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|t_{\zeta+1} - t_\zeta\|^2 &= \|\mathcal{F}(t_\zeta, s_\zeta) - \mathcal{G}(t_{\zeta-1}, s_{\zeta-1})\|^2 \\
&\leq \psi\bigg(\frac{\|t_\zeta - t_{\zeta-1}\|^2 + \|s_\zeta - s_{\zeta-1}\|^2}{2}, \frac{\|t_\zeta - \mathcal{F}(t_\zeta, s_\zeta)\|^2 + \|t_{\zeta-1} - \mathcal{G}(t_{\zeta-1}, s_{\zeta-1})\|^2}{2}, \\
&\quad \frac{\|s_\zeta - \mathcal{F}(s_\zeta, t_\zeta)\|^2 + \|s_{\zeta-1} - \mathcal{G}(s_{\zeta-1}, t_{\zeta-1})\|^2}{2}, \\
&\quad \frac{\|s_{\zeta-1} - \mathcal{G}(s_\zeta, t_\zeta)\|^2 + \|s_\zeta - \mathcal{F}(s_{\zeta-1}, t_{\zeta-1})\|^2}{2} \bigg) \\
&= \psi\bigg(\frac{\|t_\zeta - t_{\zeta-1}\|^2 + \|s_\zeta - s_{\zeta-1}\|^2}{2}, \frac{\|t_\zeta - t_{\zeta+1}\|^2 + \|t_{\zeta-1} - t_\zeta\|^2}{2}, \\
&\quad \frac{\|s_\zeta - s_{\zeta+1}\|^2 + \|s_{\zeta-1} - s_\zeta\|^2}{2}, \frac{\|s_{\zeta-1} - s_{\zeta+1}\|^2 + \|s_\zeta - s_\zeta\|^2}{2} \bigg) \\
&\leq \psi\bigg(\frac{\|t_\zeta - t_{\zeta-1}\|^2 + \|s_\zeta - s_{\zeta-1}\|^2}{2}, \frac{\|t_\zeta - t_{\zeta+1}\|^2 + \|t_{\zeta-1} - t_\zeta\|^2}{2}, \\
&\quad \frac{\|s_\zeta - s_{\zeta+1}\|^2 + \|s_{\zeta-1} - s_\zeta\|^2}{2}, \|s_\zeta - s_{\zeta+1}\|^2 + \|s_{\zeta-1} - s_\zeta\|^2 \bigg).
\end{aligned}$$

Using 1 of Definition 10 , i.e., the first property of $\psi \in \Psi$-family, there exists a $\lambda \in (0,1)$ such that

$$\|s_{\zeta+1} - s_\zeta\|^2 + \|t_{\zeta+1} - t_\zeta\|^2 \leq \lambda(\|s_\zeta - s_{\zeta-1}\|^2 + \|t_\zeta - t_{\zeta-1}\|^2).$$

In the same way, we will obtain

$$\|s_{\zeta+1} - s_\zeta\|^2 + \|t_{\zeta+1} - t_\zeta\|^2 \leq \lambda^2(\|s_{\zeta-1} - s_{\zeta-2}\|^2 + \|t_{\zeta-1} - t_{\zeta-2}\|^2).$$

Iteratively,

$$\|s_{\zeta+1} - s_\zeta\|^2 + \|t_{\zeta+1} - t_\zeta\|^2 \leq \lambda^\zeta(\|s_1 - s_0\|^2 + \|t_1 - t_0\|^2) \text{ for all } \zeta \geq 1.$$

From the triangular inequality and using Lemma 1 for any positive integer c, we may write

$$\begin{aligned}
&\|s_\zeta - s_{\zeta+c}\|^2 + \|t_\zeta - t_{\zeta+c}\|^2 \\
&\leq (\|s_\zeta - s_{\zeta+1}\| + \|s_{\zeta+1} - s_{\zeta+2}\| + ... + \|s_{\zeta+c-1} - s_{\zeta+c}\|)^2 \\
&+ (\|t_\zeta - t_{\zeta+1}\| + \|t_{\zeta+1} - t_{\zeta+2}\| + ... + \|t_{\zeta+c-1} - t_{\zeta+c}\|)^2 \\
&\leq c\{(\|s_\zeta - s_{\zeta+1}\|^2 + \|s_{\zeta+1} - s_{\zeta+2}\|^2 + ... + \|s_{\zeta+c-1} - s_{\zeta+c}\|^2) \\
&+ (\|t_\zeta - t_{\zeta+1}\|^2 + \|t_{\zeta+1} - t_{\zeta+2}\|^2 + ... + \|t_{\zeta+c-1} - t_{\zeta+c}\|^2)\} \\
&= c\{(\|s_\zeta - s_{\zeta+1}\|^2 + \|t_\zeta - t_{\zeta+1}\|^2) + (\|s_{\zeta+1} - s_{\zeta+2}\|^2 + \|t_{\zeta+1} - t_{\zeta+2}\|^2) + ... \\
&+ (\|s_{\zeta+c-1} - s_{\zeta+c}\|^2 + \|t_{\zeta+c-1} - t_{\zeta+c}\|^2)\} \\
&\leq c(\lambda^\zeta + \lambda^{\zeta+1} + \lambda^{\zeta+2} .....\lambda^{\zeta+c-1})(\|s_1 - s_0\|^2 + \|t_1 - t_0\|^2) \\
&\leq \frac{c\lambda^\zeta}{1-\lambda}(\|s_1 - s_0\|^2 + \|t_1 - t_0\|^2),
\end{aligned}$$

this shows

$$\lim_{\zeta\to\infty}(\|s_\zeta - s_{\zeta+c}\|^2 + \|t_\zeta - t_{\zeta+c}\|^2) = 0,$$

by using the fact $\lambda \in (0,1)$. Hence,

$$\lim_{\zeta\to\infty}\|s_\zeta - s_{\zeta+c}\|^2 = 0$$

and

$$\lim_{\zeta\to\infty}\|t_\zeta - t_{\zeta+c}\|^2 = 0.$$

As $\mathcal{K}$ is closed, there exist $s,t \in \mathcal{K}$ such that $\{s_\zeta\} \to s$ and $\{t_\zeta\} \to t$. Now, by using the continuity of $\mathcal{F}$ and $\mathcal{G}$, we have

$$s = \lim_{\zeta\to\infty} s_{\zeta+1} = \lim_{\zeta\to\infty}\mathcal{F}(s_\zeta, t_\zeta) = \mathcal{F}(s,t) = \lim_{\zeta\to\infty}\mathcal{G}(s_{\zeta+1}, t_{\zeta+1}) = \mathcal{G}(s,t),$$

also

$$t = \lim_{\zeta\to\infty} t_{\zeta+1} = \lim_{\zeta\to\infty}\mathcal{F}(t_\zeta, s_\zeta) = \mathcal{F}(t,s) = \lim_{\zeta\to\infty}\mathcal{G}(t_{\zeta+1}, s_{\zeta+1}) = \mathcal{G}(t,s).$$

This shows that (s,t) is a common coupled fixed point of $\mathcal{F}$ and $\mathcal{G}$. □

Corollary 1. *Let $\mathcal{K}$ be a closed subset of a Hilbert space $\mathcal{M}$ and define $\mathcal{F}, \mathcal{G} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ such that*

1. *$\mathcal{F}$ and $\mathcal{G}$ are continuous;*
2. *$\mathcal{F}, \mathcal{G}$ satisfy the following contraction condition,*

$$\|\mathcal{F}^{\alpha}(s,t)-\mathcal{G}^{\beta}(s',t')\|^2 \leq \psi\Bigg(\frac{\|s-s'\|^2+\|t-t'\|^2}{2}, \frac{\|s-\mathcal{F}^{\alpha}(s,t)\|^2+\|s'-\mathcal{G}^{\beta}(s',t')\|^2}{2}, \frac{\|t-\mathcal{F}^{\alpha}(t,s)\|^2+\|t'-\mathcal{G}^{\beta}(t',s')\|^2}{2}, \frac{\|t'-\mathcal{G}^{\beta}(t,s)\|^2+\|t-\mathcal{F}^{\alpha}(t',s')\|^2}{2}\Bigg),$$

where $\psi \in \Psi$-family and α, β are some positive integers.

Then, $\mathcal{F}$ and $\mathcal{G}$ have a common coupled fixed point in $\mathcal{K} \times \mathcal{K}$.

Corollary 2. *Let $\mathcal{K}$ be a closed subset of a Hilbert space $\mathcal{M}$, and define a continuous map $\mathcal{F} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ which satisfies the ψ-contraction; then, it has a unique coupled fixed point in $\mathcal{K} \times \mathcal{K}$.*

Proof. The proof of this result follows from Theorem 1 by setting $\mathcal{G}(s',t') = \mathcal{F}(s',t')$. For uniqueness, let $(\check{s},\check{t}) \in \mathcal{K} \times \mathcal{K}$, be another coupled fixed point of $\mathcal{F}$ such that $(\check{s},\check{t}) \neq (s,t)$. Consider

$$\begin{aligned}
\|s-\check{s}\|^2 &= \|\mathcal{F}(s,t)-\mathcal{F}(\check{s},\check{t})\|^2 \\
&\leq \psi\Bigg(\frac{\|s-\check{s}\|^2+\|t-\check{t}\|^2}{2}, \frac{\|s-\mathcal{F}(s,t)\|^2+\|\check{s}-\mathcal{F}(\check{s},\check{t})\|^2}{2}, \\
&\qquad \frac{\|t-\mathcal{F}(t,s)\|^2+\|\check{t}-\mathcal{F}(\check{t},\check{s})\|^2}{2}, \frac{\|\check{t}-\mathcal{F}(t,s)\|^2+\|t-\mathcal{F}(\check{t},\check{s})\|^2}{2}\Bigg) \\
&= \psi\Bigg(\frac{\|s-\check{s}\|^2+\|t-\check{t}\|^2}{2}, 0, 0, \|\check{t}-t\|^2\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
\|t-\check{t}\|^2 &= \|\mathcal{F}(t,s)-\mathcal{F}(\check{t},\check{s})\|^2 \\
&\leq \psi\Bigg(\frac{\|t-\check{t}\|^2+\|s-\check{s}\|^2}{2}, \frac{\|t-\mathcal{F}(t,s)\|^2+\|\check{t}-\mathcal{F}(\check{t},\check{s})\|^2}{2}, \\
&\qquad \frac{\|s-\mathcal{F}(s,t)\|^2+\|\check{s}-\mathcal{F}(\check{s},\check{t})\|^2}{2}, \frac{\|\check{s}-\mathcal{F}(s,t)\|^2+\|s-\mathcal{F}(\check{s},\check{t})\|^2}{2}\Bigg) \\
&= \psi\Bigg(\frac{\|t-\check{t}\|^2+\|s-\check{s}\|^2}{2}, 0, 0, \|\check{s}-s\|^2\Bigg),
\end{aligned}$$

Using (2) as $\psi \in \Psi$, we obtain

$$\|s-\check{s}\|^2+\|t-\check{t}\|^2 \leq \lambda(\|s-\check{s}\|^2+\|t-\check{t}\|^2).$$

This gives

$$\|s-\check{s}\|^2+\|t-\check{t}\|^2 = 0,$$

as $\lambda \in (0,1)$. So, $\|s-\check{s}\|^2 = 0$ and $\|t-\check{t}\|^2 = 0$, that is, $s = \check{s}$ and $t = \check{t}$, which contradicts our assumption. Hence, (s,t) is a unique coupled fixed point of $\mathcal{F}$. □

Example 1. *Consider $\mathcal{K} = [0, 0.5]$ to be a closed subset of a Hilbert space $\mathbb{R}$. Define $\mathcal{F} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ such that*

$$\mathcal{F}(s,t) = \begin{cases} \frac{1}{2}(s-t), & \text{if } s \geq t \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we define $\psi : \mathbb{R}^4 \to \mathbb{R}^+$ such that $\psi(s,t,\omega_1,\omega_2) = s + \max\{t,\omega_1,\omega_2\}$.

It is easy to observe that if $s < t$, then $\psi-$contraction is trivially satisfied by $\mathcal{F}$. Consider $s \geq t$; then,

$$\begin{aligned}
&\psi\Bigg(\frac{\|s-s^{'}\|^2+\|t-t^{'}\|^2}{2}, \frac{\|s-\mathcal{F}(s,t)\|^2+\|s^{'}-\mathcal{F}(s^{'},t^{'})\|^2}{2},\\
&\frac{\|t-\mathcal{F}(t,s)\|^2+\|t^{'}-\mathcal{F}(t^{'},s^{'})\|^2}{2}, \frac{\|t^{'}-\mathcal{F}(t,s)\|^2+\|t-\mathcal{F}(t^{'},s^{'})\|^2}{2}\Bigg)\\
&=\psi\Bigg(\frac{\|s-s^{'}\|^2+\|t-t^{'}\|^2}{2}, \frac{\|s+t\|^2+\|s^{'}+t^{'})\|^2}{4},\\
&\frac{\|s+t\|^2+\|s^{'}+t^{'})\|^2}{4}, \frac{\|2t^{'}-(t-s)\|^2+\|2t-(t^{'}-s^{'})\|^2}{4}\Bigg)\\
&=\frac{\|s-s^{'}\|^2+\|t-t^{'}\|^2}{2}+\max\Bigg(\frac{\|s+t\|^2+\|s^{'}+t^{'}\|^2}{4}, \frac{\|s+t\|^2+\|s^{'}+t^{'})\|^2}{4}\\
&\frac{\|2t^{'}-(t-s)\|^2+\|2t-(t^{'}-s^{'})\|^2}{4}\Bigg)\\
&=\frac{\|s-s^{'}\|^2+\|t-t^{'}\|^2}{2}+\max\Bigg(\frac{\|s+t\|^2+\|s^{'}+t^{'}\|^2}{4}, \frac{\|2t^{'}-(t-s)\|^2+\|2t-(t^{'}-s^{'})\|^2}{4}\Bigg)\\
&\geq \frac{\|s-s^{'}\|^2+\|t-t^{'}\|^2}{2}\\
&\geq \frac{\|(s-s^{'})+(t-t^{'})\|^2}{2}\\
&=\|\mathcal{F}(s,t)-\mathcal{F}(s',t')\|^2
\end{aligned}$$

Here, we have used the fact that the inequality holds with both possible choices of maximum value of above mentioned function. Hence, all the conditions of Corollary (2) are satisfied, proving that $(0,0)$ is a coupled fixed point $\mathcal{F}$.

Remark 1. *If $\omega_1 = \omega_2$ and $s = t$, then the defined coupled implicit relation in Definition 10 would be restricted to the following implicit relation.*
Let $\psi : \mathbb{R}^4 \to \mathbb{R}^+$ be a continuous function and it is non-decreasing in the fourth argument; then, it will satisfy implicit relation for all $s, \omega_1 > 0$, i.e., if

1. $s \leq \psi\Big(\omega_1, \frac{s+\omega_1}{2}, \frac{s+\omega_1}{2}, s+\omega_1\Big)$
   or
2. $s \leq \psi\Big(\omega_1, 0, 0, \omega_1\Big)$.

   Then, there exists a real number $\lambda \in (0,1)$ such that $s \leq \lambda(\omega_1)$.

Theorem 2. *Let $\mathcal{K}$ be a closed subset of a Hilbert space $\mathcal{M}$ and $\mathcal{F} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a ψ-contraction then $(\{s_\zeta, t_\zeta\})$ is coupled asymptotically $\mathcal{F}$-regular and $\mathcal{F}$ has a unique coupled fixed point in $\mathcal{K} \times \mathcal{K}$ if and only if $\mathcal{F}$ is continuous at its coupled fixed point.*

Proof. Let $(s,t) \in \mathcal{K} \times \mathcal{K}$ be a coupled fixed point of $\mathcal{F}$ that is $\mathcal{F}(s,t) = s, \mathcal{F}(t,s) = t$. Consider two sequences $\{s_\zeta,\}, \{t_\zeta\} \in \mathcal{M}$ such that $s_\zeta \to s, t_\zeta \to t$, and the pair $(\{s_\zeta,\}, \{t_\zeta\})$ is asymptotically regular with respect to $\mathcal{F}$ this means

$$\lim_{\zeta\to\infty} \|s_\zeta - \mathcal{F}(s_\zeta, t_\zeta)\| = 0 \text{ and } \lim_{\zeta\to\infty} \|t_\zeta - \mathcal{F}(t_\zeta, s_\zeta)| = 0.$$

Then,

$$\begin{aligned}
\|\mathcal{F}(s_\zeta,t_\zeta)-\mathcal{F}(s,t)\|^2 \leq \psi\bigg(&\frac{\|s_\zeta-s\|^2+\|t_\zeta-t\|^2}{2},\frac{\|s_\zeta-\mathcal{F}(s_\zeta,t_\zeta)\|^2+\|s-\mathcal{F}(s,t)\|^2}{2},\\
&\frac{\|t_\zeta-\mathcal{F}(t_\zeta,s_\zeta)\|^2+\|t-\mathcal{F}(t,s)\|^2}{2},\\
&\frac{\|t-\mathcal{F}(t_\zeta,s_\zeta)\|^2+\|t_\zeta-\mathcal{F}(t,s)\|^2}{2}\bigg)\\
=\psi\bigg(&\frac{\|\mathcal{F}(s_\zeta,t_\zeta)-\mathcal{F}(s,t)\|^2+\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2}{2},0,0,\\
&\frac{\|\mathcal{F}(t,s)-\mathcal{F}(t_\zeta,s_\zeta)\|^2+\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2}{2}\bigg)\\
=\psi\bigg(&\frac{\|\mathcal{F}(s_\zeta,t_\zeta)-\mathcal{F}(s,t)\|^2+\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2}{2},0,0,\\
&\|\mathcal{F}(t,s)-\mathcal{F}(t_\zeta,s_\zeta)\|^2\bigg).
\end{aligned}$$

Similarly, one can easily obtain

$$\begin{aligned}
\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2 \leq \psi\bigg(&\frac{\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2+\|\mathcal{F}(s_\zeta,t_\zeta)-\mathcal{F}(s,t)\|^2}{2},0,0,\\
&\|\mathcal{F}(s,t)-\mathcal{F}(s_\zeta,t_\zeta)\|^2\bigg).
\end{aligned}$$

Now, using 2 from Definition 2 of $\psi \in \Psi$-family, we obtain

$$\begin{aligned}
&\|\mathcal{F}(s_\zeta,t_\zeta)-\mathcal{F}(s,t)\|^2+\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2\\
&\leq \lambda(\|\mathcal{F}(s_\zeta,t_\zeta)-\mathcal{F}(s,t)\|^2+\|\mathcal{F}(t_\zeta,s_\zeta)-\mathcal{F}(t,s)\|^2).
\end{aligned}$$

Using $\lambda \in (0,1)$ and taking $\lim \zeta \to \infty$,

$$\mathcal{F}(s_\zeta,t_\zeta) \to \mathcal{F}(s,t) \text{ and } \mathcal{F}(t_\zeta,s_\zeta) \to \mathcal{F}(t,s).$$

Thus, $\mathcal{F}$ is continuous at its coupled fixed point.

For the other side, assume that $\mathcal{F}$ is continuous at $(s,t) \in \mathcal{M} \times \mathcal{M}$; then, from Theorem 1 $\mathcal{F}$ has a unique coupled fixed point. Now, let $\{s_\zeta\}, \{t_\zeta\}$ be two sequences such that $s_\zeta \to s$ and $t_\zeta \to t$; then,

$$\mathcal{F}(s_\zeta,t_\zeta) \to \mathcal{F}(s,t) \text{ and } \mathcal{F}(t_\zeta,s_\zeta) \to \mathcal{F}(t,s) \text{ as } \zeta \to \infty.$$

In addition, we have

$$\|s_\zeta-\mathcal{F}(s_\zeta,t_\zeta)\| \to \|s-\mathcal{F}(s,t)\| = 0,$$

and

$$\|t_\zeta-\mathcal{F}(t_\zeta,s_\zeta)\| \to \|t-\mathcal{F}(t,s)\| = 0.$$

This gives,

$$\lim_{\zeta\to\infty}\|s_\zeta-\mathcal{F}(s_\zeta,t_\zeta)\| = 0 \text{ and } \lim_{\zeta\to\infty}\|t_\zeta-\mathcal{F}(t_\zeta,s_\zeta)\| = 0.$$

□

Theorem 3. *Let $\mathcal{K}$ be a closed subset of a Hilbert space $\mathcal{M}$ and $\mathcal{F}_\zeta : \mathcal{K} \to \mathcal{K}$ be a sequence of self mappings and $\{\mathcal{F}_\zeta\}$ converges pointwise to a self map $\mathcal{F}$. Additionally, $\{\mathcal{F}_\zeta\}$ satisfies*

$$\|\mathcal{F}_\zeta p - \mathcal{F}_\zeta p'\|^2 \leq \psi\Bigg(\|p - p'\|^2, \frac{\|p - \mathcal{F}_\zeta p\|^2 + \|p' - \mathcal{F}_\zeta p'\|^2}{2}, \frac{\|p - \mathcal{F} p\|^2 + \|p' - \mathcal{F} p'\|^2}{2}, \frac{\|p - \mathcal{F}_\zeta p'\|^2 + \|p' - \mathcal{F} p\|^2}{2}\Bigg),$$

for all $p, p' \in \mathcal{M}$. If $\{\mathcal{F}_\zeta\}$ has a fixed point s_ζ and $\mathcal{F}$ has a fixed point $\check{s}$, then the sequence $\{s_\zeta\}$ converges to $\check{s}$.

Proof. Let $s_1 \in \mathcal{K}$ such that for $\mathcal{F}_\zeta : \mathcal{K} \to \mathcal{K}$ we have $\mathcal{F}_1(s_1) = s_1$. Similarly, $\mathcal{F}_2(s_2) = s_2$, and iteratively, we will obtain

$$\mathcal{F}_\zeta s_\zeta = s_\zeta \ \ \forall\, \zeta \in \mathbb{N}.$$

Consider

$$\begin{aligned}
\|\check{s} - s_\zeta\|^2 &= \|\mathcal{F}\check{s} - \mathcal{F}_\zeta s_\zeta\|^2 \\
&\leq \|\mathcal{F}\check{s} - \mathcal{F}_\zeta \check{s}\|^2 + \|\mathcal{F}_\zeta \check{s} - \mathcal{F}_\zeta s_\zeta\|^2 + 2\langle \mathcal{F}\check{s} - \mathcal{F}_\zeta \check{s}, \mathcal{F}_\zeta \check{s} - \mathcal{F}_\zeta s_\zeta \rangle \\
&\leq \|\mathcal{F}\check{s} - \mathcal{F}_\zeta \check{s}\|^2 + \psi\Bigg(\|\check{s} - s_\zeta\|^2, \frac{\|\check{s} - \mathcal{F}_\zeta \check{s}\|^2 + \|s_\zeta - \mathcal{F}_\zeta s_\zeta\|^2}{2}, \\
&\qquad \frac{\|\check{s} - \mathcal{F}\check{s}\|^2 + \|s_\zeta - \mathcal{F} s_\zeta\|^2}{2}, \frac{\|\check{s} - \mathcal{F}_\zeta s_\zeta\|^2 + \|s_\zeta - \mathcal{F}\check{s}\|^2}{2}\Bigg) \\
&\qquad + 2\langle \mathcal{F}\check{s} - \mathcal{F}_\zeta \check{s}, \mathcal{F}_\zeta \check{s} - \mathcal{F}_\zeta s_\zeta \rangle.
\end{aligned}$$

Now, by using the fact that $\mathcal{F}_\zeta \check{s} \to \mathcal{F}\check{s}$ and $\zeta \to \infty$ we obtain

$$\lim_{\zeta \to \infty} \|\check{s} - s_\zeta\|^2 \leq \lim_{\zeta \to \infty} \psi\Big(\|\check{s} - s_\zeta\|^2, 0, 0, \|\check{s} - s_\zeta\|\Big).$$

Using 2 of ψ defined in Remark 1, we have

$$\lim_{\zeta \to \infty} \|\check{s} - s_\zeta\|^2 \leq \lambda \lim_{\zeta \to \infty} \|\check{s} - s_\zeta\|^2,$$

which proves $s_\zeta \to \check{s}$ as $\zeta \to \infty$. □

4. Well-Posedness Theorem

The concept of the well-posedness of a fixed-point problem has captured the attention of various scholars, which can be observed in [30–34]. One can also see the most recent work performed by Dong Ji et al. [35] ensuring suitable conditions for coupled problems using Mann's iteration scheme. Now, we demonstrate the well-posedness of a coupled fixed-point problem of self-mapping in Corollary 2.

Definition 12. *If we define a self map $\mathcal{F}$ on a Hilbert space $\mathcal{M}$, then the fixed-point problem of $\mathcal{F}$ is said to be a well-posed problem if*

1. *$\mathcal{F}$ has a unique fixed point $s_0 \in \mathcal{M}$;*
2. *For a sequence $\{s_\zeta\} \in \mathcal{M}$ if $\lim_{\zeta \to \infty} \|s_\zeta - \mathcal{F}(s_\zeta)\| = 0$, then $\lim_{\zeta \to \infty} \|s_\zeta - s_0\| = 0$.*

Definition 13. *Let $\mathcal{M}$ be a Hilbert space and define $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. A coupled fixed-point problem on $\mathcal{M} \times \mathcal{M}$ for a self-mapping $\mathcal{F}$ is said to be a well-posed problem if the following conditions are satisfied,*

1. *$\mathcal{F}$ has a unique coupled fixed point;*

2. *For asymptotically $\mathcal{F}$-regular sequences $\{s_\zeta\}, \{t_\zeta\} \in \mathcal{M}$*

$$\check{s} = \lim_{\zeta\to\infty} s_\zeta \text{ and } \check{t} = \lim_{\zeta\to\infty} t_\zeta,$$

where $(\check{s}, \check{t})$ is a coupled fixed points of $\mathcal{F}$.

Theorem 4. *Let $\mathcal{K}$ be a closed subset of a Hilbert space $\mathcal{M}$ and define $\mathcal{F} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ such that*

1. *$\mathcal{F}$ is continuous at its coupled fixed point;*
2. *$\mathcal{F}$ is a ψ-contraction;*
3. *For any sequences $\{s_\zeta\}, \{t_\zeta\}$ and $(\check{s}, \check{t}) \in \mathcal{C}(\mathcal{F}, \mathcal{K} \times \mathcal{K})$, we have*

$$\lim_{\zeta\to\infty} \|\check{s} - \mathcal{F}(s_\zeta, t_\zeta)\| = 0 = \lim_{\zeta\to\infty} \|\check{t} - \mathcal{F}(t_\zeta, s_\zeta)\|.$$

Then, the coupled fixed-point problem of $\mathcal{F}$ is well-posed.

Proof. From Corollary (2), $\mathcal{F}$ has a unique coupled fixed point, say $(s_0, t_0) \in \mathcal{C}(\mathcal{F}, \mathcal{K} \times \mathcal{K})$. Let for the sequences $\{s_\zeta\}, \{t_\zeta\}$, we have

$$\lim_{\zeta\to\infty} \|\mathcal{F}(s_\zeta, t_\zeta) - s_\zeta\| = 0 = \lim_{\zeta\to\infty} \|\mathcal{F}(t_\zeta, s_\zeta) - t_\zeta\|,$$

such that $(s_0, t_0) \neq (s_\zeta, t_\zeta)$ for any $\zeta \in \mathbb{N}$. Using $\mathcal{F}(s_0, t_0) = s_0$ and $\mathcal{F}(t_0, s_0) = t_0$, we may write

$$\begin{aligned}
\|s_0 - s_\zeta\|^2 &= \|\mathcal{F}(s_0, t_0) - s_\zeta\|^2 \\
&\leq \|\mathcal{F}(s_0, t_0) - \mathcal{F}(s_\zeta, t_\zeta)\|^2 + \|\mathcal{F}(s_\zeta, t_\zeta) - s_\zeta\|^2 \\
&\quad + 2\langle \mathcal{F}(s_0, t_0) - \mathcal{F}(s_\zeta, t_\zeta), \mathcal{F}(s_\zeta, t_\zeta) - s_\zeta \rangle \\
&\leq \psi\Bigg(\frac{\|s_0 - s_\zeta\|^2 + \|t_0 - t_\zeta\|^2}{2}, \frac{\|s_0 - \mathcal{F}(s_0, t_0)\|^2 + \|s_\zeta - \mathcal{F}(s_\zeta, t_\zeta)\|^2}{2}, \\
&\quad \frac{\|t_0 - \mathcal{F}(t_0, s_0)\|^2 + \|t_\zeta - \mathcal{F}(t_\zeta, s_\zeta)\|^2}{2}, \\
&\quad \frac{\|t_\zeta - \mathcal{F}(t_0, s_0)\|^2 + \|t_0 - \mathcal{F}(t_\zeta, s_\zeta)\|^2}{2} \Bigg) + \|\mathcal{F}(s_\zeta, t_\zeta) - s_\zeta\|^2 \\
&\quad + 2\langle \mathcal{F}(s_0, t_0) - \mathcal{F}(s_\zeta, t_\zeta), \mathcal{F}(s_\zeta, t_\zeta) - s_\zeta \rangle,
\end{aligned}$$

as

$$\lim_{\zeta\to\infty} \|s_0 - \mathcal{F}(s_\zeta, t_\zeta)\| = 0 = \lim_{\zeta\to\infty} \|t_0 - \mathcal{F}(t_\zeta, s_\zeta)\|,$$

for $(s_0, t_0) \in \mathcal{C}(\mathcal{F}, \mathcal{K} \times \mathcal{K})$, so we obtain

$$\lim_{\zeta\to\infty} \|s_0 - s_\zeta\|^2 \leq \lim_{\zeta\to\infty} \psi\left(\frac{\|s_0 - s_\zeta\|^2 + \|t_0 - t_\zeta\|^2}{2}, 0, 0, \|t_\zeta - t_0\|^2 \right).$$

Similarly, one can easily obtain

$$\lim_{\zeta\to\infty} \|t_0 - t_\zeta\|^2 \leq \lim_{\zeta\to\infty} \psi\left(\frac{\|t_0 - t_\zeta\|^2 + \|s_0 - s_\zeta\|^2}{2}, 0, 0, \|s_\zeta - s_0\|^2 \right).$$

Then,

$$\lim_{\zeta\to\infty} \|s_0 - s_\zeta\|^2 + \lim_{\zeta\to\infty} \|t_0 - t_\zeta\|^2 \leq \lambda(\lim_{\zeta\to\infty} \|s_0 - s_\zeta\|^2 + \lim_{\zeta\to\infty} \|t_0 - t_\zeta\|^2).$$

Thus,

$$\lim_{\zeta\to\infty} s_\zeta = s_0 \text{ and } \lim_{\zeta\to\infty} t_\zeta = t_0,$$

which completes the proof. □

5. Conclusions and Future Work

In this article, we presented some important results for the existence and uniqueness of coupled fixed points of self-mappings in a Hilbert space. We also offered a well-posedness scheme for coupled problems using generalized contraction conditions via implicit relation. One can observe that this novel extension in Hilbert spaces generalizes some results presented by Pitchaimani et al. [22] and K.S Kim [21] by restricting ψ in our Definition 10 from $\mathbb{R}^3$ to $\mathbb{R}^+$. In the future, one can expand such functions satisfying an implicit relation from $\mathbb{R}^n$ to $\mathbb{R}^+$, where $n \geq 5$. After investigating more suitable properties of such functions, one can check the validity of our proven results. Expanding these results in 2-Banach spaces would also be an appreciative effort.

Author Contributions: D.-e.-S.S.: conceptualization, supervision, writing—original draft; I.U.: investigation, writing—review and editing; S.B.: investigation, writing—review and editing; A.A.: conceptualization, writing—original draft; N.M.: writing—original draft, supervision, methodology. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors A. ALoqaily and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrals. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Anwar, M.; Sagheer, D.E.S.; Ali, R. Fixed Point Theorems of Wardowski Type Mappings in S_b-Metric Spaces. *Thai J. Math.* **2022**, *20*, 945–956.
3. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequalities Appl.* **2014**, *2014*, 38. [CrossRef]
4. Suzuki, T. A generalized Banach contraction principle characterizes metric completeness. *Proc. Am. Math. Soc.* **2008**, *136*, 1861–1869. [CrossRef]
5. Ćirić, L.B. A generalization of Banach's contraction principle. *Proc. Am. Math. Soc.* **1974**. *45*, 267–273. [CrossRef]
6. Anwar, M.; Shehwar, D.; Ali, R.; Hussain, N. Wardowski Type α-F-Contractive Approach for Nonself Multivalued Mappings. *U.P.B. Sci. Bull. Ser. A* **2020**, *82*, 69–78.
7. Kamran, T.; Postolache, M.; Ghiura, A.; Batul, S.; Ali, R. The Banach contraction principle in C∗algebra-valued b-metric spaces with the application. *Fixed Point Theory Appl.* **2016**, *2016*, 10. [CrossRef]
8. Aslam, M.; Aydi, H.; Batul, S.; Naz, A. Coincidence Point Results on Relation Theoretic (F_w, R)-g-Contractions and Applications. *J. Funct. Spaces* **2021**, *2021*, 1–10. [CrossRef]
9. Batul, S.; Sagheer, D.; Anwar, M.; Aydi, H.; Parvaneh, V. Fuzzy Fixed Point Results of Fuzzy Mappings on b-metric Spaces via $(\alpha*$-$F)$-contractions. *Adv. Math. Phys.* **2022**, *2022*, 4511632 [CrossRef]
10. Aloqaily, A.; Sagheer, D.E.S.; Urooj, I.; Batul, S.; Mlaiki, N. Solving Integral Equations via Hybrid Interpolative $\mathcal{RI}$-Type Contractions in b Metric Spaces. *Symmetry* **2023**, *15*, 465. [CrossRef]
11. Aslam, M.S.; Chowdhury, M.S.R.; Guran, L.; Manzoor, I.; Abdeljawad, T.; Santina, D.; Mlaiki, N. Complex-valued double controlled metric like spaces with applications to fixed point theorems and Fredholm type integral equations. *AIMS Math.* **2023**, *8*, 4944–4963. [CrossRef]
12. Zada, M.B.; Sarwar, M.; Abdeljawad, T.; Mukheimer, A. Coupled Fixed Point Results in Banach Spaces with Applications. *Mathematics* **2021**, *9*, 2283. [CrossRef]
13. Alamgir, N.; Kiran, Q.; Aydi, H.; Mukheimer, A. A Mizoguchi–Takahashi Type Fixed Point Theorem in Complete Extended b-Metric Spaces. *Mathematics* **2019**, *7*, 478. [CrossRef]
14. Al-Rawashdeh, A.; Hassen, A.; Abdelbasset, F.; Sehmim, S.; Shatanawi, W. On common fixed points for $\alpha - F-$contractions and applications. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3445–3458. [CrossRef]
15. Lalescu, T.; Picard, É. Introduction à la Théorie des équations Intégrales. Library Science A Hermann: Paris, France, 1912.
16. Browder, F.E. Fixed-point theorems for noncompact mappings in Hilbert space. *Proc. Natl. Acad. Sci. USA* **1965**, *53*, 1272–1276. [CrossRef]
17. Takahashi, W.; Yao, J.C. Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces. *Taiwan. J. Math.* **2011**, *15*, 457–472. [CrossRef]

18. Kocourek, P.; Takashi, W.; Yao, J.C. Fixed point and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. *Taiwan. J. Math.* **2010**, *14*, 2497–2511. [CrossRef]
19. Beg, I.; Shahzad, N. Applications of the proximity map to random fixed point theorems in Hilbert spaces. *J. Math. Anal. Appl.* **1995**, *196*, 606–613. [CrossRef]
20. Ma, Y.K.; Kavitha, K.; Albalawi, W.; Shukla, A.; Nisar, K.S.; Vijayakumar, V. An analysis of the approximate controllability of Hilfer fractional neutral differential systems in Hilbert spaces. *Alex. Eng. J.* **2022**, *61*, 7291–7302. [CrossRef]
21. Kim, K.S. Coupled fixed point theorems under new coupled implicit relation in Hilbert spaces. *Demonstr. Math.* **2022**, *55*, 81–89. [CrossRef]
22. Pitchaimani, M.; Kumar, D.R. On the construction of fixed point theory under implicit relation in Hilbert spaces. *Nonlinear Funct. Anal. Appl.* **2016**, *21*, 513–522.
23. Kim, K.S. 2020. A Constructive scheme for common coupled fixed point problems in Hilbert space. *Mathematics* **2020**, *8*, 1717. [CrossRef]
24. Browder, F.E.; Petryshyn, W.V. Construction of fixed points of nonlinear mappings in Hilbert space. *J. Math. Anal. Appl.* **1967**, *20*, 197–228. [CrossRef]
25. Veerapandi, T.; Kumar, S.A. Common fixed point theorems of a sequence of mappings on Hilbert space. *Bull. Cal. Math. Soc.* **1999**, *91*, 299–308.
26. Cui, H.; Wang, F. Iterative methods for the split common fixed point problem in Hilbert spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 78. [CrossRef]
27. Wang, F. A new iterative method for the split common fixed point problem in Hilbert spaces. *Optimization* **2017**, *66*, 407–415. [CrossRef]
28. Guo, D.; Lakshmikantham, V. Coupled fixed points of nonlinear operators with applications. *Nonlinear Anal. Theory, Methods Appl.* **1987**, *11*, 623–632. [CrossRef]
29. Popa, V. Some fixed point theorems for compatible mappings satisfy an implicit relation. *Demonstr. Math.* **1999**, *32*, 157–164.
30. Reich, S.A.; Zaslavski, J. Well-Posedness of Fixed Point Problems. *Far East J. Math. Sci. Special* **2001**, *20*, 393–401. [CrossRef]
31. De Blasi, F.S.; Myjak, J. Sur la porosité de l'ensemble des contractions sans point fixe. *CR Acad. Sci. Paris* **1989**, *308*, 51–54.
32. Păcurar, M; Rus, I.A. Fixed point theory for cyclic ϕ-contractions. *Nonlinear Anal. Theory, Methods Appl.* **2010**, *72*, 1181–1187. [CrossRef]
33. Lahiri, B.K.; Das, P. Well-posedness and porosity of a certain class of operators. *Demonstr. Math.* **2005**, *38*, 169–176. [CrossRef]
34. Popa, V. Well-posedness of fixed point problem in orbitally complete metric spaces. *Stud. Cerc. St. Ser. Mat. Univ.* **2006**, *16*, 18–20.
35. Ji, D.; Li, C.; Cui, Y. Fixed Point Theorems for Mann's Iteration Scheme in Convex Gb-Metric Spaces with an Application. *Axioms* **2023**, *12*, 108. [CrossRef]