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New Class of K-G-Type Symmetric Second Order Vector Optimization Problem

Chetan Swarup [1], Ramesh Kumar [2], Ramu Dubey [2,*] and Dowlath Fathima [3]

[1] Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh-Male Campus, Riyadh 13316, Saudi Arabia; c.swarup@seu.edu.sa

[2] Department of Mathematics, J.C. Bose University of Science and Technology, YMCA, Faridabad 121 006, India; rameshgoswami2619@gmail.com

[3] Basic Sciences Department, College of Science and Theoretical Studies, Saudi Electronic University, Jeddah 23442, Saudi Arabia; d.fathima@seu.edu.sa

* Correspondence: rdubeyjiya@gmail.com

Abstract: In this paper, we present meanings of K-G_f-bonvexity/K-G_f-pseudobonvexity and their generalization between the above-notice functions. We also construct various concrete non-trivial examples for existing these types of functions. We formulate K-G_f-Wolfe type multiobjective second-order symmetric duality model with cone objective as well as cone constraints and duality theorems have been established under these aforesaid conditions. Further, we have validates the weak duality theorem under those assumptions. Our results are more generalized than previous known results in the literature.

Keywords: K-G_f-pseudobonvexity; second-order; K-G_f-Wolfe type; efficient solution; multiobjective programming; arbitrary cones; strong duality; generalized assumptions

MSC: 90C26; 90C30; 90C32; 90C46

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1. Introduction

The field of optimization theory has progressed far beyond anyone's expectations. Due to its wide variety of uses, it has made its way into all disciplines of science and engineering. When approximations are utilized, one of the most important practical applications of duality is that it provides bounds on the value of the objective functions because there are more factors involved, second-order duality has a greater computational benefit than first-order duality. For intriguing applications and breakthroughs in multiobjective optimization, we refer to [1] , and the references cited therein. Dorn [2] presented the primary symmetric duality definition for quadratic programming in 1965. Dantzig et al. [3] and Mond [4] proposed a pair of symmetric dual Duality plays a vital role in investigating nonlinear programming problem solutions. Several writers have proposed several duality models, such as Wolfe dual [5] and Mond-Weir dual [6]. Nanda and Das [7] introduced four different forms of duality models for the nonlinear programming problem with cone constraints. The work of Bazaraa and Goode [8] and Hanson and Mond [9] inspired these findings.

Mangasian [10] established the duality theorem in the context of a second-order dual problem in nonlinear programming, where none of the constraints imposed convexity restrictions on all functions. Mond [11] introduced second-order symmetric dual models and established second-order symmetric duality theorems under second-order convexity conditions for the first time. In mathematical programming, Hasnson [12] defined the second-order invexity of a differentiable function and studied it. In 1999, Mishra [13] proposed a pair of second-order vector symmetric dual multiobjective models for arbitrary cones based on the Wolfe and Mond-Weir types. In addition 2006, ref. [14] a couple of Mond–Weir type second-order symmetric duality multiobjective calculations for cone

second-order pseudoinvex and emphatically cone second-order pseudoinvex algorithm were presented. A couple of Mond–Weir type second-order symmetric dual multiobjective projects over discretion cones is created under pseudoinvexity/$K\breve{}F$-convexity assumptions by Gulati [15], which is as:
Primal(MP):

$$\begin{aligned} \text{K-minimize} \quad & \psi(\iota,\kappa) \\ \text{subject to} \quad & \\ & -\Big(\nabla_\kappa(\lambda^T\psi)(\iota,\kappa)+\nabla_{\kappa\kappa}(w^T\phi)(\iota,\kappa)p\Big) \in C_2^*, \\ & \kappa^T\Big(\nabla_\kappa(\lambda^T\psi)(\iota,\kappa)+\nabla_{\kappa\kappa}(w^T\phi)(\iota,\kappa)p\Big) \geqq 0, \\ & \lambda \in intK^*, \quad \iota \in C_1 \end{aligned}$$

Dual(MD):

$$\begin{aligned} \text{K-maximize} \quad & \psi(\mu,\nu) \\ \text{subject to} \quad & \\ & \Big(\nabla_\iota(\lambda^T\psi)(\mu,\nu)+\nabla_{\iota\iota}(w^T\phi)(\mu,\nu)p\Big) \in C_1^*, \\ & \mu^T\Big(\nabla_\iota(\lambda^T\psi)(\mu,\nu)+\nabla_{\iota\iota}(w^T\phi)(\mu,\nu)r\Big) \leqq 0, \\ & \lambda \in intK^*, \quad \iota \in C_2, \end{aligned}$$

where,

(i) $R_1 \subseteq \mathbb{R}^n, R_2 \subseteq \mathbb{R}^m$ are open sets,
(ii) $\psi, \phi : R_1 \times R_2 \to \mathbb{R}^k$ is a twice differentiable function of ι and κ, is a differentiable function of ι and κ,
(iii) $\lambda \in \mathbb{R}^k, w \in \mathbb{R}^q, p \in \mathbb{R}^m$ and $r \in \mathbb{R}^n$,
(iv) for i=1,2, $C_i \subset S_i$ is a closed convex cone with non-empty interior and C_i^* is its positive polar cone.
Aside from them, a number of other researchers are working in this field. For additional information, see [16–20].
In this paper be start by defining in section 2, K-G_f-bonvexity as well as pseduobonvexity and construct non-trivial numerical examples for clear understanding the concept introduced by authors. We identify several examples lying exclusively K-G_f-bonvex and not in the class of K-invex function with respect to same η already exist in the literature. We illustrate an example which is K-G_f-pseudobonvex but not K-G_f-bonvex with respect to same η. In the next section, we formulate a new pair of multiobjective symmetric second order K-G_f-primal-dual models over arbitrary cone and drive duality results under K-G_f-bonvex as well as K-G_f-pseudobonvex assumptions. We, also construct a non-trivial example for validate the weak duality theorem presented in the paper. we also introduced geometry figure for clear understanding the concept through figure.

2. Preliminaries and Definitions

In this paper, we used $\mathbb{R}^n$ for n-dimensional Euclidean space and $\mathbb{R}^n_+$ for semi-positive orthant. Also, here C_1 and C_2 used for closed convex cone $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, with non-void interiors. For a real-valued twice differentiable function $g(\varphi,\vartheta)$ described on an open set in $\mathbb{R}^n \times \mathbb{R}^m$, indicate by $\nabla_\varphi g(\bar\varphi,\bar\vartheta)$ the gradient vector of g with respect to a at $(\bar\varphi,\bar\vartheta)$, $\nabla_{\varphi\varphi} g(\bar\varphi,\bar\vartheta)$ the Hessian matrix with respect to φ an at $(\bar\varphi,\bar\vartheta)$.
Throughout the paper $\tilde{N} = \{1,2,...,k\}$, $\tilde{O} = \{1,2,...,m\}$.

A differentiable function $f : X \times Y \to R^k$, $\eta_1 : X \times Y \to R^k$, $\eta_2 : X \times Y \to R^k$,

$G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$, $G_{f_i} : I_{f_i}(X) \to R$ is range f_i for $i = \tilde{N}$. Also, K is used for pointed convex cone with non-void interiors in $\mathbb{R}^k$, for $\vartheta, z \in \mathbb{R}^k$ and we specify cone orders with respect to K as follows:

$$\vartheta \leqq z \iff z - \vartheta \in K; \;\; \vartheta \leq z \iff z - \vartheta \in K \backslash \{0\}; \;\; \vartheta < z \iff z - \vartheta \in intK.$$

Let $f : X \to \mathbb{R}^k$ be a differentiable function defined on open set $\phi \neq X \subseteq \mathbb{R}^n$ and $I_{f_i}(X), i \in \tilde{N}$ be the range of f_i.

Consider the following multiobjective programming problem with cone objective as well as constraints as :

(MP) K-min $f(\varphi)$
subject to

$$\varphi \in X^0 = \Big\{\varphi \in S : g(\varphi) \in Q\Big\}.$$

where $S \subseteq \mathbb{R}^n, f : S \to \mathbb{R}^k, g : S \to \mathbb{R}^m$. Q is a closed convex cone with a non-empty interior in $\mathbb{R}^m$.

Definition 1 ([21]). *$\bar{\varphi} \in X^0$ is a weak efficient solution of (MP), $\nexists$ $\varphi \in X$ such that*

$$f(\bar{\varphi}) - f(\varphi) \in intK.$$

Definition 2 ([21]). *$\bar{\varphi} \in X^0$ is an efficient solution of (MP), $\nexists$ $\varphi \in X$ such that*

$$f(\bar{\varphi}) - f(\varphi) \in K \setminus \{0\}.$$

Now, we consider the following multiobjective programming with cone objective and cone constraints as:

(GMP) $K-min\ G_f(f(z))$

$$subject\ to\ z \in Z^0 = \Big\{z \in S : -G_g(g(z)) \in Q\Big\}.$$

Definition 3 ([21]). *$\bar{z} \in Z^0$ is a weak efficient solution of (GMP), $\nexists\, z \in Z^0$ s.t.*

$$G_f(f(\bar{z})) - G_f(f(z)) \in intK.$$

Definition 4 ([21]). *$\bar{z} \in Z^0$ is a efficient solution of (GMP), $\nexists$ $z \in Z^0$ s.t. $G_f(f(\bar{z})) - G_f(f(z)) \in K \backslash \{0\}$.*

Definition 5 ([21]). *The positive polar cone C_i^* of C_i (i=1,2) is defined as $C_i^* = \Big\{z : \varphi^T z \geqq 0, \forall \varphi \in C_1\Big\}$.*
Suppose that $S_1 \subseteq \mathbb{R}^n$ and $S_2 \subseteq \mathbb{R}^m$ are open sets such that

$$C_1 \times C_2 \subset S_1 \times S_2.$$

A differentiable function $f : X \to R^k$ and G_f such that every component G_{f_i} is strictly increasing on the range of I_{f_i}.

Definition 6. *If* $\exists\ G_f$ *and* η *such that* $\forall\ \varphi \in X$ *and* $p_i \in R^n$, *we have*

$$\Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi}f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)$$
$$+\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi}f_1(\delta)\Big\}p_1\Big], ..., G_{f_k}(f_k(\varphi)) - G_{f_k}(f_k(\delta)) + \tfrac{1}{2}p_k^T\Big[G''_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)(\nabla_\varphi f_k(\delta))^T$$
$$+G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi}f_k(\delta)\Big]p_k - \eta^T(\varphi,\delta)\Big[G'_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta) + \Big\{G''_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)(\nabla_\varphi f_k(\delta))^T + G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi}f_k(\delta)\Big\}p_k\Big]\Big\} \in K,$$

then f *is* K-G_f*-bonvex at* $\delta \in X$ *with respect to* η.

Definition 7. *If* $\exists\ G_f$ *and* η *such that* $\forall\ \varphi \in X$ *and* $p_i \in R^m$, *we have*

$$\Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi}f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)$$
$$+\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi}f_1(\delta)\Big\}p_1\Big], ..., G_{f_k}(f_k(\varphi)) - G_{f_k}(f_k(\delta)) + \frac{1}{2}p_k^T\Big[G''_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)(\nabla_\varphi f_k(\delta))^T$$
$$+ G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi}f_k(\delta)\Big]p_k - \eta^T(\varphi,\delta)\Big[G'_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta) + \Big\{G''_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)(\nabla_\varphi f_k(\delta))^T + G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi}f_k(\delta)\Big\}p_k\Big]\Big\} \in -K,$$

then f *is* K-G_f*-boncave at* $\delta \in X$ *with respect to* η.
Generalized the above definitions on two variable, as follows,

Definition 8. *If* $\exists$ *and* G_f *and* η_1 *such that* $\forall\ \varphi \in X$ *and* $q_i \in R^n$, *we have*

$$\Big\{G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\delta,\ell)) + \frac{1}{2}q_1^T\Big[G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big]q_1 - \eta_1^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta,\ell))$$
$$\nabla_\varphi f_1(\delta,\ell) + \{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\}q_1\Big], ..., G_{f_k}(f_k(\varphi,\ell)) - G_{f_k}(f_k(\delta,\ell))$$
$$+ \frac{1}{2}q_k^T\Big[G''_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell)(\nabla_\varphi f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi}f_k(\delta,\ell)\Big]q_k$$
$$- \eta_1^T(\varphi,\delta)\Big[G'_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta,\ell) + \Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell)(\nabla_\varphi f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi}f_k(\delta,\ell)\Big\}q_k\Big]\Big\} \in K,$$

then, f *is* K-G_f*-bonvex in the first variable at* $\delta \in X$ *for fixed* $\ell \in Y$ *with* η_1,
and
If $\exists\ G_f\ \eta_2$ *such that* $\forall\ \vartheta \in Y$ *and* $p_i \in R^m$, *we have*

$$\Big\{G_{f_1}(f_1(\delta,\vartheta)) - G_{f_1}(f_1(\delta,\ell)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta,\ell))\nabla_\vartheta f_1(\delta,\ell)(\nabla_\vartheta f_1(\delta,\ell))^T + G'_{f_1}f_1(\delta,\ell)\nabla_{\vartheta\vartheta}f_1(\delta,\ell)\Big]p_1 - \eta_2^T(\ell,\vartheta)\Big[G'_{f_1}(f_1(\delta,\ell))$$
$$\nabla_\vartheta f_1(\delta,\ell) + \{G''_{f_1}(f_1(\delta,\ell))\nabla_\vartheta f_1(\delta,\ell)(\nabla_\vartheta f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta}f_1(\delta,\ell)\}p_1\Big], ..., G_{f_k}(f_k(\delta,\vartheta)) - G_{f_k}(f_k(\delta,\ell))$$
$$+ \frac{1}{2}p_k^T\Big[G''_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell)(\nabla_\vartheta f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta}f_k(\delta,\ell)\Big]p_k$$
$$- \eta_2^T(\ell,\vartheta)\Big[G'_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell) + \Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell)(\nabla_\vartheta f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta}f_k(\delta,\ell)\Big\}p_k\Big]\Big\} \in K,$$

then, f *is* K-G_f*-bonvex in the second variable at* $\ell \in Y$ *for fixed* $\delta \in X$ *with* η_2.

Definition 9. *If $\exists\ G_f$ and η_1 such that $\forall\ \varphi \in X$ and $q_i \in R^n$, we have*

$$\Big\{G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\delta,\ell)) + \frac{1}{2}q_1^T\Big[G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi} f_1(\delta,\ell)\Big]q_1 - \eta_1^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta,\ell))$$
$$\nabla_\varphi f_1(\delta,\ell) + \{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi} f_1(\delta,\ell)\}q_1\Big], \ldots, G_{f_k}(f_k(\varphi,\ell)) - G_{f_k}(f_k(\delta,\ell))$$
$$+ \frac{1}{2}q_k^T\Big[G''_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell)(\nabla_\varphi f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi} f_k(\delta,\ell)\Big]q_k$$
$$- \eta_1^T(\varphi,\delta)\Big[G'_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta,\ell) + \Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell)(\nabla_\varphi f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi} f_k(\delta,\ell)\Big\}q_k\Big]\Big\} \in -K,$$

then, f is K-G_f-boncave in the first variable at $\delta \in X$ for fixed $\ell \in Y$ with respect to η_1, and
If $\exists\ G_f$ and η_2 such that $\forall\ \vartheta \in Y$ and $p_i \in R^m$, we have

$$\Big\{G_{f_1}(f_1(\delta,\vartheta)) - G_{f_1}(f_1(\delta,\ell)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta,\ell))\nabla_\vartheta f_1(\delta,\ell)(\nabla_\vartheta f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta} f_1(\delta,\ell)\Big]p_1 - \eta_2^T(\ell,\vartheta)\Big[G'_{f_1}(f_1(\delta,\ell))$$
$$\nabla_\vartheta f_1(\delta,\ell) + \{G''_{f_1}(f_1(\delta,\ell))\nabla_\vartheta f_1(\delta,\ell)(\nabla_\vartheta f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta} f_1(\delta,\ell)\}p_1\Big], \ldots, G_{f_k}(f_k(\delta,\vartheta)) - G_{f_k}(f_k(\delta,\ell))$$
$$+ \frac{1}{2}p_k^T\Big[G''_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell)(\nabla_\vartheta f_k(\delta,\ell))^T$$
$$- \eta_2^T(\ell,\vartheta)[G'_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell) + \Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell)(\nabla_\vartheta f_k(\delta,\ell))^T + G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta} f_k(\delta,\ell)\Big\}p_k\Big]\Big\} \in -K,$$

then function f is K-G_f-boncave in the second variable at $\ell \in Y$ for fixed $\delta \in X$ with respect to η_2.

Example 1. *Let $X = [1,2] \subseteq \mathbb{R}, n = m = 1$ and $k = 2$. Consider $f : X \to \mathbb{R}^2$ be defined by*

$$f(\varphi) = \Big(f_1(\varphi), f_2(\varphi)\Big),$$

where,

$$f_1(\varphi) = \varphi sin\left(\frac{1}{\varphi}\right), \;\; f_2(\varphi) = cos\varphi.$$

Next, $G_f : (G_{f_1}, G_{f_2}) : \mathbb{R} \to \mathbb{R}^2$ defined by

$$G_{f_1} = t^2, \;\; G_{f_2} = t^4.$$

Let $K = \Big\{(\varphi,\vartheta); \varphi \geqq 0$ and $\vartheta \geqq 0\Big\}$ and $\eta : X \times X \to \mathbb{R}$ be given by

$$\eta(\varphi,\delta) = (1 - \delta^2).$$

Now, we have to claim that f is $K - G_f$-bonvex, for this, we have driven that the following expression as

$$\Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)$$
$$+\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\}p_1\Big], G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T$$
$$+ G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big]p_2 - \eta^T(\varphi,\delta)[G'_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta) + \{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\}p_2]\Big\} \in K.$$

Let

$$\prod = \Big\{ G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2} p_1^T \Big[G''_{f_1}(f_1(\delta)) \nabla_\varphi f_1(\delta) (\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta)) \nabla_{\varphi\varphi} f_1(\delta) \Big] p_1 - \eta^T(\varphi,\delta) \Big[G'_{f_1}(f_1(\delta)) \nabla_\varphi f_1(\delta) + \Big\{ G''_{f_1}(f_1(\delta)) \nabla_\varphi f_1(\delta) (\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta)) \nabla_{\varphi\varphi} f_1(\delta) \Big\} p_1 \Big], G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2} p_2^T \Big[G''_{f_2}(f_2(\delta)) \nabla_\varphi f_2(\delta) (\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta)) \nabla_{\varphi\varphi} f_2(\delta) \Big] p_2 - \eta^T(\varphi,\delta) \Big[G'_{f_2}(f_2(\delta)) \nabla_\varphi f_2(\delta) + \Big\{ G''_{f_2}(f_2(\delta)) \nabla_\varphi f_2(\delta) (\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta)) \nabla_{\varphi\varphi} f_2(\delta) \Big\} p_2 \Big] \Big\}.$$

Substituting the values of $f_1, f_2, G_{f_1}, G_{f_2}$ *and* η*, we obtain*

$$\prod = \Big\{ \varphi^2 sin^2 \tfrac{1}{\varphi^2} - \delta^2 sin^2 \frac{1}{\delta^2} + \frac{1}{2} p^2 \Big[2\Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big)^2 + 2\delta sin\frac{1}{\delta} \Big(-\frac{1}{\delta^3} sin\frac{1}{\delta} \Big) \Big] - (1-\delta^2) \Big[2\delta sin\frac{1}{\delta} \Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big) + p \Big[2\Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big)^2 + 2\delta sin\frac{1}{\delta} \Big(-\frac{1}{\delta^3} sin\frac{1}{\delta} \Big) \Big] \Big], cos^4\varphi - cos^4\delta + \frac{1}{2} p^2 [12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta)] - (1-\delta^2)[4cos^3\delta(-sin\delta) + p(12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta))] \Big\}.$$

Now, we consider

$$\Psi = \varphi^2 sin^2 \tfrac{1}{\varphi^2} - \delta^2 sin^2 \frac{1}{\delta^2} + \frac{1}{2} p^2 \Big[2\Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big)^2 + 2\delta sin\frac{1}{\delta} \Big(-\frac{1}{\delta^3} sin\frac{1}{\delta} \Big) \Big] - (1-\delta^2) \Big[2\delta sin\frac{1}{\delta} \Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big) + p \Big[2\Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big)^2 + 2\delta sin\frac{1}{\delta} \Big(-\frac{1}{\delta^3} sin\frac{1}{\delta} \Big) \Big] \Big].$$

Let us apply the following ansatz:

$$\Psi = \Psi_1 + \Psi_2 \ \ \Big(say \Big),$$

consider

$$\Phi = \Big\{ cos^4\varphi - cos^4\delta + \frac{1}{2} p^2 \Big[12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta) \Big] - (1-\delta^2) \Big[4cos^3\delta(-sin\delta) + p\Big(12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta) \Big) \Big] \Big\} \in K.$$

The above expression breaks in Φ_1 *and* Φ_2 *(say) as follows:*

$$\Phi = \Phi_1 + \Phi_2,$$

where

$$\Psi_1 = \varphi^2 sin^2 \frac{1}{\varphi^2} - \delta^2 sin^2 \frac{1}{\delta^2} - (1-\delta^2) \Big[2\delta sin\frac{1}{\delta} \Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big) \Big].$$

It is easily verified from Figure 1, we have

$$\Psi_1 \geqq 0, \ \ \forall\ \varphi, \delta \in X.$$

$$\Psi_2 = \frac{1}{2} p^2 \Big[2\Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big)^2 + 2\delta sin\frac{1}{\delta} \Big(-\frac{1}{\delta^3} sin\frac{1}{\delta} \Big) \Big] + p \Big[2\Big(sin\frac{1}{\delta} - \frac{1}{\delta} cos\frac{1}{\delta} \Big)^2 + 2\delta sin\frac{1}{\delta} \Big(-\frac{1}{\delta^3} sin\frac{1}{\delta} \Big) \Big].$$

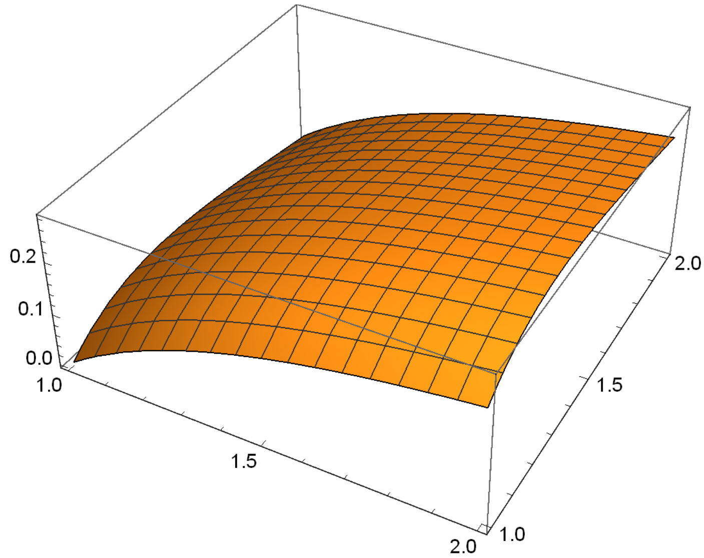

Figure 1. $\Psi_1 = \left\{\varphi^2 sin^2\frac{1}{\varphi^2} - \delta^2 sin^2\frac{1}{\delta^2} - (1-\delta^2)\left[2\delta sin\frac{1}{\delta}\left(sin\frac{1}{\delta} - \frac{1}{\delta}cos\frac{1}{\delta}\right)\right]\right\}$.

It is clear from Figure 2, we obtain

$$\Psi_2 \geqq 0,\ \forall\ \delta \in X \text{ and } p \in \left[-\frac{1}{10^{10}}, -1\right].$$

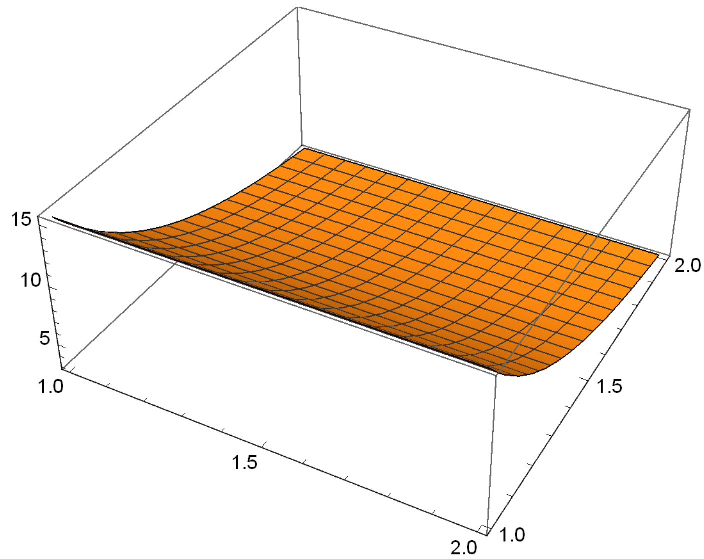

Figure 2. $\Psi_2 = \frac{1}{2}p^2\left[2\left(sin\frac{1}{\delta} - \frac{1}{\delta}cos\frac{1}{\delta}\right)^2 + 2\delta sin\frac{1}{\delta}\left(-\frac{1}{\delta^3}sin\frac{1}{\delta}\right)\right] + p\left[2\left(sin\frac{1}{\delta} - \frac{1}{\delta}cos\frac{1}{\delta}\right)^2 + 2\delta sin\frac{1}{\delta}\left(-\frac{1}{\delta^3}sin\frac{1}{\delta}\right)\right]$.

Now,

$$\Phi_1 = cos^4\varphi - cos^4\delta + -(1-\delta^2)\left[4cos^3\delta(-sin\delta)\right],$$

as can be seen from Figure 3.

$$\Phi_1 \geqq 0\ \ \forall\ \varphi, \delta \in X,$$

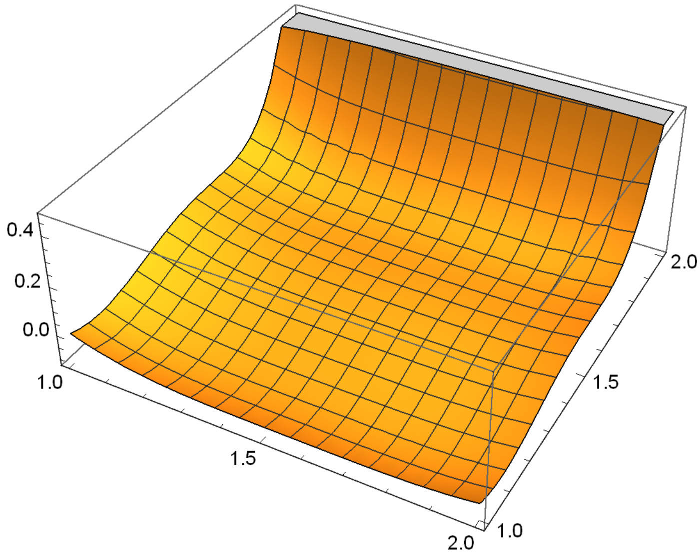

Figure 3. $\Phi_1 = \left\{cos^4\varphi - cos^4\delta + -(1-\delta^2)4cos^3\delta(-sin\delta)\right\}$.

and

$$\Phi_2 = \frac{1}{2}p^2\left[12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta) + p\left(12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta)\right)\right].$$

As can be seen from Figure 4. $\Phi_2 \geqq 0$, $\forall\delta \in X$ and $p_1, p_2 \in [-\frac{1}{10^{10}}, -1]$. (From Figure 4).

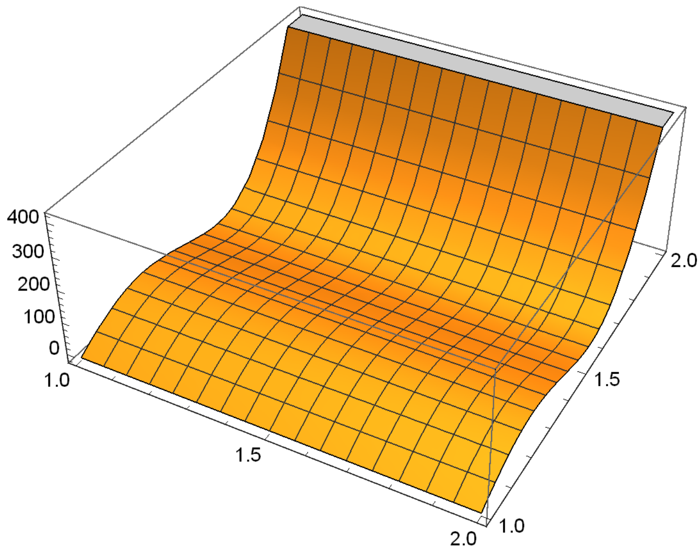

Figure 4. $\Phi_2 = \frac{1}{2}p^2\left[12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta) + p\left(12cos^2\delta(-sin\delta)^2 + 4cos^3\delta(-cos\delta)\right)\right]$.

Hence, $\Psi \geqq 0$ and $\Phi \geqq 0$. This gives $\psi + \phi \geqq 0$. Thus, we can find that $(\Psi, \Phi) \in K$.

Hence, f is K-G_f-bonvex function at (Ψ, Φ) w.r.t. η.

We will show that f is not invex. For this it is either

$$f_1(\varphi) - f_1(\delta) - \eta^T(\varphi, \delta)\nabla_\varphi f_1(\delta) \ngeqq 0$$

or

$$f_2(\varphi) - f_2(\delta) - \eta^T(\varphi, \delta)\nabla_\varphi f_2(\delta) \ngeqq 0.$$

Since $f_1(\varphi) - f_1(\delta) - \eta^T(\varphi,\delta)\nabla_\varphi f_1(\delta) = \varphi sin\frac{1}{\varphi} - \delta sin\frac{1}{\delta} - (1-\delta^2)sin\frac{1}{\delta} - \frac{1}{\delta}cos\frac{1}{\delta} \ngeq 0$, is not $\forall\varphi,\delta \in X$ as can be seen from Figure 5. Also, $f_2(\varphi) - f_1(\delta) - \eta^T(\varphi,\delta)\nabla_\varphi f_2(\delta) = cos\varphi - cos\delta + (1-\delta^2)sin\delta \ngeq 0$, is not $\forall\varphi,\delta \in X$ as can be seen from Figure 6.

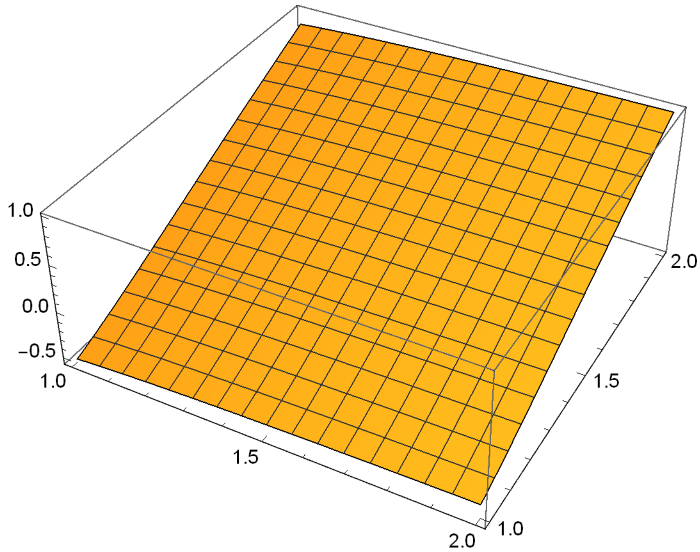

Figure 5. $\varphi sin\frac{1}{\varphi} - \delta sin\frac{1}{\delta} - (1-\delta^2)sin\frac{1}{\delta} - \frac{1}{\delta}cos\frac{1}{\delta}$.

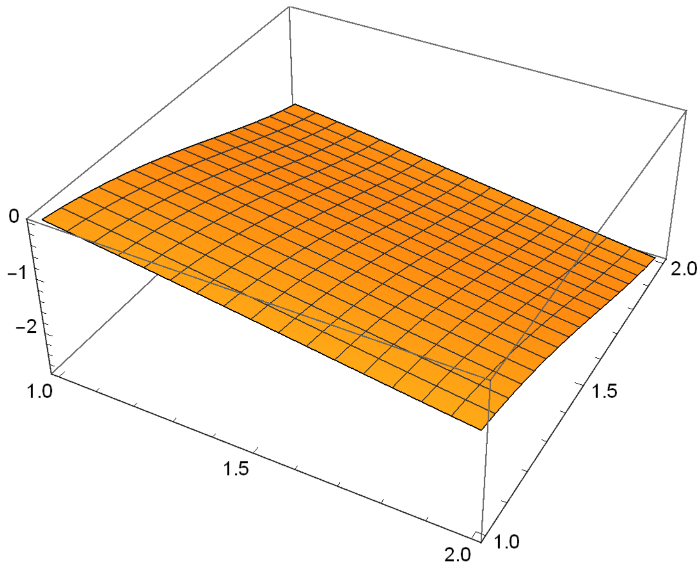

Figure 6. $cos\varphi - cos\delta + (1-\delta^2)sin\delta$.

Therefore, from the above example, it shows that f is K-G_f-bonvex, but it is not invex with respect to same η.

Definition 10. *If $\exists\, G_f$ and η such that $\forall\ \varphi \in X$ and $q_i \in R^n$, we have*

$$\eta^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + q_1\Big\{G''_{f_1}(f_1(\delta))(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}, ..., G'_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta) + q_k\Big\{G''_{f_k}(f_k(\delta))(\nabla_\varphi f_k(\delta))^T + G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi} f_k(\delta)\Big\}\Big\} \in K \Rightarrow \Big[G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}q_1^T\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}q_1, ..., G_{f_k}(f_k(\varphi)) - G_{f_k}(f_k(\delta)) + \frac{1}{2}q_k^T\Big\{G''_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)(\nabla_\varphi f_k(\delta))^T + G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi} f_k(\delta)\Big\}q_k\Big] \in K,$$

then, f is G_f-pseudobonvex at $\delta \in X$ with η.

Definition 11. *If $\exists\, G_f$ and η such that $\forall\, \varphi \in X$ and $q_1 \in R^n$, we have*

$$\eta^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)+q_1\Big\{G''_{f_1}(f_1(\delta))(\nabla_\varphi f_1(\delta))^T+G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi}f_1(\delta)\Big\},...,G'_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)+q_k\Big\{G''_{f_k}(f_k(\delta))(\nabla_\varphi f_k(\delta))^T + G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi}f_k(\delta)\Big\}\Big\}\in -K \Rightarrow \Big[G_{f_1}(f_1(\varphi))-G_{f_1}(f_1(\delta))+\frac{1}{2}q_1^T\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T+G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi}f_1(\delta)\Big\}q_1, ..., G_{f_k}(f_k(\varphi))-G_{f_k}(f_k(\delta))+\tfrac{1}{2}q_k^T\Big\{G''_{f_k}(f_k(\delta))\nabla_\varphi f_k(\delta)(\nabla_\varphi f_k(\delta))^T+G'_{f_k}(f_k(\delta))\nabla_{\varphi\varphi}f_k(\delta)\Big\}q_k\Big]\in -K,$$

then f is G_f-pseudoboncave at $\delta \in X$ with respect to η.

We generalized the above definition as follows:

Definition 12. *If $\exists\, G_f$ and η_1 such that $\forall\, \varphi \in X$ and $q_i \in R^n$, we have*

$$\eta_1^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)+q_1\Big\{G''_{f_1}(f_1(\delta,\ell))(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\},...,G'_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell) + q_k\Big\{G''_{f_k}(f_k(\delta,\ell))(\nabla_\varphi f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi}f_k(\delta,\ell)\Big\}\Big\}\in K$$
$$\Rightarrow \Big[G_{f_1}(f_1(\varphi,\ell))-G_{f_1}(f_1(\delta,\ell))+\frac{1}{2}q_1^T\Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\}q_1,...,G_{f_k}(f_k(\varphi,\ell)) - G_{f_k}(f_k(\delta,\ell))+\frac{1}{2}q_k^T\Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell)(\nabla_\varphi f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi}f_k(\delta,\ell)\Big\}q_k\Big]\in K,$$

then f is K-G_f-bonvex in the first variable at $\delta \in X$ for fixed $\ell \in Y$ with η_1,

and

if $\exists\, G_f$ and η_2 such that $\forall\, \vartheta \in Y$ and $p_i \in R^m$, we have

$$\eta_2^T(\delta,\vartheta)\Big\{G'_{f_1}(f_1(\delta,\vartheta))\nabla_\vartheta f_1(\delta,\ell)+\Big\{G''_{f_1}(f_1(\delta,\ell))(\nabla_\vartheta f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta}f_1(\delta,\ell)\ell\}p_1,...,G'_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell) + p_k\Big\{G''_{f_k}(f_k(\delta,\ell))(\nabla_\vartheta f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta}f_k(\delta,\ell)\Big\}\Big\}\in K$$
$$\Rightarrow \Big[G_{f_1}(f_1(\delta,\vartheta))-G_{f_1}(f_1(\delta,\ell))+\frac{1}{2}p_1^T\Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\vartheta f_1(\delta,\ell)(\nabla_\vartheta f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta}f_1(\delta,\ell)\Big\}p_1,...,G_{f_k}(f_k(\delta,\vartheta)) - G_{f_k}(f_k(\delta,\ell))+\frac{1}{2}p_k^T\Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell)(\nabla_\vartheta f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta}f_k(\delta,\ell)\Big\}p_k\Big]\in K,$$

then f is K-G_f-bonvex in the second variable at $\ell \in Y$ for fixed $\delta \in X$ with η_2.

Definition 13. *If $\exists\, G_f$ and η_1 such that $\forall\, \varphi \in X$ and $q_i \in R^n$, we have*

$$\eta_1^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)+q_1\Big\{G''_{f_1}(f_1(\delta,\ell))(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\},...,G'_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell) + q_k\Big\{G''_{f_k}(f_k(\delta,\ell))(\nabla_\varphi f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi}f_k(\delta,\ell)\Big\}\Big\}\in -K$$
$$\Rightarrow \Big[G_{f_1}(f_1(\varphi,\ell))-G_{f_1}(f_1(\delta,\ell))+\frac{1}{2}q_1^T\Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\}q_1,...,G_{f_k}(f_k(\varphi,\ell)) - G_{f_k}(f_k(\delta,\ell))+\frac{1}{2}q_k^T\Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\varphi f_k(\delta,\ell)(\nabla_\varphi f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\varphi\varphi}f_k(\delta,\ell)\Big\}q_k\Big]\in -K,$$

then f is K-G_f-bonvex in the first variable at $\delta \in X$ for fixed $\ell \in Y$ with η_1,

and

If $\exists\, G_f$ and η_2 such that $\forall\, \vartheta \in Y$ and $p_i \in R^m$, we have

$$\eta_2^T(\delta,\vartheta)\Big\{G'_{f_1}(f_1(\delta,\vartheta))\nabla_\vartheta f_1(\delta,\ell)+\Big\{G''_{f_1}(f_1(\delta,\ell))(\nabla_\vartheta f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta}f_1(\delta,\ell)\Big\}p_1,...,G'_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell) + p_k\Big\{G''_{f_k}(f_k(\delta,\ell))(\nabla_\vartheta f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta}f_k(\delta,\ell)\Big\}\Big\}\in -K$$
$$\Rightarrow \Big[G_{f_1}(f_1(\delta,\vartheta))-G_{f_1}(f_1(\delta,\ell))+\frac{1}{2}p_1^T\Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\vartheta f_1(\delta,\ell)(\nabla_\vartheta f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\vartheta\vartheta}f_1(\delta,\ell)\Big\}p_1,...,G_{f_k}(f_k(\delta,\vartheta)) - G_{f_k}(f_k(\delta,\ell))+\frac{1}{2}p_k^T\Big\{G''_{f_k}(f_k(\delta,\ell))\nabla_\vartheta f_k(\delta,\ell)(\nabla_\vartheta f_k(\delta,\ell))^T+G'_{f_k}(f_k(\delta,\ell))\nabla_{\vartheta\vartheta}f_k(\delta,\ell)\Big\}p_k\Big]\in -K.$$

then f is K-G_f-boncave in the second variable at $\ell \in Y$ for fixed $\delta \in X$ with respect to η_2.

Remark 1. *If $G_f(t) = t$, then above definition reduces in $K-\eta$-pseudo bonvex w.r.t. η,*

$$\eta^T(\varphi,\delta)\Big[\nabla_\varphi f_1(\delta) + \nabla_{\varphi\varphi} f_1(\delta) q_1, ..., \nabla_\varphi f_k(\delta) + \nabla_{\varphi\varphi} f_k(\delta) q_k\Big] \in K$$

$$\Rightarrow \Big[f_1(\varphi) - f_1(\delta) + \frac{1}{2} q_1^T \nabla_{\varphi\varphi} f_1(\delta) q_1, ..., f_k(\varphi) - f_k(\delta) + \frac{1}{2} q^T \nabla_{\varphi\varphi} f_k(\delta) q_k\Big] \in K.$$

Example 2. *Let $X = [-10, 10]$ and $K = \Big\{(\varphi, \vartheta) : \varphi \geqq 0, \varphi \leqq \vartheta\Big\}$. Consider the function $f : X \to \mathbb{R}^2$ defined by*

$$f(\varphi) = (f_1(\varphi), f_2(\varphi)),$$

where

$$f_1(\varphi) = sin\varphi, \quad f_2(\varphi) = e^\varphi$$

Define $G_f = (G_{f_1}, G_{f_2}) : R^2 \to R$ given by

$$G_{f_1} = t^2, G_{f_2} = t^3, \eta = \varphi^2 - \delta^2, \text{ and } q_1 = q_2 \in [2, \infty].$$

We have to claim that function f is K-G_f-pseudobonvex at point δ, i.e.,

$$\eta^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + q_1\Big\{G''_{f_1}(f_1(\delta))(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}, G'_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta) + q_2\Big\{G''_{f_2}(f_2(\delta))(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}\Big\} \in K \Rightarrow \Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2} q_1^T\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}q_1, G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \tfrac{1}{2} q_2^T\Big\{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}q_2\Big\} \in K.$$

Consider

$$\tau = \eta^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + q_1\Big\{G''_{f_1}(f_1(\delta))(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}, G'_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta) + q_2\Big\{G''_{f_2}(f_2(\delta))(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}\Big\}.$$

Putting the values of $f_1, f_2, G_{f_1}, G_{f_2}$ and η, we have

$$\tau = (\varphi^2 - \delta^2)\Big(sin2\delta + 2q_1(cos\delta - sin^2\delta),\ 3e^{3\delta} + 9e^{2\delta} q_2\Big).$$

At the point $\delta = 0$, the value of above expression becomes

$$\tau = \Big\{2\varphi^2 q_1,\ 3\varphi^2(1 + 3q_2)\Big\}, \quad \forall\, q_1 = q_2 \in [2, \infty)$$

Obviously,

$$\tau = \Big\{2\varphi^2 q_1,\ 3\varphi^2(1 + 3q_2)\Big\} \in K.$$

Next, consider

$$\Psi = \Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2} q_1^T \{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\} q_1,\ G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2} q_2^T \{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\} q_2\}.$$

Putting the values of f_1, f_2, G_{f_1}, G_{f_2} and η, we have

$$\Psi = \Big\{sin^2\varphi - sin^2\delta + \frac{1}{2} q_1^2 (2cos^2\delta - 2sin^2\delta),\ e^{3\varphi} - e^{3\delta} + \frac{9}{2} q_2^2 e^{3\delta}\Big\}.$$

The value of above expression at the point $\delta = 0$, we get

$$\Psi = \Big\{sin^2\varphi + q_1^2,\ e^{3\varphi} + \frac{9}{2} q_2^2 - 1\Big\} \in K.$$

From the Figure 7. We can easily observe that the value of φ-coordinate always less than ϑ-coordinate in K, so $\varphi \in K$.

Hence, f is K-G_f-pseudobonvex at the point $\delta = 0$ with respect to η.

Next,

$$\Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + \Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}p_1\Big], G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2} f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\Big]p_2 - \eta^T(\varphi,\delta)\Big[G'_{f_2} f_2(\delta)\nabla_\varphi f_2(\delta) + \{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2} f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\}p_2\Big]\Big\} \notin K.$$

Let

$$\Psi = \Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + \{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\}p_1\Big], G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2} f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\Big]p_2 - \eta^T(\varphi,\delta)\Big[G'_{f_2} f_2(\delta)\nabla_\varphi f_2(\delta) + \Big\{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2} f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\Big\}p_2\Big]\Big\}.$$

Substituting the values of $f_1, f_2, G_{f_1}, G_{f_2}$ and η, we obtain

$$\Psi = \Big\{sin^2\varphi - sin^2\delta + p_1^2(cos^2\delta - sin^2\delta) - (\varphi^2 - \delta^2)(sin2\delta + 2p_1(cos\delta - sin^2\delta)),\ e^{3\varphi} - e^{3\delta} + \frac{9}{2}p_2^2 e^{3\delta} - (\varphi^2 - \delta^2)(3e^{3\delta} + 9e^{2\delta}p_2)\Big\}.$$

At the point $\delta = 0$, it follows that

$$\Psi = \Big\{sin^2\varphi + p_1^2 - 2p_1\varphi^2,\ e^{3\varphi} + \frac{9}{2}p_2^2 - 1 - \varphi^2(3 + 9p_2)\Big\}, \quad p_1 = p_2 \in [2,\infty).$$

Take particular point $\varphi = -\frac{\pi}{2}$ and $p_1 = p_2 = 2 \in [2,\infty)$, we obtain,

$$\Psi = (-4.86,\ -34.80) \notin K.$$

Hence, f is K-G_f-pseudobonvex, but it is not K-G_f-bonvex at $\delta = 0$ with respect to η.

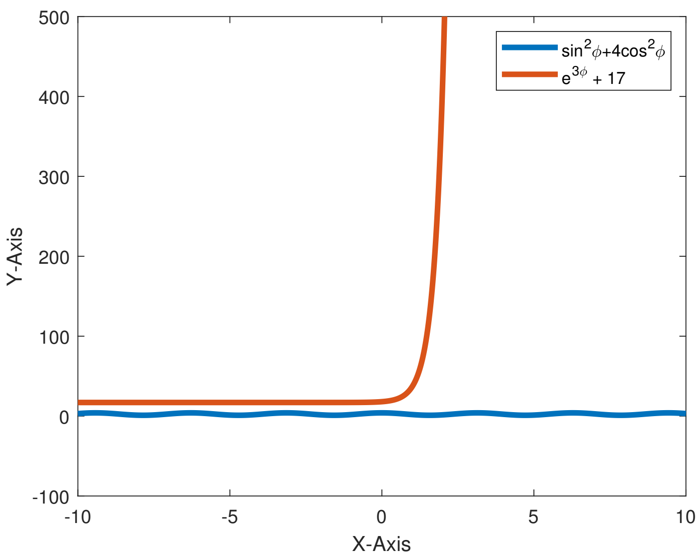

Figure 7. $\left(sin^2\varphi + 4cos^2\varphi, e^{3\varphi} + 17\right)$.

In the following example, we showed that the function f is K-G_f-pseudobonvex, but it is not K-G_f-bonvex function with same η.

Example 3. *Let* $X = \left[0, \frac{\pi}{2}\right]$ *and* $K = \{(\varphi, \vartheta) : \varphi \geqq 0, \vartheta \geqq \varphi\}$*. Consider* $G_f = (G_{f_1}, G_{f_2}) : R^2 \to R$ *and* $f : X \to \mathbb{R}^2$ *given by*

$$f(\varphi) = (f_1(\varphi), f_2(\varphi)),$$

where

$$f_1(\varphi) = sin\varphi, \quad f_2(\varphi) = \varphi,$$

$$G_{f_1} = t, \;\; G_{f_2} = t^2.$$

Define $\eta : X \times X \to R^n$ *given by*

$$\eta(\varphi, \delta) = \varphi - \delta \text{ and } q_1, \; q_2 \in [1, \infty].$$

Solution: In this example, we will try to derive that f is K-G_f-pseudobonvex i.e.,

$$\eta^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + q_1\Big\{G''_{f_1}(f_1(\delta))(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}, G'_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta) + q_2\Big\{G''_{f_2}(f_2(\delta))(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}\Big\} \in K \Rightarrow \Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}q_1^T\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}q_1, G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \tfrac{1}{2}q_2^T\Big\{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}q_2\Big\} \in K.$$

Consider

$$\Pi_1 = \eta^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta) + q_1\Big\{G''_{f_1}(f_1(\delta))(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}, G'_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta) + q_2\Big\{G''_{f_2}(f_2(\delta))(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}\Big\}.$$

Putting the values of $f_1, f_2, G_{f_1}, G_{f_2}$ and η, we have

$$\Pi_1 = \{(\varphi - \delta)cos\delta, \; (\varphi - \delta)(2\delta + 2q_2)\}.$$

The value of above expression at the point $\delta = 0$, we get

$$\Pi_1 = \{\varphi, \; 2\delta q_2\} \in K.$$

Next, let

$$\Pi_2 = \Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \tfrac{1}{2}q_1^T\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}q_1, \;\; G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \tfrac{1}{2}q_2^T\Big\{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}(f_2(\delta))\nabla_{\varphi\varphi} f_2(\delta)\Big\}q_2\Big\}.$$

Putting the values of $f_1, \; f_2, \; G_{f_1}, \; G_{f_2}$ and η, we have

$$\Pi_2 = \left\{sin\varphi - sin\delta + \frac{1}{2}q_1^2(-sin\delta), \varphi - \delta + q_2^2\right\}.$$

After simplifying and the value at $\delta = 0$, it follows that

$$\Pi_2 = \left\{sin\varphi, \varphi + q_2^2\right\} \in K.$$

Hence, f is K-G_f-pseudobonvex at the point $\delta = 0$ with respect to η.

Next,

$$\Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)$$
$$+\Big\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big\}p_1\Big],$$
$$G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T$$
$$+G'_{f_2}f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\Big]p_2 - \eta^T(\varphi,\delta)\Big[G'_{f_2}f_2(\delta)\nabla_\varphi f_2(\delta) + \{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\}p_2\Big]\Big\} \notin K.$$

Let

$$\Psi = \Big\{G_{f_1}(f_1(\varphi)) - G_{f_1}(f_1(\delta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\Big]p_1 - \eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)$$
$$+\{G''_{f_1}(f_1(\delta))\nabla_\varphi f_1(\delta)(\nabla_\varphi f_1(\delta))^T + G'_{f_1}(f_1(\delta))\nabla_{\varphi\varphi} f_1(\delta)\}p_1\Big], G_{f_2}(f_2(\varphi)) - G_{f_2}(f_2(\delta)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T$$
$$+G'_{f_2}f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\Big]p_2 - \eta^T(\varphi,\delta)\Big[G'_{f_2}f_2(\delta)\nabla_\varphi f_2(\delta) + \Big\{G''_{f_2}(f_2(\delta))\nabla_\varphi f_2(\delta)(\nabla_\varphi f_2(\delta))^T + G'_{f_2}f_2(\delta)\nabla_{\varphi\varphi} f_2(\delta)\Big\}p_2\Big]\Big\}.$$

Substituting the values of $f_1, f_2, G_{f_1}, G_{f_2}$ and η, we obtain

$$\Psi = \Big\{sin\varphi - sin\delta + \frac{1}{2}p_1^2(-sin\delta) - (\varphi-\delta)p_1 cos\delta,\ \varphi^2 + p_2^2 - 2(\varphi-\delta)p_2\Big\}.$$

At the point $\delta = 0$, it follows that

$$\Psi = \Big\{sin\varphi - p_1\varphi, (\varphi - p_2)^2\Big\} \notin K.$$

Hence, f is K-G_f-pseudobonvex, but it is not K-G_f-bonvex at $\delta = 0$ with respect to η.

3. K-G_f-Wolfe Type Second-Order Symmetric Primal-Dual Pair with Cones

The study of second-order duality is more significant due to computational advantage over first order duality as it provides tighter bounds for the objective functions, when approximation is used.

The motivated by [21–27] several researches in this area, we formulated a new type K-G_f-Wolfe type primal dual pair, with cone objectives as well as cone constraint as follows:
Primal Problem (GWPP):

$$K\text{-min } L(\varphi,\vartheta,\lambda,p) = \Big\{L_1(\varphi,\vartheta,\lambda,p), L_2(\varphi,\vartheta,\lambda,p), L_3(\varphi,\vartheta,\lambda,p), ..., L_k(\varphi,\vartheta,\lambda,p)\Big\},$$

where

$$L_i(\varphi,\vartheta,\lambda,p) = G_{f_i}(f_i(\varphi,\vartheta)) - \vartheta^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta) + \Big\{G''_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta)(\nabla_\vartheta f_i(\varphi,\vartheta))^T$$
$$+G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\varphi,\vartheta} f_i(\varphi,\vartheta)\Big\}p_i\Big] - \tfrac{1}{2}\textstyle\sum_{i=1}^{k}\lambda_i p_i\Big\{G''_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta)(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i,$$

subject to

$$-\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta) + \Big\{G''_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta)(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i\Big] \in C_2^*, \quad (1)$$

$$\lambda^T e_k = 1,\ \ \lambda \in \text{int}K^*,\ \ \varphi \in C_1. \quad (2)$$

Dual Problem (GWDP):

$$K\text{-max } M(\delta,\ell,\lambda,q) = \Big\{M_1(\delta,\ell,q), M_2(\delta,\ell,\lambda,q), M_3(\delta,\ell,\lambda,q), ..., M_k(\delta,\ell,\lambda,q)\Big\},$$

where

$$M_i(\delta,\ell,\lambda,q) = G_{f_i}(f_i(\delta,\ell)) - \delta^T \sum_{i=1}^{k} \lambda_i \Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell) + \Big\{G''_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)(\nabla_\varphi f_i(\delta,\ell))^T$$
$$+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\}q_i\Big] - \tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i q_i \Big[G''_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)(\nabla_\varphi f_i(\delta,\ell))^T + G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big]q_i,$$

subject to

$$\sum_{i=1}^{k} \lambda_i \Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell) + \Big\{G''_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)(\nabla_\varphi f_i(\delta,\ell))^T + G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\}q_i\Big] \in C_1^*, \tag{3}$$

$$\lambda^T e_k = 1, \;\; \lambda \in \mathrm{int}K^*, \;\; \delta \in C_2, \tag{4}$$

where, for $i \in \tilde{Q}$,

- $f_i : R_1 \times R_2 \to R$, is a differential function of φ and ϑ, $e_k = (1, 1, ..., 1)^T \in R^k$,
- q_i and p_i are vectors in R^n and R^m, respectively and $\lambda \in R^k$.

Let V^* and W^* be the sets of feasible solutions of (GWPP) and (GWDP) respectively.

Theorem 1 (Weak duality). *Let $(\varphi,\vartheta,\lambda,p) \in V^*$ and $(\delta,\ell,\lambda,q) \in W^*$. Let, for $i \in \tilde{N}$*

(i) $\Big\{f_1(.,\ell), f_2(.,\ell), ..., f_k(.,\ell)\Big\}$ *be K-G_{f_i}-bonvex at δ w.r.t. η_1,*
(ii) $\Big\{f_1(\varphi,.), f_2(\varphi,.), ..., f_k(\varphi,.)\Big\}$ *be K-G_{f_i}-boncave in ϑ w.r.t. η_2,*
(iii) $\eta_1(\varphi,\delta) + \delta \in C_1, \forall\, (\varphi,\delta) \in C_1 \times C_2$,
(iv) $\eta_2(\ell,\vartheta) + \vartheta \in C_2, \forall\, (\ell,\vartheta) \in C_1 \times C_2$,
Then, $L(\varphi,\vartheta,\lambda,p) - M(\delta,\ell,\lambda,q) \notin -K\backslash\{0\}$.

Proof. If possible, then suppose

$$L(\varphi,\vartheta,\lambda,p) - M(\delta,\ell,\lambda,q) \in -K\backslash\{0\},$$

or

$$\Big\{G_{f_1}(f_1(\varphi,\vartheta)) - \vartheta^T \sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta) + \Big\{G''_{f_i}(f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i\Big)$$
$$-\tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i p_i^T \Big\{G''_{f_i}(f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i, ..., G_{f_k}(f_k(\varphi,\vartheta))$$
$$-\vartheta^T \textstyle\sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta) + \Big\{G''_{f_i}(f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i\Big)$$
$$-\tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i p_i^T \Big\{G''_{f_i}(f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i - G_{f_1}(f_1(\delta,\ell)) - \delta^T \textstyle\sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(\delta,\ell))$$
$$\nabla_\varphi f_i(\delta,\ell) + \Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T + G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\}q_i\Big) - \tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i q_i^T \Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T$$
$$+G'_{f_i}(f(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\}q_i, ..., G_{f_k}(f_k(\delta,\ell)) - \delta^T \textstyle\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell) + \Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T$$
$$+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\}q_i) - \tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i q_i^T \Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T + G'_{f_i}(f(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\}q_i\Big\} \in -K\backslash\{0\}.$$

Since $\lambda \in \mathrm{int}K^*$, we get

$$\begin{aligned}&\sum_{i=1}^{k} \lambda_i \Big\{G_{f_i}(f_i(\varphi,\vartheta)) - \vartheta^T \sum_{i=1}^{k} \lambda_i \Big[G'_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta) + \Big\{G''_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta)(\nabla_\vartheta f_i(\varphi,\vartheta))^T\\ &+G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\}p_i\Big] - \delta^T \textstyle\sum_{i=1}^{k} \lambda_i \Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell) + \Big\{G''_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)\\ &(\nabla_\varphi f_i(\delta,\ell))^T\Big\}\Big] - \tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i p_i^T \Big\{G''_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta f_i(\varphi,\vartheta)(\nabla_\vartheta f_i(\varphi,\vartheta))^T + G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_i(\varphi,\vartheta)\Big\} - \Big\{G_{f_i}(f_i(\delta,\ell))\\ &+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\delta,\ell)\Big\} - \tfrac{1}{2}\textstyle\sum_{i=1}^{k} \lambda_i q_i^T \Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T + G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi} f_i(\varphi,\vartheta)\Big\}q_i\Big\} < 0.\end{aligned} \tag{5}$$

By hypothesis (i) and using $\lambda \in \mathrm{int}\, K^*$, we get

$$\sum_{i=1}^{k}\lambda_i\Big\{G_{f_i}(f_i(\varphi,\ell))-G_{f_i}(f_i(\delta,\ell))+\frac{1}{2}q_i^T\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}q_i$$
$$-\eta_1^T(\varphi,\delta)\Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)+\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}q_i\Big]\Big\}\geqq 0,$$

Using feasibility of dual problem (GWDP) & using dual constraints with assumption (iii), it yields

$$\Big(\eta_1(\varphi,\delta)+\delta\Big)^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)+\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi f_i(\delta,\ell)\Big\}q_i\Big]\geqq 0,$$

it implies that

$$\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(\varphi,\ell))-G_{f_i}(f_i(\delta,\ell))+\frac{1}{2}q_i^T\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}q_i\Big]$$
$$\geqq -\delta^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi(f_i(\delta,\ell))+\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}q_i\Big].\quad (6)$$

Similarly, using hypotheses (ii), (iv), feasible conditions of primal problem (GWPP), dual constraint and $\lambda\in intK^*$,
we get

$$\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(\varphi,\vartheta))-G_{f_i}(f_i(\varphi,\ell))+\frac{1}{2}p_i^T\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\vartheta f_i(\delta,\ell))(\nabla_\vartheta f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}p_i\Big]$$
$$\geqq \vartheta^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\vartheta(f_i(\delta,\ell))+\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}p_i\Big].\quad (7)$$

Now, from inequalities (6), (7) and using the fact that $\lambda^Te_k=1$, we find that

$$\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(\varphi,\vartheta))-\vartheta^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\varphi,\vartheta))\nabla_\vartheta(f_i(\varphi,\vartheta))+\Big\{G''_{f_i}(f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\delta,\ell))(\nabla_\vartheta f_i(\delta,\ell))^T+G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_i(\varphi,\vartheta)\Big\}p_i\Big]$$
$$-\tfrac{1}{2}\textstyle\sum_{i=1}^{k}\lambda_ip_i^T\Big\{G''_{f_i}(f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))(\nabla_\vartheta f_i(\varphi,\vartheta))^T+G'_{f_i}(f_i(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_i(\varphi,\vartheta)\Big\}-G_{f_i}(f_i(\delta,\ell))$$
$$-\delta^T\textstyle\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(\delta,\ell))\nabla_\varphi(f_i(\delta,\ell))+\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))\nabla_{\varphi\varphi}f_i(\delta,\ell)\Big\}\Big]$$
$$-\tfrac{1}{2}\delta^T\textstyle\sum_{i=1}^{k}\lambda_iq_i^T\Big\{G''_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))^T+G'_{f_i}(f_i(\delta,\ell))(\nabla_\varphi f_i(\delta,\ell))q_i\Big\}\Big]\geqq 0,$$

we arrive at contradiction. □

Through following example, we validate the Weak duality theorem as:

Example 4. *Let n=m=1,* $k=2$, $X=[1,2]$, $p\in[2^2,2^{10}]$, $q\in[10^{-19},10^{19}]$, $K=\Big\{(\varphi,\vartheta);\ \varphi\geqq 0,\ \varphi\geqq\vartheta\Big\}$ *and*

$-K=\Big\{(\varphi,\vartheta);\ \varphi\leq 0,\ \varphi\leq\vartheta\Big\}$, $R_1=R_2=R_+$. *Let* $f_i: R_1\times R_2\to R$ *and* G_{f_i} *for* $i=1,2$. *be defined as*

$$f_1(\varphi,\vartheta)=\varphi+cos\vartheta,\ f_2(\varphi,\vartheta)=sin\vartheta,\ G_{f_1}(t)=t^2,\ G_{f_2}(t)=t.$$

Further, let

$$\eta_1(\varphi,\delta)=\varphi\delta,\ \ \eta_2(\ell,\vartheta)=\ell-\vartheta.$$

Assume that $C_1 = C_2 = C_1^* = C_2^* = R_+$.

(GWPP) K-minimize $L(\varphi,\vartheta,\lambda,p) = \Big\{L_1(\varphi,\vartheta,\lambda,p),\ L_2(\varphi,\vartheta,\lambda,p)\Big\}$

Subject to constraints

$$\lambda_1\Big[2(\varphi+cos\vartheta)(-sin\vartheta) + \Big\{2sin^2\vartheta + 2(\varphi+cos\vartheta)(-cos\vartheta)\Big\}p_1\Big] + \lambda_2\Big[cos\vartheta - p_2 sin\vartheta\Big] \leqq 0, \tag{8}$$

$$\lambda_1+\lambda_2 = 1, \lambda_i \in \text{int}K^*, \varphi \in C_1,\ i = 1,\ 2. \tag{9}$$

(GWDP) K-maximize $M(\delta,\ell,\lambda,q) = \Big\{M_1(\delta,\ell,\lambda,q),\ M_2(\delta,\ell,\lambda,q)\Big\}$

Subject to constraints

$$\lambda_1\Big[2(\varphi+cos\vartheta)+2q_1\Big] \geqq 0, \tag{10}$$

$$\lambda_1+\lambda_2 = 1,\ \lambda_i \in \text{int}K^*,\ \varphi \in C_2,\ i = 1,\ 2. \tag{11}$$

(A1). $\Big\{f_1(.,\ell), f_2(.,\ell)\Big\}$ is K-G_f-bonvex at $\delta = 0$ w.r.t. η_1 ,$\forall\ \varphi \in S_1$, i.e.,

$$\begin{aligned}&\Big\{G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\delta,\ell)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big]p_1\\ &-\eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell) + \Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\}p_1\Big],\\ &G_{f_2}(f_2(\varphi,\ell)) - G_{f_2}(f_2(\delta,\ell)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)(\nabla_\varphi f_2(\delta,\ell))^T + G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big]p_2\\ &-\eta^T(\varphi,\delta)\Big[G'_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell) + \Big\{G''_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)(\nabla_\varphi f_2(\delta,\ell))^T + G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big\}p_2\Big]\Big\} \in K.\end{aligned} \tag{12}$$

Consider

$$\begin{aligned}\Psi = &\Big\{G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\delta,\ell)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big]p_1\\ &-\eta^T(\varphi,\delta)\Big[G'_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell) + \Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T + G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\}p_1\Big],\\ &G_{f_2}(f_2(\varphi,\ell)) - G_{f_2}(f_2(\delta,\ell)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)(\nabla_\varphi f_2(\delta,\ell))^T + G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big]p_2\\ &-\eta^T(\varphi,\delta)\Big[G'_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell) + \Big\{G''_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)(\nabla_\varphi f_2(\delta,\ell))^T + G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big\}p_2\Big]\Big\}.\end{aligned} \tag{13}$$

Putting the values of $f_1,\ f_2, G_{f_1}, G_{f_2}$ and η_1 at the point $\delta = 0$, and simplifying, we get

$$\Psi = \Big(\varphi^2 + 2\ \varphi\ cos\ell + p^2,\ \ 0\Big).$$

It is clear that

$$\Psi = \Big(\varphi^2 + 2\ \varphi\ cos\ell + p^2,\ \ 0\Big) \in K.$$

(A2). $\Big\{f_1(\varphi,.),\ f_2(\varphi,.)\Big\}$ is K-G_f-boncave at $\vartheta = 0$ w.r.t. η_2, $\ell \in S_2$,

$$\begin{aligned}&\Big\{G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\varphi,\vartheta)) + \frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta)(\nabla_\vartheta f_1(\varphi,\vartheta))^T + G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_1(\varphi,\vartheta)\Big]p_1\\ &-\eta^T(\ell,\vartheta)\Big[G'_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta) + \Big\{G''_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta)(\nabla_\vartheta f_1(\varphi,\vartheta))^T + G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_1(\varphi,\vartheta)\Big\}p_1\Big],\\ &G_{f_2}(f_2(\varphi,\ell)) - G_{f_2}(f_2(\varphi,\vartheta)) + \frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)(\nabla_\vartheta f_2(\varphi,\vartheta))^T + G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_2(\varphi,\vartheta)\Big]p_2\end{aligned}$$

$$-\eta^T(\ell,\vartheta)\Big[G'_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)+\Big\{G''_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)(\nabla_\vartheta f_2(\varphi,\vartheta))^T+G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_2(\varphi,\vartheta)\Big\}p_2\Big]\Big\}\in -K. \tag{14}$$

Let $\Psi_1=\Big\{G_{f_1}(f_1(\varphi,\ell))-G_{f_1}(f_1(\varphi,\vartheta))+\frac{1}{2}p_1^T\Big[G''_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta)(\nabla_\vartheta f_1(\varphi,\vartheta))^T+G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_1(\varphi,\vartheta)\Big]p_1$
$-\eta^T(\ell,\vartheta)\Big[G'_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta)+\Big\{G''_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta)(\nabla_\vartheta f_1(\varphi,\vartheta))^T+G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_1(\varphi,\vartheta)\Big\}p_1\Big],$
$G_{f_2}(f_2(\varphi,\ell))-G_{f_2}(f_2(\varphi,\vartheta))+\frac{1}{2}p_2^T\Big[G''_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)(\nabla_\vartheta f_2(\varphi,\vartheta))^T+G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_2(\varphi,\vartheta)\Big]p_2$

$$-\eta^T(\ell,\vartheta)\Big[G'_{f_2}(f_2(\varphi,\vartheta)\nabla_\vartheta f_2(\varphi,\vartheta)+\Big\{G''_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)(\nabla_\vartheta f_2(\varphi,\vartheta))^T+G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta}f_2(\varphi,\vartheta)\Big\}p_2\Big]\Big\}. \tag{15}$$

Putting the values of $f_1, f_2, G_{f_1}, G_{f_2}$ and η_2 at $\vartheta=0$, we obtain

$$\Psi_1=\Big((\varphi+cos\ell)^2-(\varphi+1)^2-p_1^2(\varphi+1)+2\ell(\varphi+1),\ \ sin\ell-\ell\Big).$$

$$\Psi_1=\Big((\varphi+cos\ell)^2-(\varphi+1)^2-p_1^2(\varphi+1)+2\ell(\varphi+1),\ \ sin\ell-\ell\Big)\in -K.$$

(A3). $\eta_1(\varphi,\delta)+\delta\in C_1,\ \ \forall\ \varphi\in C_1.$

(A4). $\eta_2(\ell,\vartheta)+\vartheta\in C_2,\ \ \forall\ \ell\in C_2.$

Validation: To validate Weak duality theorem it is enough to claim that any point $(\varphi,0,\lambda_1,\lambda_2,p)$ such that $\varphi\geqq 0, \lambda_1+\lambda_2=1$ are feasible to $(GWPP)$. Also, the points $(0,\ell,\lambda_1,\lambda_2,q)$ such that $\ell\geqq 0, \lambda_1+\lambda_2=1$ are feasible to $(GWDP)$. Now, at these feasible points,

$$L=(L_1,L_2)=\Big((\varphi+1)^2+\lambda_1p_1^2(\varphi+1),\ \lambda_1p_1^2(\varphi+1)\Big),$$

and

$$M=(M_1,M_2)=\Big(cos^2\ell-\lambda_1q_1^2,\ sin\ell-\lambda_1q_1^2\Big).$$

Now, calculate the value at above feasible points, we have

$$L(\varphi,\vartheta,\lambda,p)-M(\delta,\ell,\lambda,q)=\Big((\varphi+1)^2+\lambda_1p_1^2(\varphi+1)-cos^2\ell+\lambda_1q_1^2,\ \ \lambda_1p_1^2(\varphi+1)-sin\ell+\lambda_1q_1^2\Big)\notin K\backslash\{0\}. \tag{16}$$

In particular, the points $\Big(\varphi,\vartheta,\lambda_1,\lambda_2,p\Big)=\Big(1,0,\frac{1}{2},\frac{1}{2},4\Big)$ and $\Big(\delta,\ell,\lambda_1,\lambda_2,q\Big)=\Big(0,\frac{22}{14},\frac{1}{2},\frac{1}{2},2\Big)$ are feasible solutions for $(GWPP)$ and $(GWDP)$, respectively. Also

$$L(\varphi,\vartheta,\lambda,p)-M(\delta,\ell,\lambda,q)=(22,17)\notin -K\backslash\{0\}. \tag{17}$$

Hence, this validate the results.

Remark 2. *Every pseudoconvex function is convex function. On the same pattern we can proof that K-G_f-pseudobonvex is K-G_f-bonvex with respect to same η. So, above proof of Weak duality 3.2 follows on same pattern as Theorem 1.*

Theorem 2 (Weak duality). *Let $(\varphi,\vartheta,\lambda,p)\in V^*$ and $(\delta,\ell,\lambda,q)\in W^*$. Let, For $i\in\tilde{N}$*

(i) $\Big\{f_1(.,\ell),f_2(.,\ell),...,f_k(.,\ell)\Big\}$ *be K-G_f-pseudobonvex at ℓ w.r.t. η_1,*

(ii) $\Big\{f_1(\varphi,.),f_2(\varphi,.),...,f_k(\varphi,.)\Big\}$ *be K-G_f-pseudoboncave at ϑ, w.r.t. η_2,*

(iii) $\eta_1(\varphi,\delta)+\delta\in C_1, \forall\ (\varphi,\delta)\in C_1\times C_2,$

(iv) $\eta_2(\ell,\vartheta)+\vartheta \in C_2, \forall\ (\ell,\vartheta)\in C_1\times C_2,$
Then, $L(\varphi,\vartheta,\lambda,p)-M(\delta,\ell,\lambda,q)\notin -K\backslash\{0\}.$

Proof. Proof follows on same lines as Weak Duality Theorem 1. □

Example 5. *For* $n=m=1,\ k=2,\ X=[2,3],\ p\in[0,1],\ q\in[2,2^{10}],\ K=\Big\{(\varphi,\vartheta);\ \varphi\leqq 0,\ \vartheta\geqq 0,\ |\varphi|\geqq\vartheta\Big\},$

$R_1=R_2=R_+$. *Let* $f_i: R_1\times R_2\to R$ *be given as*

$$f_1(\varphi,\vartheta)=\varphi+\vartheta^2,\ f_2(\varphi,\vartheta)=1-\vartheta,\ G_{f_1}(t)=t^2,\ G_{f_2}(t)=t.$$

Further, Let

$$\eta_1(\varphi,\delta)=\varphi\delta,\ \eta_2(\ell,\vartheta)=\ell-\vartheta.$$

Assume that $C_1=C_2=C_1^*=C_2^*=R_+$.

(GWPP) K-minimize $L(\varphi,\vartheta,\lambda,p)=\Big\{L_1(\varphi,\vartheta,\lambda,p),\ L_2(\varphi,\vartheta,\lambda,p)\Big\}$

Subject to constraints

$$\lambda_1\Big[4\vartheta(\varphi+\vartheta^2)+p_1\{8\vartheta^2+4(\varphi+\vartheta^2)\}\Big]-\lambda_2\leqq 0, \tag{18}$$

$$\lambda_1+\lambda_2=1,\ \lambda_i\in \mathrm{int}K^*,\ \varphi\in C_1,\ i=1,2. \tag{19}$$

(GWDP) K-maximize $M(\delta,\ell,\lambda,q)=\Big\{M_1(\delta,\ell,\lambda,q),\ M_2(\delta,\ell,\lambda,q)\Big\}$

Subject to constraints

$$\lambda_1\Big[2(\delta+\ell^2+q)\Big]\geqq 0, \tag{20}$$

$$\lambda_1+\lambda_2=1,\ \lambda_i\in \mathrm{int}K^*,\ \delta\in C_2,\ i=1,2. \tag{21}$$

(A1). $\Big\{f_1(.,\ell),\ f_2(.,\ell)\Big\}$ is K-G_f-pseudobonvex at δ with respect to η_1 , $\varphi\in R_1$, so that

$$\eta_1^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)+p\Big\{G''_{f_1}(f_1(\delta,\ell))(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\},$$
$$G'_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)+p\Big\{G''_{f_2}(f_2(\delta,\ell))(\nabla_\varphi f_2(\delta,\ell))^T+G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big\}\Big\}\in K. \tag{22}$$

Let

$$\Pi_1=\eta_1^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)+p\Big\{G''_{f_1}(f_1(\delta,\ell))(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\},$$
$$G'_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)+p\Big\{G''_{f_2}(f_2(\delta,\ell))(\nabla_\varphi f_2(\delta,\ell))^T+G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big\}\Big\}. \tag{23}$$

Next, let

$$\Pi_2=\Big[G_{f_1}(f_1(\varphi,\ell))-G_{f_1}(f_1(\delta,\ell))+\frac{1}{2}p^T\Big\{G''_{f_1}(f_1(\delta,\ell))\nabla_\varphi f_1(\delta,\ell)(\nabla_\varphi f_1(\delta,\ell))^T+G'_{f_1}(f_1(\delta,\ell))\nabla_{\varphi\varphi}f_1(\delta,\ell)\Big\}p,$$
$$G_{f_2}(f_2(\varphi,\ell))-G_{f_2}(f_2(\delta,\ell))+\frac{1}{2}p^T\Big\{G''_{f_2}(f_2(\delta,\ell))\nabla_\varphi f_2(\delta,\ell)(\nabla_\varphi f_2(\delta,\ell))^T+G'_{f_2}(f_2(\delta,\ell))\nabla_{\varphi\varphi}f_2(\delta,\ell)\Big\}p\Big]. \tag{24}$$

After simplification, substituting the value of $f_1,\ f_2,\ G_{f_1},\ G_{f_2}$ and η_1 at $\delta = 0$, we get

$$\Pi_1 = (0,\ 0) \in K \Rightarrow \Pi_2 = (\varphi^2 - 2\varphi\ell^2 + p^2,\ 0) \in K.$$

(A2). $\Big\{f_1(\varphi,.), f_2(\varphi,.)\Big\}$ is K-G_f-pseudoboncave at ϑ with respect to η_2 for fixed φ for all $\ell \in S_2$, i.e.,

$$\begin{aligned}
&\eta_2^T(\varphi,\delta)\Big\{G'_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta) + q\Big\{G''_{f_1}(f_1(\varphi,\vartheta))(\nabla_\vartheta f_1(\varphi,\vartheta))^T + G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_1(\varphi,\vartheta)\Big\}, G'_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta) \\
&\quad + q\Big\{G''_{f_2}(f_2(\varphi,\vartheta))(\nabla_\vartheta f_2(\varphi,\vartheta))^T + G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_2(\varphi,\vartheta)\Big\}\Big\} \in K \\
&\quad \Rightarrow \Big[G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\varphi,\vartheta)) + \frac{1}{2}q^T\Big\{G''_{f_1}(f_1(\varphi,\vartheta))\ \nabla_\vartheta f_1(\varphi,\vartheta)(\nabla_\vartheta f_1(\varphi,\vartheta))^T + G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_1(\varphi,\vartheta)\Big\}q,
\end{aligned}$$

$$G_{f_2}(f_2(\varphi,\ell)) - G_{f_2}(f_2(\varphi,\vartheta)) + \frac{1}{2}q^T\Big\{G''_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)(\nabla_\vartheta f_2(\varphi,\vartheta))^T + G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_2(\varphi,\vartheta)\Big\}q\Big] \in -K. \tag{25}$$

Let
$$\begin{aligned}
\Pi_3 = \eta_2^T(\varphi,\delta)\Big\{&G'_{f_1}(f_1(\varphi,\vartheta))\nabla_\vartheta f_1(\varphi,\vartheta) + q\Big\{G''_{f_1}(f_1(\varphi,\vartheta))(\nabla_\vartheta f_1(\varphi,\vartheta))^T + G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_1(\varphi,\vartheta)\Big\}, \\
&G'_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta) + q\Big\{G''_{f_2}(f_2(\varphi,\vartheta))(\nabla_\vartheta f_2(\varphi,\vartheta))^T + G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_2(\varphi,\vartheta)\Big\}\Big\},
\end{aligned} \tag{26}$$

and

$$\Pi_4 = \Big[G_{f_1}(f_1(\varphi,\ell)) - G_{f_1}(f_1(\varphi,\vartheta)) + \frac{1}{2}q^T\Big\{G''_{f_1}(f_1(\varphi,\vartheta))\ \nabla_\vartheta f_1(\varphi,\vartheta)(\nabla_\vartheta f_1(\varphi,\vartheta))^T + G'_{f_1}(f_1(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_1(\varphi,\vartheta)\Big\}q,$$

$$G_{f_2}(f_2(\varphi,\ell)) - G_{f_2}(f_2(\varphi,\vartheta)) + \frac{1}{2}q^T\Big\{G''_{f_2}(f_2(\varphi,\vartheta))\nabla_\vartheta f_2(\varphi,\vartheta)(\nabla_\vartheta f_2(\varphi,\vartheta))^T + G'_{f_2}(f_2(\varphi,\vartheta))\nabla_{\vartheta\vartheta} f_2(\varphi,\vartheta)\Big\}q\Big]. \tag{27}$$

Substituting the value of $f_1,\ f_2,\ G_{f_1},\ G_{f_2}$ and η_2 at the point $\delta = 0$ and simplify, we get

$$\Pi_3 = \Big(4vq\varphi,\ -1\Big) \in -K \Rightarrow \Pi_4 = \Big(\ell^4 + 2\varphi\ell^2,\ -\ell\Big) \in -K.$$

(A3). $\eta_1(\varphi,\delta) + \delta \in C_1, \forall\ \varphi \in C_1$.

(A4). $\eta_2(\ell,\vartheta) + \vartheta \in C_2, \forall\ \ell \in C_2$.

Validation: To prove our result its enough to prove that any point $\Big(\varphi, 0,\ \lambda_1,\ \lambda_2,\ p\Big)$ such that $\varphi \geqq 0,\ \lambda_1 + \lambda_2 = 1$ are feasible to $(GWPP)$. Also, the points $\Big(0, \ell, \lambda_1, \lambda_2, q\Big)$ such that $\ell \geqq 0,\ \lambda_1 + \lambda_2 = 1$ are feasible to $(GWDP)$. Now, at these feasible points,

$$L = \Big(L_1,\ L_2\Big) = \Big(\varphi^2 - 2\varphi\lambda_1 p^2,\ 1 - 2\varphi\lambda_1 p^2\Big)$$

and
$$M = \Big(M_1,\ M_2\Big) = \Big(\ell^4 - \lambda_1 q^2,\ 1 - \ell - \lambda_1 q^2\Big).$$

Now at above feasible condition

$$L - M = \Big(\varphi^2 - 2\varphi\lambda_1 p^2 - \ell^4 + \lambda_1 q^2,\ \ell - 2\varphi\lambda_1 p^2 + \lambda_1 q^2\Big) \notin K\backslash\{0\}. \tag{28}$$

In particular, the points $\Big(\varphi,\ \vartheta,\ \lambda_1,\ \lambda_2,\ p\Big) = \Big(2,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ 1\Big)$ and $\Big(\delta,\ \ell,\ \lambda_1,\ \lambda_2,\ q\Big) = \Big(0,\ 2,\ \frac{1}{2},\ \frac{1}{2},\ 2\Big)$ are

feasible for $(GWPP)$ and $(GWDP)$ respectively,
Now, calculate

$$L(\varphi,\vartheta,\lambda,p) - M(\delta,\ell,\lambda,q) = (-12,2) \notin K\backslash\{0\}. \tag{29}$$

Hence, this validate the Weak duality Theorem 2.

Theorem 3 (Strong duality). *Let $(\bar{\varphi},\ \bar{\vartheta},\ \bar{\lambda},\ \bar{p}_1 = \bar{p}_2 = \bar{p}_3 = ... = \bar{p}_k)$ is an efficient solution of $(GWPP)$; fix $\lambda = \bar{\lambda}$ in $(GWDP)$ such that*

(i) *for all* $i \in \tilde{N}$, $\left[G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right]$ *is nonsingular,*

(ii) *the vector* $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_\vartheta\left[\bar{p}_i\{G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\}\bar{p}_i\right]$

$$\notin span\ \left\{G'_{f_1}(f_1(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_1(\bar{\varphi},\bar{\vartheta}), G'_{f_2}(f_2(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_2(\bar{\varphi},\bar{\vartheta}), ..., G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})\right\},$$

(iii) *the set of vectors* $\left\{G'_{f_1}(f_1(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_1(\bar{\varphi},\bar{\vartheta}), G'_{f_2}(f_2(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_2(\bar{\varphi},\bar{\vartheta}), ..., G'_{f_k}(f_k(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_k(\bar{\varphi},\bar{\vartheta})\right\}$ *are linearly independent,*

(iv) $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_\vartheta\left[\bar{p}_i\{G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\}\bar{p}_i\right] = 0 \Rightarrow \bar{p}_i = 0,\ \forall\ i$, *and*

(v) *K is closed convex pointed cone with $R^k_+ \subseteq K$.*

Then, $(\bar{\varphi},\bar{\vartheta},\bar{\lambda},\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = ... = \bar{q}_k = 0) \in W^*$ and $L(\bar{\varphi},\bar{\vartheta},\bar{p}) = M(\bar{\varphi},\bar{\vartheta},\bar{q})$. Also, if the hypotheses of Theorem 1 or Theorem 2 are satisfied for all feasible solutions for $(GWPP)$ and $(GWDP)$, then $(\bar{\varphi},\bar{\vartheta},\bar{\lambda},\bar{p})$ and $(\bar{\varphi},\bar{\vartheta},\bar{\lambda},\bar{q})$ is an efficient solution for $(GWPP)$ and $(GWDP)$, respectively.

Proof. Since $(\bar{\varphi},\bar{\vartheta},\bar{\lambda},\bar{p}_1,\bar{p}_2,\bar{p}_3,....,\bar{p}_k)$, is an efficient solution of $(GWPP)$, there exist $\alpha \in K^*,\ \beta \in C_2$ and $\bar{\eta} \in R$ such that the following Fritz -John optimality condition stated by [28] are satisfied at $(\bar{\varphi},\bar{\vartheta},\bar{\lambda},\bar{p}_1,\bar{p}_2,\bar{p}_3,...,\bar{p}_k)$:

$$\begin{aligned}
&\Big(\varphi-\bar{\varphi}\Big)^T\left[\sum_{i=1}^{k}\alpha_i\Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\varphi f_i(\bar{\varphi},\bar{\vartheta})\Big] + \sum_{i=1}^{k}\bar{\lambda}_i\Big[G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}) + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\Big]\Big[\beta-(\bar{\alpha}^Te_k)\bar{\vartheta}\Big]\right.\\
&\left.+\sum_{i=1}^{k}\bar{\lambda}_i\nabla_\varphi\Big[(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))\bar{p}_i\Big]\Big(\beta-(\bar{\alpha}^Te_k)\Big(\bar{\vartheta}+\frac{1}{2}\bar{p}_i\Big)\Big)\right] \geqq 0,\ \ \forall\ \varphi \in C_1,
\end{aligned}\tag{30}$$

$$\begin{aligned}
&\Big(\vartheta-\bar{\vartheta}\Big)^T\left\{\sum_{i=1}^{k}\alpha_i\Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})\Big] + \sum_{i=1}^{k}\bar{\lambda}_i\Big[G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\Big]\right.\\
&\quad\Big(\bar{\beta}-(\bar{\alpha}^Te_k)\bar{\vartheta}\Big) + \sum_{i=1}^{k}\bar{\lambda}_i\nabla_\vartheta\Big[(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))\bar{p}_i\Big]\\
&\quad\quad\Big[\bar{\beta}-(\bar{\alpha}^Te_k)\Big(\bar{\vartheta}+\frac{1}{2}\bar{p}_i\Big)\Big] - \sum_{i=1}^{k}\bar{\lambda}_i\Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})\\
&\left.+(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))\bar{p}_i\Big](\bar{\alpha}^Te_k)\right\} \geqq 0,\ \ \forall\ \vartheta \in R^m,
\end{aligned}\tag{31}$$

$$\begin{aligned}
&G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})\Big(\bar{\beta}-(\bar{\alpha}^Te_k)\bar{\vartheta}\Big) + \bar{\eta}e_k + \left\{\Big\{\beta-(\bar{\alpha}^Te_k)\Big(\bar{\vartheta}+\frac{1}{2}\bar{p}_1\Big)\Big\}^T\Big(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T\right.\\
&\quad+G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\Big)\bar{p}_1,\ \left\{\Big\{\beta-(\bar{\alpha}^Te_k)\Big(\bar{\vartheta}+\frac{1}{2}\bar{p}_2\Big)\Big\}^T\Big(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T\right.\\
&\quad\left\{\Big\{\beta-(\bar{\alpha}^Te_k)\Big(\bar{\vartheta}+\frac{1}{2}\bar{p}_3\Big)\Big\}^T\Big(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\Big)\bar{p}_3,\ldots,\right.\\
&\left\{\Big\{\beta-(\bar{\alpha}^Te_k)\Big(\bar{\vartheta}+\frac{1}{2}\bar{p}_3\Big)\Big\}^T\Big(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta})(\nabla_\vartheta f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\Big)\bar{p}_k\right\} = 0,
\end{aligned}\tag{32}$$

$$\left[G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right]\left((\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{p}_i + \bar{\vartheta}))\bar{\lambda}_i\right) = 0,\ i \in \tilde{N}, \tag{33}$$

$$\bar{\beta}^T \sum_{i=1}^{k} \bar{\lambda}_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}) + \left\{G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right\}\bar{p}_i\right] = 0, \tag{34}$$

$$\bar{\eta}^T\left[\bar{\lambda}^T e_k - 1\right] = 0, \tag{35}$$

$$\left(\bar{\alpha},\bar{\beta},\bar{\eta}\right) \geqq 0,\ \left(\bar{\alpha},\bar{\beta},\bar{\eta}\right) \neq 0. \tag{36}$$

Inequalities (31) and (32) can be rewritten in the following expressions:

$$\begin{aligned}
&\sum_{i=1}^{k} \alpha_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right] + \sum_{i=1}^{k} \bar{\lambda}_i \left[G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right]\\
&\left(\bar{\beta} - (\bar{\alpha}^T e_k)\bar{\vartheta}\right) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{\vartheta}\left[(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))\bar{p}_i\right]\\
&\left[\bar{\beta} - (\bar{\alpha}^T e_k)\left(\bar{\vartheta} + \frac{1}{2}\bar{p}_i\right)\right] - \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\\
&+ \left(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right)\bar{p}_i\Big](\bar{\alpha}^T e_k) = 0.
\end{aligned} \tag{37}$$

$$\begin{aligned}
&G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\left(\bar{\beta} - (\bar{\alpha}^T e_k)\bar{\vartheta}\right) + \Bigg\{\left\{\beta - (\bar{\alpha}^T e_k)\left(\bar{\vartheta} + \frac{1}{2}\bar{p}_i\right)\right\}^T\\
&\left(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right)\bar{p}_i\Bigg\} + \bar{\eta} = 0,\ i \in \tilde{N}.
\end{aligned} \tag{38}$$

Now, from hypothesis (iv), it is given that $R^k_+ \subseteq K \Rightarrow \text{int } K^* \subseteq \text{int } R^k_+$.

Obviously, $\bar{\lambda} > 0$ because $\bar{\lambda} \in \text{int } K^*$.

By hypothesis (i), (33) gives

$$\beta = (\bar{\alpha}^T e_k)(\bar{p}_i + \bar{\vartheta}),\ i \in \tilde{N}. \tag{39}$$

Suppose $\bar{\alpha} = 0$, then (39) yields $\bar{\beta} = 0$. Further, from (38) gives $\bar{\eta} = 0$. Now, we reach at contradiction (36). Hence, $\bar{\alpha} \neq 0$. Further, $\bar{\alpha} \in K^* \subseteq R^k_+$ implies

$$\bar{\alpha}^T e_k > 0. \tag{40}$$

Now, we have to claim that $\bar{p}_i = 0,\ \ i \in \tilde{N}$. Using (39) and (40) in (38), we get

$$\begin{aligned}
&\sum_{i=1}^{k} \bar{\lambda}_i \left[\nabla_{\vartheta}\left\{\frac{1}{2}\bar{p}_i(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))\bar{p}_i\right\}\right]\\
&= -\frac{1}{\mu}\sum_{i=1}^{k}\left(\alpha_i - \mu\bar{\lambda}_i\right)[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})],
\end{aligned} \tag{41}$$

By hypothesis (ii), we get

$$\sum_{i=1}^{k} \bar{\lambda}_i \left[\nabla_{\vartheta}\left\{\bar{p}_i(G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta\vartheta} f_i(\bar{\varphi},\bar{\vartheta}))\bar{p}_i\right\}\right] = 0. \tag{42}$$

Again, from hypothesis (iv), we have

$$\bar{p}_i = 0, \ \forall\, i \in \tilde{N}. \tag{43}$$

From (39) implies

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{\vartheta}. \tag{44}$$

Using (42) and (43) in (37), we obtain

$$\sum_{i=1}^{k} \left(\alpha_i - (\bar{\alpha}^T e_k)\bar{\lambda}_i\right)\left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta})\right] = 0. \tag{45}$$

From hypothesis (iii), it yields

$$\alpha_i = (\bar{\alpha}^T e_k)\bar{\lambda}_i, \ \ i \in \tilde{N}. \tag{46}$$

Using (43) and (44) in (30), we get

$$(\varphi - \bar{\varphi})^T \sum_{i=1}^{k} \alpha_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})]\right] \geqq 0.$$

Using (40), (43), (44) and (46) in (30) , we find that

$$(\varphi - \bar{\varphi})^T \sum_{i=1}^{k} \bar{\lambda}_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})]\right] \geqq 0, \ \ \forall\, \varphi \in C_1. \tag{47}$$

Let $\varphi \in C_1$. Then, $\varphi + \bar{\varphi} \in C_1$ and inequality (47) gives that

$$\bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})]\right] \geqq 0, \ \ \forall\, \varphi \in C_1. \tag{48}$$

Therefore,

$$\sum_{i=1}^{k} \bar{\lambda}_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})]\right] \in C_1^*. \tag{49}$$

Also, from (44), we obtain

$$\bar{\vartheta} = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2. \tag{50}$$

Therefore, $(\bar{\varphi}, \bar{\vartheta}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = \ldots = \bar{q}_k = 0)$ satisfies the constraint of (GWDP) and is therefore a feasible solution for the dual problem (GWDP).

Now, letting $\varphi = 0$ and $\varphi = 2\bar{\varphi}$ in (47), we obtain

$$\bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \left[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})\right] = 0. \tag{51}$$

Further, from (34), (40), (43) and (44), we get

$$\bar{\vartheta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) \Big] = 0. \tag{52}$$

Therefore, using (43), (51) and (52), we obtain

$$\begin{aligned}
&\Bigg(G_{f_1}(f_1(\bar{\varphi}, \bar{\vartheta})) - \bar{\vartheta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{p}_i \Big] \\
&- \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{p}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{p}_i \Big], \ldots, G_{f_k}(f_k(\bar{\varphi}, \bar{\vartheta})) - \bar{\vartheta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \\
&\nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{p}_i \Big] - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{p}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}))^T \\
&\qquad + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\vartheta\vartheta} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{p}_i \Big] \Bigg) \\
&= \Bigg(G_{f_1}(f_1(\bar{\varphi}, \bar{\vartheta})) - \bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big] \\
&- \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big], \ldots, G_{f_k}(f_k(\bar{\varphi}, \bar{\vartheta})) - \bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \\
&\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) + \{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \} \bar{q}_i \Big] - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T \\
&\qquad + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big] \Bigg).
\end{aligned}$$

This shows that the objective values are equal.

Finally, we have to claim that $(\bar{\varphi}, \bar{\vartheta}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = \ldots = \bar{q}_k = 0)$ is an efficient solution of $(GWDP)$.

If possible, then suppose that $(\bar{\varphi}, \bar{\vartheta}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = \ldots = \bar{q}_k = 0)$ is not an efficient solution of $(GWDP)$, then there exist $(\bar{\delta}, \bar{\ell}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = \ldots = \bar{q}_k = 0)$ is efficient solution of $(GWDP)$ such that

$$\begin{aligned}
&\Bigg(G_{f_1}(f_1(\bar{\varphi}, \bar{\vartheta})) - \bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big] \\
&- \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big], \ldots, G_{f_k}(f_k(\bar{\varphi}, \bar{\vartheta})) - \bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \\
&\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big] - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) (\nabla_{\varphi} f_i(\bar{\varphi}, \bar{\vartheta}))^T \\
&\qquad + G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi\varphi} f_i(\bar{\varphi}, \bar{\vartheta}) \Big\} \bar{q}_i \Big] - G_{f_1}(f_1(\bar{\delta}, \bar{\ell})) - \bar{\delta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi}, \bar{\vartheta})) \nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}) + \Big\{ G''_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}) (\nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}))^T \\
&+ G'_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi\varphi} f_i(\bar{\delta}, \bar{\ell}) \Big\} \bar{q}_i \Big] \\
&- \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}) (\nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi\varphi} f_i(\bar{\delta}, \bar{\ell}) \Big\} \bar{q}_i \Big], \ldots, G_{f_k}(f_k(\bar{\delta}, \bar{\ell})) - \bar{\delta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \\
&\nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}) + \Big\{ G''_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}) (\nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi\varphi} f_i(\bar{\delta}, \bar{\ell}) \Big\} \bar{q}_i \Big] - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}) (\nabla_{\varphi} f_i(\bar{\delta}, \bar{\ell}))^T \\
&\qquad + G'_{f_i}(f_i(\bar{\delta}, \bar{\ell})) \nabla_{\varphi\varphi} f_i(\bar{\delta}, \bar{\ell}) \Big\} \bar{q}_i \Big] \Bigg) \in -K \backslash \{0\}.
\end{aligned}$$

As

$$\bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta}) = \bar{\vartheta}^T \sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}) \text{ and } \bar{p}_i = 0, \quad i \in \tilde{N},$$

$$\Bigg(G_{f_1}(f_1(\bar{\varphi},\bar{\vartheta})) - \bar{\vartheta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\vartheta} f_i(\bar{\varphi},\bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi\varphi} f_i(\bar{\varphi},\bar{\vartheta}) \Big\}\bar{q}_i \Big]$$

$$-\frac{1}{2}\sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi\varphi} f_i(\bar{\varphi},\bar{\vartheta}) \Big\}\bar{q}_i \Big], ..., G_{f_k}(f_k(\bar{\varphi},\bar{\vartheta})) - \bar{\varphi}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))$$

$$\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta}) + \Big\{ G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta}))^T + G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi\varphi} f_i(\bar{\varphi},\bar{\vartheta}) \Big\}\bar{q}_i \Big] - \frac{1}{2}\sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta})(\nabla_{\varphi} f_i(\bar{\varphi},\bar{\vartheta}))^T$$

$$+ G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi\varphi} f_i(\bar{\varphi},\bar{\vartheta}) \Big\}\bar{q}_i \Big] - G_{f_1}(f_1(\bar{\delta},\bar{\ell})) - \bar{\delta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\varphi},\bar{\vartheta}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}) +$$

$$\Big\{ G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \Big\}\bar{q}_i \Big]$$

$$-\frac{1}{2}\sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \Big\}\bar{q}_i \Big], ..., G_{f_k}(f_k(\bar{\delta},\bar{\ell})) - \bar{\delta}^T \sum_{i=1}^{k} \bar{\lambda}_i \Big[G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))$$

$$\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}) + \Big\{ G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \Big\}\bar{q}_i \Big] - \frac{1}{2}\sum_{i=1}^{k} \lambda_i \bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T$$

$$+ G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \Big\}\bar{q}_i \Big] \Bigg) \in -K \backslash \{0\},$$

which contradicts the Weak duality Theorem 1 or Theorem 2. Hence, completes the proof. □

Theorem 4 (Converse duality). *Let $(\bar{\delta},\bar{\ell},\bar{\lambda},\bar{q})$ is an efficient solution of $(GWDP)$; fix $\lambda = \bar{\lambda}$ in $(GWPP)$ such that*

(i) *for all $i \in \{1,2,...,k\}$, $\Big[G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \Big]$ is non singular,*

(ii) $\displaystyle\sum_{i=1}^{k} \bar{\lambda}_i \nabla_{\varphi} \Big[\bar{q}_i \{ G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \}\bar{q}_i \Big]$
$\notin \mathit{span} \Big\{ G'_{f_1}(f_1(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_1(\bar{\delta},\bar{\ell}), G'_{f_2}(f_2(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_2(\bar{\delta},\bar{\ell}) ...., G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}) \Big\}$.

(iii) *the set of vectors $\Big\{ G'_{f_1}(f_1(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_1(\bar{\delta},\bar{\ell}), G'_{f_2}(f_2(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_2(\bar{\delta},\bar{\ell}), ..., G'_{f_k}(f_k(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_k(\bar{\delta},\bar{\ell}) \Big\}$ are linearly independent,*

(iv) $\displaystyle\sum_{i=1}^{k} \bar{\lambda}_i \nabla_{\vartheta} \Big[\bar{q}_i \Big\{ G''_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell})(\nabla_{\varphi} f_i(\bar{\delta},\bar{\ell}))^T + G'_{f_i}(f_i(\bar{\delta},\bar{\ell}))\nabla_{\varphi\varphi} f_i(\bar{\delta},\bar{\ell}) \Big\}\bar{q}_i \Big] = 0 \Rightarrow \bar{q}_i = 0, \ \forall\ i,$

(v) *K is closed convex pointed cone with $R^k_+ \subseteq K$.*

Then, $(\bar{\delta},\bar{\ell},\bar{\lambda},\bar{p} = 0)$ is a feasible solution for $(GWPP)$ and the objective values of $(GWDP)$ and $(GWPP)$ are equal. Furthermore, if the hypotheses of Theorem 1 or Theorem 2 are satisfied for all feasible solutions of $(GWDP)$ and $(GWPP)$, then $(\bar{\delta},\bar{\ell},\bar{\lambda},\bar{p} = 0)$ is an optimal solution of $(GWPP)$. Also, if the hypotheses of Theorem 1 or Theorem 2 are satisfied for all feasible solutions for $(GWDP)$ and $(GWPP)$, then $(\bar{\delta},\bar{\ell},\bar{\lambda},\bar{q})$ and $(\bar{\delta},\bar{\ell},\bar{\lambda},\bar{p})$ is an efficient solution for $(GWDP)$and $(GWPP)$, respectively.

Proof. It follows on the lines of Theorem 3. □

4. Conclusions

In this paper, we have presented a novel generalized group of definitions and illustrated various non-trivial numerical examples for existing such type of functions. Numerical examples have also been illustrated to justify the weak duality theorem. Furthermore, we have studied a new class of K-G_f-Wolfe type primal-dual model with cone objective as well as constraint and proved duality theorem under K-G_f-bonvexity and K-G_f-pseudobonvexity. This work can further be extended to higher order symmetric fractional programming problem and variational control problem over cones. This will be feature task for the researchers.

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