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Finite-Time Passivity and Synchronization for a Class of Fuzzy Inertial Complex-Valued Neural Networks with Time-Varying Delays

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Abstract: This article investigates finite-time passivity for fuzzy inertial complex-valued neural networks (FICVNNs) with time-varying delays. First, by using the existing passivity theory, several related definitions of finite-time passivity are illustrated. Consequently, by adopting a reduced-order method and dividing complex-valued parameters into real and imaginary parts, the proposed FICVNNs are turned into first-order real-valued neural network systems. Moreover, appropriate controllers and the Lyapunov functional method are established to obtain the finite-time passivity of FICVNNs with time delays. Furthermore, some essential conditions are established to ensure finite-time synchronization for finite-time passive FICVNNs. In the end, corresponding simulations certify the feasibility of the proposed theoretical outcomes.

Keywords: fuzzy inertial complex-valued neural networks; finite-time passivity; finite-time synchronization

MSC: 34K20; 93D03

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1. Introduction

As is known to all, the neural networks as a hot direction of non-linear systems have aroused the interest of experts and scholars due to their fruitful applications, including artificial intelligence, pattern recognition, associative memories, etc. Many worthy related results have been explored in recent years [1–3].

However, plenty of existing articles mainly gather in the first derivative of the states instead of the inertial term, which is the second derivative of voltage concerning time. Hence, based on the Hopfield networks, inertial neural networks (INNs) were put forward by Babcock. As mentioned in this document [4], the dynamic behaviors of second-derivative neural networks could be more sophisticated. During these years, some corresponding results on inertial neural networks (INNs) have been reported [5–9]. Tu et al. [10] discussed the issue of global dissipativity for INNs with memristor-based neutral type using the Filippov theory and LMI approach. In [11], Shanmugasundaram et al. introduced the event-triggered impulsive control mechanism to guarantee synchronization for INNs. Zhang and Cao [12] considered the INNs using inequality techniques to obtain finite-time synchronization. As to the asymptotical stabilization problem, Han et al. [13] used a direct method to analyze the Cohen–Grossberg INNs and constructed two adaptive controllers to make the proposed model realize asymptotical and adaptive stabilization. In [14], the authors were concerned with the issue of global exponential convergence for impulsive INNs and presented an exponential convergence ball with a specified convergence rate.

Admittedly, time delays can easily provoke certain undesirable and unexpected dynamical behaviors and diverse types of time delays, including proportional delay [15,16], time-varying delay [17,18], and mixed delays [19,20]. Moreover, the exchange of neural network systems depends on the current state and the paste state or the variations of the paste state. From the theoretical and particle view, time delays are deeply researched for INN instability [21,22], passivity [23,24], synchronization [11,12,16,25,26], dissipativity [27,28], and so forth.

In many circumstances, the complex-valued neural networks (CVNNs), which activate functions, connection parameters, neuron states, and so forth, are both complex-valued [29–35]. Consequently, it becomes an expansion of real-valued networks. The original intention of investigating CVNNs is to research novel dynamic behaviors and to conquer some puzzles that real-valued networks can not describe [36]. Furthermore, the inertial complex-valued neural networks (ICVNNs) have become a hot theme that has attracted some researchers' attention, especially stability and synchronization [37]. Tang and Jian [38] carry out the exponential convergence for impulsive ICVNNs by developing novel delay-dependent conditions. In [39], the authors emphasized the non-reduced order method to deal with ICVNNs holistically to discuss the question of exponential and adaptive synchronization by constructing a complex-valued feedback control input. Long and Zhang et al. observed the finite-time stabilization and fixed-time synchronization of ICVNNs by utilizing Lyapunov theory and inequality techniques applied theoretical results into practical [40,41].

As a typical and essential problem, fuzzy logic has extensively emerged because it can approximate non-linear functions with arbitrary accuracy, which can be viewed as a potential method by which to emulate human thinking and sensation [42]. Hence, Combining fuzzy logic into a neural network system has been received high concern [43]. On the other hand, as [9] reveals, stabilization and synchronization of many practical engineering systems are required in a finite-time, which lead to the previous results of asymptotic stabilization and synchronization control inoperative. Therefore, it is vital to shorten the convergence time to achieve finite-time synchronization of complex-valued neural networks. In [44], the authors dealt with the question on fixed-time stabilization of fuzzy inertial neural networks (FINNs). The issues of synchronization for FINNs have been addressed in [16,19]. Xiao et al. study the passive and passification for FINNs on time scales inspired by the LMI method and analytical approaches [45]. Furthermore, the authors of [46] developed fuzzy rules into CVNNs and established a class of fuzzy inertial complex-valued neural networks (FICVNNs) to solve the adaptive synchronization problem.

Passivity is a powerful tool for investigating the internal stability of non-linear systems. It is originally from circuit analysis methods that have received much attention from the engineering fields and dynamical neural networks. And some related passivity problems for neural networks have been published [23,24,33,35,45,47–49]. The authors in [24,35,45,48,49] studied the passivity problem for different types of neural networks. Huang et al. [33] further learned the passivity issues for CVNNs with coupled weights. Motivated by the above analysis, based on the existing passivity theory, when it comes to discussing FICVNNs with time-varying delays, some related puzzles naturally arise; for example, how to ensure the proposed FICVNNs realize finite-time passivity (FTP), finite-time input strict passivity (FTISP), as well as finite-time output strictly passivity (FTOSP)? If it can be done, what kind of Lyapunov functional and control inputs are supposed to realize the corresponding passivity goal? Is there any relation between the FTP and FTS? As far as we are concerned, few documents carried out the FTP and FTS of FICVNNs, making this work remarkable and valuable. The main contributions of this article are as follows.

First, by resorting to existing passivity definitions, three concepts of finite-time passivity are illustrated. Moreover, the neural model built in this paper is concerned with inertial terms, complex-valued parameters, fuzzy logic, and time delays, which will increase the difficulty and complexity of solving the neural systems internally stable.

Second, compared with [24,35,45,48,49], we use effective control inputs and the appropriate Lyapunov function to gain some finite-time passivity criteria for delayed FICVNNs.

Third, based on the finite-time passivity, we further discuss the finite-time synchronization issue and apply the simulations about pseudorandom number generators to support the feasibility of the obtained results.

The article parts are arranged as follows: Finite-time passivity definitions and necessary lemmas are put forward in Section 2. The analytical processes and some simulations are obtained in Section 3. The conclusions are expressed in Section 4.

2. Preliminaries

Model, Assumption, Definitions, and Lemmas

Here, $\mathbb{C}^n$ and $\mathbb{R}^n$, respectively, denote the n-dimensional complex vector space and the real vector space with n-dimensional. For any $w = w^R + iw^I \in \mathbb{C}$, i is the imaginary unit and meets $i = \sqrt{-1}$, w^R implies the real part of v, and $w^I \in \mathbb{R}$ is the imaginary part. $\aleph = \{1, 2, \ldots, n\}$, $\tau_j = \sup_{k\in\aleph}\{\tau_j(t_0), \sigma\}, t_0 \geq 0$. For any $\ell = (\ell_1, \ell_2, \ldots, \ell_n)^T \in \mathbb{R}^n$, $\|\ell\| = \sqrt{\sum_{i=1}^{n} |\ell_i|^2}$. $L_k = \max\{|L_k^-|, |L_k^+|\}, k \in \aleph$.

Now, a class of FICVNNs with time-varying delays is given:

$$\begin{aligned}\ddot{x}_k(t) = &- a_k x_k(t) - b_k \dot{x}_k(t) + \sum_{j=1}^{n} c_{kj} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} d_{kj} f_j(x_j(t - \tau_j(t))) + \bigwedge_{j=1}^{n} w_{kj} f_j(x_j(t - \tau_j(t))) \\ &+ \bigvee_{j=1}^{n} q_{kj} f_j(x_j(t - \tau_j(t))),\end{aligned} \tag{1}$$

where $k \in \aleph$, $t \geq 0$, $x_k(t) \in \mathbb{C}$ is the state of kth neural at time t, a_k and b_k are positive constants, and $c_{kj}, d_{kj} \in \mathbb{C}$ denote the feedback connection weights of system (1). $f_j(\cdot) \in \mathbb{C}$ presents the feedback function, $\tau_j(t)$ is the time-varying delay, and $0 \leq \tau_j(t) \leq \tau_j$ and $\dot{\tau}_j(t) \leq \zeta_j < 1$. w_{kj} and q_{kj} imply the fuzzy feedback MIN and MAX template. $\bigwedge$ and $\bigvee$ represent fuzzy AND, OR. The initial conditions of FIVCNNs (1) are

$$x_k(\ell) = \Omega_k^x(\ell), \dot{x}_k(\ell) = \Psi_k(\ell),\ \ell \in [t_0 - \tau_j, t_0], k \in \aleph, \tag{2}$$

where $\Omega_k(\ell) = \Omega_k^R(\ell) + i\Omega_k^I$, $\Psi_k(\ell) = \Psi_k^R + i\Psi_k^I(\ell)$, $\Omega(\ell) = (\Omega_1(\ell), \Omega_2(\ell), \ldots, \Omega_n(\ell))^T$, $\Psi(\ell) = (\Psi_1(\ell), \Psi_2(\ell), \ldots, \Psi_n(\ell))^T$, and $\Omega(\ell), \Psi(\ell) \in \mathcal{C}([t_0 - \tau_j, t_0], \mathbb{C}^n)$. With regard to active function $f_k(\cdot)$, we introduce the following assumption.

Assumption 1. *As to activation function $f_k(x)$, $x = \rho + i\vartheta$, it can be divided by the real part and the imaginary part, such as $f_k(x) = f_k^R(\rho) + if_k^I(\vartheta)$. And, for any $\imath_1, \imath_2 \in \mathbb{R}$, the real part $f_k^R(\cdot)$, as well as the imaginary part $f_k^I(\cdot)$ of the activation function $f_k(\cdot)$, can be characterized by*

$$|f_k^R(\cdot)| \leq F_k^R, \quad |f_k^I(\cdot)| \leq F_k^I,$$

$$|f_k^R(\imath_1) - f_k^R(\imath_2)| \leq \eta^R |\imath_1 - \imath_2|,$$

$$|f_k^I(\imath_1) - f_k^I(\imath_2)| \leq \eta^I |\imath_1 - \imath_2|,$$

where $F_k^R(\cdot), F_k^I(\cdot), \eta^R, \eta^I$ are non-negative constants.

For certain positive scalar $\hbar \in \mathbb{R}$, we make the variable transformation:

$$v_k(t) = \dot{x}_k(t) + \hbar_k x_k(t), k \in \aleph, \tag{3}$$

then, we have

$$\begin{cases} \dot{x}_k(t) = -\hbar_k x_k(t) + v_k(t) \\ \dot{v}_k(t) = -[a_k + \hbar_k(\hbar_k - b_k)]x_k(t) - (b_k - \hbar_k)v_k(t) \\ \qquad + \sum\limits_{j=1}^{n} c_{kj} f_j(x_j(t)) + \sum\limits_{j=1}^{n} d_{kj} f_j(x_j(t - \tau_j(t))) \\ \qquad + \bigwedge\limits_{j=1}^{n} w_{kj} f_j(x_j(t - \tau_j(t))) + \bigvee\limits_{j=1}^{n} q_{kj} f_j(x_j(t - \tau_j(t))). \end{cases} \tag{4}$$

Considering system (4) as the driver system, let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, $v(t) = (v_1(t), v_2(t), \dots, v_n(t))^T$, $A = diag\{a_1 + \hbar_1(\hbar_1 - b_1), a_2 + \hbar_2(\hbar_2 - b_2), \dots, a_n + \hbar_n(\hbar_n - b_n)\}$, $B = diag\{b_1 - \hbar_1, b_2 - \hbar_2, \dots, b_n - \hbar_n\}$, $H = diag\{\hbar_1, \hbar_2, \dots \hbar_n\}$, $f(x(t)) = (f_1(x_1(t), f_1(x_1(t), \dots, f_n(x_n(t))^T \in \mathbb{C}^n$, $f(\overline{x(t)}) = (f_1(x_1(t - \tau_1(t))), f_2(x_2(t - \tau_2(t))), \dots, f_n(x_n(t - \tau_n(t))))^T \in \mathbb{C}^n$, $C = C^R + iC^I$, $D = D^R + iD^I$, $C^R = (c^R_{kj})_{n\times n}$, $C^I = (c^I_{kj})_{n\times n}$, $D^R = (d^R_{kj})_{n\times n}$, $D^I = (d^I_{kj})_{n\times n}$, $W = (w_{kj})_{n\times n}$, $Q = (q_{kj})_{n\times n}$.

$$\begin{aligned} Wf(\overline{x(t)}) = \Big(& \bigwedge_{j=1}^{n} w_{1j} f_j(x_j(t - \tau_j(t))), \bigwedge_{j=2}^{n} w_{2j} f_j(x_j(t - \tau_j(t))), \\ & \dots, \bigwedge_{j=n}^{n} w_{nj} f_j(x_j(t - \tau_j(t)))\Big)^T, \end{aligned} \tag{5}$$

$$\begin{aligned} Qf(\overline{x(t)}) = \Big(& \bigvee_{j=1}^{n} q_{1j} f_j(x_j(t - \tau_j(t))), \bigvee_{j=1}^{n} q_{2j} f_j(x_j(t - \tau_j(t))), \\ & \dots, \bigvee_{j=1}^{n} q_{nj} f_j(x_j(t - \tau_j(t)))\Big)^T. \end{aligned} \tag{6}$$

Therefore, the matrix form of system (4) can be described by

$$\begin{cases} \dot{x}(t) = -Hx(t) + v(t) \\ \dot{v}(t) = -Ax(t) - Bv_k(t) + Cf(x(t)) + Df(\overline{x(t)}) \\ \qquad Wf(\overline{x(t)}) + Qf(\overline{x(t)}), \end{cases} \tag{7}$$

then, the matrix form of the response system is as follows:

$$\begin{cases} \dot{u}(t) = -Hu(t) + y(t) + \Delta(t) \\ \dot{y}(t) = -Au(t) - By_k(t) + Cf(u(t)) + Df(\overline{u(t)}) \\ \qquad Wf(\overline{u(t)}) + Qf(\overline{u(t)}) + m(t) + I(t), \end{cases} \tag{8}$$

where $\Delta(t), m(t)$ are control schemes; that is, $\Delta(t) = \Delta^R(t) + i\Delta^I(t)$ and $m(t) = m^R(t) + im^I(t)$. $I(t)$ represents external input which $I(t) = I^R(t) + iI^I(t)$. Based on Assumption 1, system (7) can be transformed as follows:

$$\begin{cases} \dot{x}^R(t) = -Hx^R(t) + v^R(t) \\ \dot{v}^R(t) = -Ax^R(t) - Bv^R_k(t) + C^R f^R(x^R(t)) - C^I f^I(x^I(t)) \\ \qquad +D^R f^R(\overline{x^R(t)}) - D^I f^I(\overline{x^I(t)}) + Wf^R(\overline{x^R(t)}) \\ \qquad +Qf^R(\overline{x^R(t)}), \\ \dot{x}^I(t) = -Hx^I(t) + v^I(t) \\ \dot{v}^I(t) = -Ax^I(t) - Bv^I_k(t) + C^R f^I(x^I(t)) + C^I f^R(x^R(t)) \\ \qquad +D^R f^I(\overline{x^I(t)}) - D^I f^R(\overline{x^R(t)}) + Wf^I(\overline{x^I(t)}) \\ \qquad +Qf^I(\overline{x^I(t)}), \end{cases} \tag{9}$$

and response system (8) can be represented by

$$
\begin{cases}
\dot{u}^R(t) = -Hu^R(t) + y^R(t) + \Delta^R(t) \\
\dot{y}^R(t) = -Au^R(t) - By_k^R(t) + C^R f^R(u^R(t)) - C^I f^I(u^I(t)) \\
\qquad\quad +D^R f^R(\overline{u^R(t)}) - D^I f^I(\overline{u^I(t)}) + W f^R(\overline{u^R(t)}) \\
\qquad\quad +Q f^R(\overline{u^R(t)}) + m^R(t) + I^R(t), \\
\dot{u}^I(t) = -Hu^I(t) + y^I(t) + \Delta^I(t) \\
\dot{y}^I(t) = -Au^I(t) - By_k^I(t) + C^R f^I(u^I(t)) + C^I f^R(u^R(t)) \\
\qquad\quad +D^R f^I(\overline{u^I(t)}) + D^I f^R(\overline{u^R(t)}) + W f^I(\overline{u^I(t)}) \\
\qquad\quad +Q f^I(\overline{u^I(t)}) + m^I(t) + I^I(t).
\end{cases} \tag{10}
$$

Considering $e^R(t) = u^R(t) - x^R(t)$, $e^I(t) = u^I(t) - x^I(t)$, $z^R(t) = y^R(t) - v^R(t)$, $z^I(t) = y^I(t) - v^I(t)$. $\digamma^R(e^R(t)) = f^R(u^R(t)) - f^R(x^R(t))$, $\digamma^I(e^R(t)) = f^I(u^I(t)) - f^I(x^I(t))$, $\digamma^R(\overline{e^R(t)}) = f^R(\overline{u^R(t)}) - f^R(\overline{x^R(t)}) = f^R(u^R(t-\tau(t))) - f^R(x^R(t-\tau(t)))$, $\digamma^I(\overline{e^I(t)}) = f^I(\overline{u^I(t)}) - f^I(\overline{x^I(t)}) = f^I(u^I(t-\tau(t))) - f^I(x^I(t-\tau(t)))$. Through (9) and (10), we obtain the following error system

$$
\begin{cases}
\dot{e}^R(t) = -He^R(t) + z^R(t) + \Delta^R(t) \\
\dot{z}^R(t) = -Ae^R(t) - Bz_k^R(t) + C^R \digamma^R(e^R(t)) - C^I \digamma^I(e^I(t)) \\
\qquad\quad +D^R \digamma^R(\overline{e^R(t)}) - D^I \digamma^I(\overline{e^I(t)}) + W \digamma^R(\overline{e^R(t)}) \\
\qquad\quad +Q \digamma^R(\overline{e^R(t)}) + m^R(t) + I^R(t), \\
\dot{e}^I(t) = -He^I(t) + z^I(t) + \Delta^I(t) \\
\dot{z}^I(t) = -Ae^I(t) - Bz_k^I(t) + C^R \digamma^I(e^I(t)) + C^I \digamma^R(e^R(t)) \\
\qquad\quad +D^R \digamma^I(\overline{e^I(t)}) + D^I \digamma^R(\overline{e^R(t)}) + W \digamma^I(\overline{e^I(t)}) \\
\qquad\quad +Q \digamma^I(\overline{e^I(t)}) + m^I(t) + I^I(t).
\end{cases} \tag{11}
$$

Next, we give some necessary definitions as follows.

Definition 1 ([47])**.** *Suppose that the output in system $g(t) \in \mathbb{C}^N$ and input $I(t) \in \mathbb{C}^N$ obtain finite-time passivity (FTP) for any $0 < \varepsilon < 1$ and $0 < \mu \in \mathbb{R}$, if*

$$
\dot{U}(t) + \mu(U(t))^{\varepsilon} \le (I^R(t))^T g^R(t) + (I^I(t))^T g^I(t), \tag{12}
$$

where $U(t)$ stands for a non-negative function.

Definition 2 ([47])**.** *Suppose that the output in system $g(t) \in \mathbb{C}^N$ and input $I(t) \in \mathbb{C}^N$ obtain finite-time input strict passivity (FTISP) for any $0 < \varepsilon < 1$ and $0 < \mu \in \mathbb{R}$, if*

$$
\begin{aligned}
\dot{U}(t) + \mu(U(t))^{\varepsilon} \le & (I^R(t))^T g^R(t) + (I^I(t))^T g^I(t) \\
& - \gamma_1 \Big((I^R(t))^T I^R(t) + (I^I(t))^T I^I(t) \Big),
\end{aligned} \tag{13}
$$

where $U(t)$ stands for a non-negative function.

Definition 3 ([47])**.** *Suppose that the output in system $g(t) \in \mathbb{C}^N$ and input $I(t) \in \mathbb{C}^N$ obtain finite-time input strict passivity (FTOSP) for any $0 < \varepsilon < 1$ and $0 < \mu \in \mathbb{R}$, if*

$$
\begin{aligned}
\dot{U}(t) + \mu(U(t))^{\varepsilon} \le & (I^R(t))^T g^R(t) + (I^I(t))^T g^I(t) \\
& - \gamma_2 \Big((w^R(t))^T w^R(t) + (w^I(t))^T w^I(t) \Big),
\end{aligned} \tag{14}
$$

where $U(t)$ stands for a non-negative function.

Lemma 1 ([50])**.** *It is assumed that with a continuous and non-negative function $U(t)$ which satisfies*

$$\dot{U}(t) \leq -\mu(U(t))^{\varepsilon}, t \geq 0, U(0) \geq 0$$

where $0 < \varepsilon < 1$ and $0 < \mu \in \mathbb{R}$, it can determined that,

$$(U(t))^{1-\varepsilon} \leq (U(0))^{1-\varepsilon} - \mu(1-\varepsilon)t, 0 \leq t \leq T,$$

and

$$U(t) = 0, t \leq T,$$

thus, we obtain

$$T = \frac{U(0)^{1-\varepsilon}}{\mu(1-\varepsilon)}. \tag{15}$$

Remark 1. *Many compelling results about passivity have been reported [48,49]. However, these references only pat attention to the input or output passivity problems, making the system only realize the infinite-time input or output passivity. Moreover, as [24,35,47] reveals, passivity can be an effective tool for discussing infinite-time synchronization for neural systems. But, from a practical perspective, the setting time of synchronization should be finite is more reasonable. Hence, the research in the paper is more general and extends the existing passivity theory.*

Lemma 2 ([3])**.** *For any $0 < \ell \leq 1$, $s_i \in \mathbb{R}$, $i \in \aleph$, then one has*

$$(|s_1| + |s_2| + \ldots + |s_n|)^{\ell} \leq |s_1|^{\ell} + |s_2|^{\ell} + \ldots + |s_n|^{\ell}. \tag{16}$$

Lemma 3 ([5])**.** *Suppose $\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_n(t))^T$ and $\Xi(t) = (\Xi_1(t), \Xi_2(t), \ldots, \Xi_n(t))^T$ imply two states of system (1); then, one obtains*

$$|\bigwedge_{\ell=1}^{n} \partial_{k\ell} f_{\ell}(\Xi_{\ell}) - \bigwedge_{\ell=1}^{n} \partial_{k\ell} f_{\ell}(\pi_{\ell})| \leq \sum_{\ell=1}^{n} |\partial_{k\ell}| \mid f_{\ell}(\Xi_{\ell}) - f_{\ell}(\pi_{\ell}) \mid, \tag{17}$$

$$|\bigvee_{\ell=1}^{n} \Re_{k\ell} f_{\ell}(\Xi_{\ell}) - \bigvee_{\ell=1}^{n} \Re_{k\ell} f_{\ell}(\pi_{\ell})| \leq \sum_{\ell=1}^{n} |\Re_{k\ell}| \mid f_{\ell}(\Xi_{\ell}) - f_{\ell}(\pi_{\ell}) \mid. \tag{18}$$

Remark 2. *Passivity analysis for real-valued neural networks (RVNNs) is widely observed [23,24,35,45,47–49], in which activation function, connection weight, input, and output are real-valued. Compared with RVNNs, CVNNs can viewed as a more general case because of more complex dynamic characteristics. Such symmetry detection and XOR issues are expected to be solved by CVNNs easily but cannot be settled by RVNNs [36]. In addition, compared with [23,24,33,35,49], inertial terms and fuzzy logic cases are supposed to cause complex essential impacts on dynamics behaviors for network systems. To our knowledge, few corresponding outcomes focus on the model with these elements. Therefore, it is significant to devote our effort to providing a guide for this analysis.*

Remark 3. *Suppose that an energy function $U(\cdot)$ stands for the energy stored in this system, and the energy supply bounded over finite-time intervals, then we call the system passive. As to storage function $U(\cdot)$ and supply rate $\wp(g, I)$, compared with [23,47], we develop FTP, FTISP, and FTOSP from real-valued into complex-valued, and different supply rate reveals that the dissipative of inside the system $U(t_2) - U(t_1)$ is not more than the external source $\int_{t_1}^{t_2} \wp(g(t), I(t))\mathrm{d}t$.*

3. Main Results

3.1. Finite-Time Passivity

Let the controller be $\Delta(t) = \Delta^R(t) + i\Delta^I(t)$, $m(t) = m^R(t) + im^I(t)$, and $\Delta^R(t)$, $\Delta^I(t)$, $m^R(t)$, $m^I(t)$ are constructed as follows:

$$\begin{cases} \Delta^R(t) = -\lambda^R e^R(t) - \varepsilon sign(e^R(t))|e^R(t)|^\beta, \\ m^R(t) = -\varphi^R z^R(t) - \varepsilon \sum\limits_{j=1}^{n} \left(2\int_{t-\tau_j(t)}^{t} \frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta} \mathrm{d}s \right)^{\frac{\beta+1}{2}} \frac{z^R(t)}{\|z^R(t)\|^2} \\ \qquad -\varepsilon sign(z^R(t))|z^R(t)|^\beta - G^R sign(z^R(t)) \\ \Delta^I(t) = -\lambda^I e^I(t) - \varepsilon sign(e^I(t))|e^I(t)|^\beta, \\ m^I(t) = -\varphi^I z^I(t) - \varepsilon \sum\limits_{j=1}^{n} \left(2\int_{t-\tau_j(t)}^{t} \frac{\left(\eta_j^I e_j^I(s)\right)^2}{1-\zeta} \mathrm{d}s \right)^{\frac{\beta+1}{2}} \frac{z^I(t)}{\|z^I(t)\|^2} \\ \qquad -\varepsilon sign(z^I(t))|z^I(t)|^\beta - G^I sign(z^I(t)), \end{cases} \tag{19}$$

in which $\lambda^R = diag\left(\lambda_1^R, \lambda_2^R \dots \lambda_n^R\right) \in \mathbb{R}^{n\times n}$ and $\lambda^I = diag\left(\lambda_1^I, \lambda_2^I \dots \lambda_n^I\right) \in \mathbb{R}^{n\times n}$ denote the positive definite gain matrices; $0 < \varepsilon \in \mathbb{R}$ and $0 < \beta < 1$; when $\|z(t)\| \neq 0$,

$$\begin{aligned} &G^R = diag\left(G_1^R, G_2^R, \dots G_n^R\right), G^I = diag\left(G_1^I, G_2^I, \dots G_n^I\right), \\ &|e^R(t)|^\beta = \left(|e_1^R(t)|^\beta, |e_2^R(t)|^\beta, \dots, |e_n^R(t)|^\beta\right)^T, \\ &|e^I(t)|^\beta = \left(|e_1^I(t)|^\beta, |e_2^I(t)|^\beta, \dots, |e_n^I(t)|^\beta\right)^T, \\ &sign(e^R(t)) = diag\left(sign\left(e_1^R(t)\right), sign\left(e_2^R(t)\right), \dots, sign\left(e_n^R(t)\right)\right), \\ &sign(e^I(t)) = diag\left(sign\left(e_1^I(t)\right), sign\left(e_2^I(t)\right), \dots, sign\left(e_n^I(t)\right)\right); \end{aligned}$$

$|z^R(t)|^\beta$, $|z^I(t)|^\beta$ are the same defined as $|e^R(t)|^\beta$, $|e^I(t)|^\beta$, respectively. $sign(z^R(t))$ and $sign(z^I(t))$ are the same, defined as $sign(e^R(t))$ and $sign(z^I(t))$, respectively.

What is more,

$$\begin{cases} \Delta^R(t) = -\lambda^R e^R(t) - \varepsilon sign(e^R(t))|e^R(t)|^\beta, \\ m^R(t) = 0, \\ \Delta^I(t) = -\lambda^I e^I(t) - \varepsilon sign(e^I(t))|e^I(t)|^\beta, \\ m^I(t) = 0, \end{cases} \tag{20}$$

when $\|z(t)\| = 0$.

We define the output vector $g(t) \in \mathbb{C}$ of system (11):

$$g(t) = M_1 z(t) + M_2 e(t) + M_3 I(t), \tag{21}$$

where $M_1, M_2, M_3 \in \mathbb{R}^{n\times n}$. For convenience, we let

$$\chi^R = diag\Big((\eta_1^R)^2, (\eta_2^R)^2, \dots (\eta_n^R)^2\Big), \chi^I = diag\Big((\eta_1^I)^2, (\eta_2^I)^2, \dots (\eta_n^I)^2\Big),$$
$$\Lambda = diag\Big(\frac{1}{1-\zeta_1}, \frac{1}{1-\zeta_2}, \dots, \frac{1}{1-\zeta_n}\Big),$$
$$g(t) = \Big(g_1(t), g_2(t), \dots g_n(t)\Big)^T, e^R(t) = \Big(e_1^R(t), e_2^R(t), \dots e_n^R(t)\Big)^T,$$
$$e^I(t) = \Big(e_1^I(t), e_2^I(t), \dots e_n^I(t)\Big)^T, z^R(t) = \Big(z_1^R(t), z_2^R(t), \dots z_n^R(t)\Big)^T,$$
$$z^I(t) = \Big(z_1^I(t), z_2^I(t), \dots z_n^I(t)\Big)^T, I(t) = \Big(I_1(t), I_2(t), \dots I_n(t)\Big)^T.$$

Theorem 1. *The network system (11) obtain FTP under control inputs (19) and (20) if there exist*

$$\lambda^R = Diag(\lambda_1^R, \lambda_2^R, \dots \lambda_n^R), \quad \lambda^I = Diag(\lambda_1^I, \lambda_2^I, \dots \lambda_n^R),$$
$$\varphi^R = Diag(\varphi_1^R, \varphi_2^R, \dots \varphi_n^R), \quad \varphi^I = Diag(\varphi_1^I, \varphi_2^I, \dots \varphi_n^I),$$
$$G^R = Diag(G_1^R, G_2^R, \dots G_n^R), G^I = Diag(G_1^I, G_2^I, \dots G_n^I) \in \mathbb{R}^{n\times n},$$

satisfying such conditions as

$$WF^R + QF^R - G^R \le 0, \quad WF^I + QF^I - G^I \le 0, \tag{22}$$

$$\begin{pmatrix} \Phi_1^R & \delta_1^R & \phi_1^R \\ (\delta_1^R)^T & \varpi_1^R & \Xi_1^R \\ (\phi_1^R)^T & (\Xi_1^R)^T & \theta_1^R \end{pmatrix} \le 0 \ \ and \ \ \begin{pmatrix} \Phi_1^I & \delta_1^I & \phi_1^I \\ (\delta_1^I)^T & \varpi_1^I & \Xi_1^I \\ (\phi_1^I)^T & (\Xi_1^I)^T & \theta_I^1 \end{pmatrix} \le 0, \tag{23}$$

$$\begin{pmatrix} \Pi_1^R & \phi_1^R \\ (\phi_1^R)^T & \theta_1^R \end{pmatrix} \le 0 \ \ and \ \ \begin{pmatrix} \Pi_1^I & \phi_1^I \\ (\phi_1^I)^T & \theta_1^I \end{pmatrix} \le 0, \tag{24}$$

where $\Phi_1^R = -2H + 2\chi^R + 2\chi^R\Lambda - 2\lambda^R$, $\Phi_1^I = -2H + 2\chi^I + 2\chi^I\Lambda - 2\lambda^I$, $\delta_1^R = \delta_1^I = E - A$, $\phi_1^R = \phi_1^I = \frac{M_2}{2}$, $\varpi_1^R = -2B + C^R + C^I + D^R + D^I - 2\varphi^R$, $\varpi_1^R = -2B + C^R + C^I + D^R + D^I - 2\varphi^I$, $\Xi_1^R = \Xi_1^I = \frac{2E-M_1}{2}$, $\theta_1^R = \theta_1^I = M_3$. $\Pi_1^R = -2H - 2\lambda^R$, $\Pi_1^I = -2H - 2\lambda^I$, *and E is the identity matrix.*

Proof. Case 1. When $\|z(t)\| \neq 0$, we build the Lyapunov function as follows:

$$V(t) = V_1(t) + V_2(t), \tag{25}$$

where

$$V_1(t) = (e^R(t))^T e^R(t) + (z^R(t))^T z^R(t) + 2\sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \frac{\Big(\eta_j^R e_j^R(s)\Big)^2}{1-\zeta} \mathrm{d}s, \tag{26}$$

$$V_2(t) = (e^I(t))^T e^I(t) + (z^I(t))^T z^I(t) + 2\sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \frac{\Big(\eta_j^I e_j^I(s)\Big)^2}{1-\zeta} \mathrm{d}s. \tag{27}$$

Then, when $\|z(t)\| \neq 0$, we arrange the derivative of $V_1(t)$ as

$$\begin{aligned}\dot{V}_1(t) =& 2(e^R(t))^T(-He^R(t) + z^R(t) - \lambda^R e^R(t) - \varepsilon sign(e^R(t)) \\ & \times |e^R(t)|^\beta) + 2(z^R(t))^T\Big(- Ae^R(t) - Bz_k^R(t)\end{aligned}$$

$$
\begin{aligned}
&+ C^R F^R(e^R(t)) - C^I F^I(e^I(t)) + D^R F^R(\overline{e^R(t)}) \\
&- D^I F^I(\overline{e^I(t)}) + W F^R(\overline{e^R(t)}) + Q F^R(\overline{e^R(t)}) - \varphi^R z^R(t) \\
&- G^R sign(z^R(t) - \varepsilon sign(z^R(t))|z^R(t)|^\beta + I^R(t) \\
&- \varepsilon \sum_{j=1}^{n} \left(2\int_{t-\tau_j(t)}^{t} \frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta} \mathrm{d}s\right)^{\frac{\beta+1}{2}} \frac{z^R(t)}{\|z^R(t)\|^2}\Bigg) \\
&+ 2(e^R(t))^T \chi^R \Lambda e^R(t) - 2(\overline{e^R(t)})^T \chi^R \overline{e^R}(t).
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\dot{V}_2(t) =& 2(e^I(t))^T(-He^I(t) + z^I(t) - \lambda^I e^I(t) \\
&- \varepsilon sign(e^I(t))|e^I(t)|^\beta) + 2(z^I(t))^T \Bigg(- Ae^I(t) - Bz_k^I(t) \\
&+ C^R F^I(e^I(t)) + C^I F^R(e^R(t)) + D^R F^I(\overline{e^I(t)}) \\
&+ D^I F^R(\overline{e^R(t)}) + W F^I(\overline{e^I(t)}) + Q F^I(\overline{e^I(t)}) \\
&- \varphi^I z^I(t) - G^I sign(z^I(t) - \varepsilon sign(z^I(t))|z^I(t)|^\beta + I^I(t) \\
&- \varepsilon \sum_{j=1}^{n} \left(2\int_{t-\tau_j(t)}^{t} \frac{\left(\eta_j^R e_j^I(s)\right)^2}{1-\zeta} \mathrm{d}s\right)^{\frac{\beta+1}{2}} \frac{z^I(t)}{\|z^I(t)\|^2} + I^I(t)\Bigg) \\
&+ 2(e^I(t))^T \chi^I \Lambda e^I(t) - 2(\overline{e^I(t)})^T \chi^I \overline{e^I}(t).
\end{aligned} \tag{29}
$$

under Assumption 1, from (28), then

$$
\begin{aligned}
2(z^R(t))^T C^R F^R(e^R(t)) &= 2\sum_{k=1}^{n} |z^R(t)|c_k^R \eta_k^R |e_k^R(t)| \\
&\leq \sum_{k=1}^{n} (z^R(t))^2 (c_k^R)^2 + \sum_{k=1}^{n} (\eta_k^R)^2 (e_k^R(t))^2 \\
&= (z^R(t))^T C^R z^R(t) + (e^R(t))^T \chi^R e^R(t).
\end{aligned} \tag{30}
$$

In addition, according to Lemma 3, one has

$$
2(z^R(t))^T W F^R(\overline{e^R(t)}) \leq 2|(z^R(t))^T| W F^R, \tag{31}
$$

$$
2(z^R(t))^T Q F^R(\overline{e^R(t)}) \leq 2|(z^R(t))^T| Q F^R. \tag{32}
$$

Moreover,

$$
-2(z^R(t))^T C^I F^I(e^I(t)) \leq (z^R(t))^T C^I z^R(t) + (e^I(t))^T \chi^I e^I(t), \tag{33}
$$

$$
2(z^R(t))^T D^R F^R(\overline{e^R(t)}) \leq (z^R(t))^T D^R z^R(t) + (\overline{e^R(t)})^T \chi^R \overline{e^R(t)}, \tag{34}
$$

$$
-2(z^R(t))^T D^I F^I(\overline{e^I(t)}) \leq (z^R(t))^T D^I z^R(t) + (\overline{e^I(t)})^T \chi^I \overline{e^I(t)}. \tag{35}
$$

Likewise, from Lemma 2, one has

$$
\begin{aligned}
(e^R(t))^T sign(e^R(t))|e^R(t)|^\beta &= \sum_{k=1}^{n} |e_k^R(t)|^{\beta+1} \\
&\geq \sum_{k=1}^{n} ((e_k^R(t))^2)^{\frac{\beta+1}{2}} \\
&= ((e^R(t))^T e^R(t))^{\frac{\beta+1}{2}},
\end{aligned} \tag{36}
$$

$$(z^R(t))^T sign(z^R(t))|z^R(t)|^\beta \geq ((z^R(t))^T z^R(t))^{\frac{\beta+1}{2}}. \tag{37}$$

Similarly,

$$2(z^I(t))^T C^R F^I(e^I(t)) \leq (z^I(t))^T C^R z^I(t) + (e^I(t))^T \chi^I e^I(t), \tag{38}$$

$$2(z^I(t))^T C^I F^R(e^R(t)) \leq (z^I(t))^T C^I z^I(t) + (e^R(t))^T \chi^R e^R(t), \tag{39}$$

$$2(z^I(t))^T D^R F^I(\overline{e^I(t)}) \leq (z^I(t))^T D^R z^I(t) + (\overline{e^I(t)})^T \chi^I \overline{e^I(t)}, \tag{40}$$

$$2(z^I(t))^T D^I F^R(\overline{e^R(t)}) \leq (z^I(t))^T D^I z^I(t) + (\overline{e^R(t)})^T \chi^R \overline{e^R(t)}, \tag{41}$$

$$(e^I(t))^T sign(e^I(t))|e^I(t)|^\beta \geq ((e^I(t))^T e^I(t))^{\frac{\beta+1}{2}}, \tag{42}$$

$$(z^I(t))^T sign(z^I(t))|z^I(t)|^\beta \geq ((z^I(t))^T z^I(t))^{\frac{\beta+1}{2}}. \tag{43}$$

What is more,

$$2(z^I(t))^T W F^I(\overline{e^I(t)}) \leq 2|(z^I(t))^T| W F^I, \tag{44}$$

$$2(z^I(t))^T Q F^I(\overline{e^I(t)}) \leq 2|(z^I(t))^T| Q F^I. \tag{45}$$

Because of (30)–(35), it is arranged by

$$\begin{aligned} \dot{V}_1(t) \leq& (e^R(t))^T(-2H - 2\lambda^R + 2\chi^R + 2\chi^R\Lambda)e^R(t) \\ &+ (e^R(t))^T[2(E-A)]z^R(t) + (z^R(t))^T(-2B + C^R + C^I \\ &+ D^R + D^I - 2\varphi^R)z^R(t) + 2|(z^R(t))^T|(WF^R + QF^R \\ &- G^R) + 2(z^R(t))^T I^R - \varepsilon\sum_{j=1}^n \left(2\int_{t-\tau_j(t)}^t \frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta} \mathrm{d}s\right)^{\frac{\beta+1}{2}} \\ &- 2\varepsilon((e^R(t))^T e^R(t))^{\frac{\beta+1}{2}} - 2\varepsilon((z^R(t))^T z^R(t))^{\frac{\beta+1}{2}}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} \dot{V}_2(t) \leq& (e^I(t))^T(-2H - 2\lambda^I + 2\chi^I + 2\chi^I\Lambda)e^I(t) \\ &+ (e^I(t))^T[2(E-A)]z^I(t) + (z^I(t))^T(-2B + C^R + C^I \\ &+ D^R + D^I - 2\varphi^I)z^I(t) + 2|(z^I(t))^T|(WF^I + QF^I \\ &- G^I) + 2(z^I(t))^T I^I - \varepsilon\sum_{j=1}^n \left(2\int_{t-\tau_j(t)}^t \frac{\left(\eta_j^I e_j^I(s)\right)^2}{1-\zeta} \mathrm{d}s\right)^{\frac{\beta+1}{2}} \\ &- 2\varepsilon((e^I(t))^T e^I(t))^{\frac{\beta+1}{2}} - 2\varepsilon((z^I(t))^T z^I(t))^{\frac{\beta+1}{2}}. \end{aligned} \tag{47}$$

Furthermore,

$$\dot{V}(t) - \left(\left(I^R(t)\right)^T g^R(t) + \left(I^I(t)\right)^T g^I(t)\right)$$

$$\leq \left(\Gamma^R(t)\right)^T \begin{pmatrix} \Phi_1^R & \delta_1^R & \phi_1^R \\ (\delta_1^R)^T & \omega_1^R & \Xi_1^R \\ (\phi_1^R)^T & (\Xi_1^R)^T & \theta_1^R \end{pmatrix} \Gamma^R(t) - 2\varepsilon\left(\left(e^R(t)\right)^T e^R(t)\right)^{\frac{\beta+1}{2}}$$

$$-2\varepsilon\left(\left(z^R(t)\right)^T z^R(t)\right)^{\frac{\beta+1}{2}} - \varepsilon\sum_{j=1}^n \left(2\int_{t-\tau_j(t)}^t \frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta} \mathrm{d}s\right)^{\frac{\beta+1}{2}}$$

$$+\left(\Gamma^I(t)\right)^T \begin{pmatrix} \Phi_1^I & \delta_1^I & \phi_1^I \\ (\delta_1^I)^T & \omega_1^I & \Xi_1^I \\ (\phi_1^I)^T & (\Xi_1^I)^T & \theta_1^I \end{pmatrix} \Gamma^I(t) - 2\varepsilon\left(\left(e^I(t)\right)^T e^I(t)\right)^{\frac{\beta+1}{2}}$$

$$-2\varepsilon\left(\left(z^I(t)\right)^T z^I(t)\right)^{\frac{\beta+1}{2}} - \varepsilon\sum_{j=1}^n \left(2\int_{t-\tau_j(t)}^t \frac{\left(\eta_j^I e_j^I(s)\right)^2}{1-\zeta} \mathrm{d}s\right)^{\frac{\beta+1}{2}}$$

$$+2\left|\left(z^R(t)\right)^T\right|(WF^R + QF^R - G^R) + 2\left|\left(z^I(t)\right)^T\right|(WF^I + QF^I - G^I)$$

$$\begin{aligned}
&\leq -2\varepsilon\Bigg(\left(e^R(t)\right)^T e^R(t) + \left(z^R(t)\right)^T z^R(t) + \left(e^I(t)\right)^T e^I(t) + \left(z^I(t)\right)^T z^I(t) \\
&\quad + \sum_{j=1}^n \left(\int_{t-\tau_j(t)}^t \frac{\eta_j^R e_j^R(s)\Big)^2}{1-\zeta} \mathrm{d}s + \sum_{j=1}^n \left(\int_{t-\tau_j(t)}^t \frac{\eta_j^R e_j^R(s)\Big)^2}{1-\zeta} \mathrm{d}s\Bigg)^{\frac{\beta+1}{2}} \\
&= -2\varepsilon\Big(V_1(t) + V_2(t)\Big)^{\frac{\beta+1}{2}} \\
&= -2\varepsilon(V(t))^{\frac{\beta+1}{2}},
\end{aligned} \tag{48}$$

where

$$\Gamma^R(t) = \left(e^R(t), z^R(t), (I^R(t)\right)^T \mathit{and}\ \Gamma^I(t) = \left(e^I(t), z^R(t), (I^I(t)\right)^T.$$

Consequently, it concludes that

$$\left(I^R(t)\right)^T g^R(t) + \left(I^I(t)\right)^T g^I(t) \geq \dot{V}(t) + 2\varepsilon(V(t))^{\frac{\beta+1}{2}} = \dot{V}(t) + \widetilde{\varepsilon}(V(t))^{\widetilde{\beta}}. \tag{49}$$

where $\widetilde{\varepsilon} = 2\varepsilon$, $\widetilde{\beta} = \frac{\beta+1}{2}$, $0 < \widetilde{\varepsilon} \in \mathbb{R}$ and $0 < \widetilde{\beta} < 1$.

Case 2. When $\|z(t)\| = 0$, the Lyapunov function is

$$\widetilde{V}(t) = \left(e^R(t)\right)^T e^R(t) + \left(e^I(t)\right)^T e^I(t). \tag{50}$$

Arranging the derivative of $\widetilde{V}(t)$, one has

$$\begin{aligned}
\dot{\widetilde{V}}(t) &= 2\left(e^R(t)\right)^T(-He^R(t) - \lambda^R e^R(t) - \varepsilon sign(e^R(t))|e^R(t)|^\beta) \\
&\quad + 2\left(e^I(t)\right)^T(-He^I(t) - \lambda^I e^I(t) - \varepsilon sign(e^I(t))|e^I(t)|^\beta) \\
&\leq \left(e^R(t)\right)^T(-2H - 2\lambda^R)e^R(t) + \left(e^I(t)\right)^T(-2H - 2\lambda^I)e^I(t)
\end{aligned}$$

$$-2\varepsilon\left(\left(e^R(t)\right)^Te^R(t)\right)^{\frac{\beta+1}{2}}-2\varepsilon\left(\left(e^I(t)\right)^Te^I(t)\right)^{\frac{\beta+1}{2}}. \tag{51}$$

Moreover,

$$\begin{aligned}
&\dot{\widetilde{V}}(t)-\left(\left(I^R(t)\right)^Tg^R(t)+\left(I^I(t)\right)^Tg^I(t)\right)\\
=&\dot{\widetilde{V}}(t)-\left(\left(I^R(t)\right)^TM_2e^R(t)+\left(I^R(t)\right)^TM_3I^R(t))\right.\\
&\left.+\left(I^I(t)\right)^TM_2e^I(t)+\left(I^I(t)\right)^TM_3I^I(t)\right)\\
\leq&\left(\daleth^R(t)\right)^T\begin{pmatrix}\Pi_1^R & \phi_1^R\\ (\phi_1^R)^T & \theta_1^R\end{pmatrix}\daleth^I(t)-2\varepsilon\left(\left(e^R(t)\right)^Te^R(t)\right)^{\frac{\beta+1}{2}}\\
&+\left(\daleth^I(t)\right)^T\begin{pmatrix}\Pi_1^I & \phi_1^I\\ (\phi_1^I)^T & \theta_1^I\end{pmatrix}\daleth^I(t)-2\varepsilon\left(\left(e^I(t)\right)^Te^I(t)\right)^{\frac{\beta+1}{2}}\\
\leq&-2\varepsilon\left(\left(e^R(t)\right)^Te^R(t)+\left(e^I(t)\right)^Te^I(t)\right)^{\frac{\beta+1}{2}}=\dot{V}(t)+\widetilde{\varepsilon}(\widetilde{V}(t))^{\widetilde{\beta}},
\end{aligned}\tag{52}$$

where $\daleth^R(t)=\left(e^R(t),(I^R(t)\right)^T$ and $\daleth^I(t)=\left(e^I(t),(I^I(t)\right)^T$.

Based on the above analysis, according to Definition 1, system (11) can realize FTP under controller (19). At the instant time t when $\|z(t)\|=0$, we also can infer that system (11) achieves FTP by controller (20). Therefore, system (9) can reach FTP under control schemes (19) and (20). □

Theorem 2. *Under the condition of Theorem 1, the network (11) can realize FTISP through controllers (19) and (20) if there are*

$$WF^R+QF^R-G^R\leq 0,\quad WF^I+QF^I-G^I\leq 0, \tag{53}$$

$$\begin{pmatrix}\Phi_1^R & \delta_1^R & \phi_1^R\\ (\delta_1^R)^T & \varpi_1^R & \Xi_1^R\\ (\phi_1^R)^T & (\Xi_1^R)^T & \theta_2^R\end{pmatrix}\leq 0\ \ and\ \ \begin{pmatrix}\Phi_1^I & \delta_1^I & \phi_1^I\\ (\delta_1^I)^T & \varpi_1^I & \Xi_1^I\\ (\phi_1^I)^T & (\Xi_1^I)^T & \theta_2^I\end{pmatrix}\leq 0, \tag{54}$$

$$\begin{pmatrix}\Pi_1^R & \phi_1^R\\ (\phi_1^R)^T & \theta_2^R\end{pmatrix}\leq 0\ \ and\ \ \begin{pmatrix}\Pi_1^I & \phi_2^I\\ (\phi_1^I)^T & \theta_2^I\end{pmatrix}\leq 0, \tag{55}$$

where Φ_1^R, Φ_1^I, δ_1^R, ϕ_1^R, ϖ_1^R, ϖ_1^R, Ξ_1^R, Π_1^I, Π_1^R have the same meanings as in Theorem 1, and $\theta_2^R=\theta_2^I=\gamma_1E+\theta_1^R$.

Proof. Building the same Lyapunov function as (25), combining with (46) and (47), we can carry out

$$\begin{aligned}
&\dot{V}(t)-(I^R(t))^Tg^R(t)+(I^I(t))^Tg^I(t)\\
&\quad+\gamma_1\left((I^R(t))^TI^R(t)+(I^I(t))^TI^I(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq (e^R(t))^T(-2H - 2\lambda^R + 2\chi^R + 2\chi^R\Lambda)e^R(t) \\
&\quad + (e^R(t))^T[2(E-A)]z^R(t) + (z^R(t))^T(-2B + C^R + C^I \\
&\quad + D^R + D^I - 2\varphi^R)z^R(t) + 2|(z^R(t))^T|(WF^R + QF^R \\
&\quad - G^R) + 2(z^R(t))^T I^R - \varepsilon\sum_{j=1}^{n}\left(2\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta}\mathrm{d}s\right)^{\frac{\beta+1}{2}} \\
&\quad - 2\varepsilon((e^R(t))^T e^R(t))^{\frac{\beta+1}{2}} - 2\varepsilon((z^R(t))^T z^R(t))^{\frac{\beta+1}{2}} \\
&\quad + (e^I(t))^T(-2H - 2\lambda^I + 2\chi^I + 2\chi^I\Lambda)e^I(t) \\
&\quad + (e^I(t))^T[2(E-A)]z^I(t) + (z^I(t))^T(-2B + C^R + C^I \\
&\quad + D^R + D^I - 2\varphi^I)z^I(t) + 2|(z^I(t))^T|(WF^I + QF^I \\
&\quad - G^I) + 2(z^I(t))^T I^I - \varepsilon\sum_{j=1}^{n}\left(2\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^I e_j^I(s)\right)^2}{1-\zeta}\mathrm{d}s\right)^{\frac{\beta+1}{2}} \\
&\quad - 2\varepsilon((e^I(t))^T e^I(t))^{\frac{\beta+1}{2}} - 2\varepsilon((z^I(t))^T z^I(t))^{\frac{\beta+1}{2}} \\
&\quad + \gamma_1\Big((I^R(t))^T I^R(t) + (I^I(t))^T I^I(t)\Big) - \Big[\Big(I^R(t)\Big)^T M_1 z^R(t) \\
&\quad + \Big(I^R(t)\Big)^T M_2 e^R(t) + \Big(I^R(t)\Big)^T M_3 I^R(t) + \Big(I^I(t)\Big)^T M_1 z^I(t) \\
&\quad + \Big(I^I(t)\Big)^T M_2 e^I(t) + \Big(I^I(t)\Big)^T M_3 I^I(t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big(\Gamma^R(t)\Big)^T \begin{pmatrix} \Phi_1^R & \delta_1^R & \phi_1^R \\ (\delta_1^R)^T & \varpi_1^R & \Xi_1^R \\ (\phi_1^R)^T & (\Xi_1^R)^T & \theta_2^R \end{pmatrix} \Gamma^R(t) - 2\varepsilon\left(\Big(e^R(t)\Big)^T e^R(t)\right)^{\frac{\beta+1}{2}} \\
&-2\varepsilon\left(\Big(z^R(t)\Big)^T z^R(t)\right)^{\frac{\beta+1}{2}} - \varepsilon\sum_{j=1}^{n}\left(2\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta}\mathrm{d}s\right)^{\frac{\beta+1}{2}} \\
&+\Big(\Gamma^I(t)\Big)^T \begin{pmatrix} \Phi_1^I & \delta_1^I & \phi_1^I \\ (\delta_1^I)^T & \varpi_1^I & \Xi_1^I \\ (\phi_1^I)^T & (\Xi_1^I)^T & \theta_2^I \end{pmatrix} \Gamma^I(t) - 2\varepsilon\left(\Big(e^I(t)\Big)^T e^I(t)\right)^{\frac{\beta+1}{2}} \\
&-2\varepsilon\left(\Big(z^I(t)\Big)^T z^I(t)\right)^{\frac{\beta+1}{2}} - \varepsilon\sum_{j=1}^{n}\left(2\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^I e_j^I(s)\right)^2}{1-\zeta}\mathrm{d}s\right)^{\frac{\beta+1}{2}} \\
&+2\Big|\Big(z^R(t)\Big)^T\Big|(WF^R + QF^R - G^R) + 2\Big|\Big(z^I(t)\Big)^T\Big|(WF^I + QF^I - G^I)
\end{aligned}
$$

$$
\begin{aligned}
&\leq -2\varepsilon\Bigg(\Big(e^R(t)\Big)^T e^R(t) + \Big(z^R(t)\Big)^T z^R(t) + \Big(e^I(t)\Big)^T e^I(t) + \Big(z^I(t)\Big)^T z^I(t) \\
&\quad + \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta}\mathrm{d}s + \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta}\mathrm{d}s\Bigg)^{\frac{\beta+1}{2}} \\
&= -2\varepsilon(V(t))^{\frac{\beta+1}{2}} = -\widetilde{\varepsilon}(V(t))^{\widetilde{\beta}}.
\end{aligned} \tag{56}
$$

Then, one attains

$$\left(I^R(t)\right)^T g^R(t) + \left(I^I(t)\right)^T g^I(t) - \gamma_1\Big((I^R(t))^T I^R(t) + (I^I(t))^T I^I(t)\Big)$$
$$\geq \dot{V}(t) + \widetilde{\varepsilon}(V(t))^{\widetilde{\beta}},$$

where $\widetilde{\varepsilon} = 2\varepsilon$, $\widetilde{\beta} = \frac{\beta+1}{2}$, $0 < \widetilde{\varepsilon} \in \mathbb{R}$, $0 < \widetilde{\beta} < 1$. Consequently, system (11) can achieve FTISP under controller (19). When $\|z(t)\| = 0$, the proving procedure has a resemblance to Theorem 1 and is omitted here. So, based on Definition 2, the system (11) can obtain FTISP through controller (19) and (20). □

Theorem 3. *Under the condition of Theorem 1, the network (11) can realize FTOSP through controllers (19) and (20) if there are*

$$WF^R + QF^R - G^R \leq 0, \quad WF^I + QF^I - G^I \leq 0, \tag{57}$$

$$\begin{pmatrix} \Phi_2^R & \delta_2^R & \phi_2^R \\ (\delta_2^R)^T & \omega_2^R & \Xi_2^R \\ (\phi_2^R)^T & (\Xi_2^R)^T & \theta_3^R \end{pmatrix} \leq 0 \ \ and \ \ \begin{pmatrix} \Phi_2^I & \delta_2^I & \phi_2^I \\ (\delta_2^I)^T & \omega_2^I & \Xi_2^I \\ (\phi_2^I)^T & (\Xi_2^I)^T & \theta_3^I \end{pmatrix} \leq 0, \tag{58}$$

$$\begin{pmatrix} \Pi_2^R & \phi_2^R \\ (\phi_2^R)^T & \theta_3^R \end{pmatrix} \leq 0 \ \ and \ \ \begin{pmatrix} \Pi_2^I & \phi_2^I \\ (\phi_2^I)^T & \theta_3^I \end{pmatrix} \leq 0, \tag{59}$$

where $\Phi_2^R = \Phi_1^R + \gamma_2 M_2^T M_2$, $\Phi_2^I = \Phi_2^I + \gamma_2 M_2^T M_2$, $\delta_2^R = \delta_1^I = \delta_1^R + \frac{\gamma_2 M_1^T M_2}{2}$, $\phi_2^R = \phi_1^I = \phi_1^R + \frac{\gamma_2 M_2^T M_3 - M_2}{2}$, $\omega_2^R = \omega_1^R + \gamma_2 M_1^T M_1$, $\omega_2^I = \omega_1^I + \gamma_2 M_1^T M_1$, $\Xi_2^R = \Xi_2^I = \Xi_1^R + \frac{\gamma_2 M_1^T M_3 - M_1}{2}$, $\theta_3^R = \theta_3^I = \theta_1^R + \gamma_2 M_3^T M_3 - M_3$. $\Pi_2^R = \Pi_1^R + \gamma_2 M_2^T M_2$, $\Pi_2^I = \Pi_1^I + \gamma_2 M_2^T M_2$.

Proof. Considering the same Lyapunov function as (25), combined with (46) and (47), it follows that

$$\begin{aligned}
&\dot{V}(t) - (I^R(t))^T g^R(t) + (I^I(t))^T g^I(t) \\
&\qquad + \gamma_2\Big((w^R(t))^T w^R(t) + (w^I(t))^T w^I(t)\Big) \\
&\leq (e^R(t))^T(-2H - 2\lambda^R + 2\chi^R + 2\chi^R\Lambda)e^R(t) \\
&\quad + (e^R(t))^T[2(E-A)]z^R(t) + (z^R(t))^T(-2B + C^R + C^I \\
&\quad + D^R + D^I - 2\varphi^R)z^R(t) + 2|(z^R(t))^T|(WF^R + QF^R \\
&\quad - G^R) + 2(z^R(t))^T I^R - \varepsilon\sum_{j=1}^{n}\left(2\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^R e_j^R(s)\right)^2}{1-\zeta}\mathrm{d}s\right)^{\frac{\beta+1}{2}} \\
&\quad - 2\varepsilon((e^R(t))^T e^R(t))^{\frac{\beta+1}{2}} - 2\varepsilon((z^R(t))^T z^R(t))^{\frac{\beta+1}{2}} \\
&\quad + (e^I(t))^T(-2H - 2\lambda^I + 2\chi^I + 2\chi^I\Lambda)e^I(t) \\
&\quad + (e^I(t))^T[2(E-A)]z^I(t) + (z^I(t))^T(-2B + C^R + C^I \\
&\quad + D^R + D^I - 2\varphi^I)z^I(t) + 2|(z^I(t))^T|(WF^I + QF^I \\
&\quad - G^I) + 2(z^I(t))^T I^I - \varepsilon\sum_{j=1}^{n}\left(2\int_{t-\tau_j(t)}^{t}\frac{\left(\eta_j^I e_j^I(s)\right)^2}{1-\zeta}\mathrm{d}s\right)^{\frac{\beta+1}{2}} \\
&\quad - 2\varepsilon((e^I(t))^T e^I(t))^{\frac{\beta+1}{2}} - 2\varepsilon((z^I(t))^T z^I(t))^{\frac{\beta+1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&-\Big[\Big(I^R(t)\Big)^T M_1 z^R(t) + \Big(I^R(t)\Big)^T M_2 e^R(t) + \Big(I^R(t)\Big)^T M_3 I^R(t)\\
&+\Big(I^I(t)\Big)^T M_1 z^I(t) + \Big(I^I(t)\Big)^T M_2 e^I(t) + \Big(I^I(t)\Big)^T M_3 I^I(t)\Big]\\
&+\Big[\gamma_2\Big(z^R(t)\Big)^T M_1^T M_1 z^R(t) + \gamma_2\Big(z^R(t)\Big)^T M_1^T M_2 e^R(t)\\
&+\gamma_2\Big(z^R(t)\Big)^T M_1^T M_3 I^R(t) + \gamma_2\Big(e^R(t)\Big)^T M_2^T M_2 e^R(t)\\
&+\gamma_2\Big(e^R(t)\Big)^T M_2^T M_3 I^R(t) + \gamma_2\Big(I^R(t)\Big)^T M_3^T M_3 I^R(t)\\
&+\gamma_2\Big(z^I(t)\Big)^T M_1^T M_1 z^I(t) + \gamma_2\Big(z^I(t)\Big)^T M_1^T M_2 e^I(t)\\
&+\gamma_2\Big(z^I(t)\Big)^T M_1^T M_3 I^I(t) + \gamma_2\Big(e^I(t)\Big)^T M_2^T M_2 e^I(t)\\
&+\gamma_2\Big(e^I(t)\Big)^T M_2^T M_3 I^I(t) + \gamma_2\Big(I^I(t)\Big)^T M_3^T M_3 I^I(t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big(\Gamma^R(t)\Big)^T \begin{pmatrix} \Phi_2^R & \delta_2^R & \phi_2^R \\ (\delta_2^R)^T & \varpi_2^R & \Xi_2^R \\ (\phi_2^R)^T & (\Xi_2^R)^T & \theta_3^R \end{pmatrix} \Gamma^R(t) - 2\varepsilon\bigg(\Big(e^R(t)\Big)^T e^R(t)\bigg)^{\frac{\beta+1}{2}}\\
&-2\varepsilon\bigg(\Big(z^R(t)\Big)^T z^R(t)\bigg)^{\frac{\beta+1}{2}} - \varepsilon\sum_{j=1}^{n}\Big(2\int_{t-\tau_j(t)}^{t}\frac{\Big(\eta_j^R e_j^R(s)\Big)^2}{1-\zeta}\mathrm{d}s\Big)^{\frac{\beta+1}{2}}\\
&+\Big(\Gamma^I(t)\Big)^T \begin{pmatrix} \Phi_2^I & \delta_2^I & \phi_2^I \\ (\delta_2^I)^T & \varpi_2^I & \Xi_2^I \\ (\phi_2^I)^T & (\Xi_2^I)^T & \theta_3^I \end{pmatrix} \Gamma^I(t) - 2\varepsilon\bigg(\Big(e^I(t)\Big)^T e^I(t)\bigg)^{\frac{\beta+1}{2}}\\
&-2\varepsilon\bigg(\Big(z^I(t)\Big)^T z^I(t)\bigg)^{\frac{\beta+1}{2}} - \varepsilon\sum_{j=1}^{n}\Big(2\int_{t-\tau_j(t)}^{t}\frac{\Big(\eta_j^I e_j^I(s)\Big)^2}{1-\zeta}\mathrm{d}s\Big)^{\frac{\beta+1}{2}}\\
&+2\Big|\Big(z^R(t)\Big)^T\Big|(WF^R + QF^R - G^R) + 2\Big|\Big(z^I(t)\Big)^T\Big|(WF^I + QF^I - G^I)\\
&\leq -2\varepsilon\bigg(\Big(e^R(t)\Big)^T e^R(t) + \Big(z^R(t)\Big)^T z^R(t) + \Big(e^I(t)\Big)^T e^I(t) + \Big(z^I(t)\Big)^T z^I(t)\\
&+\sum_{j=1}^{n}\Big(\int_{t-\tau_j(t)}^{t}\frac{\Big(\eta_j^R e_j^R(s)\Big)^2}{1-\zeta}\mathrm{d}s + \sum_{j=1}^{n}\Big(\int_{t-\tau_j(t)}^{t}\frac{\Big(\eta_j^R e_j^R(s)\Big)^2}{1-\zeta}\mathrm{d}s\bigg)^{\frac{\beta+1}{2}}\\
&= -2\varepsilon(V(t))^{\frac{\beta+1}{2}} = -\widetilde{\varepsilon}(V(t))^{\widetilde{\beta}},
\end{aligned} \tag{60}
$$

where $\widetilde{\varepsilon} = 2\varepsilon$, $\widetilde{\beta} = \frac{\beta+1}{2}$, $0 < \widetilde{\varepsilon} \in \mathbb{R}$, $0 < \widetilde{\beta} < 1$. Then, one derives

$$
\begin{aligned}
&(I^R(t))^T g^R(t) + (I^I(t))^T g^I(t) + \gamma_2\Big((w^R(t))^T w^R(t) + (w^I(t))^T w^I(t)\Big)\\
&\geq \dot{V}(t) + \widetilde{\varepsilon}(V(t))^{\widetilde{\beta}}.
\end{aligned}
$$

Consequently, the system (11) can achieve FTOSP under controller (19). When $\|z(t)\| = 0$, the proving procedure has a resemblance to Theorem 1 which is omitted here. So, based on Definition 2, the system (11) can obtain FTISP through controllers (19) and (20). □

3.2. Finite-Time Synchronization

Theorem 4. *Suppose that $\mathbb{U}(t)$: $[0,+\infty] \to [0,+\infty]$ is a differentiable continuous function which has*

$$\omega(\|x(t)\|_2) \leq \mathbb{U}(t),$$

where $\omega : [0,+\infty] \to [0,+\infty]$ stands for a strictly monotonically continuous increasing function, for $r > 0$, $\omega(r)$ is positive with $\omega(0) = 0$. If network system (11) obtains FTP (FTISP, FTOSP) through control inputs (19) and (20), the networks (9) and (10) reach FTS based on controllers (19) and (20).

Proof. If system (11) realizes FTP under controllers (19) and (20), there exist $0 < \widetilde{\varepsilon} \in \mathbb{R}$ and $0 < \widetilde{\beta} < 1$, such that

$$\left(I^R(t)\right)^T g^R(t) + \left(I^I(t)\right)^T g^I(t) \geq \dot{\mathbb{U}}(t) + \widetilde{\varepsilon}(\mathbb{U}(t))^{\widetilde{\beta}}. \tag{61}$$

Considering $I(t) = 0$, we have

$$\dot{\mathbb{U}}(t) \leq -\widetilde{\varepsilon}(\mathbb{U}(t))^{\widetilde{\beta}}.$$

From Lemma 1, we obtain $\mathbb{U}(t) = 0$ for $t \geq T^\sharp$, $T^\sharp = \frac{\mathbb{U}(0)^{1-\widetilde{\varepsilon}}}{\widetilde{\beta}(1-\widetilde{\varepsilon})}$.

Because of

$$\omega(\|x(t)\|_2) \leq \mathbb{U}(t),$$

one attains

$$\omega(\|x(t)\|_2) \leq \mathbb{U}(t) = 0,$$

where $t \geq T^\sharp$. Then, we can derive $\|x(t)\|_2 = 0$. Namely, the system (9) and (10) arrive at FTS under controller (19) and (20). Similarly, when system (11) obtains FTISP and FTOSP, we can deduce that system (9) and (10) reach FTS under control inputs (19) and (20). □

Corollary 1. *If there exist*

$$\begin{aligned}
&\lambda^R = Diag(\lambda_1^R, \lambda_2^R, \ldots \lambda_n^R), \quad \lambda^I = Diag(\lambda_1^I, \lambda_2^I, \ldots \lambda_n^R), \\
&\varphi^R = Diag(\varphi_1^R, \varphi_2^R, \ldots \varphi_n^R), \quad \varphi^I = Diag(\varphi_1^I, \varphi_2^I, \ldots \varphi_n^I), \\
&G^R = Diag(G_1^R, G_2^R, \ldots G_n^R), G^I = Diag(G_1^I, G_2^I, \ldots G_n^I) \in \mathbb{R}^{n\times n},
\end{aligned}$$

satisfying such conditions as

$$WF^R + QF^R - G^R \leq 0, \quad WF^I + QF^I - G^I \leq 0, \tag{62}$$

$$\begin{pmatrix} \Phi_1^R & \delta_1^R & \phi_1^R \\ (\delta_1^R)^T & \varpi_1^R & \Xi_1^R \\ (\phi_1^R)^T & (\Xi_1^R)^T & \theta_1^R \end{pmatrix} \leq 0 \quad and \quad \begin{pmatrix} \Phi_1^I & \delta_1^I & \phi_1^I \\ (\delta_1^I)^T & \varpi_1^I & \Xi_1^I \\ (\phi_1^I)^T & (\Xi_1^I)^T & \theta_I^1 \end{pmatrix} \leq 0, \tag{63}$$

$$\begin{pmatrix} \Pi_1^R & \phi_1^R \\ (\phi_1^R)^T & \theta_1^R \end{pmatrix} \leq 0 \quad and \quad \begin{pmatrix} \Pi_1^I & \phi_1^I \\ (\phi_1^I)^T & \theta_1^I \end{pmatrix} \leq 0, \tag{64}$$

where Φ_1^R, Φ_1^I, δ_1^R, ϕ_1^R, ϖ_1^R, ϖ_1^R, Ξ_1^R, Π_1^I, Π_1^R and θ_2^R are same as in Theorem 1, then FICVNN (1) is a finite-time synchronization based on controllers (19) and (20).

Remark 4. *Commonly, it is not easy to deal with passivity and synchronization in real-valued neural network systems, let alone solve the problem of FTP and FTS with complex-valued parameters. Furthermore, owing to inertial terms and fuzzy logic, traditional ways can not directly deal with the FTP and FTS of FICVNNs. To conquer these points, in this paper, some suitable control inputs and novel Lyapunov functionals are divided into the real part and the imaginary part to ensure that FICVNNs are addressed to obtain passivity and synchronization in a finite time interval.*

3.3. Example

Example 1. *Think about the delayed FICVNNs as follows:*

$$\begin{aligned}\ddot{x}_k(t) =& - a_k x_k(t) - b_k \dot{x}_k(t) + \sum_{j=1}^{2} c_{kj} f_j(x_j(t)) \\ &+ \sum_{j=1}^{2} d_{kj} f_j(x_j(t-\tau_j(t))) + \bigwedge_{j=1}^{2} w_{kj} f_j(x_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{2} q_{kj} f_j(x_j(t-\tau_j(t))).\end{aligned} \tag{65}$$

Through Formulae (3) and (4), the matrix form of system (65) is demonstrated by

$$\begin{cases} \dot{x}(t) = -Hx(t) + v(t) \\ \dot{v}(t) = -Ax(t) - Bv_k(t) + Cf(x(t)) + Df(\overline{x(t)}) \\ \qquad Wf(\overline{x(t)}) + Qf(\overline{x(t)}), \end{cases} \tag{66}$$

where $A = diag(-0.3, 0.2)$, $B = diag(1.1, 1.2)$, $H = diag(1, 1)$, $C = (c_{kj})_{2\times2}$, $c_{11} = -0.6 - 1.2i$, $c_{12} = -8.1 + 5.1i$, $c_{21} = -3.4 - 1.9i$, $c_{22} = -550 - 550i$, $D = (d_{kj})_{2\times2}$, $d_{11} = -1.8 + 1.3i$, $d_{12} = -3.9 - 6.9i$, $d_{21} = -0.3 - 4.3i$, $d_{12} = -536 - 536i$, $W = (w_{kj})_{2\times2}$, $w_{11} = 0.5$, $w_{12} = -1$, $w_{21} = 0.1$, $w_{22} = -1$, $Q = (q_{kj})_{2\times2}$, $w_{11} = 0.1$, $w_{12} = -0.2$, $w_{21} = -0.3$, $w_{22} = 0.1$, $\tau_j(t) = \frac{e^t}{1+e^t}$, $\eta^R = \eta^I = 1$, $f_j(\cdot) = \tanh(Re(\cdot)) + i\tanh(Im(\cdot))$, $j = 1, 2$. The initial conditions of the system (65) are selected as $x_1(0) = 10 + 10i$, $v_1(0) = 2.5 + 0.9i$, $x_2(0) = 0.6 + 0.2i$, $v_1(0) = 0.4 + 6i$. Then, Figures 1–3 show the trajectories of states $x_k(t), v_k(t), k = 1, 2$ without control. Moreover, consider system (66) as a drive system; the response system is

$$\begin{cases} \dot{u}(t) = -Hu(t) + y(t) + \Delta(t) \\ \dot{y}(t) = -Au(t) - By_k(t) + Cf(u(t)) + Df(\overline{u(t)}) \\ \qquad Wf(\overline{u(t)}) + Qf(\overline{u(t)}) + m(t) + I(t), \end{cases} \tag{67}$$

where $\Delta(t)$, m(t) are controllers given in formula (19), I(t) is external input, $I_1(t) = 3.9\cos(t) + 5.4\cos(t)i$, $I_2(t) = 5.2\cos(t) + 9.8\cos(t)i$. We select M_1, M_2, and M_3 as follows:

$$M_1 = \begin{pmatrix} 4 & 0.1 \\ 2 & 0.5 \end{pmatrix}, M_2 = \begin{pmatrix} -2 & 0.2 \\ -3 & 0.2 \end{pmatrix}, M_3 = \begin{pmatrix} -32 & 11 \\ -18 & 9 \end{pmatrix}.$$

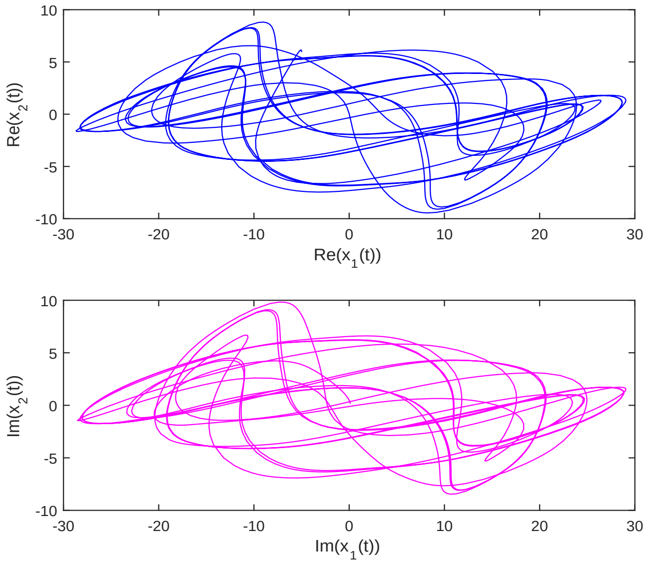

Figure 1. Bule line stands for transient behavior of variables $Re(x_k(t))$, $k = 1, 2$ and red line stands for transient behavior of variables $Im(x_k(t)$, $k = 1, 2$ of FICVNNs (65).

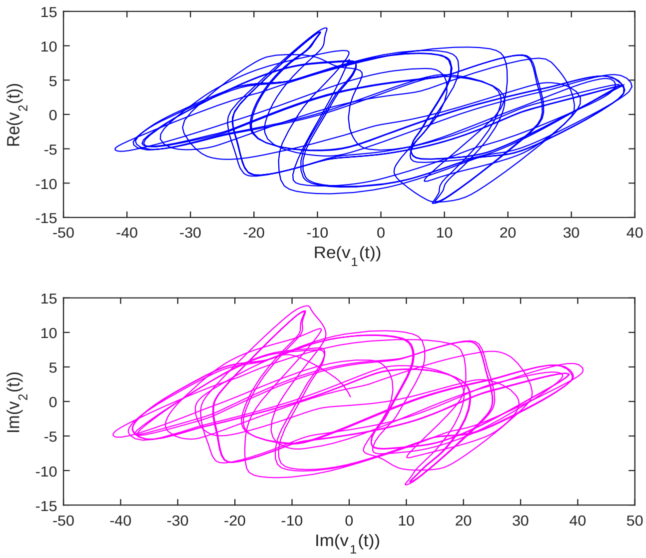

Figure 2. Bule line stands for transient behavior of variables $Rev_k(t)$, $k = 1, 2$ and red line stands for transient behavior of variables $Im(v_k(t))$, $k = 1, 2$of FICVNNs (65).

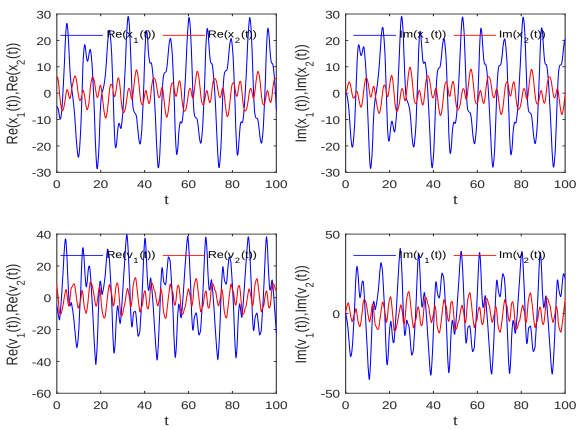

Figure 3. Bule line stands for state trajectory of variables $Re(x_k(t))$, $Re(v_k(t))$, $k = 1, 2$ and red line stands for state of trajectory of variables $Im(x_k(t))$, $Im(v_k(t))$, $k = 1, 2$ of FICVNNs (65) without control.

The rest of parameters are the same as in system (66). In addition, the parameters in (19) are selected as $\varepsilon = 0.5$, $\beta = 0.5$, $\zeta = 0.1$, $\chi = diag(0.1+0.6i, 1+0.8i)$, and $G = diag(70+27i, 60+5i)$. Take $\lambda = diag(5+2.6i, 5+2.6i)$ and $\varphi = diag(9+15.6i, 9+15.6i)$, which satisfy the condition of Theorem 1. Through Theorem 1, the network in (65) obtains FTP under controller (19). Take $\gamma_1 = 0.2$, $\lambda = diag(12+8.3i, 12+8.3i)$, and $\varphi = diag(18+0.5i, 18+0.5i)$, which satisfy the condition of Theorem 2; (65) achieves FTISP under controller (19). Take $\gamma_2 = 0.8$, $\lambda = diag(21.4+9.1i, 21.4+9.1i)$, and $\varphi = diag(15+0.9i, 15+0.9i)$, and one can satisfy the condition of Theorem 3. In terms of Theorem 3, system (65) can realize the FTOSP under controller (19). Above all, the simulation of dynamical changes for state error $e(t)$, $z(t)$, input $I(t)$, and output $g(t)$ are given in Figure 4.

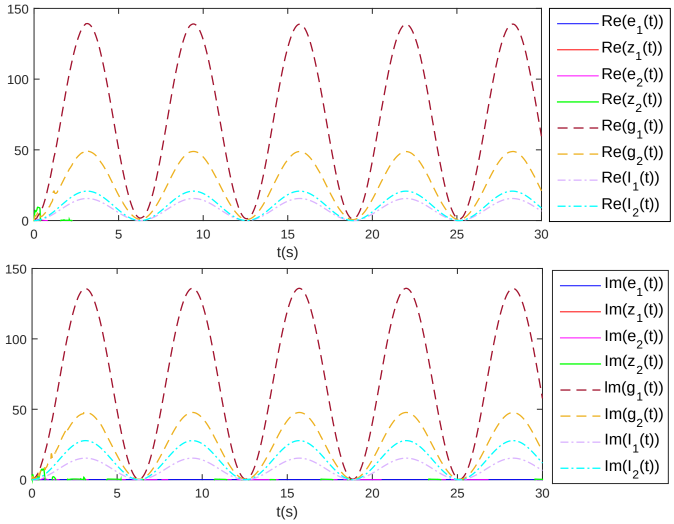

Figure 4. The curves of error states $e_k(t)$, $z_k(t)$, external input $I_k(t)$, and output $g_k(t)$, $k = 1,2$ under controller (19).

Let $e_j^R(t) = u_j^R(t) - x_j^R(t)$, $e_j^I(t) = u_j^I(t) - x_j^I(t)$, $z_j^R(t) = y_j^R(t) - v_j^R(t)$, $z_j^I(t) = y_j^I(t) - v_j^I(t)$, $j = 1,2$. Figure 5 denotes the trajectories of state error $Re(e_j(t))$, $Re(z_j(t))$, $j = 1,2$ and $Im(e_j(t))$, $Im(z_j(t))$, $j = 1,2$ with controller (19) when (65) is finite-time passive. The state trajectories of state error $Re(e_j(t))$, $Re(z_j(t))$, $j = 1,2$ and $Im(e_j(t))$, $Im(z_j(t))$, $j = 1,2$, respectively, are illustrated in Figure 6.

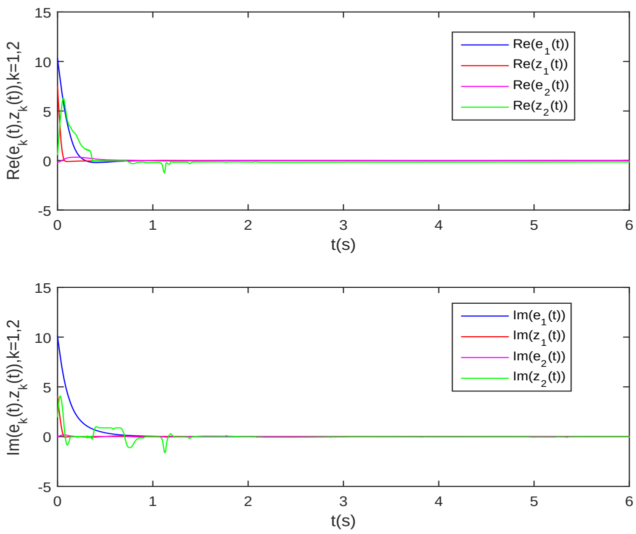

Figure 5. Trajectory of error states $e_k(t)$, $z_k(t)$, $k = 1,2$ under controller (19).

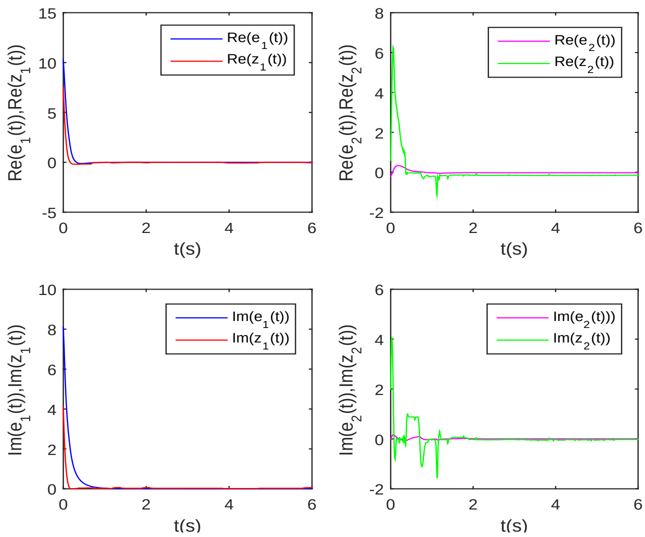

Figure 6. Trajectories of error $e_k(t), z_k(t), k = 1, 2$ with controller(19).

According to Corollary 1, the network (65) with the above parameters can realize FTS under controller (19), and the setting time is computed as $T^* = 5.904$. As Figure 7 reveals, when time increases to 5.904, the corresponding simulation curves of the state error $Re(e_j(t)), Re(z_j(t)), j = 1, 2$ and $Im(e_j(t)), Im(z_j(t)), j = 1, 2$ tend to 0. This demonstrates that network (65) can achieve finite-time synchronization.

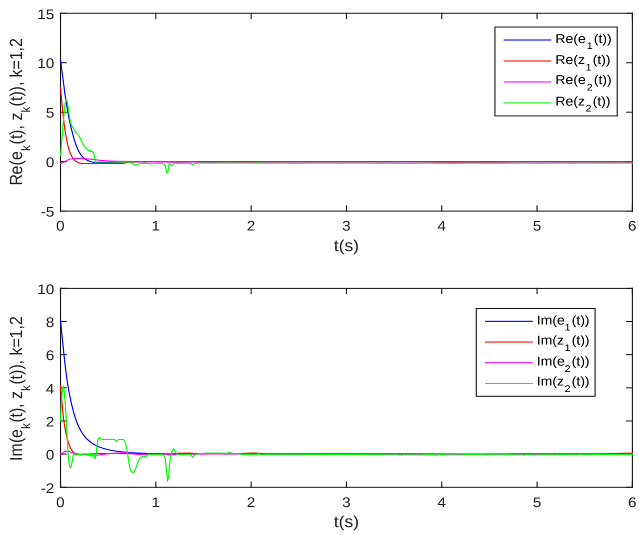

Figure 7. The synchronization curve of error states $e_k(t), z_k(t), k = 1, 2$ with controller (19).

Example 2. *The application example of FICVNNs (65) relates to the pseudorandom number generator (PRNG) [43]. Let a sequence of pseudorandom number* $k(\check{t}) = \vartheta(o_1(\check{t}), o_2(\check{t}))$, $\check{t} \in [\check{t}_{start}, \check{t}_{end}]$, *and* $[\check{t}_{start}, \check{t}_{end}]$ *stand for the operating time interval; then, one has*

$$\vartheta(o_1(\check{t}), o_2(\check{t})) = \begin{cases} 1, & o_1(\check{t}) \leq o_2(\check{t}), \\ 0, & o_1(\check{t}) > o_2(\check{t}), \end{cases} \tag{68}$$

where

$$o_1(\check{t}) = \frac{x_1^R(\check{t})}{max_{\check{t} \in [\check{t}_{start}, \check{t}_{end}]} x_2^R(\check{t})},$$

$$o_2(\check{t}) = \frac{x_2^I(\check{t})}{max_{\check{t} \in [\check{t}_{start}, \check{t}_{end}]} x_2^I(\check{t})}.$$

Consequently, by using the same parameters as in Example 1, because of the chaotic features of the FICVNNs, the PRNG is produced in Figure 8. Then, Figure 9 explains that s(t) is the original transmission signal. So, through the transformation $p(t) = s(t) \otimes k(t)$*, we can obtain the encrypted signal in Figure 10.*

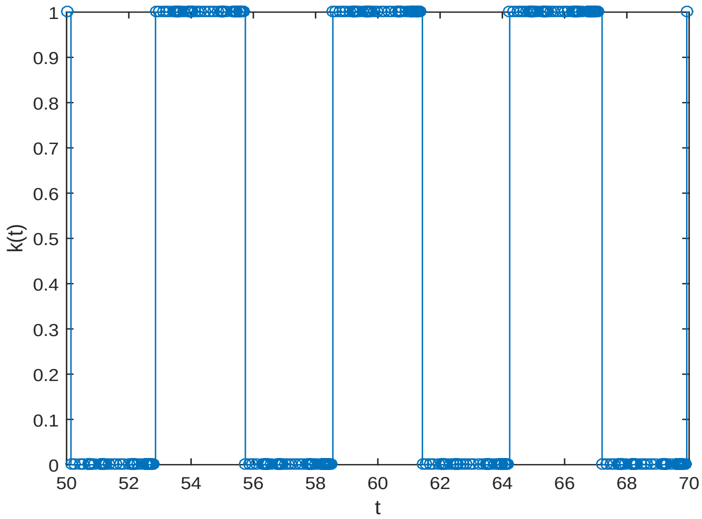

Figure 8. PRNG produced by FICVNNs.

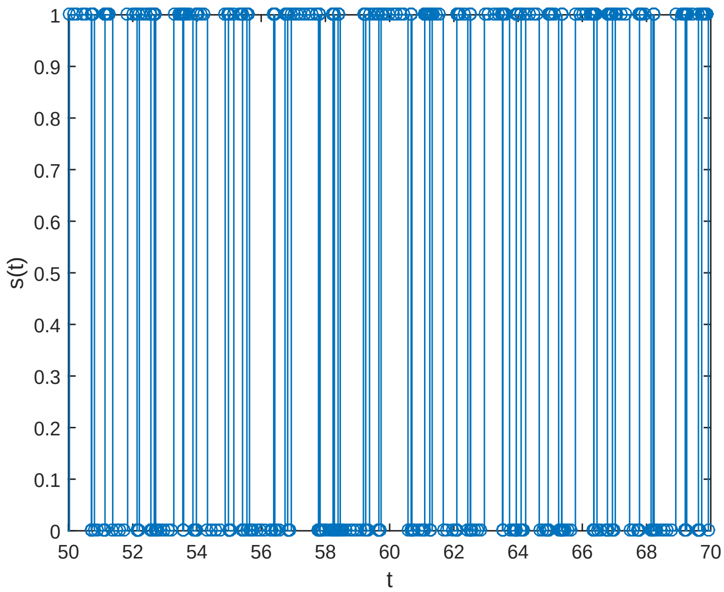

Figure 9. Original signals.

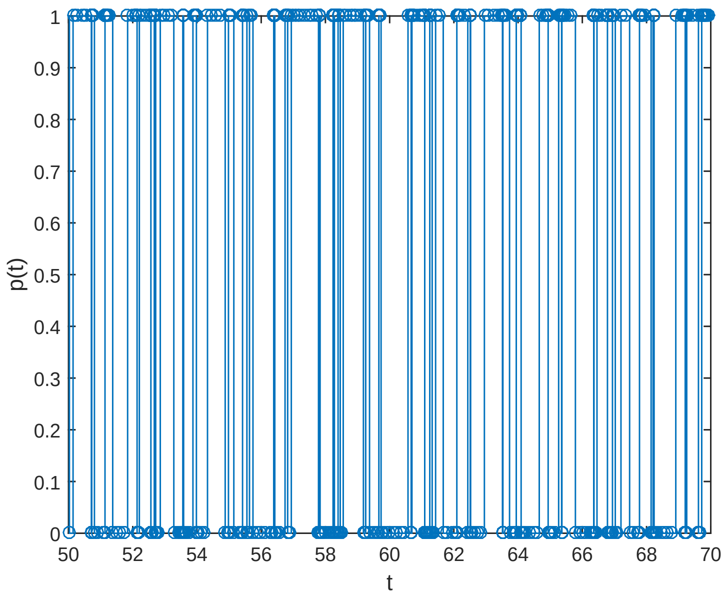

Figure 10. Encrypted signals.

4. Conclusions

This article investigated the finite-time passivity and finite-time synchronization of the proposed FICVNNs with time delays. By transforming the second-order complex-valued model into first-order real-valued differential systems, the Lyapunov functional method and some novel controllers are explored to guarantee the FTP, FTISP, and FTOSP of FICVNNs. Furthermore, based on finite-time passive FICVNNs, finite-time synchronization has been investigated. Finally, some numerical simulations are provided to confirm the theory results.

Some existing works have discussed the passive properties of neural network systems that can maintain the system's internal stability. For example, the infinite-time passivity or infinite-time synchronization of neural networks is generally considered by resorting to passivity theory [24,35,49]. Compared with the above literature, the neural model built in this paper, with inertial terms, complex-valued parameters, fuzzy logic, and time delays, enriches and expands previous results. It is supposed to create a foundation for observing the creative controllers for fixed-time synchronization or predefined-time synchronization in neural networks [9,51]. Compared with [33], we can also develop the related results to fit inertial neural networks, the parameters of the systems state-dependently switching. This is an essential and valuable direction in investigating various kinds of neural networks with more control methods in the future.

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