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The Bifurcations of Completely Integrable Holonomic Systems of First-Order Differential Equations

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Abstract: As an application of the Legendrian singularity theory, we classify the bifurcations of a holonomic first-order differential equation with a complete integral. The equations satisfy that the one-parameter integral diagrams are $\mathcal{R}^+$-simple and stable. Using this result, the parametric differential equation models in electrical power systems and engineering can be studied.

Keywords: legendrian singularity; bifurcation; holonomic first-order differential equation; complete integral

MSC: 35F20; 58K40; 58K50

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1. Introduction

In 1975, V. V. Lychagin introduced the application of singularity theory to the study of the geometric theory of partial differential equations within the framework of contact geometry [1]. In the following thirty years, there was significant progress in the study of geometric partial differential equations and their geometric solutions [2–12].

In recent years, due to problems from many branches of mathematics and other disciplines, singularity theory has experienced rapid development and widespread application. In bifurcation theory, studying the existence and number of branch solutions of parameterized nonlinear differential equations is a very important issue. It is also meaningful to study the impact of nonlinear terms on the existence and number of branch solutions of equations. When the nonlinear term in the parameterized equation is a differentiable mapping germ, we can naturally use the singularity theory of differentiable mappings to study the branching behavior.

M. Golubitsky and D. Schaeffer have used singularity theory and group theory methods to study bifurcation problems in [13,14]. This type of research mainly includes (1) the unfolding of bifurcation problems, studying the change in the state of bifurcation problems under general perturbation; (2) the identification of bifurcation problems, exploring under what conditions a bifurcation problem is equivalent to a given standard form; (3) the classification of bifurcation problems; and (4) the application of bifurcation theory in physics and chemistry. For example, M. Golubitsky and W.F. have used singularity theory to study the bifurcation of the number of periodic solutions near equilibrium points in a first-order autonomous differential equation, and obtained its normal forms and identification conditions in [15]; H.W. Broer and others have studied the bifurcation of solutions of the inverted pendulum equation and its normal forms and universal unfoldings in [16]; D.Y. Du and Y. Tang have studied the bifurcation of solutions of differential-difference-algebraic equations in [17].

Nonlinear nth-order systems are a very important class of mathematical model that appear in a wide range of practical applications. There has been a significant amount of work on the bifurcation problems of nonlinear second-order systems. For instance, in [18], several types of nonlinear second-order systems with boundary value problems have been studied in terms of the existence and number of branch solutions. As an important research tool, Legendrian unfoldings theory has been widely used in the study of differential equations, differential geometry, variational calculus, and mathematical physics [2,3,5,6,8,9,12,19–26]. Based on S. Izumiya and M. Takahashi's studies, this article uses Legendrian unfoldings theory to classify the bifurations of completely integrable holonomic first-order differential equation germs, which satisfy that the corresponding one-parameter integral diagrams are $\mathcal{R}^+$-simple and stable.

2. Preliminary Theorem

In this section, we give the classification of the generating family of one-parameter complete Legendrian unfolding, and will utilize some general results of $S.P$-$\mathcal{K}$-equivalence of function germs [2,3,27]. The related terminologies are as in [3,10,11].

We define the equivalence relation of one-parameter unfoldings of holonomic equations as follows. Let F and F' be one-parameter unfoldings of holonomic equations associated with f and f', respectively, then F and F' are *equivalent* if the diagram

$$\begin{array}{ccccc}
(\mathbb{R}^{n+1}\times\mathbb{R},\mathbf{0}) & \xrightarrow{F} & (J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R},(z,\mathbf{0})) & \xrightarrow{\pi\times id} & (\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathbf{0}) \\
\psi\downarrow & & \downarrow\Phi & & \downarrow\phi \\
(\mathbb{R}^{n+1}\times\mathbb{R},\mathbf{0}) & \xrightarrow[F']{} & (J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R},(z',\mathbf{0})) & \xrightarrow[\pi\times id]{} & (\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathbf{0})
\end{array}$$

commutes; here, ψ, Φ and ϕ are germs of diffeomorphisms, which satisfy the following forms

$$\begin{aligned}
\psi(u_1,\ldots,u_n,u_{n+1},t) &= (\psi_1(u_1,\ldots,u_n,u_{n+1},t),\varphi(t)),\\
\Phi(x_1,\ldots,x_n,y,p_1,\ldots,p_n,t) &= (\widehat{\phi}_t((x_1,\ldots,x_n,y,p_1,\ldots,p_n),\varphi(t)),\\
\phi(x_1,\ldots,x_n,y,t) &= (\phi_1(x_1,\ldots,x_n,y,t),\phi_2(x_1,\ldots,x_n,y,t),\varphi(t)),
\end{aligned}$$

where $\widehat{\phi}_t$ is the unique contact lift of ϕ_t. Here, $\phi_t = \phi|_{\mathbb{R}^n\times\mathbb{R}\times\{t\}} : (\mathbb{R}^n\times\mathbb{R}\times\{t\},\mathbf{0}) \to (\mathbb{R}^n\times\mathbb{R}\times\{\varphi(t)\},\mathbf{0})$.

Let $(\widehat{\mu},G)$ and $(\widehat{\mu'},G')$ be one-parameter unfoldings of integral diagrams. Then, $(\widehat{\mu},G)$ and $(\widehat{\mu'},G')$ are *equivalent* if the diagram

$$\begin{array}{ccc}
(\mathbb{R}^{n+1}\times\mathbb{R},\mathbf{0}) & \xrightarrow{(\mu,G)} & (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathbf{0}) \\
\psi\downarrow & & \downarrow\phi \\
(\mathbb{R}^{n+1}\times\mathbb{R},\mathbf{0}) & \xrightarrow[(\mu',G')]{} & (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathbf{0})
\end{array}$$

commutes for some diffeomorphism germs ψ and ϕ, where

$$\begin{aligned}
\psi(u_1,\ldots,u_n,u_{n+1},t) &= (\psi_1(u_1,\ldots,u_n,u_{n+1},t),\varphi(t)),\\
\phi(s,x_1,\ldots,x_n,y,t) &= (\kappa(s,t),\phi_1(x_1,\ldots,x_n,y,t),\phi_2(x_1,\ldots,x_n,y,t),\varphi(t)).
\end{aligned}$$

If $\kappa(s,t) = s$ in the above second equality, then (μ,G) and (μ',G') are called *strictly equivalent*.

Lemma 1 (V. V. Goryunov [27]). *Let $f : (\mathbb{R} \times \mathbb{R}^k, \mathbf{0}) \to (\mathbb{R}, 0)$ be an S.P-$\mathcal{K}$-simple germ. Then, f is stably S-$\mathcal{K}$-equivalent to one of germs in the following list:*

$$\begin{array}{llll} \mathrm{A}_\ell\ (\ell \geq 1): & \pm s \pm q_1^{\ell+1}; & \mathrm{B}_\ell\ (\ell \geq 2): & \pm s^\ell \pm q_1^2; \\ \mathrm{C}_\ell\ (\ell \geq 3): & q_1^\ell \pm s q_1; & \mathrm{D}_\ell\ (\ell \geq 4): & \pm s + (q_1^2 q_2 \pm q_2^{\ell-1}); \\ \mathrm{E}_6: & \pm s + (q_1^3 \pm q_2^4); & \mathrm{E}_7: & \pm s + (q_1^3 + q_1 q_2^3); \\ \mathrm{E}_8: & \pm s + (q_1^3 + q_2^5); & \mathrm{F}_4: & q_1^3 \pm s^2. \end{array}$$

Theorem 1. *Let $\widetilde{G} : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k, \mathbf{0}) \to (\mathbb{R}, 0)$ be the generating family of one-parameter complete Legendrian unfolding $\ell_{(\mu,F)}$ satisfying that $\ell_{(\mu,F)}$ is one-parameter P-Legendre stable and the corresponding one-parameter integral diagram $(\mu, \pi \circ f)$ is $\mathcal{R}^+$-simple and stable. Then, $\widetilde{G}$ is stably one-parameter S.P$^+$-$\mathcal{K}$-equivalent to one of the members in the following list:*

a_ℓ: $\pm s \pm q_1^{\ell+1} + \sum_{i=1}^{\ell-1} x_i q_1^i - y\ (1 \leq \ell \leq n+1)$;

$\widetilde{(\mathrm{a}_\ell)_\mathrm{j}}$: $\pm s \pm q_1^{\ell+1} + \sum_{i=1}^{j-1} x_i q_1^i + y q_1^j + \sum_{i=j+1}^{\ell-1} x_{i-1} q_1^i - y\ (3 \leq \ell \leq n+2,\ 2 \leq j \leq \ell-1)$;

b_ℓ: $\pm s^\ell \pm q_1^2 + \sum_{i=1}^{\ell-1} x_i s^i - y\ (2 \leq \ell \leq n+1)$;

$(\mathrm{b}_\ell)_\mathrm{j}$: $\pm s^\ell \pm q_1^2 + \sum_{i=1}^{j-1} x_i s^i + \beta(\mathbf{x}, t) s^j + \sum_{i=j+1}^{\ell-1} x_{i-1} s^i - y$,
where $\beta \in \mathfrak{M}_{(\mathbf{x},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0\ (3 \leq \ell \leq n+2,\ 2 \leq j \leq \ell-1)$;

$\widetilde{(\mathrm{b}_{\mathrm{n}+3})_\mathrm{j}}$: $\pm s^{n+3} \pm q_1^2 + \sum_{i=1}^{j-1} x_i s^i + \beta(\mathbf{x}, t) s^j + \sum_{i=j+1}^{n+1} x_{i-1} s^i - y$, *where* $\beta \in \mathfrak{M}_{(\mathbf{x},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0\ (2 \leq j \leq n+1)$;

c_ℓ: $q_1^\ell \pm s q_1 + \sum_{i=1}^{\ell-1} x_i q_1^i - y\ (3 \leq \ell \leq n+1)$;

$(\mathrm{c}_\ell)_\mathrm{j}$: $q_1^\ell \pm s q_1 + \sum_{i=1}^{j-1} x_i q_1^i + \beta(\mathbf{x}, t) q_1^j + \sum_{i=j+1}^{\ell-1} x_{i-1} q_1^i - y$,
where $\beta \in \mathfrak{M}_{(\mathbf{x},t)}$, $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$ *and* $(\partial\beta/\partial r_0)(\mathbf{0}) \neq 0$, $r_0 = 1$ *or* $\ell - 2\ (4 \leq \ell \leq n+2,\ 2 \leq j \leq \ell-2)$;

$(\mathrm{c}_\ell)_{\ell-1}$: $q_1^\ell \pm s q_1 + \sum_{i=1}^{\ell-2} x_i q_1^i + \beta(\mathbf{x}, t) q_1^{\ell-1} - y$, *where* $\beta \in \mathfrak{M}_{(\mathbf{x},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0\ (3 \leq \ell \leq n+2)$;

$\widetilde{(\mathrm{c}_{\mathrm{n}+3})_\mathrm{j}}$: $q_1^{n+3} \pm s q_1 + x_1 q_1 + \sum_{i=1}^{j-1} x_i q_1^{i+1} + \beta(\mathbf{x}, t) q_1^{j+1} + \sum_{i=j+1}^{n+1} x_{i-1} q_1^{i+1} - y$,
where $\beta \in \mathfrak{M}_{(\mathbf{x},t)}$, $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$ *and* $(\partial\beta/\partial x_n)(\mathbf{0}) \neq 0\ (1 \leq j \leq n)$;

$\widetilde{(\mathrm{c}_{\mathrm{n}+3})_{\mathrm{n}+1}}$: $q_1^{n+3} \pm s q_1 + x_1 q_1 + \sum_{i=1}^{n} x_i q_1^{i+1} + \beta(\mathbf{x}, t) q_1^{n+2} - y$, *where* $\beta \in \mathfrak{M}_{(\mathbf{x},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$;

d_ℓ: $\pm s + (q_1^2 q_2 \pm q_2^{\ell-1}) + x_1 q_1 + x_2 q_2 + x_3 q_1^2 + \sum_{i=4}^{\ell-1} x_i q_2^{i-2} - y\ (4 \leq \ell \leq n+1)$;

$\tilde{\mathrm{d}}_\ell$: $\pm s + (q_1^2 q_2 \pm q_2^{\ell-1}) + x_1 q_1 + x_2 q_2 + y q_1^2 + \sum_{i=4}^{\ell-1} x_{i-1} q_2^{i-2} - y\ (4 \leq \ell \leq n+2)$;

$\widetilde{(\mathrm{d}_\ell)_\mathrm{j}}$: $\pm s + (q_1^2 q_2 \pm q_2^{\ell-1}) + x_1 q_1 + x_2 q_2 + x_3 q_1^2 + \sum_{i=4}^{j-1} x_i q_2^{i-2} + y q_2^{j-2} + \sum_{i=j+1}^{\ell-1} x_{i-1} q_2^{i-2} - y$
$(5 \leq \ell \leq n+2,\ 4 \leq j \leq \ell-1)$;

$\mathrm{e}_6^{\mathrm{n}}$ $(\mathrm{n} \geq 5)$: $\pm s + (q_1^3 \pm q_2^4) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + x_4 q_1 q_2 + x_5 q_1 q_2^2 - y;$

$\widetilde{(\mathrm{e}_6^{\mathrm{n}})}_1$ $(\mathrm{n} \geq 4)$: $\pm s + (q_1^3 \pm q_2^4) + x_1 q_1 + x_2 q_2 + y q_2^2 + x_3 q_1 q_2 + x_4 q_1 q_2^2 - y;$

$\widetilde{(\mathrm{e}_6^{\mathrm{n}})}_2$ $(\mathrm{n} \geq 4)$: $\pm s + (q_1^3 \pm q_2^4) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + y q_1 q_2 + x_4 q_1 q_2^2 - y;$

$\widetilde{(\mathrm{e}_6^{\mathrm{n}})}_3$ $(\mathrm{n} \geq 4)$: $\pm s + (q_1^3 \pm q_2^4) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + x_4 q_1 q_2 + y q_1 q_2^2 - y;$

$\mathrm{e}_7^{\mathrm{n}}$ $(\mathrm{n} \geq 6)$: $\pm s + (q_1^3 + q_1 q_2^3) + x_1 q_1 + x_2 q_2 + x_3 q_1^2 + x_4 q_2^2 + x_5 q_1 q_2 + x_6 q_1^2 q_2 - y;$

$\widetilde{(\mathrm{e}_7^{\mathrm{n}})}_1$ $(\mathrm{n} \geq 5)$: $\pm s + (q_1^3 + q_1 q_2^3) + x_1 q_1 + x_2 q_2 + y q_1^2 + x_3 q_2^2 + x_4 q_1 q_2 + x_5 q_1^2 q_2 - y;$

$\widetilde{(\mathrm{e}_7^{\mathrm{n}})}_2$ $(\mathrm{n} \geq 5)$: $\pm s + (q_1^3 + q_1 q_2^3) + x_1 q_1 + x_2 q_2 + x_3 q_1^2 + y q_2^2 + x_4 q_1 q_2 + x_5 q_1^2 q_2 - y;$

$\widetilde{(\mathrm{e}_7^{\mathrm{n}})}_3$ $(\mathrm{n} \geq 5)$: $\pm s + (q_1^3 + q_1 q_2^3) + x_1 q_1 + x_2 q_2 + x_3 q_1^2 + x_4 q_2^2 + y q_1 q_2 + x_5 q_1^2 q_2 - y;$

$\widetilde{(\mathrm{e}_7^{\mathrm{n}})}_4$ $(\mathrm{n} \geq 5)$: $\pm s + (q_1^3 + q_1 q_2^3) + x_1 q_1 + x_2 q_2 + x_3 q_1^2 + x_4 q_2^2 + x_5 q_1 q_2 + y q_1^2 q_2 - y;$

$\mathrm{e}_8^{\mathrm{n}}$ $(\mathrm{n} \geq 7)$: $\pm s + (q_1^3 + q_2^5) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + x_4 q_1 q_2 + x_5 q_2^3 + x_6 q_1 q_2^2 + x_7 q_1 q_2^3 - y;$

$\widetilde{(\mathrm{e}_8^{\mathrm{n}})}_1$ $(\mathrm{n} \geq 6)$: $\pm s + (q_1^3 + q_2^5) + x_1 q_1 + x_2 q_2 + y q_2^2 + x_3 q_1 q_2 + x_4 q_2^3 + x_5 q_1 q_2^2 + x_6 q_1 q_2^3 - y;$

$\widetilde{(\mathrm{e}_8^{\mathrm{n}})}_2$ $(\mathrm{n} \geq 6)$: $\pm s + (q_1^3 + q_2^5) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + y q_1 q_2 + x_4 q_2^3 + x_5 q_1 q_2^2 + x_6 q_1 q_2^3 - y;$

$\widetilde{(\mathrm{e}_8^{\mathrm{n}})}_3$ $(\mathrm{n} \geq 6)$: $\pm s + (q_1^3 + q_2^5) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + x_4 q_1 q_2 + y q_2^3 + x_5 q_1 q_2^2 + x_6 q_1 q_2^3 - y;$

$\widetilde{(\mathrm{e}_8^{\mathrm{n}})}_4$ $(\mathrm{n} \geq 6)$: $\pm s + (q_1^3 + q_2^5) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + x_4 q_1 q_2 + x_5 q_2^3 + y q_1 q_2^2 + x_6 q_1 q_2^3 - y;$

$\widetilde{(\mathrm{e}_8^{\mathrm{n}})}_5$ $(\mathrm{n} \geq 6)$: $\pm s + (q_1^3 + q_2^5) + x_1 q_1 + x_2 q_2 + x_3 q_2^2 + x_4 q_1 q_2 + x_5 q_2^3 + x_6 q_1 q_2^2 + y q_1 q_2^3 - y;$

$(\mathrm{f}_4^{\mathrm{n}})_1$ $(\mathrm{n} \geq 3)$: $q_1^3 \pm s^2 + x_1 q_1 + x_2 s + x_3 s q_1 - y;$

$(\mathrm{f}_4^{\mathrm{n}})_2$ $(\mathrm{n} \geq 2)$: $q_1^3 \pm s^2 + x_1 q_1 + x_2 s + \omega(\mathbf{x}, t) s q_1 - y$, *where* $\omega \in \mathfrak{M}_{(\mathbf{x},t)}$, $(\partial\omega/\partial t)(\mathbf{0}) \neq 0$ *and* $(\partial\omega/\partial x_2)(\mathbf{0}) \neq 0$.

Proof. By Theorem 3.3 in [10] and the definition of $\mathcal{R}^+$-simple and stable one-parameter integral diagram, the generating family $\widetilde{G}$ of $\ell_{(\mu,F)}$ is the P-$\mathcal{K}$-versal deformation of $g = \widetilde{G}|_{t=0}$ and the $S.P^+$-$\mathcal{K}$-versal deformation of the $S.P$-$\mathcal{K}$-simple function germ $g_0 = \widetilde{G}|_{\mathbb{R}\times\mathbf{0}\times\mathbb{R}^k}$. Furthermore, by Theorem 4.2 in [10] and Lemma 4, there exists a diffeomorphism germ $\phi : (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbf{0}) \to (\mathbb{R}^{n+2}, \mathbf{0})$ such that $\widetilde{G}$ is stably one-parameter $S.P^+$-$\mathcal{K}$-equivalent to one of the germs in the following forms:

(1) $\mathrm{a}_\ell(\mathbf{w})$: $\pm s \pm q_1^{\ell+1} + \sum_{i=1}^{\ell-1} v_i(\mathbf{w}) q_1^i + \sum_{i=\ell}^{n+2} v_i(\mathbf{w})$ $(1 \leq \ell \leq n+2);$

(2) $\widetilde{\mathrm{a}_{\mathrm{n+3}}(\mathbf{w})}$: $\pm s \pm q_1^{n+4} + \sum_{i=1}^{n+2} v_i(\mathbf{w}) q_1^i;$

(3) $\mathrm{b}_\ell(\mathbf{w})$: $\pm s^\ell \pm q_1^2 + \sum_{i=1}^{\ell-1} v_i(\mathbf{w}) s^i + \sum_{i=\ell}^{n+2} v_i(\mathbf{w})$ $(2 \leq \ell \leq n+2);$

(4) $\widetilde{\mathrm{b}_{\mathrm{n+3}}(\mathbf{w})}$: $\pm s^{n+3} \pm q_1^2 + \sum_{i=1}^{n+1} v_i(\mathbf{w}) s^i + v_{n+2}(\mathbf{w});$

(5) $\mathrm{c}_\ell(\mathbf{w})$: $q_1^\ell \pm s q_1 + \sum_{i=1}^{\ell-1} v_i(\mathbf{w}) q_1^i + \sum_{i=\ell}^{n+2} v_i(\mathbf{w})$ $(3 \leq \ell \leq n+2);$

(6) $\widetilde{\mathrm{c}_{\mathrm{n+3}}(\mathbf{w})}$: $q_1^{n+3} \pm s q_1 + \sum_{i=1}^{n+1} v_i(\mathbf{w}) q_1^{i+1} + v_{n+2}(\mathbf{w});$

(7) $\mathrm{d}_\ell(\mathbf{w})$: $\pm s + (q_1^2 q_2 \pm q_2^\ell) + v_1(\mathbf{w}) q_1 + v_2(\mathbf{w}) q_2 + v_3(\mathbf{w}) q_1^2 + \sum_{i=4}^{\ell-1} v_i(\mathbf{w}) q_2^{i-2} + \sum_{i=\ell}^{n+2} v_i(\mathbf{w})$ $(4 \leq \ell \leq n+2);$

(8) $\widetilde{\mathrm{d}_{\mathrm{n+3}}(\mathbf{w})}$: $\pm s + (q_1^2 q_2 \pm q_2^{n+2}) + v_1(\mathbf{w}) q_1 + v_2(\mathbf{w}) q_2 + v_3(\mathbf{w}) q_1^2 + \sum_{i=4}^{n+2} v_i(\mathbf{w}) q_2^{i-2};$

(9) $\mathrm{e}_6^{\mathrm{n}}(\mathbf{w}):\quad \pm s+(q_1^3\pm q_2^4)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_2^2+v_4(\mathbf{w})q_1q_2+v_5(\mathbf{w})q_1q_2^2+\sum_{i=6}^{n+2}v_i(\mathbf{w})\ (n\geq 4);$

(10) $\widetilde{\mathrm{e}_6^3(\mathbf{w})}:\quad \pm s+(q_1^3\pm q_2^4)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_2^2+v_4(\mathbf{w})q_1q_2+v_5(\mathbf{w})q_1q_2^2;$

(11) $\mathrm{e}_7^{\mathrm{n}}(\mathbf{w}):\quad \pm s+(q_1^3+q_1q_2^3)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_1^2+v_4(\mathbf{w})q_2^2+v_5(\mathbf{w})q_1q_2+v_6(\mathbf{w})q_1^2q_2+\sum_{i=7}^{n+2}v_i(\mathbf{w})$ $(n\geq 5);$

(12) $\widetilde{\mathrm{e}_7^4(\mathbf{w})}:\quad \pm s+(q_1^3+q_1q_2^3)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_1^2+v_4(\mathbf{w})q_2^2+v_5(\mathbf{w})q_1q_2+v_6(\mathbf{w})q_1^2q_2;$

(13) $\mathrm{e}_8^{\mathrm{n}}(\mathbf{w}):\quad \pm s+(q_1^3+q_2^5)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_2^2+v_4(\mathbf{w})q_1q_2$ $+v_5(\mathbf{w})q_2^3+v_6(\mathbf{w})q_1q_2^2+v_7(\mathbf{w})q_1q_2^3+\sum_{i=8}^{n+2}v_i(\mathbf{w})\ (n\geq 6);$

(14) $\widetilde{\mathrm{e}_8^5(\mathbf{w})}:\quad \pm s+(q_1^3+q_2^5)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_2^2+v_4(\mathbf{w})q_1q_2+v_5(\mathbf{w})q_2^3+v_6(\mathbf{w})q_1q_2^2+v_7(\mathbf{w})q_1q_2^3;$

(15) $\mathrm{f}_4^{\mathrm{n}}(\mathbf{w}):\quad q_1^3\pm s^2+v_1(\mathbf{w})q_1+v_2(\mathbf{w})s+v_3(\mathbf{w})sq_1+\sum_{i=4}^{n+2}v_i(\mathbf{w})\ (n\geq 2);$

(16) $\widetilde{\mathrm{f}_4^1(\mathbf{w})}:\quad q_1^3\pm s^2+v_1(\mathbf{w})q_1+v_2(\mathbf{w})sq_1+v_3(\mathbf{w})\,,$

where $\mathbf{w}=(\mathbf{x},t,y)$. Since $\widetilde{G}$ is of the form $G-y$, we fix y (for example, $(\partial v_{n+2}/\partial y)(\mathbf{0})\neq 0$) and perform a local coordinate change, so that $v_i(\mathbf{w})=v_i(\mathbf{x},t)$ $(i=1,\ldots,n+1)$ and $v_{n+2}(\mathbf{w})=-y$. Therefore, we classify these germs by one-parameter $S.P^+$-$\mathcal{K}$-equivalence under the condition P-$\mathcal{K}$-cod$(g)\leq 1$, where $g=\widetilde{G}|_{t=0}$ and G satisfies the following immersion condition (c.f. [10])

$$\operatorname{rank}\begin{pmatrix}\dfrac{\partial^2 G}{\partial s\partial\mathbf{q}} & \dfrac{\partial G}{\partial s} & \dfrac{\partial^2 G}{\partial s\partial\mathbf{x}}\\ \dfrac{\partial^2 G}{\partial\mathbf{q}\partial\mathbf{q}} & \mathbf{0} & \dfrac{\partial^2 G}{\partial\mathbf{x}\partial\mathbf{q}}\end{pmatrix}(\mathbf{0})=k+1. \tag{1}$$

The germs of cases (2), (8), (10), (12) and (14) are not the P-$\mathcal{K}$-versal deformation of $g=\widetilde{G}|_{t=0}$. Case (16) does not satisfy the immersion condition (3.1). Hence, they do not all appear in the list.

Cases (1), (7), (9), (11) and (13) can be considered by a similar method, and cases (3)–(6) and (15) also can be treated by a similar method. Therefore, because of too many types and the rather tedious process, we only discuss the cases (9) and (15) here.

Since (9) is also one-parameter $S.P^+$-$\mathcal{K}$-equivalent to $\pm s+(q_1^3\pm q_2^4)+v_1(\mathbf{w})q_1+v_2(\mathbf{w})q_2+v_3(\mathbf{w})q_2^2+v_4(\mathbf{w})q_1q_2+v_5(\mathbf{w})q_1q_2^2+\sum_{i=1}^{n+2}v_i(\mathbf{w})$, by the immersion condition,we can assume $(\partial v_1/\partial x_1)(\mathbf{0})\neq 0$ and $(\partial v_2/\partial x_2)(\mathbf{0})\neq 0$. All the transformations performed are local coordinate changes, so we have the following situations.

If $(\partial v_j/\partial t)(\mathbf{0})\neq 0$ and $(\partial v_m/\partial y)(\mathbf{0})\neq 0$, where $6\leq j,m\leq n+2$ and $j\neq m$, then (9) is one-parameter $S.P$-$\mathcal{K}$-equivalent to $\mathrm{e}_6^{\mathrm{n}}$ $(\mathrm{n}\geq 5)$.

If $(\partial v_j/\partial t)(\mathbf{0})\neq 0$ and $(\partial v_m/\partial y)(\mathbf{0})\neq 0$, where $3\leq j\leq 5$ and $6\leq m\leq n+2$, then (9) is one-parameter $S.P$-$\mathcal{K}$-equivalent to one of the following germs:

$$\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+\beta(\mathbf{x},t)q_2^2+x_3q_1q_2+x_4q_1q_2^2-y;$$
$$\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+x_3q_2^2+\beta(\mathbf{x},t)q_1q_2+x_4q_1q_2^2-y;$$
$$\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+x_3q_2^2+x_4q_1q_2+\beta(\mathbf{x},t)q_1q_2^2-y\,,$$

where $(\partial\beta/\partial t)(\mathbf{0})\neq 0$. They are P-$\mathcal{K}$-versal deformations if and only if there exists $i_0:5\leq i_0\leq n$ such that $(\partial\beta/\partial x_{i_0})(\mathbf{0})\neq 0$. Therefore, they are all one-parameter $S.P$-$\mathcal{K}$-equivalent to $\mathrm{e}_6^{\mathrm{n}}$ $(\mathrm{n}\geq 5)$.

If $(\partial v_m/\partial y)(\mathbf{0}) \neq 0, 3 \leq m \leq 5$, then (9) is one-parameter $S.P^+$-$\mathcal{K}$-equivalent to either of $\widetilde{(\mathrm{e}^{\mathrm{n}}_6)}_1$, $\widetilde{(\mathrm{e}^{\mathrm{n}}_6)}_2$ and $\widetilde{(\mathrm{e}^{\mathrm{n}}_6)}_3$ $(n \geq 4)$.

For case (15), by the immersion condition, we can suppose $(\partial v_1/\partial x_1)(\mathbf{0}) \neq 0$ and $(\partial v_2/\partial x_2)(\mathbf{0}) \neq 0$. Since the form of $\widetilde{G}$ is $G - y$ and ϕ is a diffeomorphism germ, we have the following situations.

If $(\partial v_j/\partial t)(\mathbf{0}) \neq 0$, $4 \leq j \leq n+2$, then (15) is one-parameter $S.P$-$\mathcal{K}$-equivalent to $(\mathrm{f}^{\mathrm{n}}_4)_1$ $(\mathrm{n} \geq 3)$, and it is a P-$\mathcal{K}$-versal deformation.

If $(\partial v_3/\partial t)(\mathbf{0}) \neq 0$, then (15) is one-parameter $S.P$-$\mathcal{K}$-equivalent to

$$q_1^3 \pm s^2 + x_1 q_1 + x_2 s + \beta(\mathbf{x}, t) s q_1 - y,$$

where $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$. It is P-$\mathcal{K}$-versal deformation if and only if there exists $i_0 : 2 \leq i_0 \leq n$ such that $(\partial\beta/\partial x_{i_0})(\mathbf{0}) \neq 0$. In the case of $3 \leq i_0 \leq n$, (15) is also one-parameter $S.P$-$\mathcal{K}$-equivalent to $(\mathrm{f}^{\mathrm{n}}_4)_1$ $(\mathrm{n} \geq 3)$. In the case of $i_0 = 2$, the normal form $(\mathrm{f}^{\mathrm{n}}_4)_2$ $(\mathrm{n} \geq 2)$ can be obtained. □

3. Classification Theorem

In this section, we give a generic classification of one-parameter unfoldings of complete integral holonomic equations which have $\mathcal{R}^+$-simple and stable one-parameter integral diagrams and prove it. The classification theorem is as follows.

Theorem 2. *For a generic one-parameter unfolding of a holonomic equation with complete integral*

$$(\mu, F) : (\mathbb{R}^{n+1} \times \mathbb{R}, \mathbf{0}) \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R}$$

such that $(\mu, \pi \circ f)$ is $\mathcal{R}^+$-simple and stable, the one-parameter unfolding of the integral diagram $(\mu, (\pi \times id) \circ F)$ is strictly equivalent to one of the germs in the following list:

SDA_1: $\mu = u_{n+1},\ G = (u_1, \ldots, u_n, u_{n+1}, t)$;

SDA_2: $\mu = \frac{2}{3}u_1^3 + u_{n+1},\ G = (u_1^2, u_2, \ldots, u_{n+1}, t)$;

SDA_ℓ: $\mu = \frac{\ell}{\ell+1}u_1^{\ell+1} \pm (\sum_{i=1}^{\ell-2} \frac{i}{i+1} u_{i+1} u_1^{i+1} + u_{n+1}),\ G = \left(\pm u_1^\ell + \sum_{i=1}^{\ell-2} u_{i+1} u_1^i, u_2, \ldots, u_{n+1}, t\right)$ $(3 \leq \ell \leq n+1)$;

$\widetilde{(\mathrm{SDA}_\ell)}_{\mathrm{j}}$: $\mu = \frac{\ell}{\ell+1}u_1^{\ell+1} \pm (\sum_{i=1}^{\ell-2} \frac{i}{i+1} u_{i+1} u_1^{i+1} + u_j) + \alpha \circ G, G = (\pm u_1^\ell + \sum_{i=1}^{\ell-2} u_{i+1} u_1^i, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n+1}, u_j, t)$,
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(2 \leq j \leq \ell - 1,\ 3 \leq \ell \leq n+2)$;

SDB_2: $\mu = u_{n+1} - \frac{1}{2}u_1,\ G = (u_1, \ldots, u_n, u_{n+1}^2, t)$;

SDB_ℓ: $\mu = u_{n+1},\ G = \left(u_1, \ldots, u_n, \pm u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i, t\right)$ $(3 \leq \ell \leq n+1)$;

$(\mathrm{SDB}_\ell)_{\mathrm{j}}$: $\mu = u_{n+1},\ G = \left(u_1, \ldots, u_n, \pm u_{n+1}^\ell + \sum_{i=1}^{j-1} u_i u_{n+1}^i + \beta(\mathbf{u}_{\overline{1,n}}, t) u_{n+1}^j + \sum_{i=j+1}^{\ell-1} u_{i-1} u_{n+1}^i, t\right)$,
where $\beta \in \mathfrak{M}_{(\mathbf{u}_{\overline{1,n}},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$ $(2 \leq j \leq \ell - 1,\ 3 \leq \ell \leq n+2)$;

$\widetilde{(\mathrm{SDB}_{\mathrm{n+3}})}_{\mathrm{j}}$: $\mu = u_{n+1} + \alpha \circ G,\ G = \left(u_1, \ldots, u_n, \pm u_{n+1}^{n+3} + \sum_{i=1}^{j-1} u_i u_{n+1}^i + \beta(\mathbf{u}_{\overline{1,n}}, t) u_{n+1}^j + \sum_{i=j+1}^{n+1} u_{i-1} u_{n+1}^i, t\right)$,
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$, $\beta \in \mathfrak{M}_{(\mathbf{u}_{\overline{1,n}},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$ $(2 \leq j \leq n+1)$;

$(\mathrm{SDC}_\ell)_1$: $\mu = \mp\left(\ell u_{n+1}^{\ell-1} + \sum_{i=1}^{\ell-2} \frac{(i+1)(\ell-1)}{i} u_{i+1} u_{n+1}^i + u_1\right),\ G = \left(u_1, \ldots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-2} u_{i+1} u_{n+1}^{i+1}, t\right)$ $(3 \leq \ell \leq n+1)$;

$$(\mathrm{SDC}_\ell)_{\mathrm{j}}: \quad \mu = \mp\Big(\ell u_{n+1}^{\ell-1} + u_1 + \sum_{i=1}^{j-2} \frac{(i+1)(\ell-1)}{i} u_{i+1}u_{n+1}^i + j\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^{j-1} + \sum_{i=j}^{\ell-2} \frac{(i+1)(\ell-1)}{i} u_i u_{n+1}^i\Big),$$

$$G = \Big(u_1, \ldots, u_n, u_{n+1}^\ell + \sum_{i=1}^{j-2} u_{i+1}u_{n+1}^{i+1} + \frac{j-1}{\ell-1}\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^j + \sum_{i=j}^{\ell-2} u_i u_{n+1}^{i+1}, t\Big),$$

where $\beta \in \mathfrak{M}_{(\mathbf{u}_{\overline{1,n}},t)}$, $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$, $(\partial\beta/\partial u_{r_0})(\mathbf{0}) \neq 0$, $r_0 = 1$ *or* $\ell-2$ $(2 \leq j \leq \ell-2,\ 4 \leq \ell \leq n+2)$;

$$(\mathrm{SDC}_\ell)_{\ell-1}: \quad \mu = \mp\Big(\ell u_{n+1}^{\ell-1} + u_1 + \sum_{i=1}^{\ell-3} \frac{(i+1)(\ell-1)}{i} u_{i+1}u_{n+1}^i + (\ell-1)\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^{\ell-2}\Big),$$

$$G = \Big(u_1, \ldots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-3} u_{i+1}u_{n+1}^{i+1} + \frac{\ell-2}{\ell-1}\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^{\ell-1}, t\Big),$$

where $\beta \in \mathfrak{M}_{(\mathbf{u}_{\overline{1,n}},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$ $(3 \leq \ell \leq n+2)$;

$$(\widetilde{\mathrm{SDC}_{\mathrm{n+3}}})_{\mathrm{j}}: \quad \mu = \mp\Big((n+3)u_{n+1}^{n+2} + (n+2)u_1 + \sum_{i=1}^{j-1} \frac{(i+1)(n+2)}{i} u_i u_{n+1}^i + (j+1)\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^j + \sum_{i=j+1}^{n+1} \frac{(i+1)(n+2)}{i} u_{i-1}u_{n+1}^i\Big) + \alpha \circ G,$$

$$G = \Big(u_1, \ldots, u_n, u_{n+1}^{n+3} + \sum_{i=1}^{j-1} u_i u_{n+1}^{i+1} + \frac{j}{n+2}\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^{j+1} + \sum_{i=j+1}^{n+1} u_{i-1}u_{n+1}^{i+1}, t\Big),$$

where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$, $\beta \in \mathfrak{M}_{(\mathbf{u}_{\overline{1,n}},t)}$, $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$ *and* $(\partial\beta/\partial u_n)(\mathbf{0}) \neq 0$ $(1 \leq j \leq n)$;

$$(\widetilde{\mathrm{SDC}_{\mathrm{n+3}}})_{\mathrm{n+1}}: \quad \mu = \mp\Big((n+3)u_{n+1}^{n+2} + (n+2)u_1 + \sum_{i=1}^{n} \frac{(i+1)(n+2)}{i} u_i u_{n+1}^i + (n+2)\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^{n+1}\Big) + \alpha \circ G,$$

$$G = \Big(u_1, \ldots, u_n, u_{n+1}^{n+3} + \sum_{i=1}^{n} u_i u_{n+1}^{i+1} + \frac{n+1}{n+2}\beta(\mathbf{u}_{\overline{1,n}}, t)u_{n+1}^{n+2}, t\Big),$$

where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$, $\beta \in \mathfrak{M}_{(\mathbf{u}_{\overline{1,n}},t)}$ *and* $(\partial\beta/\partial t)(\mathbf{0}) \neq 0$;

$$\mathrm{SDD}_\ell: \quad \mu = \pm 2u_1^2u_2 + (\ell-2)u_2^{\ell-1} + u_1^2u_3 + \sum_{i=4}^{\ell-1}(i-3)u_i u_2^{i-2} + u_{n+1},$$

$$G = \Big(u_1u_2 + u_1u_3, u_1^2 \pm (\ell-1)u_2^{\ell-2} + \sum_{i=4}^{\ell-1}(i-2)u_i u_2^{i-3}, u_3, \ldots, u_{n+1}, t\Big)\ (4 \leq \ell \leq n+1);$$

$$(\widetilde{\mathrm{SDD}_\ell})_3: \quad \mu = \pm 2u_1^2u_2 + (\ell-2)u_2^{\ell-1} + \sum_{i=4}^{\ell-1}(i-3)u_{i-1}u_2^{i-2} + u_1^2u_{n+1} + u_{n+1} + \alpha \circ G,$$

$$G = \Big(u_1u_2 + u_1u_{n+1}, u_1^2 \pm (\ell-1)u_2^{\ell-2} + \sum_{i=4}^{\ell-1} u_{i-1}u_2^{i-3}, u_3, \ldots, u_{n+1}, t\Big),$$

where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(4 \leq \ell \leq n+2)$;

$$(\widetilde{\mathrm{SDD}_\ell})_{\mathrm{j}}: \quad \mu = \pm 2u_1^2u_2 + (\ell-2)u_2^{\ell-1} + u_1^2u_3 + \sum_{i=4}^{\ell-1}(i-3)u_i u_2^{i-2} + u_j + \alpha \circ G,$$

$$G = \Big(u_1u_2 + u_1u_3, u_1^2 \pm (\ell-1)u_2^{\ell-2} + \sum_{i=4}^{\ell-1}(i-2)u_i u_2^{i-3}, u_3, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n+1}, u_j, t\Big),$$

where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(4 \leq j \leq \ell-1,\ 5 \leq \ell \leq n+2)$;

$$\mathrm{SDE}_6^{\mathrm{n}}: \quad \mu = 3u_2^4 + 2u_1u_2^2u_5 + u_1u_2u_4 + 2u_1^3 + u_2^2u_3 + u_{n+1},$$

$$G = (3u_1^2 + u_2u_4 + u_2^2u_5, 4u_2^3 + 2u_2u_3 + u_1u_4 + 2u_1u_2u_5, u_3, \ldots, u_{n+1}, t)\ (n \geq 5);$$

$$(\widetilde{\mathrm{SDE}_6^{\mathrm{n}}})_1: \quad \mu = 3u_2^4 + 2u_1u_2^2u_4 + u_1u_2u_3 + 2u_1^3 + u_2^2u_{n+1} + u_{n+1} + \alpha \circ G,$$

$$G = (3u_1^2 + u_2u_3 + u_2^2u_4, 4u_2^3 + 2u_1u_2u_4 + 2u_2u_{n+1} + u_1u_3, u_3, \ldots, u_{n+1}, t),\ \text{where } \alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}\ (n \geq 4);$$

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_6})_2: \quad \mu = 3u_2^4 + 2u_1u_2^2u_4 + u_1u_2u_{n+1} + 2u_1^3 + u_2^2u_3 + u_{n+1} + \alpha \circ G,$
$G = (3u_1^2 + u_2^2u_4 + u_2u_{n+1}, 4u_2^3 + 2u_1u_2u_4 + 2u_2u_3 + u_1u_{n+1}, u_3, \ldots, u_{n+1}, t),$ *where* $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 4)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_6})_3: \quad \mu = 3u_2^4 + 2u_1u_2^2u_{n+1} + 2u_1^3 + u_1u_2u_4 + u_2^2u_3 \pm u_{n+1} + \alpha \circ G,$
$G = (u_2^2u_{n+1} + 3u_1^2 + u_2u_4, 4u_2^3 + 2u_1u_2u_{n+1} + u_1u_4 + 2u_2u_3, u_3, \ldots, u_{n+1}, t),$ *where* $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 4)$;

$\mathrm{SDE}^{\mathrm{n}}_7: \quad \mu = 2u_1^2u_2u_6 + 3u_1u_2^3 + u_2^2u_4 + u_1^2u_3 + 2u_1^3 + u_1u_2u_5 + u_{n+1},$
$G = (u_2^3 + 2u_1u_2u_6 + 3u_1^2 + 2u_1u_3 + u_2u_5, 3u_1u_2^2 + u_1^2u_6 + 2u_2u_4 + u_1u_5, u_3, \ldots, u_{n+1}, t)$ $(n \geq 6)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_7})_1: \quad \mu = 2u_1^2u_2u_5 + 3u_1u_2^3 + u_1u_2u_4 + 2u_1^3 + u_2^2u_3 + u_1^2u_{n+1} + u_{n+1} + \alpha \circ G,$
$G = (u_2^3 + 2u_1u_2u_5 + 3u_1^2 + 2u_1u_{n+1} + u_2u_4, 3u_1u_2^2 + u_1^2u_5 + 2u_2u_3 + u_1u_4, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 5)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_7})_2: \quad \mu = 2u_1^2u_2u_5 + 3u_1u_2^3 + u_1^2u_3 + 2u_1^3 + u_1u_2u_4 + u_2^2u_{n+1} + u_{n+1} + \alpha \circ G,$
$G = (u_2^3 + 2u_1u_2u_5 + 3u_1^2 + 2u_1u_3 + u_2u_4, 3u_1u_2^2 + u_1^2u_5 + 2u_2u_{n+1} + u_1u_4, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 5)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_7})_3: \quad \mu = 2u_1^2u_2u_5 \pm 3u_1u_2^3 + u_1^2u_3 + 2u_1^3 + u_2^2u_4 + u_1u_2u_{n+1} + u_{n+1} + \alpha \circ G,$
$G = (u_2^3 + 2u_1u_2u_5 + 3u_1^2 + 2u_1u_3 + u_2u_{n+1}, u_1^2u_5 \pm 3u_1u_2^2 + u_1u_{n+1} + 2u_2u_4, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 5)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_7})_4: \quad \mu = 2u_1^2u_2u_{n+1} + 3u_1u_2^3 + u_1^2u_3 + 2u_1^3 + u_2^2u_4 + u_1u_2u_5 + u_{n+1} + \alpha \circ G,$
$G = (u_2^3 + 2u_1u_2u_{n+1} + 3u_1^2 + 2u_1u_3 + u_2u_5, u_1^2u_{n+1} + 3u_1u_2^2 + u_1u_5 + 2u_2u_4, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 5)$;

$\mathrm{SDE}^{\mathrm{n}}_8: \quad \mu = 4u_2^5 + 3u_1u_2^3u_7 + 2u_2^3u_5 + 2u_1u_2^2u_6 + u_1u_2u_4 + 2u_1^3 + u_2^2u_3 + u_{n+1},$
$G = (u_2^3u_7 + u_2^2u_6 + 3u_1^2 + u_2u_4, 5u_2^4 + 3u_1u_2^2u_7 + 2u_1u_2u_6 + 3u_2^2u_5 + 2u_2u_3 + u_1u_4, u_3, \ldots, u_{n+1}, t)$ $(n \geq 7)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_8})_1: \quad \mu = 4u_2^5 + 3u_1u_2^3u_6 + 2u_2^3u_4 + 2u_1u_2^2u_5 + u_1u_2u_3 + 2u_1^3 + u_2^2u_{n+1} + u_{n+1} + \alpha \circ G,$
$G = (u_2^3u_6 + u_2^2u_5 + 3u_1^2 + u_2u_3, 5u_2^4 + 3u_1u_2^2u_6 + 3u_2^2u_4 + u_1u_2u_5 + 2u_2u_{n+1} + u_1u_3, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 6)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_8})_2: \quad \mu = 4u_2^5 + 3u_1u_2^3u_6 + 2u_1u_2^2u_5 + 2u_2^3u_4 + 2u_1^3 + u_1u_2u_{n+1} + u_2^2u_3 + u_{n+1} + \alpha \circ G,$
$G = (u_2^3u_6 + u_2^2u_5 + 3u_1^2 + u_2u_{n+1}, 5u_2^4 + 3u_1u_2^2u_6 + 2u_1u_2u_5 + 3u_2^2u_4 + u_1u_{n+1} + 2u_2u_3, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 6)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_8})_3: \quad \mu = 4u_2^5 + 3u_1u_2^3u_6 + 2u_1u_2^2u_5 + 2u_2^3u_{n+1} + u_1u_2u_4 + 2u_1^3 + u_2^2u_3 + u_{n+1} + \alpha \circ G,$
$G = (u_2^3u_6 + u_2^2u_5 + 3u_1^2 + u_2u_4, 5u_2^4 + 3u_1u_2^2u_6 \pm 3u_2^2u_{n+1} + 2u_1u_2u_5 + 2u_2u_3 + u_1u_4, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 6)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_8})_4: \quad \mu = 4u_2^5 + 3u_1u_2^3u_6 \pm 2u_1u_2^2u_{n+1} + 2u_2^3u_5 + 2u_1^3 + u_1u_2u_4 + u_2^2u_3 + u_{n+1} + \alpha \circ G,$
$G = (u_2^3u_6 \pm u_2^2u_{n+1} + 3u_1^2 + u_2u_4, 5u_2^4 + 3u_1u_2^2u_6 \pm 2u_1u_2u_{n+1} + 3u_2^2u_5 + u_1u_4 + 2u_2u_3, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 6)$;

$(\widetilde{\mathrm{SDE}^{\mathrm{n}}_8})_5: \quad \mu = 4u_2^5 + 2u_2^3u_5 + 3u_1u_2^3u_{n+1} + 2u_1u_2^2u_6 + 2u_1^3 + u_1u_2u_4 + u_2^2u_3 + u_{n+1} + \alpha \circ G,$
$G = (u_2^3u_{n+1} + u_2^2u_6 + 3u_1^2 + u_2u_4, 5u_2^4 + 3u_1u_2^2u_{n+1} + 2u_1u_2u_6 + 3u_2^2u_5 + u_1u_4 + 2u_2u_3, u_3, \ldots, u_{n+1}, t),$
where $\alpha \in \mathfrak{M}_{(\mathbf{x},y,t)}$ $(n \geq 6)$;

$(\mathrm{SDF}^{\mathrm{n}}_4)_1: \quad \mu = u_{n+1},\ G = \big(3u_1^2 + u_3u_{n+1}, u_2, \ldots, u_n, u_1u_3 + u_2 \pm 2u_{n+1}, t\big)$ $(n \geq 3)$;

$(\mathrm{SDF}^{\mathrm{n}}_4)_2: \quad \mu = u_{n+1},\ G = \big(3u_1^3 + tu_{n+1} - \beta(X_1, u_2, \ldots, u_{n+1}, t), u_2, \ldots, u_n, u_2u_{n+1} - 2u_1^3 \pm u_{n+1}^2, t\big),$
where $X_1 = -3u_1^3 - tu_{n+1}$*, and* β *is a smooth function germ such that* $\beta(\mathbf{0}) = 0$ *and satisfies the condition* $\beta(X_1, u_2, \ldots, u_{n+1}, t) = -\gamma(X_1 + \beta(X_1, u_2, \ldots, u_{n+1}, t), u_2, \ldots, u_n, t)u_{n+1}$ *for* $\gamma \in \mathfrak{M}_{(\mathbf{x}',t)}$ *and* $(\partial\gamma/\partial x_2')(\mathbf{0}) \neq 0$ $(n \geq 2)$.

Here, $\mathfrak{M}_{(\mathbf{x},y,t)}$ is the unique maximal ideal of the set of all smooth function germs $(\mathbb{R}^{n+1}\times\mathbb{R},\mathbf{0})\to(\mathbb{R},0)$, and denote $(u_1,\ldots,u_n)$ by $\mathbf{u}_{\overline{1,n}}$ for the simpleness of mark.

Proof. The set of one-parameter P-Legendrian stable one-parameter complete Legendrian unfoldings is an open and dense subset in $\mathrm{L}(U\times V, J^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}))$.Therefore, Theorem 2 gives a generic classification of one-parameter complete Legendrian unfoldings, which satisfy the above condition and have the $\mathcal{R}^+$-simple and stable one-parameter integral diagrams under the one-parameter $S.P^+$-$\mathcal{K}$ (or, $S.P$-$\mathcal{K}$)-Legendrian equivalence relation.

Let (μ,f) be a one-parameter family of holonomic equations with complete integral, such that the corresponding one-parameter complete Legendrian unfolding $\ell_{(\mu,F)}$ is one-parameter P-Legendrian stable and $(\mu,\pi\circ f)$ is $\mathcal{R}^+$-simple and stable. The generating family $\widetilde{G}(s,\mathbf{x},t,y,\mathbf{q})$ of $\ell_{(\mu,F)}$ is stably one-parameter $S.P^+$-$\mathcal{K}$-equivalent to one of the members in the list of Theorem 1.

The generating family of types a_ℓ $(1\le\ell\le n+1)$, b_ℓ $(2\le\ell\le n+1)$, $(\mathrm{b}_\ell)_\mathrm{j}$ $(3\le\ell\le n+2, 2\le j\le\ell-1)$, c_ℓ $(3\le\ell\le n+1)$, $(\mathrm{c}_\ell)_\mathrm{j}$ $(4\le\ell\le n+2, 2\le j\le\ell-2)$, $(\mathrm{c}_\ell)_{\ell-1}$ $(3\le\ell\le n+2)$, d_ℓ $(4\le\ell\le n+1)$, e_6^n $(n\ge5)$, e_7^n $(n\ge6)$, e_8^n $(n\ge7)$, $(\mathrm{f}_4^\mathrm{n})_1$ $(n\ge3)$ and $(\mathrm{f}_4^\mathrm{n})_2$ $(n\ge2)$ are $S.P$-$\mathcal{K}$-versal deformations of the corresponding germs, so the corresponding integral diagrams have no functional moduli. However, the germs of other cases are $S.P^+$-$\mathcal{K}$-versal deformations but not $S.P$-$\mathcal{K}$-versal deformations. Hence, the corresponding integral diagrams have functional moduli.

Next, we detect the corresponding normal forms of one-parameter integral diagrams.

e_6^n (n $\ge$ 5): Choose $\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+x_3q_2^2+x_4q_1q_2+x_5q_1q_2^2$ as a generalized phase family. Then,

$$\frac{\partial G}{\partial q_1}=3q_1^2+x_1+x_4q_2+x_5q_2^2,$$
$$\frac{\partial G}{\partial q_2}=\pm4q_2^3+x_2+2x_3q_2+x_4q_1+2x_5q_1q_2.$$

By the above two formulas and

$$y=\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+x_3q_2^2+x_4q_1q_2+x_5q_1q_2^2,$$

we have

$$\begin{aligned}C(G)=&(\pm(y+x_3q_2^2+2q_1^3+x_4q_1q_2+2x_5q_1q_2^2)+3q_2^4,-3q_1^2-x_4q_2-x_5q_2^2,\\&\mp4q_2^3-2x_3q_2-x_4q_1-2x_5q_1q_2,x_3,\ldots,x_n,t,q_1,q_2),\\ \mathcal{L}_G=&(\pm(y+x_3q_2^2+2q_1^3+x_4q_1q_2+2x_5q_1q_2^2)+3q_2^4,-3q_1^2-x_4q_2-x_5q_2^2,\\&\mp4q_2^3-2x_3q_2-x_4q_1-2x_5q_1q_2,x_3,\ldots,x_n,t,y,\ q_1,q_2,q_2^2,q_1q_2,q_1q_2^2,0,\ldots,0).\end{aligned}$$

Define a diffeomorphism germ by $u_1=-q_1$, $u_2=-q_2$, $u_3=-x_3$, $u_4=-x_4$, $u_i=x_i$ $(5\le i\le n)$ and $u_{n+1}=-y$, and define the transformation $X_i=-x_i$ $(1\le i\le4)$, $X_j=x_j$ $(5\le j\le n)$, $Y=-y$ and $T=t$, then (μ,G) is strictly equivalent to the normal form $\mathrm{SDE}_6^\mathrm{n}$ $(n\ge5)$.

$(\widetilde{\mathrm{e}_6^\mathrm{n}})_1$ (n $\ge$ 4): Since $1-q_2^2$ is a unit in $\mathcal{E}_{(s,\mathbf{x},t,y,q_1,q_2)}$, we have

$$\begin{aligned}&\left\langle\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+yq_2^2+x_3q_1q_2+x_4q_1q_2^2-y\right\rangle_{\mathcal{E}_{(s,\mathbf{x},t,y,q_1,q_2)}}\\=&\left\langle\frac{\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+x_3q_1q_2+x_4q_1q_2^2}{1-q_2^2}-y\right\rangle_{\mathcal{E}_{(s,\mathbf{x},t,y,q_1,q_2)}}.\end{aligned}$$

Hence we can choose a function germ

$$G(s,\mathbf{x},t,q_1,q_2)=\frac{\pm s+(q_1^3\pm q_2^4)+x_1q_1+x_2q_2+x_3q_1q_2+x_4q_1q_2^2}{1-q_2^2}$$

as a generalized phase family. Then, the normal form can be detected by the above method.

$(\mathrm{f}_4^{\mathrm{n}})_2$ (n ≥ 2): Choose $q_1^3\pm s^2+x_1q_1+x_2s+\omega(\mathbf{x},t)sq_1$ as a generalized phase family. So,

$$\frac{\partial G}{\partial q_1}=3q_1^2+x_1+\omega(\mathbf{x},t)s.$$

By this formula and $y=q_1^3\pm s^2+x_1q_1+x_2s+\omega(\mathbf{x},t)sq_1$, we have $y=-2q_1^3\pm s^2+x_2s$ and $3q_1^2+x_1+\omega(\mathbf{x},t)s=0$. Since $(\partial\omega/\partial t)(\mathbf{0})\neq 0$, there exists a function germ $\phi(x,t)$ such that $\omega(\mathbf{x},t)=t+\phi(\mathbf{x},t)$. Set $X_1=x_1+\phi(\mathbf{x},t)s$ and $X_i=x_i$ $(2\leq i\leq n)$, then there exists a function germ $\psi(s,\mathbf{X},t)$ such that $x_1=X_1+\psi(s,\mathbf{X},t)$ by the inverse function theorem. Therefore, the following equation holds:

$$\psi(s,\mathbf{X},t)=-\phi(X_1+\psi(s,\mathbf{X},t),X_2,\ldots,X_n,t)s \tag{2}$$

Also, $X_1=-3q_1^3-ts$ and $y=-2q_1^3\pm s^2+x_2s$. Then, we have

$$C(G)=\Big\{(s,X_1+\psi(s,\mathbf{X},t),X_2,\ldots,X_n,t,q_1)\,\Big|\,X_1=-3q_1^2-ts,X_i=x_i,2\leq i\leq n\Big\},$$

$$\mathcal{L}_G=\Big(s,X_1+\psi(s,\mathbf{X},t),X_2,\ldots,X_n,t,-2q_1^3\pm s^2+x_2s,\pm 2s+x_2,0,s,0,\ldots,0\Big).$$

Set $u_1=q_1$, $u_i=x_i$ $(2\leq i\leq n)$ and $u_{n+1}=s$, and define the transformation of $\mathbb{R}^n\times\mathbb{R}$ by $X_1=-x_1$, $X_i=x_i$ $(2\leq i\leq n)$, $Y=y$ and $T=t$. Then, we replace ψ and ϕ by β and γ, respectively. Finally, the normal form $(\mathrm{SDF}_4^{\mathrm{n}})_2$ $(n\geq 2)$ is obtained.

Because the calculations of other types are the same as those of the above cases, but rather a tedious, then we omit the detail. □

4. Conclusions

The bifurcation theory has been applied to connect the dynamics of quantum systems and classical mechanical systems. It can be applied to atomic systems, molecular systems, harmonic tunneling diodes, laser dynamics, and has also been applied to many theoretical examples that are difficult to handle in experiments, such as kicked rotors and coupled quantum wells. The main reason for connecting bifurcations in quantum systems and classical mechanical motion equations is that the signature of classical mechanical orbits becomes larger during bifurcation, as proposed by Martin Gutzwiller in his research on quantum chaos. Many bifurcations have been studied to connect classical mechanics and quantum mechanics, such as saddle-node bifurcations, Hopf bifurcations, cusp bifurcations, period-doubling bifurcations, reconnection bifurcations, tangent bifurcations, and point bifurcations.

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