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Article

Parameter Conditions for Boundedness of Two Integral Operators in Weighted Lebsgue Space and Calculation of Operator Norms

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Abstract: Firstly, Hilbert-type integral inequalities with best constant factors are established for two non-homogeneous kernels. Then, by utilizing the relationship between the Hilbert-type inequality and the integral operator of same kernel, the parameter conditions for two integral operators with non-homogeneous kernels in weighted Lebesgue space to be bounded and the formula for calculating the operator norm are obtained.

Keywords: weighted Lebesgue space; integral operator; non-homogeneous kernel; bounded operator; operator norm; Hilbert-type inequality

MSC: 26D15; 47A07

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1. Introduction

In 1925, Hardy obtained the famous Hilbert inequality in [1] : If $\frac{1}{p}+\frac{1}{q}=1\ (p>1)$, $f\in L_p(0,+\infty)$, $g\in L_q(0,+\infty)$, then

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y\le\frac{\pi}{\sin(\pi/p)}||f||_p||g||_q,$$

where the constant factor $\pi/\sin(\pi/p)$ is the best value. Due to the important application of the Hilbert inequality in the study of operator theory, it has been widely studied by scholars in various countries, and a series of research results have been obtained. Yang et al. [2–5] discussed Hilbert inequalities with independent parameters, while Rassias et al. [6–10] studied fully planar, high-dimensional, generalized, and improved Hilbert-type inequalities.

In order to promote the study of the Hilbert inequality further, scholars have extended the Lebesgue space to a weighted Lebesgue space: if $r>1$, and $\alpha\in\mathbb{R}$, then

$$L_r^\alpha(0,+\infty)=\left\{f(x):||f||_{r,\alpha}=\left(\int_0^{+\infty}x^\alpha|f(x)|^r\mathrm{d}x\right)^{1/r}<+\infty\right\}$$

is called a weighted Lebesgue space, with x^α as the weight. When $\alpha=0$, $L_p^\alpha(0,+\infty)$ becomes the usual Lebesgue space. If $K(x,y)\ge 0$, then [11]

$$\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\le M||f||_{p,\alpha}||g||_{q,\beta},\tag{1}$$

is said to be a Hilbert-type inequality, $K(x,y)$ is the kernel of inequality (1), and M is the constant factor.

In 1934, Hardy et al. [12] abstractly discussed the Hilbert-type inequality for the non-homogeneous kernel $K(x,y) = h(xy)$: let $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$ and denote

$$W(s) = \int_0^{+\infty} h(u)u^{s-1}\mathrm{d}u.$$

If $W(1/p) < +\infty$, then

$$\int_0^{+\infty}\int_0^{+\infty} h(xy)f(x)g(y)\mathrm{d}x\mathrm{d}y \leq W(1/p)||f||_{p,p-2}||g||_q,$$

where the constant factor $W(1/p)$ is the optimal value.

Adiyasuren et al. [13–17] discussed Hilbert-type inequalities and their optimal constant factors for specific homogeneous and non-homogeneous kernels, Vukovic et al. [18,19] provided improvements and generalizations of several Hilbert-type inequalities, and Zhao et al. [20,21] considered inverse Hilbert-type inequalities. The results of this literature indicate that, in order for the Hilbert-type inequality to hold and obtain the optimal constant factor, several parameters need to have some relationship. Therefore, parameters in the kernel $K(x,y)$ of (1) and parameters p, q, α, β etc. in weighted Lebesgue space must meet certain conditions to ensure that (1) holds and the constant factor is optimal. In order to make the conclusions obtained have universal significance and theoretical value, it is an inevitable trend to explore abstract kernel functions. In 2008, Hong [22] first discussed the problem of optimal matching parameters of Hilbert-type inequalities in series form for the abstract λ-order homogeneous kernel, and gave sufficient conditions for optimal matching parameters, and later proved that the conditions were also necessary [23]. In 2017, Hong [24] further discussed the construction conditions of Hilbert-type integral inequalities for abstract homogeneous kernels and obtained the calculation formula for the optimal constant factor, pushing the theoretical research of Hilbert-type inequalities to a new stage and obtaining important applications in operator theory. At present, the Hilbert-type inequality and its applications have formed a relatively complete theoretical system [25].

In this article, we will use the construction theorem of Hilbert-type integral inequalities with non-homogeneous kernels in [25] to obtain the construction conditions and calculation formulas for two optimal Hilbert-type integral inequalities with non-homogeneous kernels. This will generalize the results of Rassias, Yang, and Raigorodskii [26]. Then, we apply the obtained results to discuss parameter conditions for the boundedness of corresponding integral operators in weighted Lebesgue space and the calculation of operator norm.

Rassias, Yang, and Raigorodskii [26] discussed the optimal Hilbert-type integral inequality of non-homogeneous kernels

$$K_0(x,y) = |\ln xy| \prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\alpha/s}}{(\max\{xy, c_k\})^{(\lambda+\alpha)/s}}.$$

The matching form of parameters in $K_0(x,y)$ is rather special, and it is difficult for the obtained optimal inequality to show the law of each parameter, and it is difficult to obtain more applications. In this paper, we discuss the more general non-homogeneous kernels

$$K_1(x,y) = \left|\ln\left(x^{\lambda_1}y^{\lambda_2}\right)\right| \prod_{k=1}^{s} \frac{\left(\min\{x^{\lambda_1}y^{\lambda_2}, c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}y^{\lambda_2}, c_k\}\right)^{\sigma_2}},\ \lambda_1\lambda_2 > 0,$$

$$K_2(x,y) = \left|\ln\left(x^{\lambda_1}/y^{\lambda_2}\right)\right| \prod_{k=1}^{s} \frac{\left(\min\{x^{\lambda_1}/y^{\lambda_2}, c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}/y^{\lambda_2}, c_k\}\right)^{\sigma_2}},\ \lambda_1\lambda_2 > 0,$$

and find out the necessary and sufficient condition for the best matching parameters.

2. Preliminary Lemmas

Lemma 1 ([25]). *Suppose that* $\frac{1}{p}+\frac{1}{q}=1\ (p>1), \alpha, \beta \in \mathbb{R}$, $K(x,y)=G(x^{\lambda_1}y^{\lambda_2})\geq 0$, $\lambda_1\lambda_2>0$, *and*

$$W_1(\beta,q)=\int_0^{+\infty}G(t^{\lambda_2})t^{-\frac{\beta+1}{q}}\mathrm{d}t<+\infty, W_2(\alpha,p)=\int_0^{+\infty}G(t^{\lambda_1})t^{-\frac{\alpha+1}{p}}\mathrm{d}t<+\infty.$$

(i) There exists a constant $M>0$, *the necessary and sufficient condition for Hilbert-type inequality*

$$\int_0^{+\infty}\int_0^{+\infty}G(x^{\lambda_1}y^{\lambda_2})f(x)g(y)\mathrm{d}x\mathrm{d}y\leq M||f||_{p,\alpha}||g||_{q,\beta} \tag{2}$$

to be true is that $\frac{\alpha}{\lambda_1 p}-\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}-\frac{1}{\lambda_2 p}$, *where* $f\in L_p^{\alpha}(0,+\infty), g\in L_q^{\beta}(0,+\infty)$.

(ii) If $\frac{\alpha}{\lambda_1 p}-\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}-\frac{1}{\lambda_2 p}$, *then the best possible constant factor of* (2) *is*

$$\inf\{M\}=\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}(W_0=|\lambda_1|W_2(\alpha,p)=|\lambda_2|W_1(\beta,q)).$$

Lemma 2 ([25]). *Assume that* $\frac{1}{p}+\frac{1}{q}=1\ (p>1), \alpha, \beta\in\mathbb{R}$, $K(x,y)=G(x^{\lambda_1}/y^{\lambda_2})\geq 0$, $\lambda_1\lambda_2>0$, *and*

$$\overline{W}_1(\beta,q)=\int_0^{+\infty}G(t^{-\lambda_2})t^{-\frac{\beta+1}{q}}\mathrm{d}t<+\infty, \overline{W}_2(\alpha,p)=\int_0^{+\infty}G(t^{\lambda_1})t^{-\frac{\alpha+1}{p}}\mathrm{d}t<+\infty.$$

(i) There exists a constant $M>0$, *the necessary and sufficient condition for Hilbert-type inequality*

$$\int_0^{+\infty}\int_0^{+\infty}G(x^{\lambda_1}/y^{\lambda_2})f(x)g(y)\mathrm{d}x\mathrm{d}y\leq M||f||_{p,\alpha}||g||_{q,\beta} \tag{3}$$

to be true is that $\frac{\alpha}{\lambda_1 p}+\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}$, *where* $f\in L_p^{\alpha}(0,+\infty), g\in L_q^{\beta}(0,+\infty)$.

(ii) If $\frac{\alpha}{\lambda_1 p}+\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}$, *then the best possible constant factor of* (3) *is*

$$\inf\{M\}=\frac{\overline{W}_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\overline{W}_0=|\lambda_1|\overline{W}_2(\alpha,p)=|\lambda_2|\overline{W}_1(\beta,q)\right).$$

Lemma 3. *Let* $0\leq a<b\leq+\infty, c\in\mathbb{R}$. *Then*

$$\int_a^b t^c\ln t\mathrm{d}t=\begin{cases}\frac{1}{c+1}\left(b^{c+1}\ln b-a^{c+1}\ln a-\frac{b^{c+1}-a^{c+1}}{c+1}\right), & 0<a<b<+\infty,\\ \frac{1}{c+1}\left(b^{c+1}\ln b-\frac{1}{c+1}b^{c+1}\right), & 0=a<b<+\infty, c+1>0,\\ \frac{1}{c+1}\left(-a^{c+1}\ln a+\frac{1}{c+1}a^{c+1}\right), & 0<a<b=+\infty, c+1<0.\end{cases}$$

Proof. Integration by parts yields that

$$\begin{aligned}\int_a^b t^c\ln t\mathrm{d}t&=\frac{1}{c+1}\int_a^b\ln t\mathrm{d}t^{c+1}\\ &=\frac{1}{c+1}\left(t^{c+1}\ln t|_a^b-\int_a^b t^c\mathrm{d}t\right)\\ &=\frac{1}{c+1}\left(b^{c+1}\ln b-a^{c+1}\ln a-\frac{1}{c+1}(b^{c+1}-a^{c+1})\right).\end{aligned} \tag{4}$$

Notice that $\lim_{t\to0^+}t^{c+1}\ln t=0$ for $c+1>0$, $\lim_{t\to+\infty}t^{c+1}\ln t=0$ for $c+1<0$, the desired result is obtained from (4). □

3. Two Hilbert-Type Integral Inequalities with Non-Homogeneous Kernels

Theorem 1. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1\ (p>1), \alpha,\beta\in\mathbb{R}$, $\lambda_1\lambda_2>0, s, s_0\in\mathbb{N}_+$, $1\le s_0\le s$, $0=c_0<c_1\ \le\cdots\le c_{s_0}\le 1\le c_{s_0+1}\le\cdots\le c_s$, $s\lambda_2\sigma_1>\frac{\beta}{q}-\frac{1}{p}$, $s\lambda_2\sigma_2<\frac{1}{p}-\frac{\beta}{q}$, $s\lambda_1\sigma_1>\frac{\alpha}{p}-\frac{1}{q}$, $s\lambda_1\sigma_2<\frac{1}{q}-\frac{\alpha}{p}$, *and*

$$K_1(x,y)=\left|\ln\left(x^{\lambda_1}y^{\lambda_2}\right)\right|\prod_{k=1}^{s}\frac{\left(\min\{x^{\lambda_1}y^{\lambda_2},c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}y^{\lambda_2},c_k\}\right)^{\sigma_2}},$$

$$W_1(\beta,q)=\int_0^{+\infty}K_1(1,t)t^{-\frac{\beta+1}{q}}\mathrm{d}t<+\infty, W_2(\alpha,p)=\int_0^{+\infty}K_1(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t<+\infty.$$

(i) There exists a constant $M_1>0$, for any $f\in L_p^{\alpha}(0,+\infty), g\in L_q^{\beta}(0,+\infty)$, the necessary and sufficient condition for Hilbert-type inequality

$$\int_0^{+\infty}\int_0^{+\infty}K_1(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\le M_1||f||_{p,\alpha}||g||_{q,\beta} \tag{5}$$

to be true is that $\frac{\alpha}{\lambda_1 p}-\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}-\frac{1}{\lambda_2 p}$;

(ii) If $\frac{\alpha}{\lambda_1 p}-\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}-\frac{1}{\lambda_2 p}$, *then the best possible constant factor of (5) is*

$$\begin{aligned}
M_0(\beta,q) &= \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\int_0^{+\infty}K_1(1,t)t^{-\frac{\beta+1}{q}}\mathrm{d}t\\
&= -\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\frac{1}{\prod_{i=1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1}c_1^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1}\\
&\quad\times\left(\ln c_1+\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1}\right)\\
&\quad-\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\prod_{i=1}^{s}c_i^{\sigma_1}\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)-s\sigma_2}c_s^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)-s\sigma_2}\\
&\quad\times\left(\ln c_s+\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)-s\sigma_2}\right)\\
&\quad-\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\sum_{n=1}^{s_0-1}\frac{\prod_{i=1}^{n}c_i^{\sigma_1}}{\prod_{i=n+1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\\
&\quad\times\left[c_{n+1}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_{n+1}-c_n^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_n\right.\\
&\quad-\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}(c_{n+1}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\\
&\quad\left.-c_n^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n})\right]\\
&\quad+\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\sum_{n=s_0+1}^{s-1}\frac{\prod_{i=1}^{s}c_i^{\sigma_1}}{\prod_{i=n+1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\\
&\quad\times\left[c_{n+1}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_{n+1}-c_n^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_n\right.\\
&\quad-\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}(c_{n+1}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}
\end{aligned}$$

$$
\begin{aligned}
&- c_n^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\Big)\Bigg] \\
&+\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\frac{\prod_{i=1}^{s_0}c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0} \\
&\times\Bigg[c_{s_0}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}\ln c_{s_0}+c_{s_0+1}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}\ln c_{s_0+1} \\
&-\frac{1}{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}\Big(c_{s_0}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0} \\
&+c_{s_{0+1}}^{\frac{1}{\lambda_2}\left(1-\frac{\beta+1}{q}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}-2\Big)\Bigg].
\end{aligned}
$$

Proof. By Lemma 1, we only need to prove that

$$
W_1(\beta,q)=\int_0^{+\infty}K_1(1,t)t^{-\frac{\beta+1}{q}}\mathrm{d}t<+\infty, W_2(\alpha,p)=\int_0^{+\infty}K_1(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t<+\infty,
$$

and calculate the value of $\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}(|\lambda_2|W_1(\beta,q))$.

Denote $\tau=-\frac{1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)-1$, it follows that

$$
\begin{aligned}
W_1(\beta,q) &= \int_0^{+\infty}\left|\ln t^{\lambda_2}\right|\prod_{k=1}^{s}\frac{\left(\min\{t^{\lambda_2},c_k\}\right)^{\sigma_1}}{\left(\max\{t^{\lambda_2},c_k\}\right)^{\sigma_2}}t^{-\frac{\beta+1}{q}}\mathrm{d}t \\
&= \frac{1}{|\lambda_2|}\int_0^{+\infty}|\ln u|\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\sigma_1}}{(\max\{u,c_k\})^{\sigma_2}}u^{-\frac{1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)-1}\mathrm{d}u \\
&= \frac{-1}{|\lambda_2|}\int_0^{1}\ln u\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\sigma_1}}{(\max\{u,c_k\})^{\sigma_2}}u^{\tau}\mathrm{d}u \\
&\quad+\frac{1}{|\lambda_2|}\int_1^{+\infty}\ln u\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\sigma_1}}{(\max\{u,c_k\})^{\sigma_2}}u^{\tau}\mathrm{d}u.
\end{aligned}
$$

Since $s\lambda_2\sigma_1>\frac{\beta}{q}-\frac{1}{p}$, we have $\tau+s\sigma_1+1>0$. It follows from Lemma 3 that

$$
\begin{aligned}
&-\int_0^{1}\ln u\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\sigma_1}}{(\max\{u,c_k\})^{\sigma_2}}u^{\tau}\mathrm{d}u \\
=&-\int_0^{c_1}\ln u\frac{u^{s\sigma_1}}{\prod_{i=1}^{s}c_i^{\sigma_2}}u^{\tau}\mathrm{d}u+\sum_{n=1}^{s_0-1}\int_{c_n}^{c_{n+1}}\ln u\frac{u^{(s-n)\sigma_1}\prod_{i=1}^{n}c_i^{\sigma_1}}{u^{n\sigma_2}\prod_{i=n+1}^{s}c_i^{\sigma_2}}u^{\tau}\mathrm{d}u \\
&+\int_{c_{s_0}}^{1}\ln u\frac{u^{(s-s_0)\sigma_1}\prod_{i=1}^{s_0}c_i^{\sigma_1}}{u^{s_0\sigma_2}\prod_{i=s_0+1}^{s}c_i^{\sigma_2}}u^{\tau}\mathrm{d}u \\
=&-\frac{1}{\prod_{i=1}^{s}c_i^{\sigma_2}}\int_0^{c_1}u^{\tau+s\sigma_1}\ln u\mathrm{d}u-\sum_{n=1}^{s_0-1}\frac{\prod_{i=1}^{n}c_i^{\sigma_1}}{\prod_{i=n+1}^{s}c_i^{\sigma_2}}\int_{c_n}^{c_{n+1}}u^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n}\ln u\mathrm{d}u \\
&-\frac{\prod_{i=1}^{s_0}c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s}c_i^{\sigma_2}}\int_{c_{s_0}}^{1}u^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n}\ln u\mathrm{d}u
\end{aligned}
$$

$$\begin{aligned}
&= -\frac{1}{\prod_{i=1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1+1} c_1^{\tau+s\sigma_1+1}\left(\ln c_1 - \frac{1}{\tau+s\sigma_1+1}\right) \\
&\quad -\sum_{n=1}^{s_0+1} \frac{\prod_{i=1}^{n} c_i^{\sigma_1}}{\prod_{i=1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \\
&\quad \times\Big[c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\ln c_{n+1} - c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\ln c_n \\
&\quad -\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}(c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} - c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1})\Big] \\
&\quad +\frac{\prod_{i=1}^{s_0} c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}\Big[c_{s_0}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}\ln c_{s_0} \\
&\quad +\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}(1-c_{s_0}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1})\Big].
\end{aligned}$$

Since $s\lambda_2\sigma_2 < \frac{1}{p} - \frac{\beta}{q}$, we obtain $\tau - s\sigma_2 + 1 < 0$. It follows from Lemma 3 that

$$\begin{aligned}
&\int_1^{+\infty} \ln u \prod_{k=1}^{s} \frac{(\min\{u,c_k\})^{\sigma_1}}{(\max\{u,c_k\})^{\sigma_2}} u^{\tau} \mathrm{d}u \\
&= \int_1^{c_{s_0+1}} \ln u \frac{u^{(s-s_0)\sigma_1}\prod_{i=1}^{s_0} c_i^{\sigma_1}}{u^{s_0\sigma_2}\prod_{i=s_0+1}^{s} c_i^{\sigma_2}} u^{\tau}\mathrm{d}u + \sum_{n=s_0+1}^{s-1}\int_{c_n}^{c_{n+1}} \ln u \frac{u^{(s-n)\sigma_1}\prod_{i=1}^{n} c_i^{\sigma_1}}{u^{n\sigma_2}\prod_{i=n+1}^{s} c_i^{\sigma_2}} u^{\tau}\mathrm{d}u \\
&\quad + \int_{c_s}^{+\infty} \ln u \frac{\prod_{i=1}^{s} c_i^{\sigma_1}}{u^{s\sigma_2}} u^{\tau}\mathrm{d}u \\
&= \frac{\prod_{i=1}^{s_0} c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s} c_i^{\sigma_2}} \int_1^{c_{s_0+1}} u^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0}\ln u\mathrm{d}u \\
&\quad + \sum_{n=s_0+1}^{s-1} \frac{\prod_{i=1}^{n} c_i^{\sigma_1}}{\prod_{i=n+1}^{s} c_i^{\sigma_2}} \int_{c_n}^{c_{n+1}} u^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n}\ln u\mathrm{d}u \\
&\quad + \prod_{i=1}^{s} c_i^{\sigma_1} \int_{c_s}^{+\infty} u^{\tau-s\sigma_2}\ln u\mathrm{d}u \\
&= \frac{\prod_{i=1}^{s_0} c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}\Big[c_{s_0+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}\ln c_{s_0+1} \\
&\quad -\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}\left(c_{s_0+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}-1\right)\Big] \\
&\quad + \sum_{n=s_0+1}^{s-1} \frac{\prod_{i=1}^{n} c_i^{\sigma_1}}{\prod_{i=n+1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \\
&\quad \times\Big[c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\ln c_{n+1} - c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\ln c_n \\
&\quad -\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}(c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} - c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1})\Big] \\
&\quad -\prod_{i=1}^{s} c_i^{\sigma_1} \frac{1}{\tau-s\sigma_2+1} c_s^{\tau-s\sigma_2+1}\left(\ln c_s - \frac{1}{\tau-s\sigma_2+1}\right).
\end{aligned}$$

In summary, it is concluded that

$$\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}(|\lambda_2|W_1(\beta,q)) = \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} W_1(\beta,q)$$

$$
\begin{aligned}
= & -\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} \frac{1}{\prod_{i=1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1+1} c_1^{\tau+s\sigma_1+1}\left(\ln c_1-\frac{1}{\tau+s\sigma_1+1}\right) \\
& -\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} \prod_{i=1}^{s} c_i^{\sigma_1} \frac{1}{\tau-s\sigma_2+1} c_s^{\tau-s\sigma_2+1}\left(\ln c_s-\frac{1}{\tau-s\sigma_2+1}\right) \\
& -\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} \sum_{n=1}^{s_0+1} \frac{\prod_{i=1}^{n} c_i^{\sigma_1}}{\prod_{i=1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \\
& \times\Big[c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \ln c_{n+1}-c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \ln c_n \\
& -\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\left(c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}-c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\right)\Big] \\
& +\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} \sum_{n=s_0+1}^{s-1} \frac{\prod_{i=1}^{n} c_i^{\sigma_1}}{\prod_{i=n+1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \\
& \times\Big[c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \ln c_{n+1}-c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1} \ln c_n \\
& -\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\left(c_{n+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}-c_n^{\tau+s\sigma_1-(\sigma_1+\sigma_2)n+1}\right)\Big] \\
& +\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} \frac{\prod_{i=1}^{s_0} c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s} c_i^{\sigma_2}} \frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1} \\
& \times\Big[c_{s_0}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1} \ln c_{s_0}+c_{s_0+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1} \ln c_{s_0+1} \\
& -\frac{1}{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}\Big(c_{s_0}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1} \\
& +c_{s_0+1}^{\tau+s\sigma_1-(\sigma_1+\sigma_2)s_0+1}-2\Big)\Big],
\end{aligned} \tag{6}
$$

and $W_1(\beta,q) < +\infty$. Similarly, from $s\lambda_1\sigma_1 > \frac{\alpha}{p}-\frac{1}{q}, s\lambda_1\sigma_2 > \frac{1}{q}-\frac{\beta}{p}$, it can also be calculated that $W_2(\alpha,p) < +\infty$.

Substituting $\tau = -\frac{1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)-1$ into (6) yields

$$\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}(|\lambda_2|W_1(\beta,q)) = M_0.$$

Theorem 1 holds according to Lemma 1. □

Remark 1. *Replacing* $\lambda_1,\lambda_2,\sigma_1,\sigma_2,\alpha$ *and* β *in Theorem 1 with* $1,1,\frac{\alpha}{s},\frac{\lambda+\alpha}{s}, p(1-\lambda_1)-1$ *and* $q(1-\sigma_1)-1$, *respectively, yields the conclusion in [25]. Therefore, Theorem 1 is a generalization of the conclusion in [25], and its proof process is more concise, with clearer relationships between parameters.*

Theorem 2. *Supposing* $\frac{1}{p}+\frac{1}{q}=1\ (p>1), \alpha,\beta\in\mathbb{R},\ \lambda_1\lambda_2>0, s,s_0\in\mathbb{N}_+, 1\le s_0\le s$, $0=c_0<c_1\ \le\cdots\le c_{s_0}\le 1\le c_{s_0+1}\le\cdots\le c_s, s\lambda_1\sigma_1>\frac{\alpha}{q}-\frac{1}{p}, s\lambda_1\sigma_2<\frac{1}{p}-\frac{\alpha}{q}, s\lambda_2\sigma_1>\frac{\alpha}{p}-\frac{1}{q}, s\lambda_2\sigma_2<\frac{1}{q}-\frac{\alpha}{p}$, *and*

$$K_2(x,y) = \left|\ln\left(x^{\lambda_1}/y^{\lambda_2}\right)\right| \prod_{k=1}^{s} \frac{\left(\min\{x^{\lambda_1}/y^{\lambda_2},c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}/y^{\lambda_2},c_k\}\right)^{\sigma_2}},$$

$$\overline{W}_1(\beta,q) = \int_0^{+\infty} K_2(1,t)t^{-\frac{\beta+1}{q}}\mathrm{d}t < +\infty, \overline{W}_2(\alpha,p) = \int_0^{+\infty} K_2(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t < +\infty.$$

(i) If and only if $\frac{\alpha}{\lambda_1 p}+\frac{\beta}{\lambda_2 q}=\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}$, *there exists a constant* $\overline{M}_1>0$ *such that*

$$\int_0^{+\infty}\int_0^{+\infty} K_2(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y \le \overline{M}_1||f||_{p,\alpha}||g||_{q,\beta}, \tag{7}$$

where $f \in L_p^{\alpha}(0,+\infty), g \in L_q^{\beta}(0,+\infty)$.

(ii) If $\frac{\alpha}{\lambda_1 p} + \frac{\beta}{\lambda_2 q} = \frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p}$, *then the best possible constant factor of (7) is*

$$\begin{aligned}
\overline{M}_0(\alpha,p) &= \left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}\int_0^{+\infty} K_2(t,1)t^{-\frac{a+1}{p}}\mathrm{d}t \\
&= -\left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}\frac{1}{\prod_{i=1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1}c_1^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1} \\
&\quad\times\left(\ln c_1 + \frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1}\right) \\
&\quad-\left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}\prod_{i=1}^{s}c_i^{\sigma_1}\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)-s\sigma_2}c_s^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)-s\sigma_2} \\
&\quad\times\left(\ln c_s + \frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)-s\sigma_2}\right) \\
&\quad-\left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}\sum_{n=1}^{s_0-1}\frac{\prod_{i=1}^{s}c_i^{\sigma_1}}{\prod_{i=n+1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n} \\
&\quad\times\left[c_{n+1}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_{n+1} - c_n^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_n\right. \\
&\quad-\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}(c_{n+1}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n} \\
&\quad\left.-c_n^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n})\right] \\
&\quad+\left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}\sum_{n=s_0+1}^{s-1}\frac{\prod_{i=1}^{s}c_i^{\sigma_1}}{\prod_{i=n+1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n} \\
&\quad\times\left[c_{n+1}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_{n+1} - c_n^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}\ln c_n\right. \\
&\quad-\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n}(c_{n+1}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n} \\
&\quad\left.-c_n^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)n})\right] \\
&\quad+\left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}\frac{\prod_{i=1}^{s_0}c_i^{\sigma_1}}{\prod_{i=s_0+1}^{s}c_i^{\sigma_2}}\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0} \\
&\quad\times\left[c_{s_0}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}\ln c_{s_0} + c_{s_0+1}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}\ln c_{s_0+1}\right. \\
&\quad-\frac{1}{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}(c_{s_0}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0} \\
&\quad\left.+c_{s_0+1}^{\frac{1}{\lambda_1}\left(1-\frac{a+1}{p}\right)+s\sigma_1-(\sigma_1+\sigma_2)s_0}-2)\right].
\end{aligned}$$

Proof. Using Lemmas 2 and 3, a proof similar to that of Theorem 1 yields. □

4. Necessary and Sufficient Condition for Boundedness of Two Integral Operators and Operator Norms

Assume $K(x,y) \geq 0$ and consider the integral operator T with $K(x,y)$ as the kernel:

$$T(f)(y) = \int_0^{+\infty} K(x,y) f(x) \mathrm{d}x.$$

According to the basic theory of the Hilbert-type inequality, we can see that the Hilbert-type integral inequality (1) is equivalent to the inequality

$$\|T(f)\|_{p,\beta(1-p)} \leq M||f||_{p,\alpha}.$$

with operator T. Therefore, based on Theorems 1 and 2, the following equivalent theorem for operator T can be obtained.

Theorem 3. *Assuming $\frac{1}{p}+\frac{1}{q}=1\ (p>1), \alpha,\beta \in \mathbb{R},\ \lambda_1\lambda_2 > 0, s, s_0 \in \mathbb{N}_+, 1 \leq s_0 \leq s$, $0 = c_0 < c_1 \leq \cdots \leq c_{s_0} \leq 1 \leq c_{s_0+1} \leq \cdots \leq c_s, s\lambda_2\sigma_1 > \frac{\beta}{q} - \frac{1}{p}, s\lambda_2\sigma_2 < \frac{1}{p} - \frac{\beta}{q}, s\lambda_1\sigma_1 > \frac{\alpha}{p} - \frac{1}{q}, s\lambda_1\sigma_2 < \frac{1}{q} - \frac{\alpha}{p}$, and $W_1(\beta,q) < +\infty$, $W_2(\alpha,p) < +\infty$ and $M_0(\beta,q)$ are the same as Theorem 1, and the integral operator T_1 is*

$$\begin{aligned} T_1(f)(y) &= \int_0^{+\infty} K_1(x,y) f(x) \mathrm{d}x \\ &= \int_0^{+\infty} \left|\ln\left(x^{\lambda_1} y^{\lambda_2}\right)\right| \prod_{k=1}^{s} \frac{\left(\min\{x^{\lambda_1} y^{\lambda_2}, c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1} y^{\lambda_2}, c_k\}\right)^{\sigma_2}} f(x) \mathrm{d}x. \end{aligned}$$

(i) If and only if $\frac{\alpha}{\lambda_1 p} - \frac{\beta}{\lambda_2 q} = \frac{1}{\lambda_1 q} - \frac{1}{\lambda_2 p}$, T_1 is a bounded operator from $L_p^{\alpha}(0,+\infty)$ to $L_p^{\beta(1-p)}(0,+\infty)$.

(ii) When $\frac{\alpha}{\lambda_1 p} - \frac{\beta}{\lambda_2 q} = \frac{1}{\lambda_1 q} - \frac{1}{\lambda_2 p}$, the operator norm of $T_1 : L_p^{\alpha}(0,+\infty) \to L_p^{\beta(1-p)}(0,+\infty)$ is $||T_1|| = M_0(\beta,q)$.

Theorem 4. *Supposing $\frac{1}{p}+\frac{1}{q}=1\ (p>1), \alpha,\beta \in \mathbb{R},\ \lambda_1\lambda_2 > 0, s, s_0 \in \mathbb{N}_+, 1 \leq s_0 \leq s, 0 = c_0 < c_1 \leq \cdots \leq c_{s_0} \leq 1 \leq c_{s_0+1} \leq \cdots \leq c_s, s\lambda_1\sigma_1 > \frac{\alpha}{q} - \frac{1}{p}, s\lambda_1\sigma_2 < \frac{1}{p} - \frac{\alpha}{q}, s\lambda_2\sigma_1 > \frac{\alpha}{p} - \frac{1}{q}, s\lambda_2\sigma_2 < \frac{1}{q} - \frac{\alpha}{p}$, $\overline{W}_1(\beta,q) < +\infty$, $\overline{W}_2(\alpha,p) < +\infty$ and $\overline{M}_0(\alpha,p)$ are the same as Theorem 2, and the integral operator T_2 is*

$$\begin{aligned} T_2(f)(y) &= \int_0^{+\infty} K_2(x,y) f(x) \mathrm{d}x \\ &= \int_0^{+\infty} \left|\ln\left(x^{\lambda_1}/y^{\lambda_2}\right)\right| \prod_{k=1}^{s} \frac{\left(\min\{x^{\lambda_1}/y^{\lambda_2}, c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}/y^{\lambda_2}, c_k\}\right)^{\sigma_2}} f(x) \mathrm{d}x. \end{aligned}$$

(i) If and only if $\frac{\alpha}{\lambda_1 p} + \frac{\beta}{\lambda_2 q} = \frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p}$, T_2 is a bounded operator from $L_p^{\alpha}(0,+\infty)$ to $L_p^{\beta(1-p)}(0,+\infty)$.

(ii) When $\frac{\alpha}{\lambda_1 p} + \frac{\beta}{\lambda_2 q} = \frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p}$, the operator norm of $T_2 : L_p^{\alpha}(0,+\infty) \to L_p^{\beta(1-p)}(0,+\infty)$ is $||T_2|| = \overline{M}_0(\alpha,p)$.

Taking $\alpha = \beta = 0$, the following results can be obtained from Theorems 3 and 4.

Corollary 1. *Let* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\lambda_1\lambda_2>0, s, s_0\in\mathbb{N}_+$, $1\le s_0\le s$, $0=c_0<c_1\le\cdots\le c_{s_0}\le 1\le c_{s_0+1}\le\cdots\le c_s, s\lambda_2\sigma_1>-\frac{1}{p}$, $s\lambda_2\sigma_2<\frac{1}{p}$, $s\lambda_1\sigma_1>-\frac{1}{q}, s\lambda_1\sigma_2<\frac{1}{q}$, $W_1(0,q)<+\infty$, $W_2(0,p)<+\infty$, $M_0(\beta,q)$ *and operator* T_1 *are the same as Theorem 3.*

(i) T_1 *is a bounded operator on* $L_p(0,+\infty)$ *if and only if* $\frac{1}{\lambda_1 p}=\frac{1}{\lambda_2 p}$.

(ii) When $\frac{1}{\lambda_1 q}=\frac{1}{\lambda_2 p}$, *the operator norm of* $T_1: L_p(0,+\infty)\to L_p(0,+\infty)$ *is* $||T_1||=M_0(0,q)$.

Corollary 2. *Supposing* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\lambda_1\lambda_2>0, s, s_0\in\mathbb{N}_+$, $1\le s_0\le s$, $0=c_0<c_1\le\cdots\le c_{s_0}\le 1\le c_{s_0+1}\le\cdots\le c_s$, $s\lambda_1\sigma_1>-\frac{1}{p}$, $s\lambda_1\sigma_2<\frac{1}{p}$, $s\lambda_2\sigma_1>-\frac{1}{q}, s\lambda_2\sigma_2<\frac{1}{q}$, $\overline{W}_1(0,q)<+\infty$, $\overline{W}_2(0,p)<+\infty$, $\overline{M}_0(0,p)$ *and operator* T_2 *are the same as Theorem 4.*

(i) If and only if $\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}=0$, T_2 *is a bounded operator on* $L_p(0,+\infty)$.

(ii) When $\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}=0$, *the operator norm of* $T_2: L_p(0,+\infty)\to L_p(0,+\infty)$ *is* $||T_1||=\overline{M}_0(0,p)$.

5. Conclusions

In this paper, we explore Hilbert-type inequalities and integral operators for the following two non-homogeneous kernels

$$K_1(x,y)=\left|\ln\left(x^{\lambda_1}y^{\lambda_2}\right)\right|\prod_{k=1}^{s}\frac{\left(\min\{x^{\lambda_1}y^{\lambda_2},c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}y^{\lambda_2},c_k\}\right)^{\sigma_2}},$$

$$K_2(x,y)=\left|\ln\left(x^{\lambda_1}/y^{\lambda_2}\right)\right|\prod_{k=1}^{s}\frac{\left(\min\{x^{\lambda_1}/y^{\lambda_2},c_k\}\right)^{\sigma_1}}{\left(\max\{x^{\lambda_1}/y^{\lambda_2},c_k\}\right)^{\sigma_2}}$$

in weighted Lebesgue space with $\lambda_1\lambda_2>0, c_k, \sigma_1, \sigma_2\in\mathbb{R}$, and provide the necessary and sufficient condition for optimal matching parameters (Theorems 1–4). By using the obtained conclusions, many different bounded integral operators with $K_1(x,y)$ and $K_2(x,y)$ as kernels can be constructed in weighted Lebesgue space, and the operator norm can be calculated. By taking some specific parameters, we can obtain bounded integral operators in ordinary Lebesgue space (Corollaries 1 and 2).

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